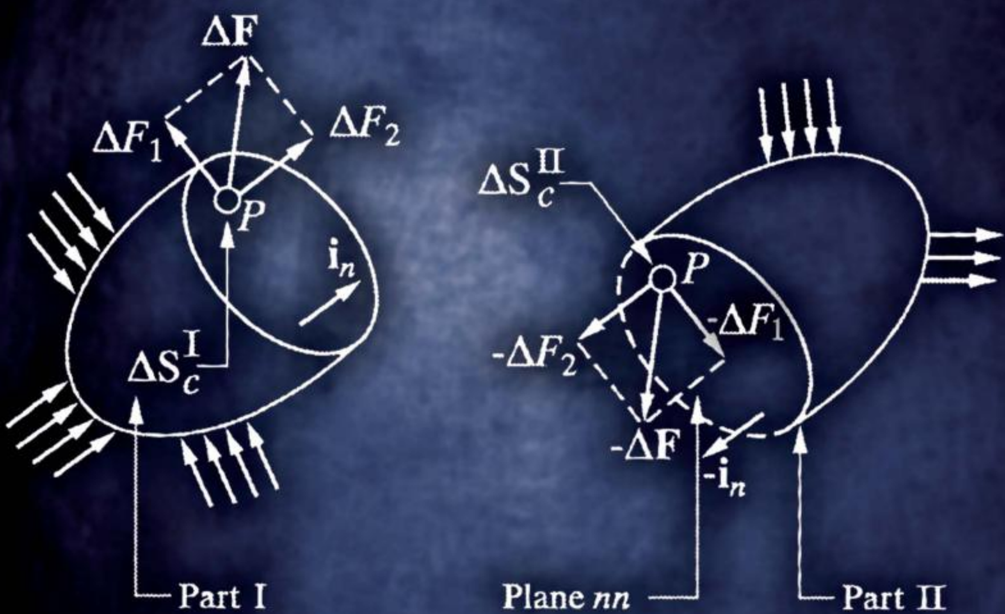
ADVANCED MECHANICS OF MATERIALS AND APPLIED ELASTICITY



$$\overset{n}{\mathbf{T}} = \lim_{\Delta S_c^{\mathrm{I}} \to 0} \frac{\Delta \mathbf{F}}{\Delta S_c^{\mathrm{I}}} \qquad \overset{-n}{\mathbf{T}} = \lim_{\Delta S_c^{\mathrm{II}} \to 0} -\frac{\Delta \mathbf{F}}{\Delta S_c^{\mathrm{II}}}$$

Anthony E. Armenàkas

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CRC Taylor & Francis
Taylor & Francis Group
Boca Raton London New York

A CRC title, part of the Taylor & Francis imprint, a member of the Taylor & Francis Group, the academic division of T&F Informa plc.

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20110713

International Standard Book Number-13: 978-1-4200-5777-5 (eBook - PDF)

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Foreword

This text is an outgrowth of the material used by the author for several decades in senior and graduate courses for students of mechanical, aerospace and civil engineering. It deals with the problem of computing the stress and displacement fields in solid bodies at two levels of approximation: the level of the linear theory of elasticity and the level of the theories of mechanics of materials. The linear theory of elasticity is based on very few assumptions and can be applied to bodies of any geometry. The theories of mechanics of materials are based on many assumptions in addition to those of the theory of elasticity and they can be applied only to bodies of certain geometries (beams, bars, shafts, frames, plates shells and thin-walled, tubular members). In this text the formulas of the theories of mechanics of materials are derived in a way that the assumptions on which they are based can be clearly understood. Moreover, wherever possible the results obtained on the basis of the theories of mechanics of materials are compared with those obtained on the basis of the theory of elasticity.

In the past, the use of the linear theory of elasticity was limited by the fact that only a few problems could be solved using the available classical methods. Thus, approximate theories like the theories of mechanics of materials were formulated for which exact solutions could be found. With the advent of the electronic computer, many problems involving bodies whose geometry does not justify the use of the theories of mechanics of materials, are formulated on the basis of the linear theory of elasticity and solved approximately with the aid of a computer. Thus, a mechanical, civil or aerospace engineer who works in the area of stress analysis and design often uses software based on the linear theory of elasticity. It is important therefore that master's level students of mechanical, aerospace and civil engineering who specialize in the area of stress analysis and design, should acquire some knowledge of applied elasticity.

The book includes 18 chapters and 7 appendices. In the first chapter a brief review of vector analysis is presented followed by a very elementary, but concise introduction to the algebra of symmetric tensors of the second rank. In the theories of mechanics of materials and elasticity one deals with quantities such as stress, strain and moments and product of inertia which are symmetric tensors of the second rank. It is desirable therefore that the student learns at the very beginning the transformation properties of such quantities as well as how to determine their stationary values.

The boundary value problems for computing the displacement and stress fields in solid bodies on the basis of the linear theory of elasticity, are formulated in Chapter 5 and applied to simple examples in Chapters 5, 6 (torsion problem) and 7 (plain stress and plain strain problems). The boundary value problems for computing the displacement and stress fields on the basis of the theory of mechanics of materials are presented in Chapters 8 and 9 for bodies made of prismatic line members, in Chapter 10 for nonprismatic line members, in Chapter 11 for curved line members, in Chapter 12 for tubular members and in Chapters 17 for plates. Part of the material presented in Chapters 8 and 9 is available in elementary texts of strength of materials. It is included in this text for completeness of our presentation and for those who need a review of this material at a slightly more advanced level than that of the elementary texts.

In this text, the boundary value problems for computing the displacement and stress fields in solid bodies are formulated using both their differential and integral forms (see Chapters 13 and 14). The latter include the principle of virtual work, Castigliano's second theorem, the theorem of minimum total potential energy, the weighted residual equation and the modified weighted residual equation. The last three are suitable for obtaining numerical solutions of boundary value problems with the aid of a computer, using the finite element method presented in Chapter 15.

With the exception of Chapter 16, where an introduction to plastic analysis of structures is presented, throughout this text, we limit our attention to bodies made from isotropic linearly elastic materials. Moreover, with the exception of Chapter 18, where an introduction to elastic instability of structures is presented, we assume that the magnitude of the deformation of each material particle of the bodies, which we are considering, is such that the change of its geometry can be approximated by its components of the strain tensor which are related to its components of displacement by linear relations. This assumption linearizes the boundary value problems involving bodies made from linearly elastic materials, that is, renders the effect linearly related to the cause and permits superposition of the results.

The author wishes to thank Dr. Nikitas Skliros for typing and laying out the final version of the manuscript and Mr. Nassos Papoutsis for drawing and labeling the figures. Moreover, the author wishes to thank Ms. Cleo Avrithy for checking the solution presented in the solution manual of the problems included at the ends of Chapters 1 to 10 and Ms. Isabella Vassilopoulou for checking the solution presented in the solution manual of the problems included at the ends of Chapters 11 and 13 to 17.

To the memory of my father

Emmanuel Anthony Armenàkas

For his sterling character, his
judgment and his dependability

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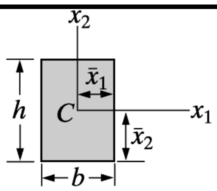
Properties of Plane Areas

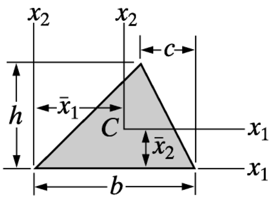
Notation:

A	=	area
$\bar{x}_1$, $\bar{x}_2$	=	coordinates of the centroid C
I_{11}, I_{22}	=	moments of inertia with respect to the x_1 and x_2 axes, respectively
I_{12}	=	product of inertia with respect to the x_1 and x_2 axes, respectively
I_1, I_2	=	moments of inertia with respect to the principal $\bar{x}_1$ and $\bar{x}_2$ axes



$$A = bh \qquad \bar{x}_1 = \frac{b}{2} \qquad \bar{x}_2 = \frac{h}{2}$$

$$I_1 = \frac{bh^3}{12} \qquad I_2 = \frac{hb^3}{12}$$

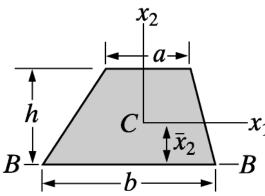


$$A = \frac{bh}{2} \qquad \bar{x}_1 = \frac{2b-c}{3} \qquad \bar{x}_2 = \frac{h}{3}$$

$$I_{11}^c = \frac{bh^3}{36} \qquad I_{22}^c = \frac{bh}{36}(b^2 - bc + c^2)$$

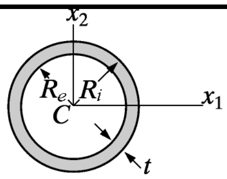
$$I_{12}^c = \frac{bh^2}{72}(b - 2c) \qquad I_{12} = \frac{bh^2}{24}(3b - 2c)$$

$$I_{11} = \frac{bh^3}{12} \qquad I_{22} = \frac{bh}{12}(3b^2 - 3bc + c^2)$$



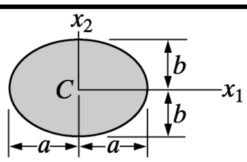
$$A = \frac{h(a + b)}{2} \qquad \bar{x}_2 = \frac{h(2a + b)}{3(a + b)}$$

$$I_{11} = \frac{h^3(a^2 + 4ab + b^2)}{36(a + b)} \qquad I_{BB} = \frac{h^3(3a + b)}{12}$$

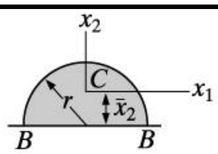


$$A = \pi(R_e^2 - R_i^2) \qquad I_p = \frac{\pi}{2}(R_e^4 - R_i^4)$$

$$I_1 = I_2 = \frac{\pi}{4} = \frac{\pi}{4}(R_e^4 - R_i^4)$$

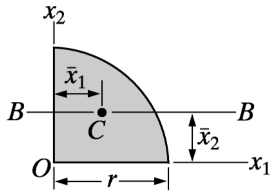


$$A = \pi ab \qquad I_1 = \frac{\pi ab^3}{4} \qquad I_2 = \frac{\pi ba^3}{4}$$



$$A = \frac{\pi r^2}{2} \qquad \bar{x}_2 = \frac{4r}{3\pi} \qquad I_{BB} = \frac{\pi r^4}{8}$$

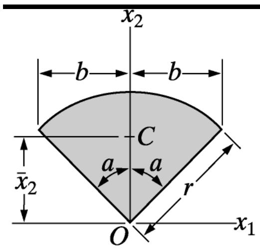
$$I_1 = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4 \qquad I_2 = \frac{\pi r^4}{8}$$



$$A = \frac{\pi r^2}{4} \qquad \bar{x}_1 = \bar{x}_2 = \frac{4r}{3\pi}$$

$$I_{11} = I_{22} = \frac{\pi r^4}{16} \qquad I_{12} = \frac{r^4}{8}$$

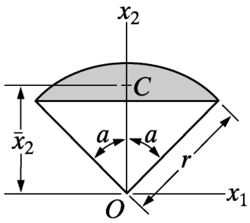
$$I_{BB} = \frac{(9\pi^2 - 64)r^4}{144\pi} \approx 0.05488 r^4$$



$$\alpha = \text{angle in radians} \leq \frac{\pi}{2}$$

$$A = \alpha r^2 \qquad b = r\sin\alpha \qquad \bar{x}_2 = \frac{2r\sin\alpha}{3\alpha}$$

$$I_1 = \frac{r^4}{4}(\alpha + \sin\alpha\,\cos\alpha) \qquad I_2 = \frac{r^4}{4}(\alpha - \sin\alpha\,\cos\alpha)$$

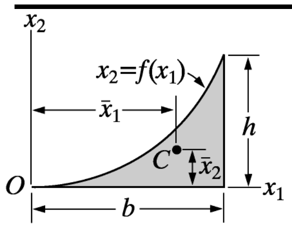


$$\alpha = \text{angle in radians} \leq \frac{\pi}{2}$$

$$A = r^2(\alpha - \sin\alpha\,\cos\alpha) \qquad \bar{x}_2 = \frac{2r}{3}\left(\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}\right)$$

$$I_1 = \frac{r^4}{4}(\alpha - \sin\alpha\,\cos\alpha + 2\sin^3\alpha\,\cos\alpha)$$

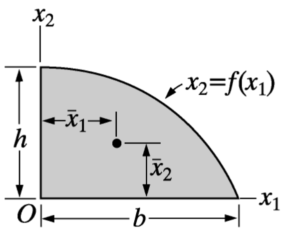
$$I_2 = \frac{r^4}{12}(3\alpha - 3\sin\alpha\,\cos\alpha - 2\sin^3\alpha\,\cos\alpha)$$



$$x_2 = f(x_1) = \frac{hx^n}{b^n} \qquad n > 0$$

$$A = \frac{bh}{n+1} \qquad \bar{x}_1 = \frac{b(n+1)}{n+2} \qquad \bar{x}_2 = \frac{h(n+1)}{2(2n+1)}$$

$$I_{11} = \frac{bh^3}{3(3n+1)} \qquad I_{22} = \frac{hb^3}{n+3} \qquad I_{12} = \frac{b^2h^2}{4(n+1)}$$



$$x_2 = f(x_1) = h\left(1 - \frac{x_1^n}{b^n}\right) \qquad n > 0 \qquad A = bh\left(\frac{n}{n+1}\right)$$

$$\bar{x}_1 = \frac{b(n+1)}{2(n+2)} \qquad \bar{x}_2 = \frac{hn}{2n+1} \qquad I_{12} = \frac{b^2h^2n^2}{4(n+1)(n+2)}$$

$$I_{11} = \frac{2bh^3n^3}{(n+1)(2n+1)(3n+1)} \qquad I_{22} = \frac{hb^3n}{3(n+3)}$$

Chapter 1

Cartesian Tensors

1.1 Vectors

In order to specify completely certain physical quantities, such as temperature, energy and mass, it is necessary to give only a real number. These quantities are referred to as *scalars*. In order to specify completely certain other physical quantities, such as force, moment, velocity and acceleration, it is necessary to give both their magnitude (a non-negative number), their direction and their sense. These quantities are referred to as *vectors*. They may be represented in a three-dimensional, Euclidian space by directed line segments (arrows) whose length is proportional to the magnitude of the vectors and whose direction and sense are those of the vectors.

Two vectors are equal if they have the same magnitude, direction and sense.

One way of denoting vector quantities is by boldface Latin letters, i.e., **a**, **b**, **c**, whereas their magnitude is represented as $|\mathbf{a}|, |\mathbf{b}|$ and $|\mathbf{c}|$, respectively.

The sum of two vectors **a** and **b**, as, for example, of two forces or of two velocities, is a vector **c**, which is specified (as shown in Fig. 1.1) by the diagonal of the parallelogram having as adjacent sides the vectors **a** and **b**. This rule of addition of two vectors is referred to as the *parallelogram rule*.

The *negative* of a vector **a** is a vector $-\mathbf{a}$ having the magnitude and direction of **a** and reverse sense.

Any vector whose magnitude is equal to unity is referred to as a *unit vector*. Every vector **a** may be expressed as the product of its magnitude and the unit vector $\mathbf{i}_n$ having the same direction and sense as the vector **a**.

$$\mathbf{a} = |\mathbf{a}|\ \mathbf{i}_n \tag{1.1}$$

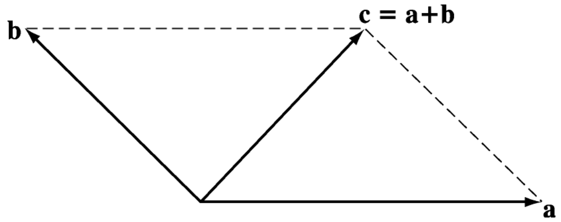


Figure 1.1 Addition of two vectors.

1.1.1 Components and Representation of a Vector

Every vector **a** can be expressed as a linear combination of three arbitrary noncoplanar linearly independent† vectors **b**, **c**, and **d**, referred to as *base vectors*. That is,

$$\mathbf{a} = m\mathbf{b} + p\mathbf{c} + q\mathbf{d} \tag{1.2}$$

where m, p and q are real numbers. The vectors $m\mathbf{b}$, $p\mathbf{c}$ and $q\mathbf{d}$ are referred to as the *components* of the vector **a** with respect to the base vectors **b**, **c** and **d**. It can be shown that the components of a vector with respect to a set of base vectors **b**, **c** and **d** are unique. That is, for any vector **a** there exists only one set of real numbers m, p and q satisfying relation (1.2). *As base vectors, we choose the three orthogonal unit vectors* $\mathbf{i}_1$, $\mathbf{i}_2$ *and* $\mathbf{i}_3$ *which lie along the positive directions of the set of right-handed*†† *rectangular system of axes (cartesian system of axes)* x_1, x_2 *and* x_3, *respectively* (see Fig. 1.2). Thus, we may represent a vector **a** as a linear combination of the three right-handed orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ as

$$\mathbf{a} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3 = \sum_{i=1}^{3} a_i\mathbf{i}_i \tag{1.3}$$

The vectors $a_1\mathbf{i}_1$, $a_2\mathbf{i}_2$ and $a_3\mathbf{i}_3$ (see Fig. 1.2) are the cartesian components of the vector **a**. However, in the sequel we will refer to the quantities a_i (i = 1, 2, 3) as the *cartesian components* of the vector **a**. It is apparent that a vector may be specified by its three

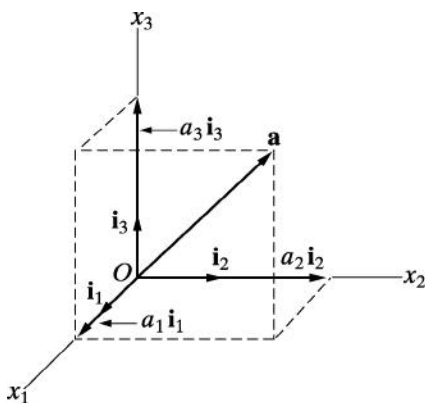


Figure 1.2 Components of a vector.

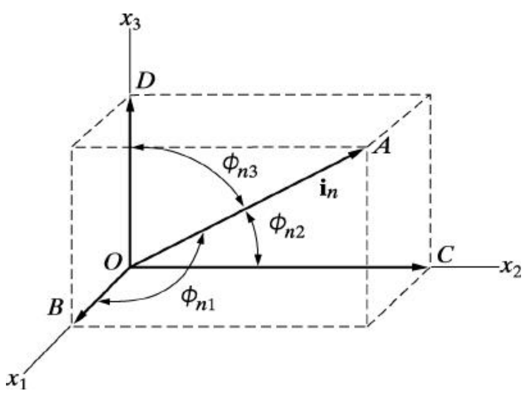


Figure 1.3 Direction cosines of a unit vector.

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† In general a set of n vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$ is said to be *linearly independent* if the relation $\Sigma \mathbf{a}_i m_i = 0$ is satisfied only when $m_i = 0$ ($i = 1,2, \ldots, n$). In the contrary case, the set of vectors is said to be *linearly dependent*.

†† We say that a set of axes x_1, x_2, x_3 is right-handed, when a right hand screw placed parallel to the x_3 axis moves in the direction of increasing x_3 when turned from the x_1 to the x_2 axis.

cartesian components. Thus, if each one of the cartesian components of a vector with respect to a rectangular system of axes is equal to the corresponding component of another vector referred to the same rectangular system of axes, the two vectors are equal; that is, their components referred to any rectangular system of axes are equal.

Referring to Fig. 1.2, from geometric considerations, it can be seen that the magnitude of the vector **a** is given by

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{1.4}$$

Consider a unit vector $\mathbf{i}_n$

$$\mathbf{i}_n = \lambda_{n1}\mathbf{i}_1 + \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3 = \sum_{i=1}^{3} \lambda_{ni}\mathbf{i}_i \tag{1.5}$$

Referring to Fig. 1.3, we may conclude that

$$\begin{aligned} \lambda_{n1} &= \cos \angle BOA = \cos \phi_{n1} \\ \lambda_{n2} &= \cos \angle COA = \cos \phi_{n2} \\ \lambda_{n3} &= \cos \angle DOA = \cos \phi_{n3} \end{aligned} \tag{1.6}$$

where $\cos \phi_{n1}$, $\cos \phi_{n2}$ and $\cos \phi_{n3}$ are called *the direction cosines of the unit vector* $\mathbf{i}_n$ or of line $\overline{OA}$. From geometric considerations we find that the direction cosines of a vector are related by the following relation:

$$\lambda_{n1}^2 + \lambda_{n2}^2 + \lambda_{n3}^2 - 1 = 0 \tag{1.7}$$

The sum of two or more vectors is the vector whose cartesian components are the sum of the corresponding cartesian components of the added vectors. For instance, if $\mathbf{a} = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3$ and $\mathbf{b} = b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3$, then their sum is a vector $\mathbf{c} = c_1\mathbf{i}_1 + c_2\mathbf{i}_2 + c_3\mathbf{i}_3$ whose components are given as

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} + \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{Bmatrix} \tag{1.8}$$

A vector may be represented as follows:

1. By the symbolic representation employed heretofore, i.e., **a**, **b**, **A**, **B**, which does not require a choice of a coordinate system.
2. By its three cartesian components with respect to a set of orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, using one of the following notations:

(a) Indicial notation, as,

$$\mathbf{a} \rightarrow a_j \qquad (j = 1, 2, 3) \tag{1.9a}$$

(b) Matrix notation, as,

$$\mathbf{a} \rightarrow \{a\} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{1.9b}$$

1.1.2 Scalar or Dot Product of Two Vectors

Let us consider two vectors **a** and **b** and let us denote the angle between them by θ ($0 \leq \theta \leq \pi$). The scalar or dot product of the two vectors is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is equal to a scalar c whose magnitude is given by the following relation:

$$c = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta = \mathbf{b} \cdot \mathbf{a} \qquad 0 \leq \theta \leq \pi \tag{1.10}$$

Notice that if $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$, then the vectors **a** and **b** are mutually perpendicular. On the basis of definition (1.10), it is apparent that the unit vectors $\mathbf{i}_i$ ($i = 1, 2, 3$) satisfy the following relations:

$$\begin{aligned} \mathbf{i}_1 \cdot \mathbf{i}_1 = \mathbf{i}_2 \cdot \mathbf{i}_2 = \mathbf{i}_3 \cdot \mathbf{i}_3 = 1 \\ \mathbf{i}_1 \cdot \mathbf{i}_2 = \mathbf{i}_2 \cdot \mathbf{i}_3 = \mathbf{i}_1 \cdot \mathbf{i}_3 = 0 \end{aligned} \tag{1.11}$$

Moreover, it can be shown that

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 > 0 \qquad \textit{if } \mathbf{a} \neq 0 \tag{1.12a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \qquad \text{(distributive law)} \tag{1.12b}$$

$$m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b}) \tag{1.12c}$$

Using equations (1.11) and (1.12b), the scalar product of two vectors may be written in terms of their cartesian components as

$$\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3) \cdot (b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3) = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{i=1}^{3} a_i b_i \tag{1.13}$$

Notice that the scalar product of two vectors can be found by matrix multiplication. That is,

$$\mathbf{a} \cdot \mathbf{b} = [a_1\ a_2\ a_3] \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \{a\}^T\{b\} = a_1b_1 + a_2b_2 + a_3b_3 \tag{1.14}$$

1.1.3 Vector or Cross Product of Two Vectors

Consider two vectors **a** and **b** and denote by θ ($0 \leq \theta \leq \pi$) the angle between them

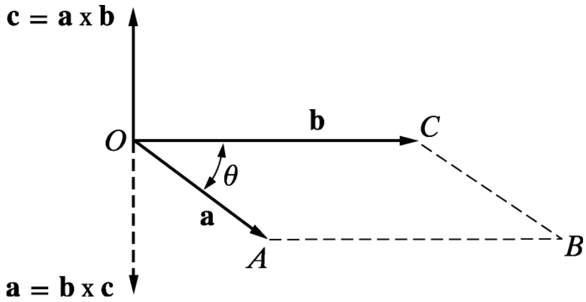


Figure 1.4 Vector product.

(see Fig. 1.4). The *vector product* or the *cross product* of the two vectors **a** and **b** is a vector **c** (see Fig. 1.4) defined by the following relation:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ (\sin\theta)\ \mathbf{i}_n \qquad 0 \le \theta \le \pi \tag{1.15}$$

where $\mathbf{i}_n$ is the unit vector normal to the plane of the vectors **a**, **b** in the sense in which the right-hand screw will move, if it is turned from **a** to **b**.

Referring to Fig. 1.4, the magnitude of the vector **c** is equal to the area of the parallelogram *OABCO*. On the basis of definition (1.15), it is apparent that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \tag{1.16}$$

Moreover, the right-handed orthogonal system of unit vectors $\mathbf{i}_i$ ($i = 1, 2, 3$) (see Fig. 1.2) satisfy the following relations:

$$\begin{array}{lll} \mathbf{i}_1 \times \mathbf{i}_1 = 0 & \mathbf{i}_2 \times \mathbf{i}_2 = 0 & \mathbf{i}_3 \times \mathbf{i}_3 = 0 \\ \mathbf{i}_1 \times \mathbf{i}_2 = \mathbf{i}_3 & \mathbf{i}_2 \times \mathbf{i}_3 = \mathbf{i}_1 & \mathbf{i}_3 \times \mathbf{i}_1 = \mathbf{i}_2 \\ \mathbf{i}_2 \times \mathbf{i}_1 = -\mathbf{i}_3 & \mathbf{i}_3 \times \mathbf{i}_2 = -\mathbf{i}_1 & \mathbf{i}_1 \times \mathbf{i}_3 = -\mathbf{i}_2 \end{array} \tag{1.17}$$

Furthermore, it can be shown that

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \tag{1.18a}$$

$$(m\ \mathbf{a}) \times \mathbf{b} = m\ (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (m\ \mathbf{b}) \tag{1.18b}$$

where m is a real number.

By direct multiplication, using relations (1.17) and (1.18a), the vector product of two vectors **a** and **b** may be expressed as follows in terms of their components, referred to the same right-handed rectangular system of axes

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i}_1 + (a_3 b_1 - a_1 b_3)\mathbf{i}_2 + (a_1 b_2 - a_2 b_1)\mathbf{i}_3 \tag{1.19}$$

where a_i and b_i ($i = 1, 2, 3$) are the component of the vectors **a** and **b** in the directions of the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$.

Relation (1.19) may be written in the following easy to remember determinant form:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.20}$$

When the components of two vectors **a** and **b** are referred to a left-handed rectangular system of axes, a minus sign must be prefixed on the right-hand side of relations (1.17), (1.19) and (1.20). In order to eliminate this difficulty, in this text *we use only right-handed rectangular systems of axes unless stated otherwise*.

1.1.4 Rotation of a Rectangular System of Axes — Transformation Matrix

Let us consider two right-handed rectangular systems of axes x'_i (i = 1, 2, 3) and x_j (j = 1, 2, 3) having the same origin at an arbitrary point O. Referring to Fig. 1.5, the direction cosines of the system of axes x'_i with respect to the system of axes x_j are defined as

$$\begin{aligned} \lambda_{11} &= \cos\phi_{11} & \lambda_{21} &= \cos\phi_{21} & \lambda_{31} &= \cos\phi_{31} \\ \lambda_{12} &= \cos\phi_{12} & \lambda_{22} &= \cos\phi_{22} & \lambda_{32} &= \cos\phi_{32} \\ \lambda_{13} &= \cos\phi_{13} & \lambda_{23} &= \cos\phi_{23} & \lambda_{33} &= \cos\phi_{33} \end{aligned}$$

or in indicial notation.

$$\lambda_{ij} = \cos\phi_{ij} \qquad (i, j = 1, 2, 3) \tag{1.21}$$

Denoting by $\mathbf{i}'_i$(i = 1, 2, 3) and $\mathbf{i}_j$(j = 1, 2, 3), the unit vectors acting along the x'_i and x_j axes, respectively, and referring to Fig. 1.5, we have

$$\begin{aligned} \mathbf{i}'_1 &= \lambda_{11}\mathbf{i}_1 + \lambda_{12}\mathbf{i}_2 + \lambda_{13}\mathbf{i}_3 \\ \mathbf{i}'_2 &= \lambda_{21}\mathbf{i}_1 + \lambda_{22}\mathbf{i}_2 + \lambda_{23}\mathbf{i}_3 \\ \mathbf{i}'_3 &= \lambda_{31}\mathbf{i}_1 + \lambda_{32}\mathbf{i}_2 + \lambda_{33}\mathbf{i}_3 \end{aligned}$$

or

$$\mathbf{i}'_i = \sum_{j=1}^{3} \lambda_{ij}\mathbf{i}_j \qquad (i = 1, 2, 3) \tag{1.22}$$

Similarly, we obtain

$$\mathbf{i}_j = \sum_{i=1}^{3} \lambda_{ij}\mathbf{i}'_i \qquad (j = 1, 2, 3) \tag{1.23}$$

The nine direction cosines may be written in a matrix form as

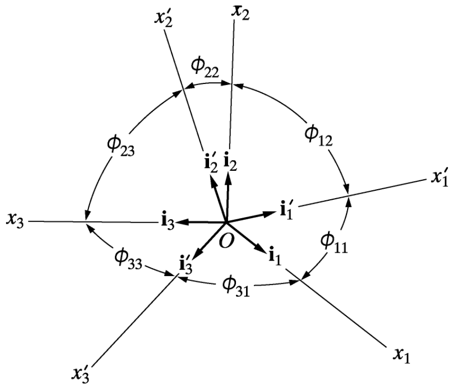


Figure 1.5 Rotation of a right-handed rectangular system of axes.

$$[\,\Lambda_s\,] = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \tag{1.24a}$$

or

$$[\,\Lambda_s\,]^{\mathrm{T}} = \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{bmatrix} \tag{1.24b}$$

The 3 x 3 matrix $[\Lambda_s]$ is referred to as the *transformation matrix* for the system of axes x_i' (i = 1, 2, 3) with respect to the system of axes x_j(j = 1, 2, 3). The nine direction cosines λ_{ij} (i,j = 1, 2, 3) specify the rectangular system of axes x_i' relative to the rectangular system of axes x_j but are not independent. They satisfy relations resulting from the orthogonality of the two systems of axes. These relations are:

$$\begin{aligned}
&\mathbf{i}_1\cdot\mathbf{i}_2 = 0 \rightarrow \sum_{i=1}^{3}\lambda_{i1}\lambda_{i2} = 0 \qquad &&\mathbf{i}_1\cdot\mathbf{i}_1 = 1 \rightarrow \sum_{i=1}^{3}\lambda_{i1}\lambda_{i1} = 1\\
&\mathbf{i}_2\cdot\mathbf{i}_3 = 0 \rightarrow \sum_{i=1}^{3}\lambda_{i2}\lambda_{i3} = 0 \qquad &&\mathbf{i}_2\cdot\mathbf{i}_2 = 1 \rightarrow \sum_{i=1}^{3}\lambda_{i2}\lambda_{i2} = 1\\
&\mathbf{i}_1\cdot\mathbf{i}_3 = 0 \rightarrow \sum_{i=1}^{3}\lambda_{i1}\lambda_{i3} = 0 \qquad &&\mathbf{i}_3\cdot\mathbf{i}_3 = 1 \rightarrow \sum_{i=1}^{3}\lambda_{i3}\lambda_{i3} = 1
\end{aligned} \tag{1.25a}$$

The above relations may be rewritten in the following form:

$$\begin{aligned}
&\mathbf{i}_1'\cdot\mathbf{i}_2' = 0 \rightarrow \sum_{i=1}^{3}\lambda_{1i}\lambda_{2i} = 0 \qquad &&\mathbf{i}_1'\cdot\mathbf{i}_1' = 1 \rightarrow \sum_{i=1}^{3}\lambda_{1i}\lambda_{1i} = 1\\
&\mathbf{i}_2'\cdot\mathbf{i}_3' = 0 \rightarrow \sum_{i=1}^{3}\lambda_{2i}\lambda_{3i} = 0 \qquad &&\mathbf{i}_2'\cdot\mathbf{i}_2' = 1 \rightarrow \sum_{i=1}^{3}\lambda_{2i}\lambda_{2i} = 1\\
&\mathbf{i}_1'\cdot\mathbf{i}_3' = 0 \rightarrow \sum_{i=1}^{3}\lambda_{1i}\lambda_{3\mathrm{i}} = 0 \qquad &&\mathbf{i}_3'\cdot\mathbf{i}_3' = 1 \rightarrow \sum_{i=1}^{3}\lambda_{3i}\lambda_{3i} = 1
\end{aligned} \tag{1.25b}$$

Referring to relations (1.24), relations (1.25) may be written in matrix form as

$$[\Lambda_s]^T [\Lambda_s] = [I] \tag{1.26a}$$

$$[\Lambda_s] [\Lambda_s]^T = [I] \tag{1.26b}$$

where $[I]$ is the 3 x 3 *unit matrix* defined by

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.27}$$

Relations (1.26) are referred to as *the conditions of orthog onality*. Linear transformations, such as (1.22) and (1.23) whose coefficients satisfy relations (1.26) are referred to as *orthogonal transformations*. The orthogonality conditions (1.26) imply that the transpose of the matrix $[\Lambda_s]$ is equal to its inverse $[\Lambda_s]^{-1}$. That is,

$$[\Lambda_s]^T = [\Lambda_s]^{-1} \tag{1.28}$$

1.1.5 Transformation of the Components of a Vector upon Ro tation of the Rectangular System of Axes to Which They Are Referred

Let us consider the vector **a** whose components relative to the rectangular systems of axes $x_i'(i = 1, 2, 3)$ and $x_j(j = 1, 2, 3)$ are a_i' and a_j, respectively. That is,

$$\mathbf{a} = \sum_{i=1}^{3} a_i' \mathbf{i}_i' = \sum_{j=1}^{3} a_j \mathbf{i}_j \tag{1.29}$$

Substituting relation (1.23) into (1.29), we obtain

$$\sum_{i=1}^{3} a_i' \mathbf{i}_i' = \sum_{i=1}^{3} \sum_{j=1}^{3} a_j \lambda_{ij} \mathbf{i}_i' \tag{1.30}$$

Consequently,

$$a_i' = \sum_{j=1}^{3} \lambda_{ij} a_j \tag{1.31}$$

Similarly, substituting relation (1.22) into (1.29), we get

$$a_j = \sum_{i=1}^{3} \lambda_{ij}\, a_i' \tag{1.32}$$

Relations (1.31) and (1.32) can be written in matrix form as

$$\begin{Bmatrix} a_1' \\ a_2' \\ a_3' \end{Bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{1.33}$$

or

$$\{ a' \} = [\Lambda_s] \{ a \}$$

and

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{bmatrix} \begin{Bmatrix} a_1' \\ a_2' \\ a_3' \end{Bmatrix} \tag{1.34}$$

or

$$\{ a \} = [\Lambda_s]^T \{ a' \}$$

1.1.6 Transformation of the Components of a Planar Vector upon Rotation of the Rectangular System of Axes to Which They Are Referred

In two-dimensional (planar) problems we encounter certain vectors such as forces and translations which act in the plane of the problem. In this section we establish the relations between the components of such vectors referred to two sets of two mutually perpendicular axes laying in the plane of the problem (see Fig. 1.6). Consider the vector **F** acting in the plane specified by the two mutually perpendicular axes x_1 and x_2 and denote its components with respect to these axes by F_1 and F_2, respectively. Moreover, consider another set of two mutually perpendicular axes x_1' and x_2' located in the plane x_1 x_2 and denote the components of vector **F** with respect to these axes by F_1' and F_2', respectively. That is,

$$\mathbf{F} = F_1\mathbf{i}_1 + F_2\mathbf{i}_2 = F_1'\mathbf{i}_1' + F_2'\mathbf{i}_2' \tag{1.35}$$

Referring to Fig. 1.6, we have

$$\begin{Bmatrix} F_1' \\ F_2' \end{Bmatrix} = \left[\Lambda_p \right] \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{1.36a}$$

and

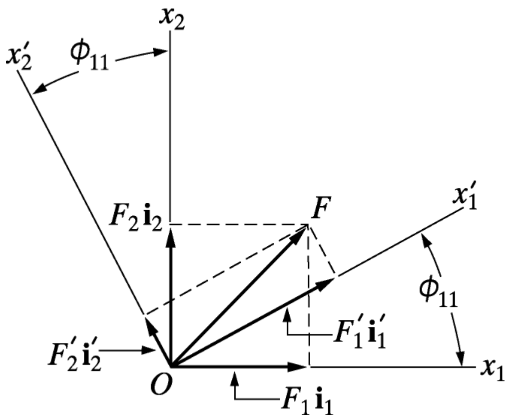


Figure 1.6 Components of a planar vector.

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = [\Lambda_P]^T \begin{Bmatrix} F_1' \\ F_2' \end{Bmatrix} \tag{1.36b}$$

where

$$[\,\Lambda_P\,] = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \cos\phi_{11} & \sin\phi_{11} \\ -\sin\phi_{11} & \cos\phi_{11} \end{bmatrix} \tag{1.37}$$

From relations (1.36a) and (1.36b) it can be seen that

$$[\,\Lambda_P\,]^{-1} = [\,\Lambda_P\,]^T \tag{1.38}$$

1.1.7 Definition of a Vector on the Basis of the Law of Transformation of Its Components

In Section 1.1, we have defined a vector as an entity possessing magnitude, direction and sense and obeying certain rules. It may be represented in the three-dimensional space by a directed line segment (arrow) whose direction is that of the vector. This definition does not associate any system of axes with a vector and, thus, it is immediately apparent that a vector has an existence independent of any system of axes in the same frame of reference. On the basis of this definition, it was established in Section 1.1.5 that the components of a vector transform, upon rotation of the rectangular system of axes to which they are referred, in accordance with the transformation relation (1.31) or (1.32). The reverse is also valid. That is, any entity defined with respect to any rectangular system of axes by an array of three numbers a_i ($i = 1, 2, 3$) satisfying relation (1.31) or (1.32) is a vector

$$\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{i}_i$$

Thus, a vector may be defined as an entity which possesses the following properties:

1. With respect to any rectangular system of axes, it is specified by an array of three numbers a_i ($i = 1, 2, 3$) — its three Cartesian components.
2. Its Cartesian components referred to any two rectangular systems of axes are related by the transformation relation (1.31) or (1.32).

1.2 Dyads

We define a *dyad* as the product **a b** of two vectors **a** and **b** which obeys the following rules:

$$\lambda(\mathbf{a}\ \mathbf{b}) = (\lambda\ \mathbf{a})\mathbf{b} = \mathbf{a}(\lambda\ \mathbf{b}) = (\mathbf{a}\ \mathbf{b})\lambda \tag{1.39a}$$

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{a}\ \mathbf{b} + \mathbf{a}\ \mathbf{c} \tag{1.39b}$$

$$(\mathbf{a} + \mathbf{b})\mathbf{c} = \mathbf{a}\ \mathbf{c} + \mathbf{b}\ \mathbf{c} \tag{1.39c}$$

$$(\mathbf{a} + \mathbf{b})\ (\mathbf{c} + \mathbf{d}) = \mathbf{a}\ \mathbf{c} + \mathbf{a}\ \mathbf{d} + \mathbf{b}\ \mathbf{c} + \mathbf{b}\ \mathbf{d} \tag{1.39d}$$

where λ is a real number. In general this product is not commutative. That is,

$$\mathbf{a}\ \mathbf{b} \neq \mathbf{b}\ \mathbf{a} \tag{1.40}$$

The following products are defined between a vector and a dyad, and between two dyads:

$$\mathbf{a} \cdot (\mathbf{b}\ \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\ \mathbf{c} \tag{1.41}$$

$$(\mathbf{b}\ \mathbf{c}) \cdot \mathbf{a} = \mathbf{b}\ (\mathbf{c} \cdot \mathbf{a}) \qquad \text{(vector)} \tag{1.42}$$

$$\mathbf{a} \times (\mathbf{b}\ \mathbf{c}) = (\mathbf{a} \times \mathbf{b})\ \mathbf{c} \tag{1.43}$$

$$(\mathbf{b}\ \mathbf{c}) \times \mathbf{a} = \mathbf{b}\ (\mathbf{c} \times \mathbf{a}) \qquad \text{(dyad)} \tag{1.44}$$

$$(\mathbf{a}\ \mathbf{b}) \cdot (\mathbf{c}\ \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\ (\mathbf{a}\ \mathbf{d}) \qquad \text{(dyad)} \tag{1.45}$$

On the basis of the above definition a dyad like a vector has an existence independent of any coordinate system. In what follows, we introduce a rectangular system of axes x_1, x_2, x_3 specified by the orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ in order to permit the use of well-known mathematical procedures. Thus, in terms of the components of the two vectors **a** and **b**, referred to the orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, using the rule (1.39d), a dyad **ab** may be written as

$$
\begin{aligned}
\mathbf{a}\ \mathbf{b} &= (a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3)\ (b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3) \\
&= a_1b_1\mathbf{i}_1\mathbf{i}_1 + a_1b_2\mathbf{i}_1\mathbf{i}_2 + a_1b_3\mathbf{i}_1\mathbf{i}_3 + a_2b_1\mathbf{i}_2\mathbf{i}_1 + a_2b_2\mathbf{i}_2\mathbf{i}_2 \\
&+ a_2b_3\mathbf{i}_2\mathbf{i}_3 + a_3b_1\mathbf{i}_3\mathbf{i}_1 + a_3b_2\mathbf{i}_3\mathbf{i}_2 + a_3b_3\mathbf{i}_3\mathbf{i}_3
\end{aligned}
$$

or

$$
\mathbf{a}\ \mathbf{b} = \sum_{i=1}^{3}\ \sum_{j=1}^{3} a_i b_j \mathbf{i}_i \mathbf{i}_j \tag{1.46}
$$

The sum of two dyads **a b** and **c d** is a dyad whose components are equal to the sum of the components of the two dyads

$$
\mathbf{a}\ \mathbf{b} + \mathbf{c}\ \mathbf{d} = \sum_{i=1}^{3}\ \sum_{j=1}^{3} a_i b_j \mathbf{i}_i \mathbf{i}_j + \sum_{i=1}^{3}\ \sum_{j=1}^{3} c_i d_j \mathbf{i}_i \mathbf{i}_j = \sum_{i=1}^{3}\ \sum_{j=1}^{3} (a_i b_j + c_i d_j) \mathbf{i}_i \mathbf{i}_j \tag{1.47}
$$

1.3 Definition and Rules of Operation of Tensors of the Second Rank

A tensor of the second rank, also known as dyadic, is a linear combination of a finite number of dyads. We denote tensors of the second rank by modified letter symbols as **A**, **B**, **e**, etc. For example,

$$
\mathbf{A} = \mathbf{a}\ \mathbf{b} + \mathbf{c}\ \mathbf{d} + \mathbf{e}\ \mathbf{f} \tag{1.48}
$$

Referring to relations (1.46) to (1.48), it is apparent that any tensor of the second rank can be written with respect to the orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ as

$$
\mathbf{A} = \sum_{i=1}^{3}\ \sum_{j=1}^{3} A_{ij} \mathbf{i}_i \mathbf{i}_j \tag{1.49}
$$

or

$$
\begin{aligned}
\mathbf{A} &= A_{11}\mathbf{i}_1\mathbf{i}_1 + A_{12}\mathbf{i}_1\mathbf{i}_2 + A_{13}\mathbf{i}_1\mathbf{i}_3 \\
&+ A_{21}\mathbf{i}_2\mathbf{i}_1 + A_{22}\mathbf{i}_2\mathbf{i}_2 + A_{23}\mathbf{i}_2\mathbf{i}_3 \\
&+ A_{31}\mathbf{i}_3\mathbf{i}_1 + A_{32}\mathbf{i}_3\mathbf{i}_2 + A_{33}\mathbf{i}_3\mathbf{i}_3
\end{aligned} \tag{1.50}
$$

The nine quantities A_{ij} $(i, j = 1, 2, 3)$ are referred to as the *cartesian components of the tensor of the second rank* with respect to the set of the orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ or with respect to the rectangular system of axes x_1, x_2, x_3 specified by these unit vectors. Any tensor of the second rank can be specified by giving its nine cartesian components with respect to a set of orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. The cartesian components A_{11}, A_{22},

A_{33} of a tensor of the second rank **A** are referred to as its *diagonal components*, while the cartesian components A_{12}, A_{13}, A_{21}, A_{23}, A_{31}, A_{32} of the tensor of the second rank **A** are referred to as its *non-diagonal components*. On the basis of the foregoing presentation, it is evident that the non-diagonal cartesian components of a tensor of the second rank **A** are associated with two orthogonal directions ($\mathbf{i}_i$ and $\mathbf{i}_j$ where $i \neq j$), whereas the diagonal cartesian components of a tensor of the second rank are associated with a single direction twice ($\mathbf{i}_i$ and $\mathbf{i}_j$ where $i = j$). Notice that the cartesian components of a vector **a** ($a_i = \mathbf{i}_i \cdot \mathbf{a}$) are associated solely with one direction. A vector is a tensor of the first rank.

The sum of two tensors of the second rank is a tensor of the second rank whose components referred to a rectangular system of axes are equal to the sum of the components of the two tensors referred to the same system of axes. That is,

$$\mathbf{D} = \mathbf{A} + \mathbf{B}$$

or

$$\sum_{i=1}^{3} \sum_{j=1}^{3} D_{ij} \mathbf{i}_i \mathbf{i}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} (A_{ij} + B_{ij}) \mathbf{i}_i \mathbf{i}_j$$

or

$$D_{ij} = A_{ij} + B_{ij} \qquad (i,j = 1,2,3) \tag{1.51}$$

Tensors of the second rank obey the following rules:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.52a}$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{D} = \mathbf{A} + (\mathbf{B} + \mathbf{D}) \tag{1.52b}$$

$$\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B} \tag{1.52c}$$

where λ and μ are real numbers. Moreover, the products of a tensor of the second rank and a vector obey the following rules:

$$(\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A} \tag{1.52d}$$

$$\mathbf{A} \cdot \mathbf{v} \neq \mathbf{v} \cdot \mathbf{A} \tag{1.53a}$$

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{v} = \mathbf{A} \cdot \mathbf{v} + \mathbf{B} \cdot \mathbf{v} \tag{1.53b}$$

$$\mathbf{A} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{A} \cdot \mathbf{a} + \mathbf{A} \cdot \mathbf{b} \tag{1.53c}$$

The dot product of a tensor of the second rank **A** and a vector **a** is a vector **c** whose components can be established from those of the tensor **A** and the vector **a** as follows:

$$
\begin{aligned}
\mathbf{c} = \mathbf{a}\cdot\mathbf{A} &= (a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3)\cdot(A_{11}\mathbf{i}_1\mathbf{i}_1 + A_{12}\mathbf{i}_1\mathbf{i}_2 + A_{13}\mathbf{i}_1\mathbf{i}_3 + A_{21}\mathbf{i}_2\mathbf{i}_1 \\
&\quad + A_{22}\mathbf{i}_2\mathbf{i}_2 + A_{23}\mathbf{i}_2\mathbf{i}_3 + A_{31}\mathbf{i}_3\mathbf{i}_1 + A_{32}\mathbf{i}_3\mathbf{i}_2 + A_{33}\mathbf{i}_3\mathbf{i}_3) \\
&= (a_1A_{11} + a_2A_{21} + a_3A_{31})\mathbf{i}_1 \\
&\quad + (a_1A_{12} + a_2A_{22} + a_3A_{32})\mathbf{i}_2 \\
&\quad + (a_1A_{13} + a_2A_{23} + a_3A_{33})\mathbf{i}_3
\end{aligned}
$$

or

$$
c_i = \sum_{j=1}^{3} a_j A_{ji} \qquad (i = 1,\, 2,\, 3) \tag{1.54}
$$

and

$$
\mathbf{b} = \mathbf{A}\cdot\mathbf{a} = \begin{pmatrix} A_{11}\mathbf{i}_1\mathbf{i}_1 + A_{12}\mathbf{i}_1\mathbf{i}_2 + A_{13}\mathbf{i}_1\mathbf{i}_3 \\ + A_{21}\mathbf{i}_2\mathbf{i}_1 + A_{22}\mathbf{i}_2\mathbf{i}_2 + A_{23}\mathbf{i}_2\mathbf{i}_3 \\ + A_{31}\mathbf{i}_3\mathbf{i}_1 + A_{32}\mathbf{i}_3\mathbf{i}_2 + A_{33}\mathbf{i}_3\mathbf{i}_3 \end{pmatrix} \cdot (a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3)
$$

$$
\begin{aligned}
&= (A_{11}a_1 + A_{12}a_2 + A_{13}a_3)\mathbf{i}_1 \\
&\quad + (A_{21}a_1 + A_{22}a_2 + A_{23}a_3)\mathbf{i}_2 \\
&\quad + (A_{31}a_1 + A_{32}a_2 + A_{33}a_3)\mathbf{i}_3
\end{aligned}
$$

or

$$
b_i = \sum_{j=1}^{3} A_{ij} a_j \qquad (i = 1,\, 2,\, 3) \tag{1.55}
$$

The dot products of a tensor of the second rank by a vector (1.54) and (1.55) may also be obtained by multiplication of the matrices of their cartesian components referred to the same rectangular system of axes. Thus,

$$
\{\, c \,\}^T = \{\, a \,\}^T[\, A \,] = \left\{ \begin{matrix} a_1A_{11} + a_2A_{21} + a_3A_{31} \\ a_1A_{12} + a_2A_{22} + a_3A_{32} \\ a_1A_{13} + a_2A_{23} + a_3A_{33} \end{matrix} \right\}^T \tag{1.56}
$$

and

$$
\{\, b \,\} = [\, A \,]\{\, a \,\} = \left\{ \begin{matrix} A_{11}a_1 + A_{12}a_2 + A_{13}a_3 \\ A_{21}a_1 + A_{22}a_2 + A_{23}a_3 \\ A_{31}a_1 + A_{32}a_2 + A_{33}A_3 \end{matrix} \right\} \tag{1.57}
$$

where

$$
[\, A \,] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.58a}
$$

$$\{\, c \,\} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} \qquad \{\, b \,\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{1.58b}$$

From relation (1.50), we may deduce that, in general, the cartesian components A_{ij} of a tensor of the second rank are given by

$$A_{ij} = \mathbf{i}_i \cdot \mathbf{A} \cdot \mathbf{i}_j \tag{1.59}$$

Moreover, the diagonal component of a tensor of the second rank **A** in the direction specified by the unit vector $\mathbf{i}_n = \lambda_{n1}\mathbf{i}_1 + \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ is given as

$$A_{nn} = \mathbf{i}_n \cdot \mathbf{A} \cdot \mathbf{i}_n \tag{1.60}$$

This relation may be rewritten in matrix form as

$$A_{nn} = \{\, n \,\}^{\mathrm{T}}[\, A \,]\{\, n \,\} \tag{1.61}$$

where $[A]$ is given by relation (1.58a) and $\{n\}$ is equal to

$$\{\, n \,\} = \begin{Bmatrix} \lambda_{n1} \\ \lambda_{n2} \\ \lambda_{n3} \end{Bmatrix} \tag{1.62}$$

The non-diagonal component of a tensor of the second rank **A** in the directions specified by the orthogonal unit vectors $\mathbf{i}_n = \lambda_{n1}\mathbf{i}_1 + \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ and $\mathbf{i}_s = \lambda_{s1}\mathbf{i}_1 + \lambda_{s2}\mathbf{i}_2 + \lambda_{s3}\mathbf{i}_3$, is given as

$$A_{ns} = \mathbf{i}_n \cdot \mathbf{A} \cdot \mathbf{i}_s \tag{1.63a}$$

$$A_{sn} = \mathbf{i}_s \cdot \mathbf{A} \cdot \mathbf{i}_n \tag{1.63b}$$

where the components A_{ns} and A_{sn} are not necessarily equal. Relations (1.63) may be written in matrix form as

$$A_{ns} = \{\, n \,\}^{T}[\, A \,]\{\, s \,\} \tag{1.64a}$$

$$A_{sn} = \{\, s \,\}^{T}[\, A \,]\{\, n \,\} \tag{1.64b}$$

where

$$\{\, s \,\} = \begin{Bmatrix} \lambda_{s1} \\ \lambda_{s2} \\ \lambda_{s3} \end{Bmatrix} \tag{1.65}$$

As in the case of vectors, tensors of the second rank may be represented as follows:

1. By the symbolic representation using modified letter symbols, as **A**, **B**, **e**, τ. This representation does not require a choice of a coordinate system.
2. By their cartesian components with respect to a set of unit orthogonal base vectors, using
   (a) Indicial notation as A_{ij}, B_{ij}, a_{ij}, b_{ij}, e_{ij}, τ_{ij}, σ_{ij}, ϵ_{ij}, ω_{ij}
   (b) Matrix notation, that is,

$$\mathbf{A} \rightarrow A_{ij} \rightarrow [\,A\,] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.66}$$

Example 1 Consider the two mutually perpendicular unit vectors $\mathbf{i}_n = 3/5\mathbf{i}_1 - 4/5\mathbf{i}_2$ and $\mathbf{i}_s = 4/5\mathbf{i}_1 + 3/5\mathbf{i}_2$. Moreover, consider the tensor

$$[\,A\,] = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix} \tag{a}$$

Find the components of the tensor A_{nn} and A_{ns}.

Solution Referring to relation (1.60), we have

$$A_{nn} = \mathbf{i}_n \cdot \mathbf{A} \cdot \mathbf{i}_s \tag{b}$$

Where

$$\mathbf{A} = -2\mathbf{i}_1\mathbf{i}_1 + 3\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_1\mathbf{i}_3 + 4\mathbf{i}_2\mathbf{i}_1 + 2\mathbf{i}_2\mathbf{i}_2 + 3\mathbf{i}_2\mathbf{i}_3 - 2\mathbf{i}_3\mathbf{i}_1 + \mathbf{i}_3\mathbf{i}_2$$

Thus,

$$\mathbf{i}_n \cdot \mathbf{A} = -\frac{6}{5}\mathbf{i}_1 + \frac{9}{5}\mathbf{i}_2 + \frac{3}{5}\mathbf{i}_3 - \frac{16}{5}\mathbf{i}_1 - \frac{8}{5}\mathbf{i}_2 - \frac{12}{5}\mathbf{i}_3 = -\frac{22}{5}\mathbf{i}_1 + \frac{1}{5}\mathbf{i}_2 - \frac{9}{5}\mathbf{i}_3$$

and

$$A_{nn} = \mathbf{i}_n \cdot \mathbf{A} \cdot \mathbf{i}_n = -\frac{66}{25} - \frac{4}{25} = -\frac{70}{25}$$

$$A_{ns} = \mathbf{i}_n \cdot \mathbf{A} \cdot \mathbf{i}_s = -\frac{88}{25} + \frac{3}{25} = -\frac{85}{25}$$

The same results may be obtained using matrix multiplication. That is,

$$A_{nn} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \\ 4 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 3/5 \\ -4/5 \\ 0 \end{Bmatrix} = \frac{1}{5} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{bmatrix} \begin{Bmatrix} -18 \\ 4 \\ 10 \end{Bmatrix} = -\frac{70}{25}$$

and

$$A_{ns} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \\ 4 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 4/5 \\ 3/5 \\ 0 \end{Bmatrix} = \frac{1}{5} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 22 \\ 5 \end{Bmatrix} = -\frac{85}{25}$$

1.3.1 Example of a Tensor of the Second Rank

In this section we give an example of a quantity which is a tensor of the second rank.

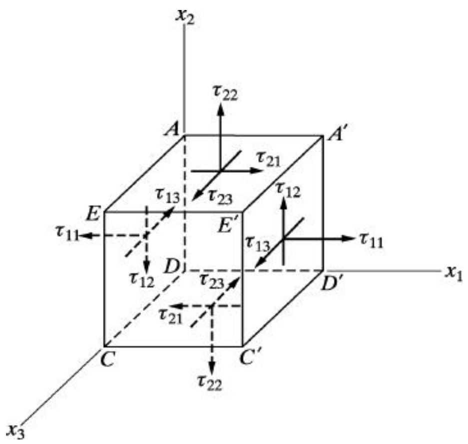
Consider a body subjected to external forces. In general the particles of this body will be stressed. In Section 2.8 we show that the state of stress of a particle of a body is completely specified by a tensor of the second rank, which referring to Fig. 1.7, it is given as

$$\boldsymbol{\tau} = \tau_{11}\mathbf{i}_1\mathbf{i}_1 + \tau_{12}\mathbf{i}_1\mathbf{i}_2 + \tau_{13}\mathbf{i}_1\mathbf{i}_3 + \tau_{21}\mathbf{i}_2\mathbf{i}_1 + \tau_{22}\mathbf{i}_2\mathbf{i}_2 + \tau_{23}\mathbf{i}_2\mathbf{i}_3 + \tau_{31}\mathbf{i}_3\mathbf{i}_1 + \tau_{32}\mathbf{i}_3\mathbf{i}_2 + \tau_{33}\mathbf{i}_3\mathbf{i}_3 \tag{1.67}$$

τ is called *the stress tensor* and $\tau_{ij}(i, j = 1, 2, 3)$ are its components with respect to the system of orthogonal axes x_1, x_2, x_3. Notice than in order to specify a component of stress we must give two directions in addition to its magnitude, namely, the direction of the normal to the plane on which it acts and the direction of the component of stress. For example, in order to specify the component of stress τ_{12}, we must give its magnitude, the direction $\mathbf{i}_1$ of the normal to the plane on which it acts and direction $\mathbf{i}_2$ in which the component of stress acts. The components of the tensor τ may also be presented in matrix form as

$$[\tau] = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \tag{1.68}$$

In what follows we present an example.



The components of stress acting on the faces *CC'EE'* and *AA'DD'* are not shown.

Figure 1.7 Components of stress acting on a particle of a body.

Example 2 The state of stress at point A of the beam of rectangular cross section is shown in Fig. b. Compute the components of stress acting on plane *aa* which is normal to the x_1x_3 plane as shown in Fig. a.

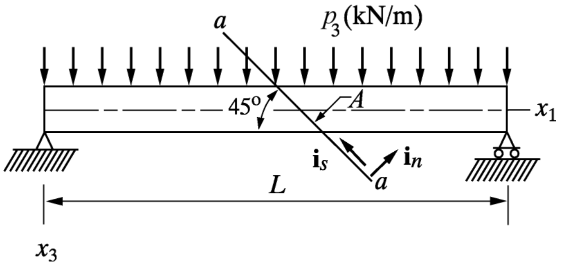


Figure a Simply supported beam of rectangular cross section.

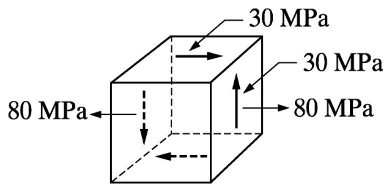


Figure b State of stress at pint *A* of the beam of Fig. a.

Solution Referring to Fig. b, the stress tensor at point *A* of the beam is

$$\boldsymbol{\tau} = 80\mathbf{i}_1\mathbf{i}_1 - 30\mathbf{i}_1\mathbf{i}_3 - 30\mathbf{i}_3\mathbf{i}_1 \quad \text{MPa} \tag{a}$$

or

$$[\tau] = \begin{bmatrix} 80 & 0 & -30 \\ 0 & 0 & 0 \\ -30 & 0 & 0 \end{bmatrix} \text{MPa} \tag{b}$$

Referring to Fig. a, the unit vector normal to the plane *aa* is

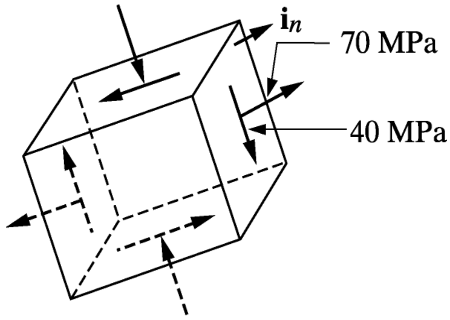


Figure c Components of stress acting on the plane normal to the unit vector $\mathbf{i}_n$.

$$\mathbf{i}_n = \frac{\sqrt{2}}{2}\mathbf{i}_1 - \frac{\sqrt{2}}{2}\mathbf{i}_3 \tag{c}$$

Moreover, the unit vector $\mathbf{i}_s$ normal to $\mathbf{i}_n$ and lying in the x_1x_3 plane is

$$\mathbf{i}_s = -\frac{\sqrt{2}}{2}\mathbf{i}_1 - \frac{\sqrt{2}}{2}\mathbf{i}_3 \tag{d}$$

Thus,

$$\mathbf{i}_n\cdot\boldsymbol{\tau} = \left(\frac{\sqrt{2}}{2}\mathbf{i}_1 - \frac{\sqrt{2}}{2}\mathbf{i}_3\right)\cdot\left(80\,\mathbf{i}_1\mathbf{i}_1 - 30\,\mathbf{i}_1\mathbf{i}_3 - 30\,\mathbf{i}_3\mathbf{i}_1\right) = 55\sqrt{2}\ \mathbf{i}_1 - 15\sqrt{2}\ \mathbf{i}_3$$

and

$$\tau_{nn} = \mathbf{i}_n\cdot\boldsymbol{\tau}\cdot\mathbf{i}_n = (55\sqrt{2}\ \mathbf{i}_1 - 15\sqrt{2}\ \mathbf{i}_3)\cdot\left(\frac{\sqrt{2}}{2}\mathbf{i}_1 - \frac{\sqrt{2}}{2}\mathbf{i}_3\right) = 70\ \text{MPa}$$

$$\tau_{ns} = \mathbf{i}_n\cdot\boldsymbol{\tau}\cdot\mathbf{i}_s = (55\sqrt{2}\ \mathbf{i}_1 - 15\sqrt{2}\ \mathbf{i}_3)\cdot\left(-\frac{\sqrt{2}}{2}\mathbf{i}_1 - \frac{\sqrt{2}}{2}\mathbf{i}_3\right) = -40\ \text{MPa}$$

The component of stress τ_{n2} acting on the plane normal to the unit vector $\mathbf{i}_n$ in the direction of the x_2 axis is equal to

$$\tau_{n2} = \mathbf{i}_n\cdot\boldsymbol{\tau}\cdot\mathbf{i}_2 = (55\sqrt{2}\ \mathbf{i}_1 - 15\sqrt{2}\ \mathbf{i}_3)\mathbf{i}_2 = 0$$

The components of stress acting on the plane normal to the unit vector $\mathbf{i}_n$ are shown in Fig. c.

1.4 Transformation of the Cartesian Components of a Tensor of the Second Rank upon Rotation of the System of Axes to Which They Are Referred

Consider a tensor of the second rank /**A** and denote by A'_{ij} and A_{ij}, its components with respect to the system of axes $x'_i (i = 1, 2, 3)$ and $x_j (j = 1, 2, 3)$, respectively. Thus, referring to relation (1.49), we have

$$\mathbf{A} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \mathbf{i}_i \mathbf{i}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} A'_{ij} \mathbf{i}'_i \mathbf{i}'_j \tag{1.69}$$

Referring to relation (1.59), we obtain

$$A'_{ij} = \mathbf{i}'_i \cdot \mathbf{A} \cdot \mathbf{i}'_j$$

Substituting relation (1.22) into the above and using relation (1.59), we get

$$A'_{ij} = \left(\sum_{k=1}^{3} \lambda_{ik} \mathbf{i}_k \right) \cdot \mathbf{A} \cdot \left(\sum_{m=1}^{3} \lambda_{jm} \mathbf{i}_m \right) = \sum_{k=1}^{3} \sum_{m=1}^{3} (\lambda_{ik} \lambda_{jm}) \; \mathbf{i}_k \cdot \mathbf{A} \cdot \mathbf{i}_m = \sum_{k=1}^{3} \sum_{m=1}^{3} \lambda_{ik} \lambda_{jm} A_{km}$$

Thus,

$$A'_{ij} = \sum_{k=1}^{3} \sum_{m=1}^{3} \lambda_{ik} \lambda_{jm} A_{km} \tag{1.70}$$

This relation specifies the transformation of the components of a tensor of the second rank upon rotation of the axes of reference. It can be expanded to give

$$\begin{aligned} A'_{11} = {} & A_{11}\lambda_{11}^2 + A_{22}\lambda_{12}^2 + A_{33}\lambda_{13}^2 + (A_{12} + A_{21})\lambda_{11}\lambda_{12} + (A_{13} + A_{31})\lambda_{11}\lambda_{13} \\ & + (A_{23} + A_{32})\lambda_{12}\lambda_{13} \end{aligned} \tag{1.71a}$$

$$\begin{aligned} A'_{12} = {} & A_{11}\lambda_{11}\lambda_{21} + A_{22}\lambda_{12}\lambda_{22} + A_{33}\lambda_{13}\lambda_{23} + A_{12}\lambda_{11}\lambda_{22} + A_{21}\lambda_{21}\lambda_{12} \\ & + A_{13}\lambda_{11}\lambda_{23} + A_{31}\lambda_{21}\lambda_{13} + A_{23}\lambda_{12}\lambda_{23} + A_{32}\lambda_{22}\lambda_{13} \end{aligned} \tag{1.71b}$$

$$\begin{aligned} A'_{13} = {} & A_{11}\lambda_{11}\lambda_{31} + A_{22}\lambda_{12}\lambda_{32} + A_{33}\lambda_{13}\lambda_{33} + A_{12}\lambda_{11}\lambda_{32} + A_{21}\lambda_{31}\lambda_{12} \\ & + A_{13}\lambda_{11}\lambda_{33} + A_{31}\lambda_{31}\lambda_{13} + A_{23}\lambda_{12}\lambda_{33} + A_{32}\lambda_{32}\lambda_{13} \end{aligned} \tag{1.71c}$$

$$\begin{aligned} A'_{21} = {} & A_{11}\lambda_{11}\lambda_{21} + A_{22}\lambda_{12}\lambda_{22} + A_{33}\lambda_{13}\lambda_{23} + A_{21}\lambda_{11}\lambda_{22} + A_{12}\lambda_{12}\lambda_{21} \\ & + A_{23}\lambda_{22}\lambda_{13} + A_{32}\lambda_{12}\lambda_{23} + A_{13}\lambda_{21}\lambda_{13} + A_{31}\lambda_{11}\lambda_{23} \end{aligned} \tag{1.71d}$$

$$\begin{aligned} A'_{22} = {} & A_{11}\lambda_{21}^2 + A_{22}\lambda_{22}^2 + A_{33}\lambda_{23}^2 + (A_{12} + A_{21})\lambda_{21}\lambda_{22} + (A_{13} + A_{31})\lambda_{21}\lambda_{23} \\ & + (A_{23} + A_{32})\lambda_{22}\lambda_{23} \end{aligned} \tag{1.71e}$$

$$\begin{aligned} A'_{23} = {} & A_{11}\lambda_{21}\lambda_{31} + A_{22}\lambda_{22}\lambda_{32} + A_{33}\lambda_{23}\lambda_{33} + A_{12}\lambda_{21}\lambda_{32} + A_{21}\lambda_{31}\lambda_{22} \\ & + A_{13}\lambda_{21}\lambda_{33} + A_{31}\lambda_{31}\lambda_{23} + A_{23}\lambda_{22}\lambda_{33} + A_{32}\lambda_{32}\lambda_{23} \end{aligned} \tag{1.71f}$$

$$A'_{31} = A_{11}\lambda_{11}\lambda_{31} + A_{22}\lambda_{12}\lambda_{32} + A_{33}\lambda_{13}\lambda_{33} + A_{32}\lambda_{33}\lambda_{12} + A_{23}\lambda_{13}\lambda_{32} + A_{31}\lambda_{33}\lambda_{11} + A_{13}\lambda_{13}\lambda_{31} + A_{21}\lambda_{32}\lambda_{11} + A_{12}\lambda_{12}\lambda_{31} \tag{1.71g}$$

$$A'_{32} = A_{11}\lambda_{21}\lambda_{31} + A_{22}\lambda_{22}\lambda_{32} + A_{33}\lambda_{23}\lambda_{33} + A_{13}\lambda_{31}\lambda_{23} + A_{31}\lambda_{21}\lambda_{33} + A_{12}\lambda_{31}\lambda_{22} + A_{21}\lambda_{21}\lambda_{32} + A_{32}\lambda_{33}\lambda_{22} + A_{23}\lambda_{23}\lambda_{32} \tag{1.71h}$$

$$A'_{33} = A_{11}\lambda^2_{31} + A_{22}\lambda^2_{32} + A_{33}\lambda^2_{33} + (A_{12}+A_{21})\lambda_{31}\lambda_{32} + (A_{13}+A_{31})\lambda_{31}\lambda_{33} + (A_{23}+A_{32})\lambda_{32}\lambda_{33} \tag{1.71i}$$

Following a procedure analogous to the one employed in obtaining the transformation relation (1.70), we may obtain

$$A_{km} = \sum_{i=1}^{3} \sum_{j=1}^{3} \lambda_{ik}\lambda_{jm} A'_{ij} \tag{1.72}$$

Relations (1.70) and (1.72) may be written in matrix form as follows

$$[\,A'\,] = [\,\Lambda_s\,]\,[\,A\,]\,[\,\Lambda_s\,]^T \tag{1.73a}$$

$$[\,A\,] = [\,\Lambda_s\,]^T\,[\,A'\,]\,[\,\Lambda_s\,] \tag{1.73b}$$

where $[\Lambda_s]$ is the transformation matrix (1.24a) and $[\Lambda_s]^T$ is its transpose.

1.5 Definition of a Tensor of the Second Rank on the Basis of the Law of Transformation of Its Components

In Section 1.3 a tensor of the second rank was defined without referring to any system of axes. Thus, it is explicitly apparent that a tensor has an existence independent of the choice of the system of axes. The system of axes (specified by the orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$) has been introduced subsequently in order to permit the use of well-known mathematical procedures. On the basis of this definition of a tensor of the second rank, it was established in Section 1.4 that its cartesian components transform upon rotation of the right-handed system of axes to which they are referred in accordance with the transformation relations (1.70) or (1.72). The reverse is also valid; that is, any physical entity defined with respect to any rectangular system of axes by an array of nine numbers A_{ij} $(i,j = 1, 2, 3)$ which satisfy relation (1.70) or (1.72) specifies a tensor of the second rank

$$\mathbf{A} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij}\mathbf{i}_i\mathbf{i}_j \tag{1.74}$$

The transformation relation (1.70) or (1.72) is often used as the basis for the definition of a tensor of the second rank as an entity which possesses the following properties:

1. With respect to any set of rectangular axes it is specified by an array of nine numbers A_{ij} $(i,j = 1, 2, 3)$ — its nine cartesian components.

2. Its cartesian components referred to any two right-handed rectangular system of axes specified by the orthogonal unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ and $\mathbf{i}'_1, \mathbf{i}'_2, \mathbf{i}'_3$ are related by the transformation relation (1.70) or (1.72).

This definition of a tensor of the second rank has the shortcoming that it is dependent upon the choice of a system of axes, while the definition presented in Section 1.3 is not.

1.6 Symmetric Tensors of the Second Rank

A symmetric tensor of the second rank is one whose components satisfy the following relation:

$$A_{ij} = A_{ji} \tag{1.75}$$

For instance, the tensor /**A** whose components with respect to a rectangular system of axes is given as

$$[\,A\,] = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & 5 \\ 3 & 5 & -3 \end{bmatrix}$$

is a symmetric tensor of the second rank. If a tensor **A** is symmetric, we have

$$\mathbf{A} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{A} \tag{1.76}$$

In Section 1.9 we show that for any symmetric tensor of the second rank, there exists at least one system of rectangular axes x_1, x_2, x_3, called *principal*, with respect to which the diagonal components of the tensor assume their stationary values. That is, one of them is a maximum and another is a minimum of the diagonal components of the tensor in any direction. Moreover in Section 1.9, we show that, with respect to the principal axes, the non-diagonal components of the tensor vanish. That is, the tensor assumes the following diagonal form:

$$[\,A\,] = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \tag{1.77}$$

In this case the diagonal components A_1, A_2 and A_3 are called the *principal components of the tensor*.

1.7 Invariants of the Cartesian Components of a Symmetric Tensor of the Second Rank

Consider a tensor of the second rank **A** and denote its components with respect to the rectangular systems of axes x_1, x_2, x_3 and x'_1, x'_2, x'_3 by A_{ik} $(i, k = 1, 2, 3)$, A_{jm} $(j, m = 1, 2, 3)$. Adding relations (1.71a), (1.71e) and (1.71i) and using relations (1.25), we obtain

$$II_1 = A'_{11} + A'_{22} + A'_{33} = A_{11} + A_{22} + A_{33} \tag{1.78}$$

That is, the sum of the three diagonal components of a tensor of the second rank is independent of the system of axes to which the components are referred. That is, it is invariant to the rotation of the axes of reference. It is referred to as *the first invariant of the tensor*. Moreover, by appropriate manipulation of relations (1.71) it may be shown that a symmetric tensor of the second rank has the following invariants:

$$\begin{aligned} II_2 &= A'_{11}A'_{22} + A'_{11}A'_{33} + A'_{22}A'_{33} - (A'_{12})^2 - (A'_{13})^2 - (A'_{23})^2 \\ &= A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33} - A_{12}^2 - A_{13}^2 - A_{23}^2 \end{aligned} \tag{1.79}$$

$$\begin{aligned} II_3 &= A'_{11}A'_{22}A'_{33} + 2A'_{12}A'_{13}A'_{23} - A'_{11}(A'_{23})^2 - A'_{22}(A'_{13})^2 - A'_{33}(A'_{21})^2 \\ &= A_{11}A_{22}A_{33} + 2A_{12}A_{13}A_{23} - A_{11}A_{23}^2 - A_{22}A_{13}^2 - A_{33}A_{21}^2 \end{aligned} \tag{1.80}$$

where II_2 and II_3 are referred to as the *second and third invariants of the tensor of the second rank*, respectively.

1.8 Stationary Values of a Function Subject to a Constraining Relation

Consider a function $f(x_1, x_2, x_3)$ having continuous first derivatives in a region R. We say that the function $f(x_1, x_2, x_3)$ assumes a stationary value at a point $P(x_1, x_2, x_3)$ in the region R if the following relation is satisfied at this point:

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i = 0 \tag{1.81}$$

A stationary value of a function could be a maximum or a minimum or a saddle point. If the variables x_1, x_2, x_3 are independent, then a necessary and sufficient condition for the satisfaction of relation (1.81) is

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0 \tag{1.82}$$

In many problems in mechanics, it is necessary to establish the stationary values of a function $f(x_1, x_2, x_3)$, when the variables are related by a constraining relation $g(x_1, x_2, x_3) = 0$. If the relation $g(x_1, x_2, x_3) = 0$ can be solved for one of the variables, say x_1, in terms of the other two, then the resulting expression may be substituted into the function $f(x_1, x_2, x_3)$ and a function $\tilde{f}(x_2, x_3)$ is obtained. In this case it is apparent that the stationary values of $f(x_1, x_2, x_3)$ under the constraining relation $g(x_1, x_2, x_3) = 0$ may be obtained from the following relation:

$$d\tilde{f} = \frac{\partial \tilde{f}}{\partial x_2} dx_2 + \frac{\partial \tilde{f}}{\partial x_3} dx_3 = 0 \tag{1.83}$$

Inasmuch as dx_2 and dx_3 are independent, the necessary and sufficient condition for the satisfaction of the above relation is

$$\frac{\partial \tilde{f}}{\partial x_2} = \frac{\partial \tilde{f}}{\partial x_3} = 0 \tag{1.84}$$

However, in certain problems, $g(x_1, x_2, x_3) = 0$ is a complicated function. In this case in order to establish the stationary values of $f(x_1, x_2, x_3)$, it is convenient to use the ingenious method proposed by Lagrange, which we describe in the sequel.

Inasmuch as the variables x_1, x_2, x_3 are related by the constraining relation $g(x_1, x_2, x_3) = 0$, the increments dx_i $(i = 1, 2, 3)$ in relation (1.81) are not independent. They are related by the following relation:

$$\sum_{i=1}^{3} \frac{\partial g}{\partial x_i} dx_i = 0 \tag{1.85}$$

Solving equation (1.85) for dx_3, we obtain

$$dx_3 = -\frac{\dfrac{\partial g}{\partial x_1}}{\dfrac{\partial g}{\partial x_3}} dx_1 - \frac{\dfrac{\partial g}{\partial x_2}}{\dfrac{\partial g}{\partial x_3}} dx_2 \tag{1.86}$$

Substituting dx_3 from the above relation into equation (1.81), we get

$$df = \left[\frac{\partial f}{\partial x_1} - \frac{\dfrac{\partial f}{\partial x_3} \dfrac{\partial g}{\partial x_1}}{\dfrac{\partial g}{\partial x_3}} \right] dx_1 + \left[\frac{\partial f}{\partial x_2} - \frac{\dfrac{\partial f}{\partial x_3} \dfrac{\partial g}{\partial x_2}}{\dfrac{\partial g}{\partial x_3}} \right] dx_2 = 0 \tag{1.87}$$

Since dx_1, dx_2 are independent, their coefficients in the above relation must vanish. Hence,

$$\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial g}{\partial x_1}} = \frac{\dfrac{\partial f}{\partial x_2}}{\dfrac{\partial g}{\partial x_2}} = \frac{\dfrac{\partial f}{\partial x_3}}{\dfrac{\partial g}{\partial x_3}} = \lambda \tag{1.88}$$

Thus, the function $f(x_1, x_2, x_3)$ assumes stationary values at the points $P(x_1, x_2, x_3)$ of the region R whose coordinates satisfy the following relation:

$$
\begin{aligned}
\frac{\partial f}{\partial x_1} - \lambda\frac{\partial g}{\partial x_1} &= 0 \\
\frac{\partial f}{\partial x_2} - \lambda\frac{\partial g}{\partial x_2} &= 0 \\
\frac{\partial f}{\partial x_3} - \lambda\frac{\partial g}{\partial x_3} &= 0 \\
g(x_1,\, x_2,\, x_3) &= 0
\end{aligned}
\tag{1.89}
$$

Consider the function

$$F(x_1,\, x_2,\, x_3,\, \lambda) = f(x_1,\, x_2,\, x_3) - \lambda g(x_1,\, x_2,\, x_3) \tag{1.90}$$

This function assumes its stationary values at points whose coordinates satisfy the following relation:

$$dF = 0 = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \frac{\partial F}{\partial x_3}dx_3 + \frac{\partial F}{\partial \lambda}d\lambda \tag{1.91}$$

Inasmuch as the function $F(x_1, x_2, x_3, \lambda)$ is not subjected to a constraint, dx_1, dx_2, dx_3 and $d\lambda$ are independent and relation (1.91) is satisfied when

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial \lambda} = 0 \tag{1.92}$$

Substituting relation (1.90) into the above relations, we get

$$
\begin{aligned}
\frac{\partial f}{\partial x_i} - \lambda\frac{\partial g}{\partial x_i} &= 0 \qquad (i = 1, 2, 3) \\
g(x_1,\, x_2,\, x_3) &= 0
\end{aligned}
\tag{1.93}
$$

These equations are identical to equations (1.89). Thus, the stationary values of $f(x_1, x_2, x_3)$ subjected to the constraining relation $g(x_1, x_2, x_3) = 0$ and the stationary values of $F(x_1, x_2, x_3)$ without a constraining relation occur at the same points. Consequently, the stationary values of $f(x_1, x_2, x_3)$ subjected to the constraining relation $g(x_1, x_2, x_3) = 0$ and the points at which they occur may be obtained from equations (1.92). The multiplying constant λ is called the *Lagrange multiplier*.

Example 3 Using the method of Lagrange multipliers, find the point on the plane specified by the following relation which is the nearest to the origin of the system of axes x_1, x_2, x_3:

$$2x_1 - 3x_2 + 5x_3 = 19 \tag{a}$$

Solution The distance $d(x_1, x_2, x_3)$ of any point $P\,(x_1, x_2, x_3)$ from the origin is given as

$$[\, d(x_1, x_2, x_3)\,]^2 = x_1^2 + x_2^2 + x_3^2 \tag{b}$$

Thus, we must establish the point at which the function d^2 assumes a stationary value under the constraining relation (a). For this purpose we form the function

$$F(x_1, x_2, x_3, \lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda(2x_1 - 3x_2 + 5x_3 - 19) \tag{c}$$

and we find x_1, x_2, x_3 and λ for which the partial derivatives of $F(x_1, x_2, x_3, \lambda)$ are all zero. That is,

$$\begin{aligned}
\frac{\partial F}{\partial x_1} &= 2x_1 - 2\lambda = 0 \\
\frac{\partial F}{\partial x_2} &= 2x_2 + 3\lambda = 0 \\
\frac{\partial F}{\partial x_3} &= 2x_3 - 5\lambda = 0 \\
\frac{\partial F}{\partial \lambda} &= -2x_1 + 3x_2 - 5x_3 + 19 = 0
\end{aligned} \tag{d}$$

From the first three of the above relations, we obtain

$$x_1 = \lambda \qquad x_2 = -\frac{3\lambda}{2} \qquad x_3 = \frac{5\lambda}{2} \tag{e}$$

Substituting relation (e) into the last of relations (d), we get

$$-2\lambda - \frac{9\lambda}{2} - \frac{25\lambda}{2} + 19 = 0$$

or

$$\lambda = 1 \tag{f}$$

Substituting result (g) into relations (e), we have

$$x_1 = 1 \qquad x_2 = -\frac{3}{2} \qquad x_3 = \frac{5}{2}$$

These are the coordinates of the point on the plane specified by relation (a) which is nearest to the origin of the system of axes x_1, x_2, x_3.

1.9 Stationary Values of the Diagonal Components of a Symmetric Tensor of the Second Rank

Consider a set of orthogonal axes x_1, x_2, x_3 specified by the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$. Referred to these axes the unit vector $\mathbf{i}_n'$ can be written as

$$\mathbf{i}_n' = \lambda_{n1}\mathbf{i}_1 + \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3 \tag{1.94}$$

where $\lambda_{ni}(i = 1, 2, 3)$ are the direction cosines of $\mathbf{i}_n'$ with respect to the set of axes x_1, x_2, x_3.

Consider a symmetric tensor of the second rank $[A]$. Referring to relation (1.70) its diagonal component A_{nn} associated with the unit vector $\mathbf{i}_n$ is equal to

$$A'_{nn} = \sum_{k=1}^{3}\sum_{m=1}^{3} \lambda_{nk}\lambda_{nm}A_{km} = A_{11}\lambda_{n1}^2 + A_{22}\lambda_{n2}^2 + A_{33}\lambda_{n3}^2 + 2A_{12}\lambda_{n1}\lambda_{n2} + 2A_{23}\lambda_{n2}\lambda_{n3} + 2A_{31}\lambda_{n1}\lambda_{n3} \tag{1.95}$$

It is apparent that the diagonal component A_{nn}' of a tensor is a function of the direction cosines λ_{n1}, λ_{n2} and λ_{n3}. However, these direction cosines are not independent; they are related [see the right column first row of relations (1.25b)] by the following relation:

$$\lambda_{n1}^2 + \lambda_{n2}^2 + \lambda_{n3}^2 - 1 = 0 \tag{1.96}$$

In what follows we employ the technique of Lagrange multipliers (see Section 1.8) in order to establish the directions $\mathbf{i}_n^{(n)}$ $(n = 1,2, ..., N)$ along which the function A_{nn}' $(\lambda_{n1}, \lambda_{n2}, \lambda_{n3})$ assumes stationary values† subjected to the constraining relation (1.96). Moreover, we establish the stationary values of the function. The unit vectors $\mathbf{i}_n^{(n)}$ may be expressed as

$$\mathbf{i}_n^{(n)} = \sum_{i=1}^{3} \lambda_{ni}^n \mathbf{i}_i \qquad (n = 1,2...N) \tag{1.97}$$

where λ_{n1}, λ_{n2} and λ_{n3} are the direction cosines of the unit vector $\mathbf{i}_n^{(n)}$.

We define the function $F(\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \lambda)$ as

$$F(\lambda_{n1}, \lambda_{n2}, \lambda_{n3}, \lambda) = A'_{nn}(\lambda_{n1}, \lambda_{n2}, \lambda_{n3}) - \lambda(\lambda_{n1}^2 + \lambda_{n2}^2 + \lambda_{n3}^2 - 1) \tag{1.98}$$

where the constant λ is the *Lagrange multiplier*. The function F assumes a stationary value for each set of direction cosines, λ_{n1}^n, λ_{n2}^n, λ_{n3}^n and multiplier λ^n $(n = 1,2...N)$. Each one of these sets of direction cosines and multiplier satisfies the following relations.

$$\frac{\partial F}{\partial \lambda_{n1}} = \frac{\partial F}{\partial \lambda_{n2}} = \frac{\partial F}{\partial \lambda_{n3}} = \frac{\partial F}{\partial \lambda} = 0 \tag{1.99}$$

Substituting relation (1.98) into (1.99) and using (1.95), we get

$$\begin{aligned}
(A_{11} - \lambda^n)\lambda_{n1}^n + A_{12}\lambda_{n2}^n + A_{13}\lambda_{n3}^n &= 0 \\
A_{21}\lambda_{n1}^n + (A_{22} - \lambda^n)\lambda_{n2}^n + A_{23}\lambda_{n3}^n &= 0 \\
A_{31}\lambda_{n1}^n + A_{32}\lambda_{n2}^n + (A_{33} - \lambda^n)\lambda_{n3}^n &= 0 \\
(\lambda_{n1}^n)^2 + (\lambda_{n2}^n)^2 + (\lambda_{n3}^n)^2 - 1 &= 0
\end{aligned} \tag{1.100}$$

† A stationary value of a function could be a maximum, a minimum or a saddle point.

Multiplying the first of relation (1.100) by λ_{n1}^{n}, the second by λ_{n2}^{n} and the third by λ_{n3}^{n} adding the resulting relations and using relation (1.96), we obtain

$$\lambda^{n} = A_{11}(\lambda_{n1}^{n})^{2} + A_{22}(\lambda_{n2}^{n})^{2} + A_{33}(\lambda_{n3}^{n})^{2} + 2A_{12}\lambda_{n1}^{n}\lambda_{n2}^{n} + 2A_{23}\lambda_{n2}^{n}\lambda_{n3}^{n} + 2A_{31}\lambda_{n1}^{n}\lambda_{n3}^{n} \tag{1.101}$$

Referring to equation (1.95), we see that relation (1.101) can be rewritten as

$$\lambda^{n} = (A_{nn}^{n})_{\text{stationary}} = A_{n} \tag{1.102}$$

Thus, the Lagrange multipliers are equal to the stationary values A_{n} of the diagonal components of the symmetric tensor of the second rank $[A]$. Using relation (1.102), relation (1.100) can be rewritten as

$$\begin{aligned} (A_{11} - A_{n})\lambda_{n1}^{n} + A_{12}\lambda_{n2}^{n} + A_{13}\lambda_{n3}^{n} &= 0 \\ A_{21}\lambda_{n1}^{n} + (A_{22} - A_{n})\lambda_{n2}^{n} + A_{23}\lambda_{n3}^{n} &= 0 \\ A_{31}\lambda_{n1}^{n} + A_{32}\lambda_{n2}^{n} + (A_{33} - A_{n})\lambda_{n3}^{n} &= 0 \end{aligned} \tag{1.103}$$

These three linear algebraic homogeneous equation, in λ_{n1}^{n}, λ_{n2}^{n} and λ_{n3}^{n} have a solution other than the trivial $\lambda_{n1}^{n} = \lambda_{n2}^{n} = \lambda_{n3}^{n} = 0$, if the determinant of the coefficients of λ_{n1}^{n}, λ_{n2}^{n} and λ_{n3}^{n} is zero. That is,

$$\begin{vmatrix} (A_{11} - A_{n}) & A_{12} & A_{13} \\ A_{12} & (A_{22} - A_{n}) & A_{23} \\ A_{13} & A_{23} & (A_{33} - A_{n}) \end{vmatrix} = 0 \tag{1.104}$$

This determinant may be expanded to yield

$$A_{n}^{3} - II_{1}A_{n}^{2} + II_{2}A_{n} - II_{3} = 0 \tag{1.105}$$

where II_1, II_2 and II_3 are the three invariants of the symmetric tensor $[A]$ of the second rank defined by relations (1.78) to (1.80). From equation (1.105) we see that the values A_n of a symmetric tensor of the second rank $[A]$ are independent of the choice of the system of axes to which the components of the tensor are referred.

It can be shown that equation (1.105) has three real *roots* A_1, A_2, A_3 *which are the three stationary values of the diagonal components of the tensor*. If the three roots of equation (1.105) are different ($A_1 \neq A_2 \neq A_3 \neq A_1$), then substituting each one of them into equations (1.103) we obtain three different sets of ratios of the direction cosines λ_{n1}^{n}, λ_{n2}^{n} and λ_{n3}^{n} ($n = 1, 2, 3$). For example $\lambda_{n2}^{n}/\lambda_{n1}^{n}$, $\lambda_{n3}^{n}/\lambda_{n1}$. We find three sets of the direction cosines λ_{n1}^{n}, λ_{n2}^{n} and λ_{n3}^{n} ($n = 1, 2, 3$), using the relation $(\lambda_{n1}^{n})^{2} + (\lambda_{n2}^{n})^{2} + (\lambda_{n3}^{n})^{2} = 0$. It can be shown that these sets of direction cosines specify three mutually perpendicular directions, referred to as the *principal directions of the tensor.*

The three roots of equation (1.105) are equal only if the given components of the tensor have the following form:

$$A_{11} = A_{22} = A_{33} \qquad A_{12} = A_{13} = A_{23} = 0 \tag{1.106}$$

In this case all directions are principal.

Finally, it can be shown that the non-diagonal components referred to the principal axes vanish. Thus, with respect to its principal axes, a symmetric tensor of the second rank assumes a diagonal form. That is,

$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \tag{1.107}$$

Moreover, it can be shown that any direction along which the non-diagonal components of a symmetric tensor vanish, is a principal direction, that is, a direction along which the normal component of the tensor assumes a stationary value.

There are several methods available for solving a cubic equation. For example one root A_1 of equation (1.105) can be found by trial and error. Equation (1.105) can then be divided by $(A_n - A_1)$ and the resulting quadratic equation can be solved by applying the general formula for the solution of such an equation. The solution of equation (1.105) can be also obtained using the following formulas†:

$$\begin{aligned} A_1 &= u + v + \frac{\Pi_1}{3} \\ A_2 &= -\frac{1}{2}(u + v) + \frac{\sqrt{-3}}{2}(u - v) + \frac{\Pi_1}{3} \\ A_3 &= -\frac{1}{2}(u + v) - \frac{\sqrt{-3}}{2}(u - v) + \frac{\Pi_1}{3} \end{aligned} \tag{1.108a}$$

where

$$\begin{aligned} u &= \left[Q + \sqrt{Q^2 + R^3}\right]^{1/3} \\ v &= \left[Q - \sqrt{Q^2 + R^3}\right]^{1/3} \end{aligned} \tag{1.108b}$$

$$3R = -\frac{\Pi_1^2}{3} + \Pi_2 \qquad 2Q = -\frac{\Pi_1 \Pi_2}{3} + \Pi_3 + \frac{2\Pi_1^3}{27} \tag{1.109}$$

† See Messai, E.E., Finding true maximum shear stress *Machine Design*, Dec. 7, 1978, p. 166-169; and Terry, E.S., *A Practical Guide to Computer Methods for Engineers,* Prentice-Hall, Englewood Cliffs, NJ, 1979.

In what follows we present an example.

Example 4 The components of a symmetric tensor of the second rank with respect to rectangular system of axis x_j $(j = 1, 2, 3)$ are

$$[\,A\,] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \tag{a}$$

Compute the principal values and the principal directions of the tensor.

Solution Referring to relation (1.78) to (1.80) the invariants of the tensor $[A]$ are

$$II_1 = 3 \qquad II_2 = -6 \qquad II_3 = -8 \tag{b}$$

Substituting the values of the invariants (b) into equation (1.105), we have

$$A_n^3 - 3A_n^2 - 6A_n + 8 = 0 \tag{c}$$

The roots of this equation are the stationary values of the diagonal components of the tensor. They may be established either by employing formulas (1.108) or by employing another method for solving cubic equations. They are

$$A_1 = 4 \quad A_2 = 1 \quad A_3 = -2 \tag{d}$$

In order to obtain the direction cosines of the principal axis corresponding to A_1 we substitute the value of A_1 in the first two of relations (1.103). That is,

$$(3 - 4)\lambda_{11}^{(1)} + \lambda_{12}^{(1)} + \lambda_{13}^{(1)} = 0$$

$$\lambda_{11}^{(1)} + (0 - 4)\lambda_{12}^{(1)} + 2\lambda_{13}^{(1)} = 0$$

From the above equations we have

$$\begin{aligned} \lambda_{12}^{(1)} &= \frac{1}{2}\lambda_{11}^{(1)} \\ \lambda_{13}^{(1)} &= \frac{1}{2}\lambda_{11}^{(1)} \end{aligned} \tag{e}$$

The orthogonality condition $\mathbf{i}_1^{(n)} \cdot \mathbf{i}_1^{(n)}$ [see relations (1.25b)], gives

$$(\lambda_{11}^{(n)})^2 + (\lambda_{12}^{(n)})^2 + (\lambda_{13}^{(n)})^2 = 1 \tag{f}$$

Substituting relations (e) into (f), we get

$$(\lambda_{11}^{(1)})^2 + \frac{(\lambda_{11}^{(1)})^2}{4} + \frac{(\lambda_{11}^{(1)})^2}{4} = 1 \tag{g}$$

Thus,

$$\lambda_{11}^{(1)} = \frac{2}{\sqrt{6}} \tag{h}$$

Substituting result (h) into relations (e), we obtain

$$\lambda_{12}^{(1)} = \frac{1}{\sqrt{6}} \qquad \lambda_{13}^{(1)} = \frac{1}{\sqrt{6}} \tag{i}$$

Similarly, using the values of A_2 and A_3 we find the following sets of direction cosines.

$$\lambda_{21}^{(2)} = -\frac{1}{\sqrt{3}} \quad \lambda_{22}^{(2)} = \frac{1}{\sqrt{3}} \quad \lambda_{23}^{(2)} = \frac{1}{\sqrt{3}} \quad \lambda_{31}^{(3)} = 0 \quad \lambda_{32}^{(3)} = -\frac{1}{\sqrt{2}} \quad \lambda_{33}^{(3)} = \frac{1}{\sqrt{2}} \tag{j}$$

Thus, the principal directions of the symmetric tensor of the second rank $[A]$ whose components with respect to the orthogonal unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are specified by the matrix (a) are

$$\mathbf{i}_1^{(1)} = \frac{1}{\sqrt{6}}(2\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3) \qquad \mathbf{i}_2^{(2)} = \frac{1}{\sqrt{3}}(-\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3) \qquad \mathbf{i}_3^{(3)} = \frac{1}{\sqrt{2}}(-\mathbf{i}_2 + \mathbf{i}_3) \tag{k}$$

1.10 Quasi Plane Form of Symmetric Tensors of the Second Rank

In this section we consider symmetric tensors of the second rank whose two, non-diagonal components vanish. We call such tensors *quasi plane symmetric tensors of the second rank*. As an example consider the symmetric tensor of the second rank whose components, with respect to the rectangular system of axes x_1, x_2, x_3, are

$$[A] = \left[\begin{array}{cc|c} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ \hline 0 & 0 & A_{33} \end{array}\right] = \left[\begin{array}{c|c} [A_P] & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0\ 0] & A_{33} \end{array}\right] \tag{1.110}$$

We call the plane specified by the axes x_1 and x_2 the plane of the tensor $[A]$. Referring to relation (1.110), we see that the x_3 axis is principal. Thus, on the basis of our discussion in the previous section there exist at least two mutually perpendicular axes in the plane $x_1 x_2$ which are principal. In the next section we establish the principal directions of quasi plane symmetric tensors of the second rank and their principal values.

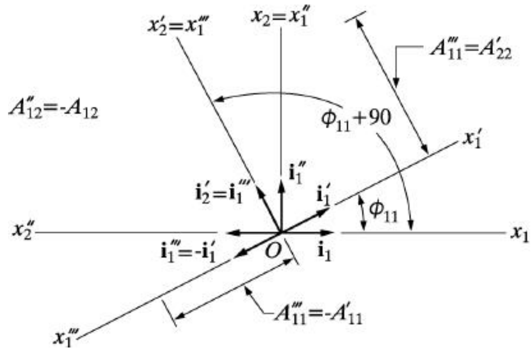


Figure 1.8 Rotation of axes.

In what follows we establish the transformation relations of the components of a quasi plane symmetric tensor of a second rank when the axes to which they are referred rotate about the axis normal to the plane of the tensor.

Consider two rectangular systems of axes x_j (1, 2, 3) and x_i' ($i = 1, 2, 3$) whose axes x_3' *and* x_3 coincide. As shown in Fig. 1.8, the x_1' and x_2' axes are located relative to the x_1 and x_2 axes by the angle ϕ_{11}. The direction cosines of the system of axes x_i' ($i = 1, 2, 3$) with respect to the system of axes x_j ($j = 1, 2, 3$) may be written in terms of the angle ϕ_{11} as

$$\lambda_{11} = \cos\phi_{11} \qquad \lambda_{12} = \sin\phi_{11} \qquad \lambda_{21} = -\sin\phi_{11} \qquad \lambda_{22} = \cos\phi_{11}$$

$$\lambda_{13} = \lambda_{23} = \lambda_{31} = \lambda_{32} = 0 \qquad \lambda_{33} = 1 \tag{1.111}$$

Thus, referring to relations (1.24a) and (1.111) the transformation matrix of the system of axes x_i' with respect to the system of axes x_j is

$$[\Lambda_s] = \left[\begin{array}{cc|c} \cos\phi_{11} & \sin\phi_{11} & 0 \\ -\sin\phi_{11} & \cos\phi_{11} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} [\Lambda_P] & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0\ 0] & 1 \end{array}\right] \tag{1.112}$$

Substituting relation (1.112) and (1.110) into (1.73a), we obtain the components of the tensor (1.110) referred to the system of axes $x_i'(i = 1, 2, 3)$. That is,

$$[A'] = \begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ A_{21}' & A_{22}' & A_{23}' \\ A_{31}' & A_{32}' & A_{33}' \end{bmatrix} = \left[\begin{array}{cc} [A_P'] & \begin{bmatrix} A_{13}' \\ A_{23}' \end{bmatrix} \\ \hline [A_{31}'\ A_{32}'] & A_{33}' \end{array}\right] = [\Lambda_s][A][\Lambda_s]^T$$

$$= \left[\begin{array}{cc} [\Lambda_P] & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0 0] & 1 \end{array}\right] \left[\begin{array}{cc} [A_P] & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0 0] & A_{33} \end{array}\right] \left[\begin{array}{cc} [\Lambda_P]^T & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0 0] & 1 \end{array}\right] = \left[\begin{array}{cc} [\Lambda_P][A_P][\Lambda_P]^T & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline [0\ \ 0] & A_{33} \end{array}\right] \tag{1.113}$$

or

$$[A_P'] = \begin{bmatrix} A_{11}' & A_{12}' \\ A_{21}' & A_{22}' \end{bmatrix} = [\Lambda_P][A_P][\Lambda_P]^T \tag{1.114}$$

and

$$A_{33}' = A_{33} \qquad A_{13}' = A_{23}' = 0 \tag{1.115}$$

The matrix $[\Lambda_P]$ is specified by relation (1.112). Relations (1.114) can be expanded to give

$$A_{11}' = \frac{1}{2}(A_{11} + A_{22}) + \frac{1}{2}(A_{11} - A_{22})\cos 2\phi_{11} + A_{12} \sin 2\phi_{11} \tag{1.116a}$$

$$A_{12}' = A_{21}' = -\frac{1}{2}(A_{11} - A_{22})\sin 2\phi_{11} + A_{12} \cos 2\phi_{11} \tag{1.116b}$$

$$A_{22}' = \frac{1}{2}(A_{11} + A_{22}) - \frac{1}{2}(A_{11} - A_{22})\cos 2\phi_{11} - A_{12} \sin 2\phi_{11} \tag{1.116c}$$

$$A_{33}' = A_{33} \qquad A_{13}' = A_{23}' = 0 \tag{1.116d}$$

Relations (1.114) or (1.116) represent the *transformation relations* for the components of a quasi plane symmetric tensor of the second rank. They transform the components of this tensor from one set of axes (x_1, x_2, x_3) to another x_1', x_2', x_3' with $x_3 = x_3'$. Referring to relation (1.116), we see that the tensor $[A]$, whose components are defined with respect to the rectangular set of axes x_1, x_2, x_3 by relation (1.110), assumes a plane form with respect to any rectangular set of axes x_1', x_2', x_3' whose axis x_3' coincides with the axis x_3. However, this tensor does not assume a plane form with respect to a rectangular set of axes x_1'', x_2'', x_3'' if the axis x_3'' does not coincide with the axes x_3.

Referring to Fig. 1.8, we see that the angle $\phi_{11} = 0°$ specifies the system of axes x_1, x_2 while the angle $\phi_{11} = 90°$ specifies the system of axes x_1'', x_2''. The unit vectors associated with these two systems of axes are related as follows:

$$\mathbf{i}_1'' = \mathbf{i}_2 \qquad \mathbf{i}_2'' = -\mathbf{i}_1 \tag{1.117}$$

Moreover, for $\phi_{11} = 90^o$ relations (1.116a) and (1.116b) give

$$A_{11}'' = A_{22} \qquad A_{12}'' = -A_{12} \tag{1.118}$$

1.11 Stationary Values of the Diagonal and the Non-Diagonal Components of Quasi Plane, Symmetric Tensors of the Second Rank

Consider a quasi plane symmetric tensor of the second rank and the system of axes x_1, x_2 in its plane. We denote the components of this tensor with respect to the x_1,x_2,x_3 axes by A_{ij} $(i, j = 1, 2, 3)(A_{13} = A_{23} = 0)$. The transformation relations (1.116) for the quasi plane form of a symmetric tensor of the second rank show that its diagonal and non-diagonal components are continuous functions of the angle ϕ_{11} . For certain values $\tilde{\phi}_{11}$ of the angle ϕ_{11}, the diagonal components of the tensor assume their stationary values. The axes $\tilde{x}_1, \tilde{x}_2$ specified by the angle $\tilde{\phi}_{11}$ are called the *principal axes* of the tensor, while the values of the diagonal components of the tensor obtained from the transformation equations (1.116a) and (1.116c) for $\phi_{11} = \tilde{\phi}_{11}$ are called its *principal values*. In order to establish the principal axes of the tensor we set the derivative of A'_{11} with respect to ϕ_{11} equal to zero. That is, referring to equation (1.116a), we have

$$\frac{dA'_{11}}{d\phi_{11}} = -(A_{11} - A_{22})\sin 2\tilde{\phi}_{11} + 2A_{12}\cos 2\tilde{\phi}_{11} = 0 \qquad (1.119)$$

If $A_{11} = A_{22}$ and $A_{12} \neq 0$, equation (1.119) yields

$$\cos 2\tilde{\phi}_{11} = 0 \qquad (1.120)$$

This equation has the following two solutions in the interval $0 \le \tilde{\phi}_{11} \le \pi$:

$$\tilde{\phi}_{11} = \frac{\pi}{4}, \frac{3\pi}{4} \qquad (1.121)$$

If $A_{11} = A_{22}$ and $A_{12} = 0$, equation (1.119) is satisfied for any value of $\tilde{\phi}_{11}$. That is, any pair of two mutually perpendicular axes in the $x_1\, x_2$ plane constitutes, with the x_3 axis, a set of principal axes. Finally, if $A_{11} \neq A_{22}$ equation (1.119) may be rewritten as

$$\tan 2\tilde{\phi}_{11} = \frac{2A_{12}}{A_{11} - A_{22}} \qquad (1.122)$$

This transcendental equation has two solutions in the interval $0 \le \tilde{\phi}_{11} \le \pi$ which differ by 90° ($\tilde{\phi}_{11}$ and $\tilde{\phi}_{11} + \pi/2$). That is, any quasi plane symmetric tensor of the second rank in the $x_1\, x_2$ plane has two mutually perpendicular principal directions in this plane. Referring to relation (1.122), we obtain

$$\cos 2\tilde{\phi}_{11} = \pm \frac{(A_{11} - A_{22})/2}{\left[\left(\frac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2\right]^{1/2}} \qquad (1.123a)$$

$$\sin 2\tilde{\phi}_{11} = \pm \frac{A_{12}}{\left[\left(\frac{A_{11} - A_{22}}{2} \right)^2 + A_{12}^2 \right]^{1/2}} \tag{1.123b}$$

Substituting relations (1.123) into (1.116), we obtain the following expressions for the stationary values A_1 and A_2 of the diagonal components of the quasi plane form (1.110) of a symmetric tensor of the second rank

$$\left. \begin{matrix} A_1 \\ A_2 \end{matrix} \right\} = \frac{1}{2}(A_{11} + A_{22}) \pm \sqrt{\left(\frac{A_{11} - A_{22}}{2} \right)^2 + A_{12}^2} \tag{1.124}$$

A_1 is the algebraically largest and A_2 is the algebraically smallest value of the diagonal components of the tensor referred to any system of axes in the $x_1\, x_2$ plane. However, a tensor of the second rank has at least a set of three mutually perpendicular principal directions. The three principal directions for the quasi plane symmetric tensor of the second rank specified by relation (1.110) are the two directions given by equation (1.122) and the x_3 axis. The two principal values of the diagonal components of the tensor are given by relations (1.124) while the third is equal to $A_3 = A_{33}$. Thus, A_1 is algebraically larger than A_2 but A_3 may be algebraically larger or smaller than either one or both A_1 or A_2.

The non-diagonal components of the quasi plane symmetric tensor (1.110) associated with the two principal directions in the $x_1\, x_2$ plane may be established by substituting relations (1.123) into equation (1.116b). *It can be shown that they are equal to zero.*

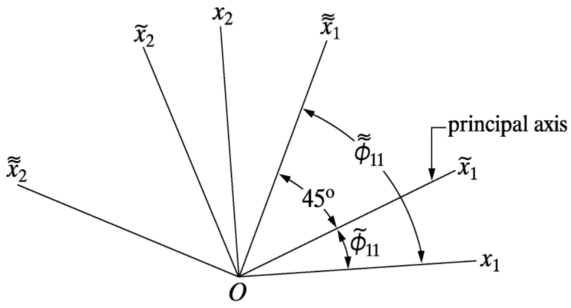
On the basis of the preceding discussion, we may conclude that for any quasi plane symmetric tensor of the second rank (1.110) there exists at least one set of three mutually perpendicular principal axes with respect to which the diagonal components of the tensor assume stationary values. Two of these axes are in the $x_1\, x_2$ plane while the third is the x_3 axis.

The values $\tilde{\tilde{\phi}}_{11}$ of the angle ϕ_{11} corresponding to the directions along which the non-diagonal components of the tensor assume their stationary values, may be obtained by differentiating relation (1.116b) and setting the derivative of A'_{12} with respect to ϕ_{11} equal to zero. That is,

$$\frac{dA'_{12}}{d\phi_{11}} = -(A_{11} - A_{22})\cos 2\tilde{\tilde{\phi}}_{11} - 2A_{11}\sin 2\tilde{\tilde{\phi}}_{11} = 0 \tag{1.125}$$

If $A_{11} = A_{22}$ and $A_{12} \neq 0$, then relation (1.125) yields

$$\sin 2\tilde{\tilde{\phi}}_{11} = 0 \qquad \text{or} \qquad \tilde{\tilde{\phi}}_{11} = 0\,,\ \frac{\pi}{2} \tag{1.126}$$



A stationary value of the non-diagonal components of the tensor occurs in the $\tilde{\tilde{x}}_1$, $\tilde{\tilde{x}}_2$ directions.

Figure 1.9 Directions in the $x_1\,x_2$ plane along which the stationary values of the non-diagonal components of the plane form (1.110) of a symmetric tensor of the second rank occur.

If $A_{11} = A_{22}$ and $A_{12} = 0$, then A'_{12} vanishes for any values of ϕ_{11}. Finally, if $A_{11} \neq A_{22}$ relation (1.125) yields.

$$\tan 2\tilde{\tilde{\phi}}_{11} = -\frac{A_{11} - A_{22}}{2A_{12}} \tag{1.127}$$

This equation has two solutions in the interval $0 < \tilde{\tilde{\phi}}_{11} < \pi$ which differ by 90° ($\tilde{\tilde{\phi}}_{11}$ and $\tilde{\tilde{\phi}}_{11} + \pi/2$). Comparing equation (1.127) with (1.122), we may conclude that the directions in the x_1, x_2 plane along which the non-diagonal components A'_{12} of the quasi plane tensor of the second rank (1.110) assume stationary values, are inclined 45° to the principal directions of this tensor in the $x_1\,x_2$ plane (see Fig. 1.9). Referring to equation (1.127), we get

$$\cos 2\tilde{\tilde{\phi}}_{11} = \pm\frac{A_{12}}{\left[\left(\dfrac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2\right]^{1/2}} \tag{1.128a}$$

$$\sin 2\tilde{\tilde{\phi}}_{11} = \pm\frac{(A_{11} - A_{22})}{2\left[\left(\dfrac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2\right]^{1/2}} \tag{1.128b}$$

Substituting relations (1.128) into (1.116b), we obtain

$$(A'_{12})_{\substack{\max\\ \min}} = \pm\left[\left(\frac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2\right]^{1/2} \tag{1.129}$$

The diagonal components of the tensor corresponding to the direction of the maximum and minimum values of the non-diagonal component A'_{12} are obtained by substituting relations (1.128) into equation (1.116a). Thus,

$$\tilde{\tilde{A}}_{11} = \tilde{\tilde{A}}_{22} = \frac{A_{11} + A_{22}}{2} \tag{1.130}$$

1.12 Mohr's Circle for Quasi Plane, Symmetric Tensors of the Second Rank

In this section we present a graphical interpretation of equations (1.116). These equations give the transformation of the components of the quasi plane form (1.110) of a symmetric tensor of the second rank from one set of mutually perpendicular axes x_1, x_2, x_3 to another obtained by rotating the set x_1, x_2, x_3 about the x_3 axis. The first two of equations (1.116) can be rewritten as

$$\begin{aligned} A'_{11} - \frac{1}{2}(A_{11} + A_{22}) &= \frac{1}{2}(A_{11} - A_{22})\cos 2\phi_{11} + A_{12}\sin 2\phi_{11} \\ A'_{12} &= -\frac{1}{2}(A_{11} - A_{22})\sin 2\phi_{11} + A_{12}\cos 2\phi_{11} \end{aligned} \tag{1.131}$$

Equations (1.131) are the parametric equations of a circle. We will prove that this is so by eliminating ϕ_{11} from equations (1.131). In order to accomplish this, we square both sides of equations (1.131), and we add and simplify the resulting expression. Thus, we obtain

$$(A'_{11} - a)^2 + (A'_{12})^2 = R^2 \tag{1.132}$$

where

$$a = \frac{1}{2}(A_{11} + A_{22}) \qquad R^2 = \left(\frac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2 \tag{1.133}$$

Taking into account that $A_{33} = A_{33}'$ and referring to relations (1.78) and (1.79), we can show that

$$2a = A_{11} + A_{22} = A'_{11} + A'_{22} \tag{1.134a}$$

$$R^2 = \left(\frac{A_{11} - A_{22}}{2}\right)^2 + A_{12}^2 = \left(\frac{A'_{11} - A'_{22}}{2}\right)^2 + \left(A'_{12}\right)^2 \tag{1.134b}$$

That is, the same values of a and R are obtained from the components of the tensor with respect to any set of orthogonal axes in the plane x_1x_2. For given values of the components of the quasi plane symmetric tensor of the second rank (A_{11}, A_{22}, A_{12}) with respect to a system of axes x_1, x_2, the quantities a and R are known constants and equation (1.132) is the familiar equation of a circle of radius R with the center at the point $(a, 0)$ plotted in the plane of the axes A'_{11}, A'_{12}. The coordinates of any point of this circle are the components A'_{11} and A'_{12} of the tensor A with respect to the set of orthogonal axes x'_1, x'_2. This circle, obtained in 1889 by the German professor Otto Mohr, is referred to in the literature as *Mohr's circle*.

Suppose the Mohr's circle shown in Fig. 1.10b was constructed from the specified values of the components A_{11}, A_{22}, A_{12} of a quasi plane symmetric tensor of the second rank referred to the set of axes x_1, x_2. Consider another set of axes x_1', x_2' in the $x_1\, x_2$ plane which, as shown in Fig. 1.10a, is obtained by rotating the x_1, x_2 axes counterclockwise by the angle ϕ_{11} about the x_3 axis. We shall determine the components A_{11}', A_{12}' of the tensor geometrically, with the aid of Mohr's circle. We designate the point on Mohr's circle whose coordinates are A_{11}, A_{12} by X_1 and the point whose coordinates are A_{11}', A_{12}' by X_1' and we suppose that X_1' is located clockwise from point X_1. *Here we are establishing the convention that if the x_1' axis is located counterclockwise from the x_1 axis, point X_1' on Mohr's circle, whose coordinates are the components A_{11}', A_{12}' of the tensor, lies on Mohr's circle clockwise from point X_1.* The angle between the radial lines to the points X_1 and X_1' of Mohr's circle is designated by θ, while the angle between the radial line to point X_1 and the A_{11}' axis is designated by ψ. From geometric considerations, referring to Fig. 1.10b, we may write

$$A_{12}' = R\sin(\psi - \theta) \tag{1.135}$$

$$A_{11}' = \frac{1}{2}(A_{11} + A_{22}) + CB = \frac{1}{2}(A_{11} + A_{22}) + R\cos(\psi - \theta) \tag{1.136}$$

$$\cos\psi = \frac{CD}{R} = \frac{1}{2R}(A_{11} - A_{22}) \tag{1.137}$$

$$\sin\psi = \frac{A_{12}}{R} \tag{1.138}$$

Expansion of the cos $(\psi - \theta)$ and sin $(\psi - \theta)$ in relations (1.135) and (1.136) and use of equations (1.137) and (1.138) result in the following relations:

$$\begin{aligned} A_{11}' &= \frac{1}{2}(A_{11} + A_{22}) + \frac{1}{2}(A_{11} - A_{22})\cos\theta + A_{12}\sin\theta \\ A_{12}' &= -\frac{1}{2}(A_{11} - A_{22})\sin\theta + A_{12}\cos\theta \end{aligned} \tag{1.139}$$

Comparing the above relations with (1.116a) and (1.116b), we see that they become identical if we set

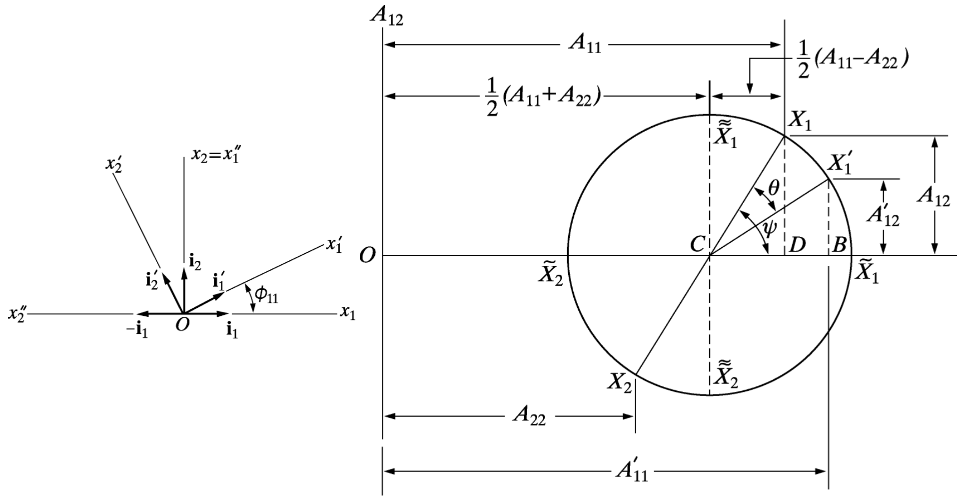
$$\theta = 2\phi_{11} \tag{1.140}$$

On the basis of foregoing presentation, we conclude that:

1. The components A_{11}', A_{12}' and A_{11}, A_{12} of the quasi plane symmetric tensor of the second rank $[A]$ referred to the axes x_1', x_2' and x_1, x_2 are the coordinates of points X_1' and X_1, respectively, of Mohr's circle.

2. If the axes x_1', x_2' are obtained by rotating the axes x_1, x_2 counterclockwise about the x_3 axis by an angle ϕ_{11}, point X_1' is located on Mohr's circle by an angle $2\phi_{11}$ clockwise from point X_1.

3. Referring to Fig. 1.10b, we note that point X_2 lies on the opposite end of the diameter



(a) Location of the x_1', x_2' axes (b) Mohr's circle

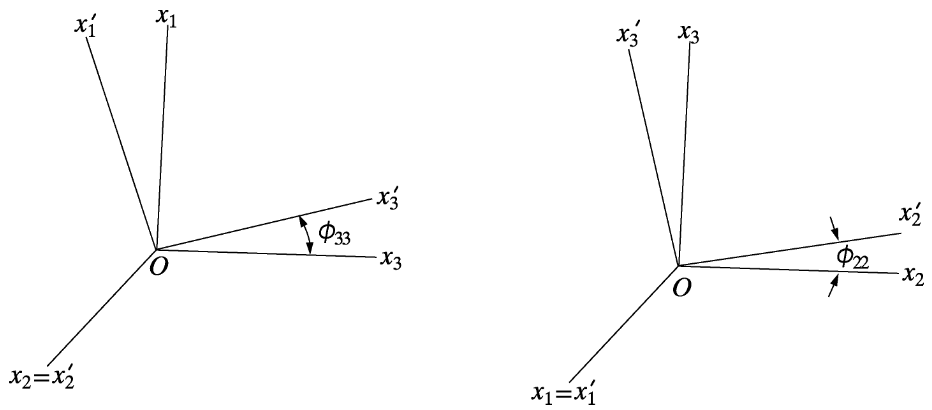
Figure 1.10 Mohr's circle.

of Mohr's circle which passes through point X_1. That is, the radial line to point X_1 must be turned clockwise by $\theta = 180°$ in order to coincide with the radial line to point X_2. Consequently, the coordinates of point X_2 on Mohr's circle represent the components A''_{11} and A''_{12} of the tensor with respect to the system of axes x''_1, x''_2 obtained by rotating the axes x_1, x_2 counterclockwise by 90° (see Fig. 1.10a). However, as discussed in Section 1.10, the components A''_{11} and A''_{12} of the tensor are equal to its components A_{22} and $-A_{12}$ respectively [see relation (1.118)]. Thus, if we know the components of a quasi plane symmetric tensor of the second rank with respect to a set of axes x_1, x_2, we can plot two points [X_1 (A_{11}, A_{12}) and X_2 (A_{22}, $-A_{12}$)] located on the same diameter of Mohr's circle. These two points and the knowledge that the center of the circle is on the A'_{11} axis are sufficient to specify Mohr's circle.

Referring to Fig. 1.10b, it is apparent that the abscissa of point $\tilde{X}_1$ of Mohr's circle represents the maximum value of the diagonal components of the tensor with respect to any direction in the $x_1\, x_2$ plane. Moreover, the ordinates of points $\tilde{\tilde{X}}_1$ and $\tilde{\tilde{X}}_2$ of Mohr's circle represent the maximum and minimum values, respectively, of the non-diagonal components of the tensor with respect to any set of axes in the $x_1 x_2$ plane.

Notice that the x_3 and x_1 axes must be taken as shown in Fig. 1.11a, when the quasi plane form of the tensor is given as

$$[\,A\,] = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{bmatrix} \tag{1.141}$$



(a) x_3 and x_1 axes (b) x_2 and x_3 axes

Figure 1.11 Right-handed axes of reference used with Mohr's circle.

In this case we denote by X_3 the point on Mohr's circle whose coordinates are the components A_{33} and A_{13} of the tensor and by X_1 the point on Mohr's circle whose coordinates are the components A_{11} and $-A_{13}$ of the tensor and we proceed as previously. Moreover, notice that the x_2 and x_3 axes must be taken as shown in Fig. 1.11b, when the quasi plane form of the tensor is given as

$$[\,A\,] = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix} \tag{1.142}$$

In this case we denote by X_2 the point on Mohr's circle whose coordinates are the components A_{22} and A_{23} of the tensor and by X_3 the point on Mohr's circle whose coordinates are the components A_{33} and $-A_{32}$ of the tensor and we proceed as previously.

The method of employing known elementary concepts in order to solve a problem which involves concepts which are more complicated and difficult to visualize is referred to in the literature as *an analogy*. In the case of Mohr's circle, elementary geometry is used to solve the problem of transformation of the components of a quasi plane symmetric tensor of the second rank. Mohr's circle could be used as a graphical solution. However, its real value is a means of visualizing the transformation of the components of the quasi plane form (1.110) of a symmetric tensor of the second rank from one set of axes to another. In what follows we illustrate the use of Mohr's circle by an example.

Example 5 The components of a tensor of the second rank referred to the system of axes x_j are given by

$$[A] = \begin{bmatrix} -0.8 & 0.8 & 0 \\ 0.8 & 0.4 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$$

(a) Find the principal values of the diagonal components of the tensor and the directions of the principal axes.

(b) Compute the components of the tensor A'_{11}, A'_{22}, A'_{12} with respect to a set of axes x'_1, x'_2, x'_3, such that x'_1, x'_2 are located in the $x_1\,x_2$ plane as shown in Fig. a.

(c) Compute the stationary values of the non-diagonal components of the tensor.

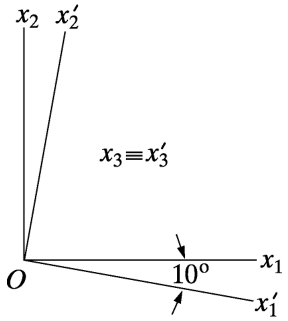


Figure a Location of the system of axes x'_i.

Solution

Part a

We draw the two axes of reference the A'_{11} axis and the A'_{12} axis, as shown in Fig. b. Then we plot point X_1 $(-0.8, 0.8)$ whose coordinates are the components of the tensor A_{11} and A_{12} and point X_2 $(0.4, -0.8)$ whose abscissa is A_{22} and its ordinate is $-A_{21}$. Points X_1 and X_2 lie on the same diameter of Mohr's circle. The center of this circle is on the A'_{11} axis at a distance $1/2\,(A_{11}+A_{22}) = -0.2$ from the origin. Moreover, from geometric consideration, the radius of this circle is

$$R = [(0.8)^2 + (0.6)^2]^{1/2} = 1.0 \tag{a}$$

The maximum and minimum values of the diagonal components of the tensor along directions located in the $x_1\,x_2$ plane may be computed by referring to Mohr's circle, shown in Fig. b. The maximum value of the diagonal components of the tensor is the abscissa of point $\tilde{X}_1$, whereas the minimum value is the abscissa of point $\tilde{X}_2$. Thus,

$$\begin{aligned} A_1 &= \overline{B\tilde{X}_1} - \overline{OB} = 1.0 - 0.2 = 0.8 \\ A_2 &= -(\overline{OB} + \overline{B\tilde{X}_2}) = -(0.2 + 1.0) = -1.2 \end{aligned} \tag{b}$$

Notice that in this example the maximum value of the diagonal components of the tensor is not $A_1 = 0.8$ but rather $A_{33} = 1.0$. The angle ϕ_{11} from the x_1 axis to the principal axis $\tilde{x}_1$ may be computed geometrically by referring to Fig. b, as

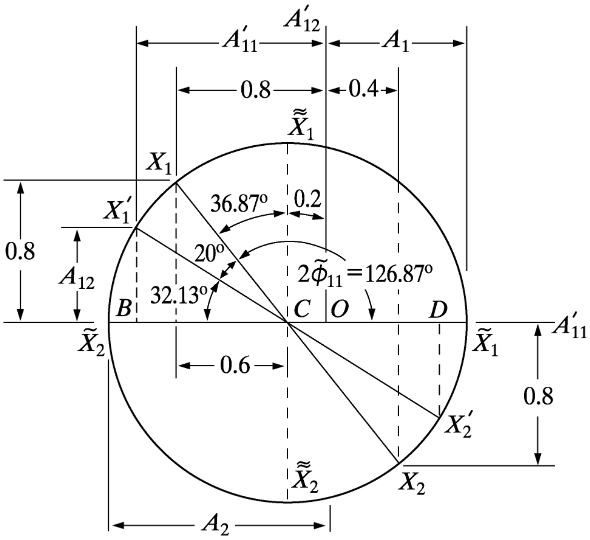


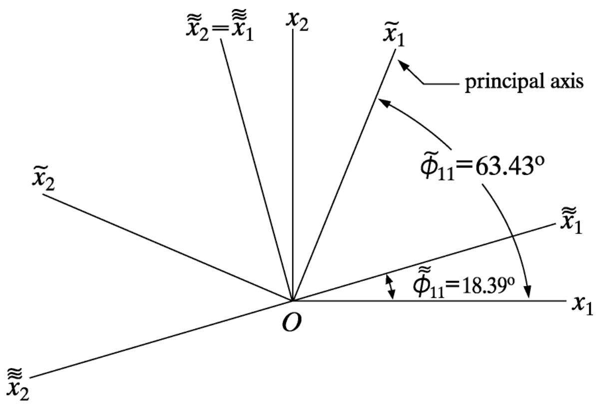
Figure b Mohr's circle.

$$\tan 2\tilde{\phi}_{11} = \tan \angle X_1 B \tilde{X}_1 = -\frac{4}{3} \tag{c}$$

Hence

$$2\tilde{\phi}_{11} = 126.87° \qquad \text{or} \qquad \tilde{\phi}_{11} = 63.43° \tag{d}$$

One of the principal directions is the $\mathbf{i}_3$, whereas the other two lie in the $x_1\,x_2$ plane. Referring to Mohr's circle of Fig. b, we see that point $\tilde{X}_1$ is located 126.87° clockwise from point X_1. Consequently, the axis $\tilde{x}_1$ associated with the principal value A_1 of the diagonal component of the tensor is located ϕ_{11} = 63.43° counterclockwise from the x_1 axis. The principal axes of the tensor are shown in Fig. c.



The non-diagonal components of the tensor assume their maximum value $(A_{12})_{max}$ with respect to the pair of axes $\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2$ and their minimum value $(A_{21})_{min}$ with respect to the pair of $\tilde{\tilde{x}}_1, \tilde{\tilde{x}}_2$ axes.

Figure c Location of the principal axes.

Part b

We locate on Mohr's circle the point X_1' counterclockwise from point X_1 so that the angle $\triangle X_1CX_1'$ is equal to 20°. The coordinates of point X_1' are equal to the components A_{11}', A_{12}' of the tensor **A** referred to the x_1', x_2' axes, obtained by 10° clockwise rotation of the x_1, x_2 axes. The components of the tensor A_{22}', A_{21}' are the abscissa and the negative of the ordinate of point X_2' of Mohr's circle. Referring to Fig. b, the components of the tensor with respect to the x_1', x_2' axes are

$$\begin{aligned} A_{11}' &= -\left(\overline{OC} + \overline{CB}\right) = -[0.2 + R\cos(33.13°)] = -1.036 \\ A_{12}' &= R\sin(33.13°) = 0.547 \\ A_{22}' &= \overline{CD} - \overline{CO} = R\cos(33.13°) - 0.2 = 0.637 \end{aligned} \tag{e}$$

Part c

Referring to Fig. b, the maximum non-diagonal components of the tensor are the ordinates of point $\tilde{\tilde{X}}_1$ and $\tilde{\tilde{X}}_2$. Thus,

$$\begin{aligned} (\tilde{\tilde{A}}_{12})_{max} &= 1.0 \\ (\tilde{\tilde{A}}_{12})_{min} &= -1.0 \end{aligned} \tag{f}$$

The axes $\approx{x}_1$, $\approx{x}_2$ and $\tilde{\tilde{x}}_1$, $\tilde{\tilde{x}}_2$ with respect to which the stationary values of non-diagonal components of the tensor occur are shown in Fig. c. Referring to Fig. b, the values of the diagonal components corresponding to the $\approx{x}_1$, $\approx{x}_2$ axes are

$$\tilde{\tilde{A}}_{11} = \tilde{\tilde{A}}_{22} = -0.2$$

1.13 Maximum Values of the Non-Diagonal Components of a Symmetric Tensor of the Second Rank

Referring to relations (1.107) and (1.110), we see that the diagonal form of a tensor of the second rank is a special case of the quasi plane form of the tensor. That is, a form which is quasi plane with respect to the three planes specified by the pairs of principal axes $\tilde{x}_1$, $\tilde{x}_2$ or $\tilde{x}_1$, $\tilde{x}_3$ or $\tilde{x}_2$, $\tilde{x}_3$. From our discussion in Section 1.11 we know that when the components of a tensor assume a plane form with respect to a pair of axes x_1, x_2, the maximum values of the non-diagonal components of the tensor with respect to any set of two mutually perpendicular axes located in the x_1x_2 plane occur with respect to a set of two axes $\tilde{\tilde{x}}_1$, $\tilde{\tilde{x}}_2$ located at 45° from the principal axes $\tilde{x}_1$, $\tilde{x}_2$ (see Fig. 1.12).

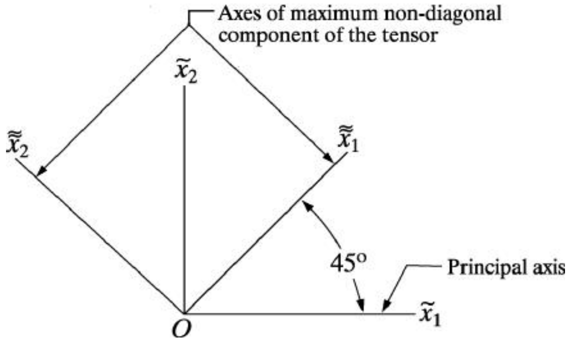


Figure 1.12 Location of the axes with respect to which the non-diagonal components of a tensor assume stationary values.

Consequently when the components of a tensor are referred to its principal axes $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, we can easily establish the maximum and minimum values of its non-diagonal components with respect to any set of axes in the $\tilde{x}_1\tilde{x}_2$, the $\tilde{x}_1\tilde{x}_3$ and the $\tilde{x}_2\tilde{x}_3$ planes. That is, referring to relation (1.129), we have

$$
\begin{aligned}
(A'_{12})_{\max} &= \pm\frac{A_1 - A_2}{2} \\
(A'_{13})_{\max} &= \pm\frac{A_1 - A_3}{2} \\
(A'_{23})_{\max} &= \pm\frac{A_2 - A_3}{2}
\end{aligned}
\tag{1.143}
$$

1.14 Problems

1. Consider points P_1 $(1, -1, 2)$, P_2 $(1, 0, -3)$ and P_3 $(-1, 2, 1)$. The units are in meters.
   (a) Determine the unit vector acting from point P_1 to point P_2.
   (b) Determine the angles $\angle P_1OP_2$ and $\angle OP_1P_2$, where O is the origin of the axes of reference.
   (c) Determine the unit vector normal to the plane specified by the points O, P_1, P_2.
   (d) Compute the volume of the parallelepiped whose edges are OP_1, OP_2 and OP_3.
   Ans. (a) $1/\sqrt{26}\ (\mathbf{i}_2 - 5\mathbf{i}_3)$ (b) 130.20°, 28.27° (c) $1/\sqrt{35}\ (3\mathbf{i}_1 + 5\mathbf{i}_2 + \mathbf{i}_3)$ (d) 8 m³

2. Consider the rectangular system of axes x'_i $(i = 1, 2, 3)$ specified with respect to the rectangular system of axes x_j by the transformation matrix

$$[\Lambda_s] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The cartesian components of a vector referred to the rectangular system of axes x_j are $\mathbf{a} = 4\mathbf{i}_1 + 3\mathbf{i}_2$. Find the components of vector $\mathbf{a}$ with respect to the rectangular system of axes x'_i. Ans. $\mathbf{a}' = 4\mathbf{i}_1 + (3\sqrt{2}/2)\mathbf{i}_2 - (3\sqrt{2}/2)\mathbf{i}_3$

3. Consider the vectors

$$\begin{aligned} \mathbf{a} &= 3\mathbf{i}_1 - 4\mathbf{i}_2 + 2\mathbf{i}_3 \quad (\text{m}) \\ \mathbf{b} &= 4\mathbf{i}_1 - 2\mathbf{i}_2 - 6\mathbf{i}_3 \quad (\text{m}) \\ \mathbf{c} &= \mathbf{i}_1 - 6\mathbf{i}_2 + \mathbf{i}_3 \quad (\text{m}) \end{aligned}$$

Find

(a) The angle between the vectors **a** and **b**

(b) The component of the vector **b** in the direction of vector **a**

(c) The area of the parallelogram whose sides are the vectors **a** and **b**

(d) The volume of the parallelepiped specified by the vectors **a**, **b** and **c**

Ans. (a) $\theta = 78.55°$ (b) 1.49 m (c) 39.5 m^2 (d) 118 m^3

4. Find the shortest distance from point $P\,(-1, 2, 0)$ to the line passing through points $A\,(-2, 1, 3)$ and $B\,(-1, 2, -1)$. Ans. 1/3

5. The transformation matrix of a right-handed cartesian system of axes x'_i ($i = 1, 2, 3$), with respect to the right-handed cartesian system of axes x_j ($j = 1, 2, 3$) is,

$$[\Lambda_s] = \begin{bmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

(a) Compute $\lambda_{31}, \lambda_{32}, \lambda_{33}$. Ans. $\lambda_{31} = 0,\ \lambda_{32} = \pm 3/5,\ \lambda_{33} = \pm 4/5$

(b) If the components of a tensor with respect to the rectangular system of axes x_j are given by

$$[A] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute its components with respect to the rectangular system of axes x'_i.

Ans. $[A'] = \begin{bmatrix} 4.68 & 0 & 2.24 \\ 0 & 2 & 0 \\ -2.76 & 0 & -0.68 \end{bmatrix}$

6. The transformation matrix of a right-handed cartesian system of axes x'_i ($i = 1, 2, 3$), with respect to the right-handed cartesian system of axes x_j ($j = 1, 2, 3$) is,

$$[\Lambda_s] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{5\sqrt{2}} & -\frac{4}{5\sqrt{2}} \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ \frac{1}{\sqrt{2}} & \frac{3}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{bmatrix}$$

(a) Find the coordinates of a point in the system of axes x'_i if its coordinates in the system of axes x_j are $P(-2, 1, 0)$. Ans. $-1.84, 0.80, -0.99$

(b) What are the components of a tensor $\mathbf{A} = 2\mathbf{i}_1\mathbf{i}_2 - \mathbf{i}_2\mathbf{i}_2 - 2\mathbf{i}_3\mathbf{i}_2$ in the system of axes x'_i.

Ans. $\begin{bmatrix} -1.26 & 2.376 & 1.26 \\ -0.17 & 0.320 & 0.17 \\ 0.06 & -0.113 & -0.06 \end{bmatrix}$

(c) What is the equation of the plane $x_1 - x_2 + 2x_3 = 1$ in the system of axes x'_i.

Ans. $-2x'_2 + \sqrt{2}x'_3 = 1$

7. The transformation matrix of a right-handed cartesian system of axes x'_i ($i = 1, 2, 3$), with respect to the right-handed Cartesian system of axes x_j ($j = 1, 2, 3$) is,

$$[\Lambda_s] = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

(a) Find the coordinates of a point in the system of axes x'_i if its coordinates in the system of axes x_j are $P(3, 2, -4)$. Ans. $(4.998, 1.513, -1.315)$

(b) What are the components of a tensor $\mathbf{A} = -2\mathbf{i}_1\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_2 - 2\mathbf{i}_2\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_1$, in the system of axes x'_i.

$$\text{Ans.} \begin{bmatrix} -0.471 & -1.288 & 0.345 \\ 1.390 & -0.658 & -0.025 \\ -0.919 & -0.503 & 2.130 \end{bmatrix}$$

(c) What is the equation of the plane $2x_1 + 3x_2 - x_3 = 1$ in the system of axes x'_i.

Ans. $1.971x'_1 + 2.8678x'_2 - 1.3749x'_3 = 1$

8. Consider the following symmetric tensors of the second rank:

$$[A] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad [B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad [C] = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 4 \end{bmatrix}$$

Using relations (1.122), (1.124), (1.127) and (1.129), compute

(a) The principal directions and the principal values of the diagonal components of these tensors and show the results on a sketch

Ans. $\tilde{A}_1 = 3+\sqrt{2}$, $\tilde{A}_2 = 3-\sqrt{2}$, $\tilde{A}_3 = 3$, $\tilde{\phi}_{11} = 67.50°$ counterclockwise

$\tilde{B}_1 = 0$, $\tilde{B}_2 = \dfrac{5+\sqrt{5}}{2}$, $\tilde{B}_3 = \dfrac{5-\sqrt{5}}{2}$, $\tilde{\phi}_{22} = 31.72°$ clockwise

$\tilde{C}_1 = 3+\sqrt{10}$, $\tilde{C}_2 = 0$, $\tilde{C}_3 = 3-\sqrt{10}$, $\tilde{\phi}_{33} = 35.78°$ clockwise

(b) The maximum values of the non-diagonal components of these tensors and show on a sketch the axes with respect to which they occur

Ans. $(A'_{12})_{\max} = \sqrt{2}$, $\tilde{\tilde{\phi}}_{11} = 22.50°$ counterclockwise

$(B'_{23})_{\max} = \dfrac{\sqrt{5}}{2}$, $\tilde{\tilde{\phi}}_{22} = 13.28°$ counterclockwise

$(C'_{13})_{\max} = \sqrt{10}$, $\tilde{\tilde{\phi}}_{33} = 9.22°$ counterclockwise

(c) The diagonal component of these tensors in the direction of the unit vector

$$\mathbf{i_n} = \frac{1}{\sqrt{3}}(\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3)$$

Ans. $A_{nn} = 11/3$ $B_{nn} = 7/3$ $C_{nn} = 4$

9. Consider the following symmetric tensors of the second rank:

$$[A] = \begin{bmatrix} -3 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 6 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & -2 \end{bmatrix} \qquad [C] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 4 & 6 \end{bmatrix}$$

$$[D] = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [E] = \begin{bmatrix} 0 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad [F] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & -3 & -12 \end{bmatrix}$$

Using Mohr's circle, compute

(a) The principal directions and the principal values of the diagonal components of these tensors and show the results on a sketch

Ans.
$$\begin{array}{llll}
\tilde{A}_1 = 2.7, & \tilde{A}_2 = -3.7, & \tilde{\phi}_{11} = 109.33° & \text{counterclockwise} \\
\tilde{B}_3 = 7, & \tilde{B}_1 = -3, & \tilde{\phi}_{33} = 108.43° & \text{counterclockwise} \\
\tilde{C}_2 = 7.657, & \tilde{C}_3 = -3.657, & \tilde{\phi}_{22} = 67.500° & \text{counterclockwise} \\
\tilde{D}_1 = -3.172, & \tilde{D}_2 = -8.828, & \tilde{\phi}_{11} = 22.500° & \text{counterclockwise} \\
\tilde{E}_1 = 0.91, & \tilde{E}_2 = -9.91, & \tilde{\phi}_{11} = 16.845° & \text{counterclockwise} \\
\tilde{F}_2 = -3, & \tilde{F}_3 = -13, & \tilde{\phi}_{22} = 18.430° & \text{clockwise}
\end{array}$$

(b) The maximum values of the non-diagonal components of these tensors and show on a sketch the axes with respect to which they occur

Ans.
$$\begin{array}{lll}
(A_{12})_{\max} = 3.2, & \tilde{\tilde{\phi}}_{11} = 64.330° & \text{counterclockwise} \\
(B_{31})_{\max} = 5, & \tilde{\tilde{\phi}}_{33} = 63.440° & \text{counterclockwise} \\
(C_{23})_{\max} = 5.657, & \tilde{\tilde{\phi}}_{22} = 22.500° & \text{counterclockwise} \\
(D_{12})_{\max} = 2.828, & \tilde{\tilde{\phi}}_{11} = 22.500° & \text{clockwise} \\
(E_{12})_{\max} = 5.41, & \tilde{\tilde{\phi}}_{11} = 28.16° & \text{clockwise} \\
(F_{23})_{\max} = 5, & \tilde{\tilde{\phi}}_{22} = 63.440° & \text{clockwise}
\end{array}$$

(c) The diagonal component of these tensors in the direction of the unit vector

$$\mathbf{i}_n = \frac{1}{\sqrt{3}}(\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3)$$

Show the results on a sketch.

Ans. $A_{nn} = -4/3$, $B_{nn} = 3.33$, $C_{nn} = -4/3$, $D_{nn} = -2.667$, $E_{nn} = -1/3$, $F_{nn} = -10/3$

10. Using Mohr's circle, compute the components of the tensors [A], [B], [D] and [E] of Problem 1.9 with respect to the directions shown in Fig. 1P10a and of the tensor [B] of problem 1.9 with respect to the direction shown in Fig. 1P10b.

Ans. $$[A'] = \begin{bmatrix} -3.4808 & 1.1640 & 0 \\ 1.1640 & 2.4808 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [D'] = \begin{bmatrix} -3.268 & -0.732 & 0 \\ -0.732 & -8.732 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [E'] = \begin{bmatrix} 0.348 & -2.40 & 0 \\ -2.40 & -9.348 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Ans. $$[B'] = \begin{bmatrix} 6.60 & 0 & 1.964 \\ 0 & 0 & 0 \\ 1.964 & 0 & -2.6 \end{bmatrix}$$

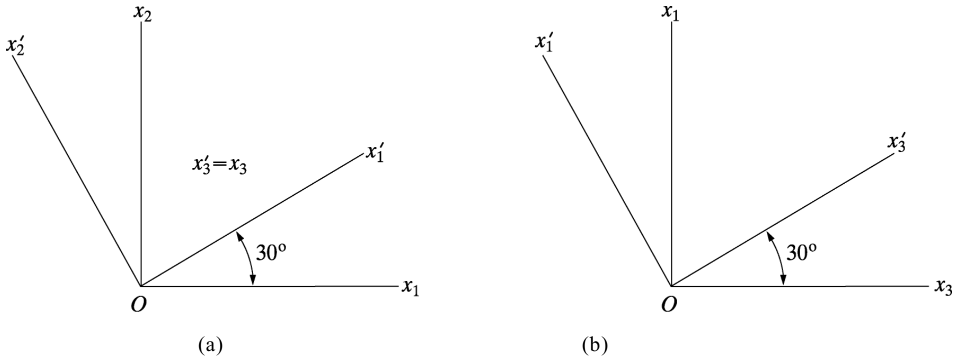


Figure 1P10

11. Consider the tensors $[A]$, and $[E]$ whose components with respect to the rectangular system of axes x_j are given in Problem 1.9. Compute their components with respect to the rectangular system of axes x_i' whose transformation matrix with respect to the rectangular system of axes x_j is

$$[\Lambda_s] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & \sin 30° \\ 0 & -\sin 30° & \cos 30° \end{bmatrix}$$

Ans. $[A'] = \frac{1}{4}\begin{bmatrix} -12 & -4\sqrt{3} & 4 \\ -4\sqrt{3} & 7 & -\sqrt{3} \\ 4 & -\sqrt{3} & 5 \end{bmatrix}$ $\quad [E'] = \frac{1}{4}\begin{bmatrix} 0 & 6\sqrt{3} & -6 \\ 6\sqrt{3} & -25 & 11\sqrt{3} \\ -6 & 11\sqrt{3} & -3 \end{bmatrix}$

12. and 13. The state of stress acting on a particle of a beam is given in Fig. 1P12. Compute at this particle:

(a) The components of stress acting on the plane normal to the unit vector

$$\mathbf{i}_n = \frac{3}{5}\mathbf{i}_1 + \frac{4}{5}\mathbf{i}_3$$

(b) The maximum normal component of stress and show on a sketch the plane on which it acts

(c) The maximum shearing component of stress and show on a sketch the plane on which it acts

(d) The normal component of stress acting on the plane normal to the unit vector

$$\mathbf{i}_n = \frac{1}{\sqrt{3}}(\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)$$

Repeat with the state of stress shown in Fig. 1P13.

Ans. 12 (a) $\tau_{nn} = 37.6$ MPa, $\tau_{ns} = 46.8$ MPa
(b) $(\tau'_{33})_{\max} = 70$ MPa, $(\tau'_{33})_{\min.} = -30$ MPa, $\tilde{\phi}_{33} = 71.57°$ counterclockwise
(c) $(\tau'_{31})_{\max} = 50$ MPa, $\tilde{\phi}_{33} = 26.57°$ counterclockwise
(d) $\tau_{nn} = 100/3$ MPa

Ans. 13 (a) $\tau_{nn} = -97.6$ MPa, $\tau_{ns} = -46.8$ MPa
(b) $(\tau_{nn})_{\max} = -30$ MPa, $(\tau_{nn})_{\min} = -80$ MPa, $\tilde{\phi}_{33} = 18.43°$ clockwise
(c) 50 MPa, $\tilde{\phi}_{33} = 63.43°$ clockwise
(d) $\tau_{nn} = -220/3$ MPa

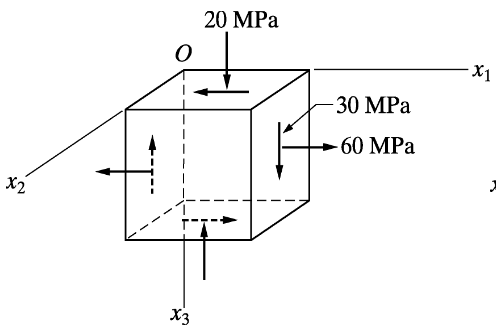


Figure 1P12

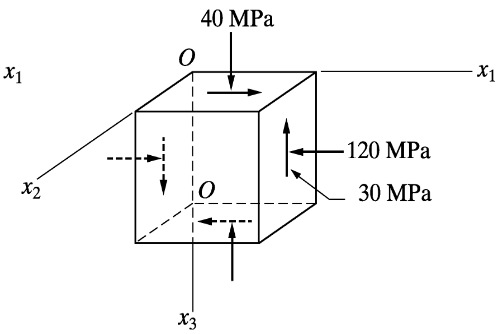


Figure 1P13

14. The components of a symmetric tensor of the second rank with respect to the system of axes x_1, x_2, x_3 are

$$[A] = \begin{bmatrix} 3 & 4 & 1 \\ 4 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Compute the stationary values of the diagonal components of the tensor and the direction cosines of the system of the axes with respect to which they occur.

Ans. $\tilde{A}_1 = 6.3863$ $\quad \tilde{A}_2 = -0.1963$ $\quad \tilde{A}_3 = -3.19$
$\lambda^{(1)}_{11} = \pm 0.7639$ $\quad \lambda^{(2)}_{21} = \pm 0.4484$ $\quad \lambda^{(3)}_{31} = \pm 0.4641$
$\lambda^{(1)}_{12} = \pm 0.5720$ $\quad \lambda^{(2)}_{22} = \mp 0.1375$ $\quad \lambda^{(3)}_{32} = \mp 0.8086$
$\lambda^{(1)}_{13} = \pm 0.2988$ $\quad \lambda^{(2)}_{23} = \mp 0.8833$ $\quad \lambda^{(3)}_{33} = \mp 0.3614$

(b) Compute the maximum value of the non-diagonal components of the tensor.

Ans. $(A_{12}')_{\max} = 3.2913$, $(A_{13}')_{\max} = 4.7882$, $(A_{21}')_{\max} = 1.4969$

15. The components of a symmetric tensor of the second rank with respect to the system of axes x_1, x_2, x_3 are

$$[A] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

(a) Compute the stationary values of the diagonal components of the tensor and the direction cosines of the system of the axes to which they are referred.

Ans. $\tilde{A}_1 = 6.3872 \quad \tilde{A}_2 = 2.3084 \quad \tilde{A}_3 = -1.6956$
$\lambda_{11}^{(1)} = 0.2114 \quad \lambda_{31}^{(2)} = 0.7636 \quad \lambda_{21}^{(3)} = 0.610$
$\lambda_{12}^{(1)} = -0.8286 \quad \lambda_{32}^{(2)} = -0.1911 \quad \lambda_{22}^{(3)} = 0.526$
$\lambda_{13}^{(1)} = -0.5185 \quad \lambda_{33}^{(2)} = 0.6168 \quad \lambda_{23}^{(3)} = -0.592$

(b) Compute the maximum value of the non-diagonal component of the tensor.

Ans. $(A_{12}')_{\max} = 2.0394$, $(A_{13}')_{\max} = 4.0414$, $(A_{21}')_{\max} = 2.002$

16. The component of a symmetric tensor of the second rank with respect to the system of axes x_1, x_2, x_3 are

$$\begin{bmatrix} 0 & 4 & 3 \\ -4 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(a) Compute the stationary values of the normal component of the tensor.

(b) Compute the component of the tensor in the direction of the axes x_1', x_2', x_3', shown in Fig. 1P16

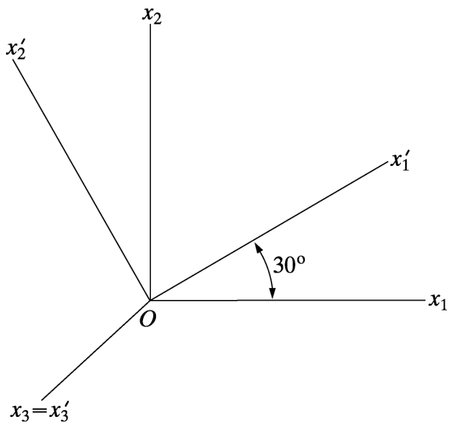


Figure 1P16

Ans. (a)

$A_1 = 0 \qquad A_2 = 5 \qquad A_3 = -5$

$$\mathbf{i}_1^{(1)} = -\frac{3}{5}\mathbf{i}_2 + \frac{4}{5}\mathbf{i}_3$$

$$\mathbf{i}_2^{(2)} = \frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{4}{5\sqrt{2}}\mathbf{i}_2 + \frac{3}{5\sqrt{2}}\mathbf{i}_3$$

$$\mathbf{i}_3^{(3)} = -\frac{1}{\sqrt{2}}\mathbf{i}_1 + \frac{4}{5\sqrt{2}}\mathbf{i}_2 + \frac{3}{5\sqrt{2}}\mathbf{i}_3$$

Ans. (b)

$$[A'] = \begin{bmatrix} 3.464 & 2 & 2.598 \\ 2 & -3.464 & -1.5 \\ 2.598 & -1.5 & 0 \end{bmatrix}$$

References

For a more detailed presentation of vector algebra, see Weatherburn, C.E., *Elementary Vector Analysis with applications to Geometry and Physics,* G. Bell and Sons Ltd., London, 1931.

Chapter 2

Strain and Stress Tensors

2.1 The Continuum Model

Bodies are composed of a large number of discrete particles (atoms, molecules) in constant motion. Solids differ from liquids and liquids from gases in the spacing of these particles and in the amplitude of their motion.

In studying the behavior of bodies the assumption is usually made that the material is distributed in the space which it occupies without leaving gaps or empty spaces. In other words, *it is assumed that at every instant of time, there is a particle at every point of the space occupied by the body at that time*. This model is referred to in the literature as *continuum* and it is used in all engineering disciplines because it is mathematically convenient. It permits integration and differentiation of the quantities describing the behavior of a body which are functions of the space coordinates. An infinitesimal portion of a continuum is called *a particle*. *In this text we study the behavior of deformable solid bodies subjected to external loads on the basis of the continuum model.*

2.2 External Loads

Consider a body initially in a reference undeformed and unstressed state of mechanical[†] and thermal[††] equilibrium at the uniform temperature T_0. In this state the body is not subjected to external loads and heat does not flow in or out of it because its temperature is uniform. Subsequently, the body is subjected to one or more of the following external loads:

1. Body forces
2. Surface forces
3. A temperature field[†††] $T(x_1, x_2, x_3)$
4. Specified components of displacements of some particles of the surface of the body

† When a body is in a state of mechanical equilibrium, its particles do not accelerate; that is, the sum of the forces acting on any portion of the body and the sum of their moments about any point vanish.

†† When a body is in a state of thermal equilibrium, heat does not flow in or out of it; that is, the temperature of all its particles is the same.

††† In general the temperature changes from particle to particle. The totality of temperatures assigned to each particle of a body is referred to as the temperature field of the body.

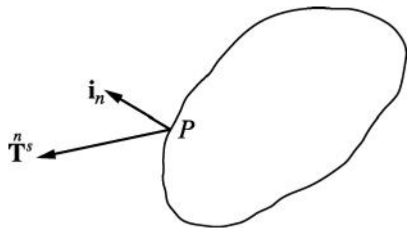


Figure 2.1 Surface traction on a particle of the surface of a body.

The points of the surface of the body where components of displacement are specified are called its *supports*. The forces exerted by the supports of a body on its particles are not known. They are called the *reactions* of the supports of the body.

The body forces are the result of the presence of a body in a force field and are distributed throughout its mass. The most important example of a body force is the weight of the body resulting from its presence in the earth's gravitational field. The measure of the body force acting on a particle of a body located at point P whose coordinates are x_1, x_2, x_3 is called the *specific body force at point P.* We denote it by $\mathbf{B}(x_1, x_2, x_3)$ and it is defined as

$$\mathbf{B} = \lim_{\Delta V \to 0} \frac{\Delta \mathbf{F}^B}{\Delta V} \tag{2.1}$$

where the volume ΔV includes point P as it approaches zero. $\Delta \mathbf{F}^B$ is the resultant body force acting on the particles of a small portion of the body of volume ΔV taken in the deformed state. However, as we shall see in Section 2.5, when the deformation of the body is small, the change of its volume due to its deformation is negligible compared to its undeformed volume. Consequently, we can approximate ΔV with the corresponding volume in the undeformed state.

The surface forces are exerted on the surface of a body through direct contact with other bodies. The measure of the surface force acting on a particle of the surface of a body located at point P whose coordinates are x_1^s, x_2^s, x_3^s is called the *surface traction at point P*. We denote it by $\overset{n}{\mathbf{T}}{}^s(x_1^s, x_2^s, x_3^s)$ and it is defined as

$$\overset{n}{\mathbf{T}}{}^s = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}^s}{\Delta S} \tag{2.2}$$

The superscript n indicates that the unit vector outward normal to the surface of the body at point P is denoted by $\mathbf{i}_n$. $\Delta \mathbf{F}^s$ is the resultant of the forces acting on a small portion of the surface of the body of area ΔS, which includes point P, as it approaches zero. In general the surface traction at point P need not be in the direction of the unit vector $\mathbf{i}_n$ (see Fig. 2.1).

Due to the application of the external loads described above the body deforms and reaches a second state of mechanical but not necessarily thermal equilibrium. We call the afore described initial reference state of equilibrium the *undeformed state* and the second state of mechanical equilibrium, the *deformed state*.

2.3 The Displacement Vector of a Particle of a Body

Consider an infinitesimal portion (particle) of a body, located in the undeformed state at point P_0 (x_1, x_2, x_3), whose position vector, referred to a fixed origin O, is designated by $\mathbf{r}_0 = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$ where $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ are unit vectors in the direction of the x_1, x_2 and x_3 axes, respectively. When the body reaches its deformed state, the particle under consideration moves to a point P (ξ_1, ξ_2, ξ_3) whose position vector, referred to the same fixed origin O, is designated by $\mathbf{r} = \xi_1\mathbf{i}_1 + \xi_2\mathbf{i}_2 + \xi_3\mathbf{i}_3$. Variation of the coordinates x_1, x_2, x_3 indicates a different particle which in the deformed state has different coordinates ξ_1, ξ_2, ξ_3. In other words, the coordinates ξ_i $(i = 1, 2, 3)$ are functions of x_1, x_2 and x_3. We assume that the functions $\xi_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ have continuous partial derivatives of any order required. Referring to Fig. 2.2, we define the *displacement vector* $\hat{\mathbf{u}} = \hat{u}_1\mathbf{i}_1 + \hat{u}_2\mathbf{i}_2 + \hat{u}_3\mathbf{i}_3$ of the particle of the body located at point P_0, prior to deformation, by the following vector equation:

$$\mathbf{r}(x_1, x_2, x_3) = \mathbf{r}_0 + \hat{\mathbf{u}}(x_1, x_2, x_3) \tag{2.3}$$

or

$$\xi_i(x_1, x_2, x_3) = x_i + \hat{u}_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3) \tag{2.4}$$

Notice that in relations (2.3) and (2.4) the coordinates of a particle in the deformed state are expressed in terms of its coordinates in the undeformed state. This is known as the *Lagrangian method* of describing the deformation of a body. In another method, the *Eulerian*, the coordinates of a particle in the undeformed state are expressed in terms of its coordinates in the deformed state. In the mechanics of solid bodies the Lagrangian method is more convenient and it is exclusively used in this text.

In general, the magnitude and direction of the displacement vector $\hat{\mathbf{u}}$ changes from

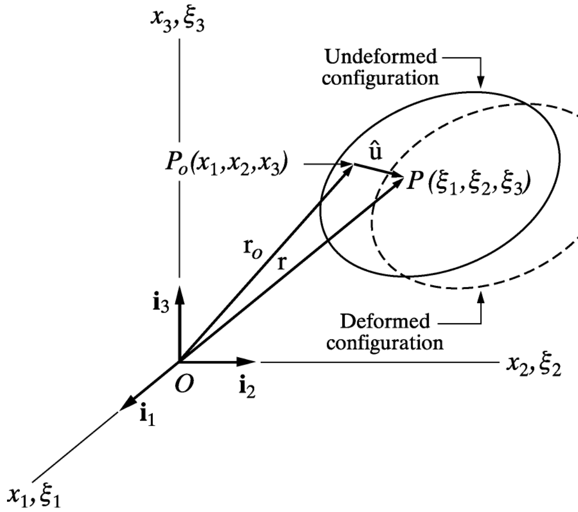


Figure 2.2 Displacement vector of a particle of a body.

particle to particle. The totality of the displacement vectors $\hat{\mathbf{u}}(x_1, x_2, x_3)$ assigned to each particle of a body is referred to as the *displacement field of the body*.

2.4 Components of Strain of a Particle of a Body

In this section we define certain quantities which specify completely the change of geometry (deformation) of a particle of a body as the body goes from its undeformed to its deformed state. These quantities are not affected by the rigid body motion of the particle. Consider a particle of a body which in the undeformed state is an orthogonal parallelepiped $P_0X_0Y_0Z_0$ with edges dx_1, dx_2, and dx_3, located at point $P_0\,(x_1, x_2, x_3)$ (see Fig. 2.3). In general, as the body goes from its undeformed to its deformed state its particles translate, rotate and deform (elongate, or shrink and distort). The translation and rotation of a particle are rigid-body motions. Material straight lines and planes of a body in its undeformed state are likely to become curves and non-planar surfaces, respectively, in its deformed state. However, *within the infinitesimal domain of a particle we assume that straight lines, planes and parallelism of straight lines and planes are preserved*. Therefore, in general, the deformed particle under consideration is a non-orthogonal parallelepiped *PXYZ* (see Fig. 2.3). Consequently, the geometry of the deformed particle is completely specified if the lengths of its three edges $\overline{PX}$, $\overline{PY}$ and $\overline{PZ}$ and its three angles $\measuredangle XPY$, $\measuredangle XPZ$ and $\measuredangle YPZ$ are specified. When the geometry of the deformed particle is known the change of its geometry due to its deformation can be established as the difference of its undeformed from its deformed geometry.

Referring to Fig. 2.3, we choose the following dimensionless quantities as the measure of the change of length of the three edges of the particle under consideration, due to its deformation

$$E_{11} = \frac{\overline{PX} - dx_1}{dx_1} \tag{2.5a}$$

$$E_{22} = \frac{\overline{PY} - dx_2}{dx_2} \tag{2.5b}$$

$$E_{33} = \frac{\overline{PZ} - dx_3}{dx_3} \tag{2.5c}$$

E_{ii} ($i = 1, 2, 3$) is called *the unit elongation or shrinkage in the direction of the axis x_i of the particle which was located in the undeformed state at point P_0*. On the basis of its definition E_{ii} represents the change of length divided by the undeformed length of a material line segment of infinitesimal length which in the undeformed state was located at point P_0 and was oriented in the direction of the axis x_i ($i = 1, 2, 3$). From relations (2.5), we see that the lengths of the edges of the deformed particle are

$$\overline{PX} = (1 + E_{11})dx_1 \tag{2.6a}$$

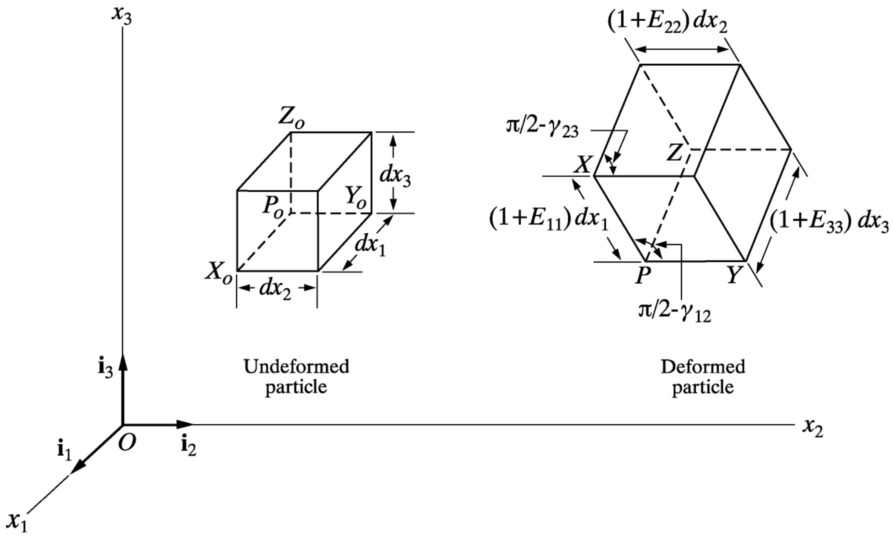


Figure 2.3 Deformation of a particle.

$$\overline{PY} = (1 + E_{22})dx_2 \tag{2.6b}$$

$$\overline{PZ} = (1 + E_{33})dx_3 \tag{2.6c}$$

Referring to Fig. 2.3, we choose the following quantities as a measure of the change of the angles of the deformed particle under consideration

$$\gamma_{12} = \gamma_{21} = \frac{\pi}{2} - \angle\ XPY \tag{2.7a}$$

$$\gamma_{13} = \gamma_{31} = \frac{\pi}{2} - \angle\ XPZ \tag{2.7b}$$

$$\gamma_{23} = \gamma_{32} = \frac{\pi}{2} - \angle\ YPZ \tag{2.7c}$$

$\tilde{a}_{ij}$ $(i, j = 1, 2, 3\ \ i \neq j)$ is called *the unit shear in the directions of the axes x_i and x_j of a particle located in the undeformed state at point P_0*. On the basis of its definition, $\tilde{a}_{ij}$ represents the change of the angle between two infinitesimal material line segments which in the undeformed state were located at point P_0 and were mutually perpendicular and oriented in the directions of the axes x_i and x_j. With the definitions (2.7) we have established the convention that positive unit shear $\tilde{a}_{ij}$ indicates reduction of the before deformation right angle.

We have defined a set of six quantities with respect to the rectangular system of axes x_1, x_2, x_3 (the three unit elongations or shrinkages E_{11}, E_{22}, E_{33} and the three unit shears $\tilde{a}_{12}, \tilde{a}_{13}, \tilde{a}_{23}$), which specify completely the change of geometry of a particle due to its deformation. Such a set of six quantities can be defined with respect to any rectangular system of axes. In what follows, we use geometry to express the unit elongations or shrinkages and the unit shears of a particle in terms of derivatives of its components of displacement. Referring to Fig. 2.4, consider two material infinitesimal line segments $\overline{P_0X_0}$ and $\overline{P_0Y_0}$ of a body directed prior to deformation, the one along the x_1 axis and the other along the x_2 axis. After deformation, these infinitesimal material line segments translate, rotate and elongate or shrink. Moreover, the angle between them changes. In Fig. 2.4 the deformed configurations of these line segments are indicated by $\overline{PX}$ and $\overline{PY}$, respectively. Referring to Fig. 2.4, we have

$$\vec{PX} = \left(dx_1 + \frac{\partial \hat{u}_1}{\partial x_1} dx_1 \right) \mathbf{i_1} + \left(\frac{\partial \hat{u}_2}{\partial x_1} dx_1 \right) \mathbf{i_2} + \left(\frac{\partial \hat{u}_3}{\partial x_1} dx_1 \right) \mathbf{i_3} \tag{2.8a}$$

$$\vec{PY} = \left(\frac{\partial \hat{u}_1}{\partial x_2} dx_2 \right) \mathbf{i_1} + \left(dx_2 + \frac{\partial \hat{u}_2}{\partial x_2} dx_2 \right) \mathbf{i_2} + \left(\frac{\partial \hat{u}_3}{\partial x_2} dx_2 \right) \mathbf{i_3} \tag{2.8b}$$

Thus, the lengths of the line segments $\overline{PX}$ and $\overline{PY}$ are

$$\overline{PX} = \sqrt{\left(dx_1 + \frac{\partial \hat{u}_1}{\partial x_1} dx_1 \right)^2 + \left(\frac{\partial \hat{u}_2}{\partial x_1} dx_1 \right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_1} dx_1 \right)^2} \tag{2.9a}$$

$$\overline{PY} = \sqrt{\left(dx_2 + \frac{\partial \hat{u}_2}{\partial x_2} dx_2 \right)^2 + \left(\frac{\partial \hat{u}_1}{\partial x_2} dx_2 \right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_2} dx_2 \right)^2} \tag{2.9b}$$

The unit elongation E_{11} is equal to

$$E_{11} = \frac{\overline{PX} - \overline{P_0X_0}}{\overline{P_0X_0}} = \frac{\overline{PX} - dx_1}{dx_1} \tag{2.10a}$$

Moreover, referring to Fig. 2.4, we see that the unit shear $\tilde{a}_{12}$ is equal to

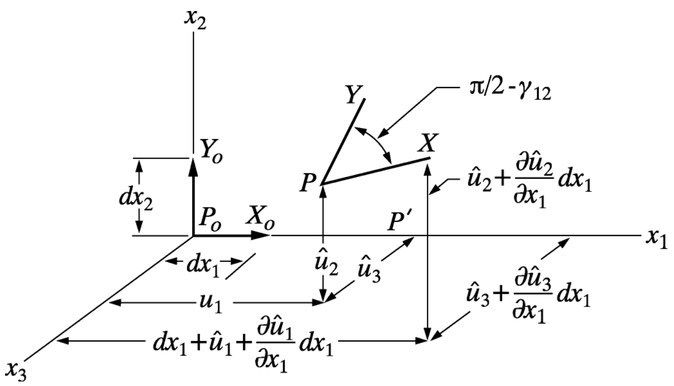


Figure 2.4 Unit elongations and unit shears.

$$\gamma_{12} = \frac{\pi}{2} - \angle XPY \tag{2.10b}$$

Substituting relation (2.9a) into relations (2.10a), we obtain

$$E_{11} = \sqrt{1 + 2\left(\frac{\partial \hat{u}_1}{\partial x_1}\right) + \left(\frac{\partial \hat{u}_1}{\partial x_1}\right)^2 + \left(\frac{\partial \hat{u}_2}{\partial x_1}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_1}\right)^2} - 1 = \sqrt{1 + 2\epsilon_{11}} - 1 \tag{2.11a}$$

Similarly, we get

$$\begin{aligned} E_{22} &= \sqrt{1 + 2\epsilon_{22}} - 1 \\ E_{33} &= \sqrt{1 + 2\epsilon_{33}} - 1 \end{aligned} \tag{2.11b}$$

Moreover, using relations (2.9) and noting by referring to relations (2.6) that $\overline{PX} = (1 + E_{11})\, dx_1$ and $\overline{PY} = (1 + E_{22})\, dx_2$, we get

$$\sin \gamma_{12} = \cos \angle XPY = \frac{\overrightarrow{PX} \cdot \overrightarrow{PY}}{\overline{PX}\,\overline{PY}} = \frac{\left(1 + \frac{\partial \hat{u}_1}{\partial x_1}\right)\frac{\partial \hat{u}_1}{\partial x_2} + \left(1 + \frac{\partial \hat{u}_2}{\partial x_2}\right)\frac{\partial \hat{u}_2}{\partial x_1} + \frac{\partial \hat{u}_3}{\partial x_1}\frac{\partial \hat{u}_3}{\partial x_2}}{(1 + E_{11})(1 + E_{22})} \tag{2.12a}$$

$$= \frac{2\epsilon_{12}}{(1 + E_{11})(1 + E_{22})} = \frac{2\epsilon_{12}}{\sqrt{(1 + 2\epsilon_{11})(1 + 2\epsilon_{22})}}$$

Similarly, we obtain

$$\sin \gamma_{13} = \frac{2\epsilon_{13}}{(1 + E_{11})(1 + E_{33})} = \frac{2\epsilon_{13}}{\sqrt{(1 + 2\epsilon_{11})(1 + 2\epsilon_{33})}} \tag{2.12b}$$

$$\sin \gamma_{23} = \frac{2\epsilon_{23}}{(1 + E_{22})(1 + E_{33})} = \frac{2\epsilon_{23}}{\sqrt{(1 + 2\epsilon_{22})(1 + 2\epsilon_{33})}} \tag{2.12c}$$

where the quantities ϵ_{ij} $(i, j = 1, 2, 3)$ in relations (2.11) and (2.12) are known as the *components of the Lagrangian strain* and are defined as:

$$\epsilon_{11} = \frac{\partial \hat{u}_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial \hat{u}_1}{\partial x_1}\right)^2 + \left(\frac{\partial \hat{u}_2}{\partial x_1}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_1}\right)^2\right] \tag{2.13a}$$

$$\epsilon_{22} = \frac{\partial \hat{u}_2}{\partial x_2} + \frac{1}{2}\left[\left(\frac{\partial \hat{u}_1}{\partial x_2}\right)^2 + \left(\frac{\partial \hat{u}_2}{\partial x_2}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_2}\right)^2\right] \tag{2.13b}$$

$$\epsilon_{33} = \frac{\partial \hat{u}_3}{\partial x_3} + \frac{1}{2}\left[\left(\frac{\partial \hat{u}_1}{\partial x_3}\right)^2 + \left(\frac{\partial \hat{u}_2}{\partial x_3}\right)^2 + \left(\frac{\partial \hat{u}_3}{\partial x_3}\right)^2\right] \tag{2.13c}$$

$$2\epsilon_{12} = \frac{\partial \hat{u}_1}{\partial x_2} + \frac{\partial \hat{u}_2}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial x_1}\frac{\partial \hat{u}_1}{\partial x_2} + \frac{\partial \hat{u}_2}{\partial x_1}\frac{\partial \hat{u}_2}{\partial x_2} + \frac{\partial \hat{u}_3}{\partial x_1}\frac{\partial \hat{u}_3}{\partial x_2} \tag{2.13d}$$

$$2\epsilon_{23} = \frac{\partial \hat{u}_2}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_2} + \frac{\partial \hat{u}_1}{\partial x_2}\frac{\partial \hat{u}_1}{\partial x_3} + \frac{\partial \hat{u}_2}{\partial x_2}\frac{\partial \hat{u}_2}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_2}\frac{\partial \hat{u}_3}{\partial x_3} \tag{2.13e}$$

$$2\epsilon_{13} = \frac{\partial \hat{u}_1}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial x_1}\frac{\partial \hat{u}_1}{\partial x_3} + \frac{\partial \hat{u}_2}{\partial x_1}\frac{\partial \hat{u}_2}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_1}\frac{\partial \hat{u}_3}{\partial x_3} \tag{2.13f}$$

Referring to relations (2.11) and (2.12), we see that the components of the Lagrangian strain of a particle specify completely its unit elongations or shrinkages and its unit shears. Consequently, they are a measure of its deformation. Moreover, it can be shown that the components of the Lagrangian strain are components of a symmeric tensor of the second rank. When the unit elongations or shrinkages and the unit shears are very small compared to unity, referring to relations (2.11) and (2.12), we have

$$\begin{aligned} E_{11} &\approx \epsilon_{11} \qquad & \gamma_{12} &\approx \sin\gamma_{12} \approx 2\epsilon_{12} \\ E_{22} &\approx \epsilon_{22} \qquad & \gamma_{13} &\approx \sin\gamma_{13} \approx 2\epsilon_{13} \\ E_{33} &\approx \epsilon_{33} \qquad & \gamma_{23} &\approx \sin\gamma_{23} \approx 2\epsilon_{23} \end{aligned} \tag{2.14}$$

From relations (2.13) we see that the components of the Lagrangian strain of a particle are non-linear functions of derivatives of its components of displacement. Consequently, they are very difficult to handle when employed in the solution of problems. In order to linearize these functions we limit our attention *to bodies whose deformation is such that the unit elongations, unit shears and rotations of their particles are very small compared to unity and, moreover, the rotations are not of a higher order of magnitude than the unit elongations or shrinkages and the unit shears*. This statement is known as *the assumption of small deformation*. It can be shown† that, on the basis of this assumption, the rates of change (gradients) of the components of displacement $[\partial \hat{u}_i/\partial x_j\ (i, j = 1, 2, 3)]$ are very small compared to unity and the squares of a gradient and the products of two gradients can be disregarded as compared to a gradient. Thus, referring to relations (2.13) and (2.14), we have

$$E_{11} \approx \epsilon_{11} \approx e_{11} \tag{2.15a}$$

$$E_{22} \approx \epsilon_{22} \approx e_{22} \tag{2.15b}$$

$$E_{33} \approx \epsilon_{33} \approx e_{22} \tag{2.15c}$$

† For a more detailed discussion, see Novozhilov, V.V., *Foundations of the Nonlinear Theory ofElasticity,* Gaylock Press, Rochester, NY, 1953.

$$\gamma_{12} \approx 2\epsilon_{12} \approx 2e_{12} \tag{2.15d}$$

$$\gamma_{13} \approx 2\epsilon_{13} \approx 2e_{13} \tag{2.15e}$$

$$\gamma_{23} \approx 2\epsilon_{23} \approx 2e_{22} \tag{2.15f}$$

where

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} \qquad e_{22} = \frac{\partial \hat{u}_2}{\partial x_2} \qquad e_{33} = \frac{\partial \hat{u}_3}{\partial x_3}$$

$$e_{21} = e_{12} = \frac{1}{2}\left(\frac{\partial \hat{u}_2}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial x_2}\right)$$

$$e_{31} = e_{13} = \frac{1}{2}\left(\frac{\partial \hat{u}_3}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial x_3}\right) \tag{2.16a}$$

$$e_{32} = e_{23} = \frac{1}{2}\left(\frac{\partial \hat{u}_2}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_2}\right)$$

Relations (2.16a) can be rewritten as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right) \qquad (i, j = 1, 2, 3) \tag{2.16b}$$

Notice that

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial \hat{u}_j}{\partial x_i} + \frac{\partial \hat{u}_i}{\partial x_j}\right) = e_{ji} \tag{2.17}$$

Consequently, the nine quantities e_{ij} $(i, j = 1, 2, 3)$ form a symmetric matrix. They are called *the components of strain of the particle with respect to the system of axes* x_1, x_2 *and* x_3. *The quantities* e_{11}, e_{22} *and* e_{33} *are called the normal components of strain of the particle in the directions of the axes* x_1, x_2 *and* x_3, *respectively, while* $e_{12} = e_{21}$, $e_{13} = e_{31}$ *and* $e_{23} = e_{32}$ *are called the shearing components of strain in the directions of the axes* x_1, x_2 *and* x_1, x_3 *and* x_2, x_3, *respectively*. The normal component of strain e_{ii} of a particle is positive, if the infinitesimal material line element elongates and negative, if it shrinks. In the range of validity of the assumption of small deformation, the normal component of strain e_{ii} $(i = 1, 2$ or $3)$ of a particle represents an approximation to the change of length due to the deformation divided by the undeformed length of an infinitesimal material line element of this particle which prior to deformation was oriented in the direction of the axis x_i $(i = 1, 2, 3)$.

2.5 Implications of the Assumption of Small Deformation

Except for Chapter 17, in this text we consider bodies which are subjected to external loads of such magnitudes that their deformation is within the range of validity of the assumption of small deformation (see Section 2.4). As a result of this assumption, the following approximations can be made:

1. The deformation of a particle of a body is completely specified by its components of strain which are related to the derivatives of the components of displacement $\hat{u}_1(x_1, x_2, x_3)$, $\hat{u}_2(x_1, x_2, x_3)$ and $\hat{u}_3(x_1, x_2, x_3)$ of the particle by the linear relations (2.16).
2. The change of length, area or volume of a segment of a body due to its deformation is negligible compared to its undeformed length, area or volume, respectively. Consequently, when we consider the equilibrium of a portion of a body (finite or infinitesimal) we do not take into account the change of its dimensions due to its deformation. That is, when we draw its free-body diagram we use its undeformed configuration. For example, we use the free-body diagram, shown in Fig. 2.5b when we ccnsider the equilibrium of point 2 of the truss of Fig. 2.5a. In this case, we have

$$\begin{aligned} \sum \bar{F}_1 &= 0 \qquad N^{(1)} = N^{(2)} \\ \sum \bar{F}_2 &= 0 \qquad N^{(1)} = N^{(2)} = \frac{P}{2\cos\alpha/2} \end{aligned} \tag{2.18}$$

It is apparent that the relation between the internal forces in the members of the truss $N^{(1)} = N^{(2)}$ and the external force P is linear. This is not true if the effect of the rotation of the members of the truss on the magnitude of their internal forces is taken into account. In this case referring to Fig. 2.5c, we have

$$\begin{aligned} \sum \bar{F}_1 &= 0 \qquad N^{(1)} = N^{(2)} \\ \sum \bar{F}_2 &= 0 \qquad N^{(1)} = N^{(2)} = \frac{P}{2\cos\beta/2} \end{aligned} \tag{2.19}$$

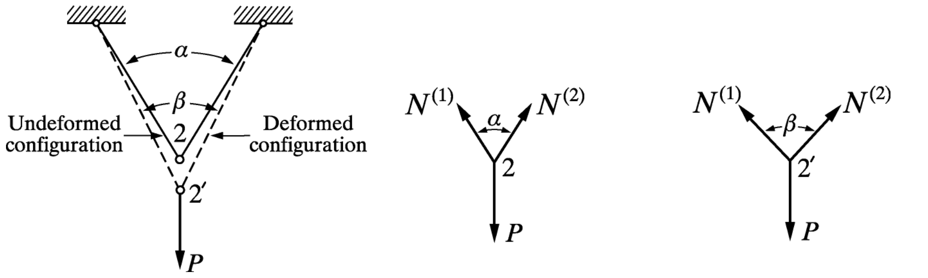
The magnitude of the angle β depends on the magnitude of the external force P. Consequently, the relation between the internal forces in the members of the truss and the external force P is not linear.

As a second example, we consider the equilibrium of a segment of a beam subjected to a transverse P_3 and an axial P_1 forces as shown in Fig. 2.6. If the effect of the change due to its deformation of the geometry of the beam on the magnitude of its internal moment is not taken into account, we use the free-body diagram of Fig. 2.6b. In this case, we have

$$\sum \bar{F}_1 = 0 \qquad N = P_1 \tag{2.20a}$$

$$\sum \bar{F}_3 = 0 \qquad Q = R^{(1)} = \frac{P_3}{3} \tag{2.20b}$$

$$\sum \bar{M}_2^{(0)} = 0 \qquad M = R^{(1)}x_1 = \frac{P_3 x_1}{3} \tag{2.20c}$$



(a) Geometry and loading of the truss

(b) Free-body diagram of joint 2 disregarding the change of the angle *a* due to the deformation

(c) Free-body diagram of joint 2 including the change of the angle *a* due to the deformation

Figure 2.5 Two-member truss subjected to an external force.

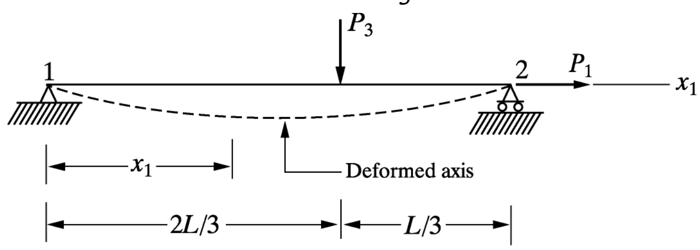
Thus, in this case, the internal forces and moment acting on a cross section of the beam are related to the external forces by a linear relation.

If the effect of the change of the geometry of the beam due to its deformation on the magnitude of its internal moment is taken into account, referring to Fig. 2.6c, we have

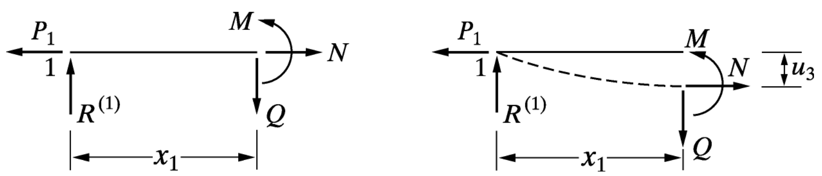
$$\sum \bar{F}_1 = 0 \qquad N = P_1 \tag{2.21a}$$

$$\sum \bar{F}_3 = 0 \qquad Q = R^{(1)} = \frac{P_3}{3} \tag{2.21b}$$

$$\sum \bar{M}_2^{(0)} = 0 \qquad M = R^{(1)}x_1 - P_1 u_3 = \frac{P_3 x_1}{3} - P_1 u_3 \tag{2.21c}$$



(a) Geometry and loading of the beam



(b) Free-body diagram of a segment of the beam using its undeformed configuration

(c) Free-body diagram of a segment of the beam using its deformed configuration

Figure 2.6 Beam subjected to transverse and axial forces.

The magnitude of the components of translation u_3 depends on the magnitude of the external forces P_1 and P_3. Consequently, as can be seen from relation (2.21c), the internal moment acting on a cross section of the beam is not related to the external forces by a linear relation.

The assumption of small deformation cannot be used for the following problems of interest to the engineer:

1. In analyzing beams subjected to transverse and axial forces when the effect of the axial forces on their bending moment cannot be neglected
2. In establishing the loading under which a structure or a group of its members reaches a state of unstable equilibrium (see Chapter 18)
3. In analyzing long cables subjected to transverse forces

2.6 Proof of the Tensorial Property of the Components of Strain

In this section we prove that the quantities e_{ij} $(i, j = 1, 2, 3)$ defined by relation (2.16) are components of a symmetric tensor of the second rank. For this purpose we consider the displacement vector by $\hat{\mathbf{u}}(x_1, x_2, x_3)$ of a particle of a body and we denote by $\hat{u}_n(x_1, x_2, x_3)$ and $\hat{u}_n'(x_1, x_2, x_3)$ its components with respect to the rectangular system of axes x_n and x_n' $(n = 1, 2, 3)$, respectively. That is,

$$\hat{\mathbf{u}} = \sum_{n=1}^{3} \hat{u}_n \mathbf{i}_n = \sum_{n=1}^{3} \hat{u}_n' \mathbf{i}_n'$$

Referring to relations (1.31), we have

$$\hat{u}_n' = \sum_{k=1}^{3} \lambda_{nk} \hat{u}_k$$

Differentiating this relation with respect to x_i, we obtain

$$\frac{\partial \hat{u}_n'}{\partial x_i} = \sum_{k=1}^{3} \lambda_{nk} \frac{\partial \hat{u}_k}{\partial x_i} \tag{2.22a}$$

Using the chain rule of differentiation, we get

$$\frac{\partial \hat{u}_n'}{\partial x_i} = \frac{\partial \hat{u}_n'}{\partial x_1'}\frac{\partial x_1'}{\partial x_i} + \frac{\partial \hat{u}_n'}{\partial x_2'}\frac{\partial x_2'}{\partial x_i} + \frac{\partial \hat{u}_n'}{\partial x_3'}\frac{\partial x_3'}{\partial x_i} = \sum_{m=1}^{3} \frac{\partial \hat{u}_n'}{\partial x_m'}\frac{\partial x_m'}{\partial x_i} \tag{2.22b}$$

The position of a point in space may be specified by a position vector $\mathbf{r}$ with respect to a fixed point O. The components of this vector with respect to a rectangular system of axes having as its origin point O are the coordinates of the point with respect to that system of axes. For example, the components of the vector $\mathbf{r}$ with respect to the rectangular systems of axes x_1, x_2, x_3 and x_1', x_2', x_3' with origin the fixed point O are

$$\mathbf{r} = x\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 = x_1'\mathbf{i}_1' + x_2'\mathbf{i}_2' + x_3'\mathbf{i}_3'$$

Referring to relations (1.31) and (1.32), the transformation relations of the coordinates of a point with respect to two rectangular systems of axes x_i' (i = 1, 2, 3) and x_j(j = 1, 2, 3) are

$$x_i' = \sum_{j=1}^{3} \lambda_{ij} x_j \tag{2.23a}$$

and

$$x_j = \sum_{i=1}^{3} \lambda_{ij} x_i' \tag{2.23b}$$

Differentiating relation (2.23a) with respect to x_j, we get

$$\frac{\partial x_i'}{\partial x_j} = \lambda_{ij} \qquad (i, j = 1, 2, 3) \tag{2.24a}$$

Similarly, by differentiating relation (2.23b) with respect to x_i', we obtain

$$\frac{\partial x_j}{\partial x_i'} = \lambda_{ij} \qquad (i, j = 1,2,3) \tag{2.24b}$$

Substituting relation (2.24a) into (2.22a), we obtain

$$\frac{\partial \hat{u}_n'}{\partial x_i} = \sum_{m=1}^{3} \lambda_{mi} \frac{\partial \hat{u}_n'}{\partial x_m'} \tag{2.25}$$

Substituting relation (2.25) into (2.22), we get

$$\sum_{k=1}^{3} \lambda_{nk} \frac{\partial \hat{u}_k}{\partial x_i} = \sum_{m=1}^{3} \lambda_{mi} \frac{\partial \hat{u}_n'}{\partial x_m'}$$

Multiplying both sides of the above relation by λ_{nj} and adding for n = 1, 2, 3, we obtain

$$\sum_{k=1}^{3} \left(\sum_{n=1}^{3} \lambda_{nj} \lambda_{nk} \right) \frac{\partial \hat{u}_k}{\partial x_i} = \sum_{m=1}^{3} \sum_{n=1}^{3} \lambda_{nj} \lambda_{mi} \frac{\partial \hat{u}_n'}{\partial x_m'}$$

Referring to relations (1.25a), we see that the $\sum_{n=1}^{3} \lambda_{nj} \lambda_{nk}$ is equal to zero if $j \neq k$ and to unity if $j = k$. Thus, the above relation reduces to

$$\frac{\partial \hat{u}_j}{\partial x_i} = \sum_{m=1}^{3} \sum_{n=1}^{3} \lambda_{mi} \lambda_{nj} \frac{\partial \hat{u}_n'}{\partial x_m'} \tag{2.26}$$

Therefore, on the basis of our discussion in Section 1.5, the nine quantities $\partial \hat{u}_i / \partial x_j$ (I, j = 1, 2, 3) transform as components of a tensor of the second rank. Similarly, we can show that

$$\frac{\partial \hat{u}_i}{\partial x_j} = \sum_{m=1}^{3} \sum_{n=1}^{3} \lambda_{mi} \lambda_{nj} \frac{\partial \hat{u}_m'}{\partial x_n'} \tag{2.27}$$

Thus, using relations (2.26) and (2.27), we get

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right) = \sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{mi}\lambda_{nj}\,\frac{1}{2}\left(\frac{\partial \hat{u}_n'}{\partial x_m'} + \frac{\partial \hat{u}_m'}{\partial x_n'}\right)$$
$$= \sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{mi}\lambda_{nj} e_{mn}' \tag{2.28}$$

Similarly, we can show that

$$e_{ij}' = \sum_{m=1}^{3}\sum_{n=1}^{3} \lambda_{im}\lambda_{jn} e_{mn} \tag{2.29}$$

Thus, the nine quantities e_{ij} $(i, j = 1, 2, 3)$ transform according to relation (1.70) and, consequently, are components of a symmetric tensor of the second rank.

2.7 Traction and Components of Stress Acting on a Plane of a Particle of a Body

Consider a body in equilibrium under the influence of external loads. Imagine that the body is cut into two parts (part I and II) by a surface S_c (not necessarily a plane) passing through point P (see Fig. 2.7). Generally, there will be a distribution of forces exerted by the particles of the one part of the surface S_c of the body on the particles of the other part of the surface S_c. On the basis of Newton's law of action and reaction, the distribution of forces on the surface S_c^{I} of part I, must be equal and opposite to the distribution of forces on the surface S_c^{II} of part II. Consider a portion ΔS_c^{I} of the surface S_c^{I} which includes point P and denote by $\mathbf{i}_n$ the unit vector outwardly normal to it at point P. The resultant of the forces acting on this portion is denoted by **ÄF**. Moreover, consider a portion ΔS_c^{II} of the surface S_c^{II}, which includes point P and denote by $-\mathbf{i}_n$ the unit vector outwardly normal to it at point P. The resultant of the forces acting on this portion is denoted by $-\Delta\mathbf{F}$. The traction, also known as the *stress vector*, at point P, acting on the surface S_c^{I} of part I is denoted by $\overset{n}{\mathbf{T}}$ and is defined as

$$\overset{n}{\mathbf{T}} = \lim_{\Delta S_c^{\mathrm{I}} \to 0} \frac{\Delta\mathbf{F}}{\Delta S_c^{\mathrm{I}}} \tag{2.30}$$

while the traction at point P, acting on the surface ΔS_c^{II} of Part II is denoted by $\overset{-n}{\mathbf{T}}$ and is defined as

$$\overset{-n}{\mathbf{T}} = \lim_{\Delta S_c^{\mathrm{II}} \to 0} -\frac{\Delta\mathbf{F}}{\Delta S_c^{\mathrm{II}}} \tag{2.31}$$

Comparing relations (2.30) and (2.31), we see that

$$\overset{n}{\mathbf{T}} = -\overset{-n}{\mathbf{T}} \tag{2.32}$$

Notice that the surface elements ΔS_c^{I} and ΔS_c^{II} have been defined when the body was in

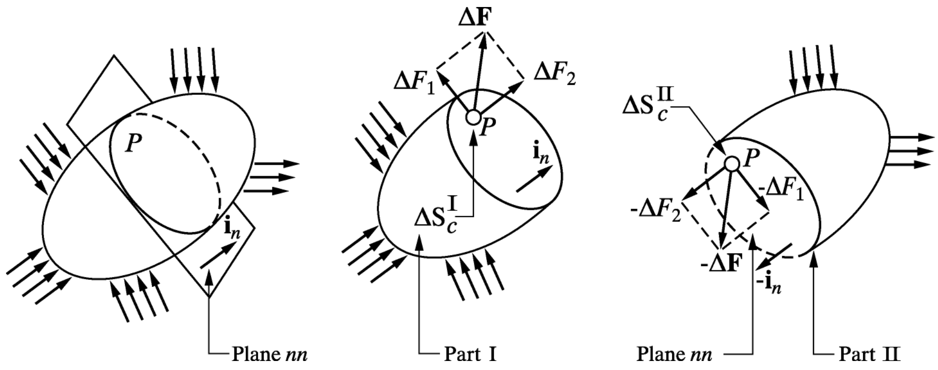


Figure 2.7 Body cut in two parts by a plane passing through point P.

the deformed state. However, for deformation within the range of validity of the assumption of small deformation the change of an area due to deformation is negligible compared to its value in the undeformed state. Therefore, it is immaterial whether the traction is referred to the deformed or undeformed state. On the basis of its definition, the traction is a vector whose magnitude is in units of force per unit area $[FL^{-2}]$. It is apparent that the traction at point P, acting on the surface $\boldsymbol{S}_{\mathrm{c}}^{\mathbf{I}}$ need not be in the direction of unit vector $\mathbf{i}_n$, normal to this surface at point P. In relations (2.30) and (2.31) $\Delta\boldsymbol{S}_{\mathrm{c}}^{\mathrm{I}}$ and $\Delta\boldsymbol{S}_{\mathrm{c}}^{\mathrm{II}}$ include point P as they approach zero. Moreover, it has been postulated that the limits in these relations exist and are independent of the way the portions of surface $\Delta\boldsymbol{S}_{\mathrm{c}}^{\mathrm{I}}$ and $\Delta\boldsymbol{S}_{\mathrm{c}}^{\mathrm{II}}$ shrink to zero. In fact, $\Delta\boldsymbol{S}_{\mathrm{c}}^{\mathrm{I}}$ and $\Delta\boldsymbol{S}_{\mathrm{c}}^{\mathrm{II}}$ could be portions of any surface through point P, whose normal at point P is the unit vector $\mathbf{i}_n$. It is apparent that the magnitude and direction of the traction $\overset{n}{\mathbf{T}}$ at a point P of a body, acting on a plane nn, generally are different than the magnitude and direction, respectively, of the traction $\overset{m}{\mathbf{T}}$ at point P, acting on another plane mm.

The traction $\overset{n}{\mathbf{T}}$ at point P of a body on a plane specified by the unit vector $\mathbf{i}_n$ may be expressed as the sum of its components in three mutually perpendicular directions specified by the unit vectors $\mathbf{i}_n$, $\mathbf{i}_t$, $\mathbf{i}_p$. That is,

$$\overset{n}{\mathbf{T}} = \tau_{nn}\,\mathbf{i}_n + \tau_{nt}\,\mathbf{i}_t + \tau_{np}\,\mathbf{i}_p \tag{2.33}$$

where $\hat{o}_{nn}$ is called *the normal component of stress* at point P acting on the plane normal to the unit vector $\mathbf{i}_n$ while $\hat{o}_{nt}$ and $\hat{o}_{np}$ are called the *shearing components of stress* at point P acting on the plane normal to the unit vector $\mathbf{i}_n$ in the directions of the unit vectors $\mathbf{i}_t$ and $\mathbf{i}_p$, respectively. Notice that the first subscript indicates the direction of the normal to the plane on which the stress acts, while the second subscript indicates the direction of the component of stress. In Fig. 2.8, the vector $\mathbf{i}_n$ is parallel to the axis x_1. In this case the normal component of stress at a point P acting on the plane normal to the x_1 axis is denoted by $\hat{o}_{11}$ and is represented schematically by an arrow acting normal to this plane, whereas the shearing components of stress at point P acting on the plane normal to the x_1 axis in the directions of the x_2 or x_3 axis are denoted by $\hat{o}_{12}$ or $\hat{o}_{13}$, respectively. They are represented by arrows on the plane normal to the x_1 axis in the directions of the x_2 or x_3 axis. Referring to relation (2.31), the tractions $\overset{i}{\mathbf{T}}$ and $\overset{-i}{\mathbf{T}}$ ($i = 1, 2, 3,$) acting on the

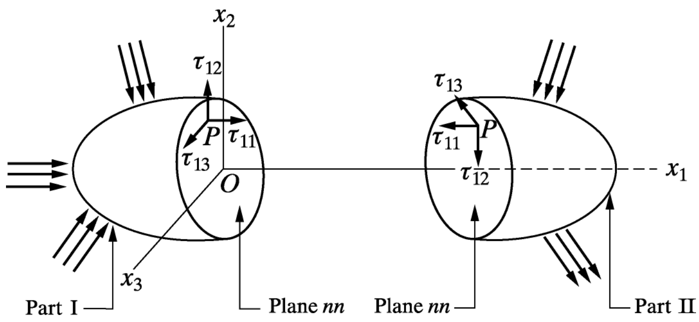


Figure 2.8 Body cut in two parts by a plane normal to the x_1 axis passing through point P.

planes of a particle which are perpendicular to the x_1, x_2, x_3 axes, are given as

$$\overset{1}{\mathbf{T}} = -\overset{-1}{\mathbf{T}} = \tau_{11}\mathbf{i}_1 + \tau_{12}\mathbf{i}_2 + \tau_{13}\mathbf{i}_3 \tag{2.34a}$$

$$\overset{2}{\mathbf{T}} = -\overset{-2}{\mathbf{T}} = \tau_{21}\mathbf{i}_1 + \tau_{22}\mathbf{i}_2 + \tau_{23}\mathbf{i}_3 \tag{2.34b}$$

$$\overset{3}{\mathbf{T}} = -\overset{-3}{\mathbf{T}} = \tau_{31}\mathbf{i}_1 + \tau_{32}\mathbf{i}_2 + \tau_{33}\mathbf{i}_3 \tag{2.34c}$$

A schematic representation of the state of stress of a particle of a body is shown in Fig. 2.9; in this figure, plane $A'E'D'C'$ of the parallelepiped represents the plane of the particle which is normal to the x_1 axis as viewed on part I of the body (see Fig. 2.8), whereas plane $AEDC$ of the parallelepiped represents the same plane of the same particle as viewed on part II of the body (see Fig. 2.8). Similarly, planes $AA'EE'$ and $DD'C'C$ represent the plane of the same particle which is normal to the x_2 axis. The components of stress at a point are assumed positive as shown in Fig. 2.9. A positive, normal component of stress is referred to as *tension*. It tends to elongate an infinitesimal material line segment oriented along the line of action of the stress. A negative normal component of stress is referred to as *compression*. It tends to shorten an infinitesimal material line segment oriented along the line of action of the stress.

2.8 Proof of the Tensorial Property of the Components of Stress

In this section we prove that the components of stress are components of a tensor of the second rank.

Consider the material tetrahedron $OABC$ shown in Fig. 2.10. This tetrahedron may be located either inside the body or on its surface. In the latter case, surface $ABC = \Delta A$ is a portion ΔA of the surface of the body. The unit vectors normal to the faces OAC, OBC, and OAB of the tetrahedron are $-\mathbf{i}_1$, $-\mathbf{i}_2$, $-\mathbf{i}_3$, respectively, whereas the unit vector normal to the face ABC is denoted by $\mathbf{i}_n$ and its components with respect to the x_i ($i = 1, 2, 3$) axes are denoted by λ_{ni}($i = 1, 2, 3$). That is,

$$\mathbf{i}_n = \lambda_{n1}\mathbf{i}_1 + \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$$

Referring to Fig. 2.10, consider the plane *OAD* which is perpendicular to line *CB* and contains the x_2 axis. It is apparent that line *CB* is perpendicular to any line in the plane *OAD* and, in particular, to line *OD*. Thus,

$$\overline{OD} = \overline{AD} \cos (\angle ODA) = \overline{AD} \cos \phi_{n2} = \overline{AD} \lambda_{n2} \tag{2.35a}$$

Moreover, referring to Fig. 2.10, we have

$$\text{area } ACB = \Delta A = \frac{(\overline{CB})(\overline{AD})}{2} \tag{2.35b}$$

Using relations (2.35a) and (2.35b), we get

$$\text{area } OBC = \frac{(\overline{CB})(\overline{OD})}{2} = \frac{(\overline{CB})(\overline{AD})\lambda_{n2}}{2} = \Delta A \lambda_{n2} = \mathbf{i}_n \cdot \mathbf{i}_2 \Delta A \tag{2.36a}$$

Similarly, we obtain

$$\begin{aligned} \text{area } OAC &= \lambda_{n1} \Delta A = \mathbf{i}_n \cdot \mathbf{i}_1 \Delta A \\ \text{area } OAB &= \lambda_{n3} \Delta A = \mathbf{i}_n \cdot \mathbf{i}_3 \Delta A \end{aligned} \tag{2.36b}$$

In Fig. 2.10, $\overset{-i}{\mathbf{T}}{}^*$ ($i = 1, 2, 3$) and $\overset{n}{\mathbf{T}}{}^*$ are the average† tractions acting on the faces of the tetrahedron which are normal to the unit vectors $-\mathrm{i}_i$ ($i = 1, 2, 3$) and $\mathbf{i}_n$, respectively. Moreover, $\mathbf{B}^*$ is the average specific body force acting on the tetrahedron. Considering the equilibrium of the tetrahedron and using relations (2.36), we get

$$\sum \text{forces} = \overset{n}{\mathbf{T}}{}^* \Delta A + \overset{-1}{\mathbf{T}}{}^* \mathbf{i}_n \cdot \mathbf{i}_1 \, \Delta A + \overset{-2}{\mathbf{T}}{}^* \mathbf{i}_n \cdot \mathbf{i}_2 \, \Delta A + \overset{-3}{\mathbf{T}}{}^* \mathbf{i}_n \cdot \mathbf{i}_3 \, \Delta A + \boldsymbol{B} \Delta V \tag{2.37}$$

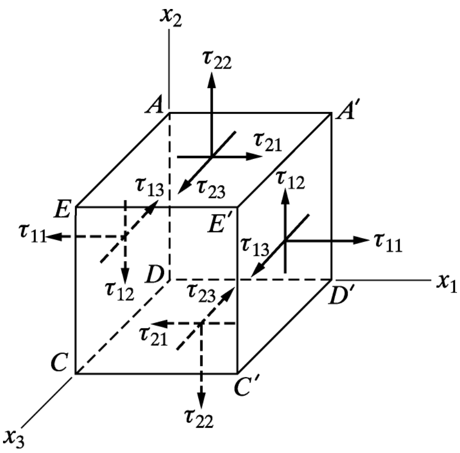


Figure 2.9 Schematic representation of the state of stress at a point.

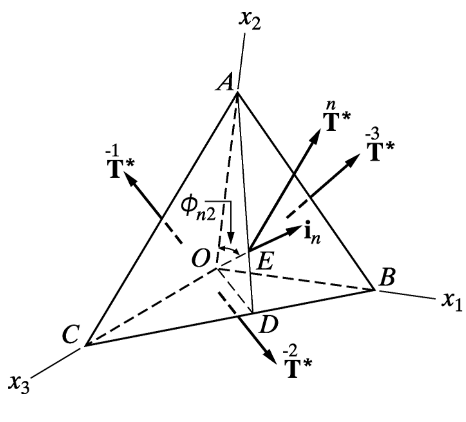


Figure 2.10 Material tetrahedron.

† The average traction $\overset{-i}{\mathbf{T}}{}^*$ is equal to the resultant force vector acting on the face *OAC* of the tetrahedron of Fig. 2.10 divided by the area of this face.

where ΔV is the volume of the tetrahedron. Notice that $\Delta V = (\Delta h)\Delta A/3)$, where $\Delta h = \overline{OE}$ is the length of the line which is perpendicular to the plane ABC from point O (see Fig. 2.10). Dividing both sides of the equation (2.37) by ΔA, we obtain

$$\overset{n}{\mathbf{T}}{}^{*} + \overset{-1}{\mathbf{T}}{}^{*}(\mathbf{i}_n \cdot \mathbf{i}_1) + \overset{-2}{\mathbf{T}}{}^{*}(\mathbf{i}_n \cdot \mathbf{i}_2) + \overset{-3}{\mathbf{T}}{}^{*}(\mathbf{i}_n \cdot \mathbf{i}_3) + \mathbf{B}^{*}\frac{\Delta h}{3} = 0 \tag{2.38}$$

In the limit as $\Delta h \to 0$ (in a way that point E moves toward point O), the tetrahedron degenerates into a particle located at point O. Moreover,

$$\lim_{\Delta h \to 0} \overset{-i}{\mathbf{T}}{}^{*} = \overset{-i}{\mathbf{T}} \quad (i = 1, 2, 3) \qquad \lim_{\Delta h \to 0} \mathbf{B}^{*} = \mathbf{B} \tag{2.39}$$

where $\overset{-i}{\mathbf{T}}$ $(i = 1, 2, 3)$ is the traction acting on the particle located at point O on the plane normal to the unit vector $-\mathbf{i}_i$ $(i = 1, 2, 3)$; $\mathbf{B}$ is the specific body force acting on the particle located at point O. Thus, in the limit as Δh approaches zero, equation (2.38) reduces to

$$-\overset{n}{\mathbf{T}} = (\mathbf{i}_n \cdot \mathbf{i}_1)\overset{-1}{\mathbf{T}} + (\mathbf{i}_n \cdot \mathbf{i}_2)\overset{-2}{\mathbf{T}} + (\mathbf{i}_n \cdot \mathbf{i}_3)\overset{-3}{\mathbf{T}} \tag{2.40}$$

From equation (2.32), we have

$$\overset{-i}{\mathbf{T}} = -\overset{i}{\mathrm{T}} \qquad (i = 1,2,3) \tag{2.41}$$

Substituting relation (2.41) into (2.40), we obtain

$$\overset{\mathbf{n}}{\mathbf{T}} = \mathbf{i}_n \cdot (\mathbf{i}_1\overset{1}{\mathbf{T}} + \mathbf{i}_2\overset{2}{\mathbf{T}} + \mathbf{i}_3\overset{3}{\mathbf{T}}) \tag{2.42}$$

Notice that the term in parentheses in relation (2.42), is the sum of dyads. Consequently, it is a tensor of the second rank (see Section 1.3). It is called the *stress tensor* of the material particle under consideration and we denote by $\hat{\mathbf{o}}$. That is

$$\boldsymbol{\tau} = \mathbf{i}_1\overset{1}{\mathbf{T}} + \mathbf{i}_2\overset{2}{\mathbf{T}} + \mathbf{i}_3\overset{3}{\mathbf{T}} \tag{2.43}$$

In Section 2.13, we show that the stress tensor is symmetric.

Using relation (2.43), relation (2.42) can be written as

$$\overset{n}{\mathbf{T}} = \mathbf{i}_n \cdot \boldsymbol{\tau} \tag{2.44}$$

Substituting relation (2.34) into (2.42), we get

$$\begin{aligned}\boldsymbol{\tau} &= \tau_{11}\mathbf{i}_1\mathbf{i}_1 + \tau_{12}\mathbf{i}_1\mathbf{i}_2 + \tau_{13}\mathbf{i}_1\mathbf{i}_3 + \tau_{21}\mathbf{i}_2\mathbf{i}_1 + \tau_{22}\mathbf{i}_2\mathbf{i}_2 + \tau_{23}\mathbf{i}_2\mathbf{i}_3 + \tau_{31}\mathbf{i}_3\mathbf{i}_1 + \tau_{32}\mathbf{i}_3\mathbf{i}_2 + \tau_{33}\mathbf{i}_3\mathbf{i}_3 \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\mathbf{i}_i\mathbf{i}_j\end{aligned} \tag{2.45}$$

Referring to Fig. 2.10, consider another set of axes x_1', x_2', x_3' with origin point O (the origin of the set of axes x_1, x_2, x_3); with respect to this set of axis, we have

$$
\begin{aligned}
\boldsymbol{\tau} &= \tau'_{11}\mathbf{i}'_1\mathbf{i}'_1 + \tau'_{12}\mathbf{i}'_1\mathbf{i}'_2 + \tau'_{13}\mathbf{i}'_1\mathbf{i}'_3 + \tau'_{21}\mathbf{i}'_2\mathbf{i}'_1 + \tau'_{22}\mathbf{i}'_2\mathbf{i}'_2 + \tau'_{23}\mathbf{i}'_2\mathbf{i}'_3 + \tau'_{31}\mathbf{i}'_3\mathbf{i}'_1 + \tau'_{32}\mathbf{i}'_3\mathbf{i}'_2 + \tau'_{33}\mathbf{i}'_3\mathbf{i}'_3 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \tau'_{ij}\mathbf{i}'_i\mathbf{i}'_j
\end{aligned}
$$

or

$$
\boldsymbol{\tau} = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\mathbf{i}_i\mathbf{i}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau'_{ij}\mathbf{i}'_i\mathbf{i}'_j \tag{2.46}
$$

$\hat{o}_{ij}(i, j = 1, 2 \text{ or } 3)$ are the components of the stress tensor acting on the three planes which are normal to the x_1, x_2, x_3 axes of the particle under consideration. Notice that each component of stress is associated with two unit vectors. The first is normal to the plane on which the component of stress acts while the second acts in the direction of the component of stress. It is clear that the stress tensor of a particle specifies completely the state of stress of the particle.

2.9 Properties of the Strain and Stress Tensors

In Section 2.6 it is shown that the nine components of strain e_{ij} $(i, j = 1, 2, 3)$ of a particle of a body referred to a rectangular system of axes x_1, x_2, x_3 are cartesian components of a symmetric tensor of the second rank called the *strain tensor*. Moreover, in Section 2.8 it is shown that the nine components of stress $\hat{o}_{ij}$ $(i, j = 1, 2, 3)$ acting on three mutually perpendicular planes of a particle of a body are components of a tensor of the second rank called the *stress tensor*. In Section 2.13, we prove that the stress tensor is symmetric. On the basis of our discussion in Sections 1.3, 1.4 and 1.9 to 1.13, the strain or stress tensors of a particle of a body have the following properties:

1. They are specified by their nine components e_{ij} $(i, j = 1, 2, 3)$ or $\hat{o}_{ij}$ $(i, j = 1, 2, 3)$ with respect to a rectangular system of axes x_1, x_2, x_3. That is, if the nine components of strain e_{ij} $(i, j = 1, 2, 3)$ or stress $\hat{o}_{ij}$ $(i, j = 1, 2, 3)$ of a particle are known with respect to a rectangular system of axes, its components with respect to any other rectangular system of axes can be obtained using the following transformation relation:

$$
e'_{ij} = \sum_{k=1}^{3}\sum_{m=1}^{3} \lambda_{ik}\lambda_{jm}e_{km} \quad \text{or} \quad \tau'_{ij} = \sum_{k=1}^{3}\sum_{m=1}^{3} \lambda_{ik}\lambda_{jm}\tau_{km} \quad (i, j = 1, 2, 3) \tag{2.47a}
$$

and

$$
e_{ij} = \sum_{k=1}^{3}\sum_{m=1}^{3} \lambda_{ki}\lambda_{mj}e'_{km} \quad \text{or} \quad \tau_{ij} = \sum_{k=1}^{3}\sum_{m=1}^{3} \lambda_{ki}\lambda_{mj}\tau'_{km} \quad (i, j = 1, 2, 3) \tag{2.47b}
$$

where λ_{ki} and λ_{mj} are the direction cosines of the x'_1, x'_2 and x'_3 axes with respect to the x_1, x_2 and x_3 axes; that is, $\ddot{e}_{12}$ is the cosine of the angle between the axes x'_1 and x_2 (see Section 1.1.4). e_{km} or τ_{km} $(k, m = 1, 2, 3)$ are the components of strain or stress, respectively, referred to the rectangular system of axes x_1, x_2, x_3 while e'_{ij} $(i, j = 1, 2, 3)$ or $\hat{o}'_{ij}$ $(i, j = 1, 2, 3)$ are the components of strain or stress, respectively, referred to the rectangular system of axes x'_1, x'_2 and x'_3. Relations (2.47) can be written in matrix form as

$$[\, e' \,] = [\, \Lambda_S \,][\, e \,][\, \Lambda_S \,]^T \qquad \text{or} \qquad [\, \tau' \,] = [\, \Lambda_S \,][\, \tau \,][\, \Lambda_S \,]^T \tag{2.48a}$$

and

$$[\, e \,] = [\, \Lambda_S \,]^T [\, e' \,][\, \Lambda_S \,] \qquad \text{or} \qquad [\, \tau \,] = [\, \Lambda_S \,]^T [\, \tau' \,][\, \Lambda_S \,] \tag{2.48b}$$

where $[\ddot{E}_S]^T$ is the transpose of the matrix $[\ddot{E}_S]$ defined by relation (1.24a) and $[e]$ and [ô] are the matrices of the components of strain and stress, respectively, referred to the axes x_1, x_2, x_3, given as

$$[\, e \,] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \qquad [\, \tau \,] = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \tag{2.49}$$

2. There exists at least one rectangular system of axes, called principal axes, with respect to which the shearing components of strain vanish, while the normal components of strain assume stationary values. That is, one is the maximum and another is the minimum of the normal components of strain with respect to any axis.

3. There exist at least three mutually perpendicular axes, called principal, which are normal to the planes on which, the shearing components of stress vanish, while the normal components of stress assume stationary values. That is, one is the maximum and another is the minimum of the normal components of stress acting on any plane.

4. The quasi plane form of the tensor $[e]$ or [ô] transforms in accordance with relations (1.116). Moreover, this transformation can be applied using Mohr's circle.

In general, the strain or stress tensors change from particle to particle. That is, its components referred to the same system of axes are functions of the space coordinates. The totality of the strain $[e(x_1, x_2, x_3)]$ or stress [ô(x_1, x_2, x_3)] tensors assigned at each point of a body is referred to as the *strain or stress field of the body, respectively*.

Referring to relation (1.60), the normal component of strain in the direction of the unit vector $\mathbf{i}_n$ and the normal component of stress acting on the plane normal to the unit vector $\mathbf{i}_n$ are equal to

$$e_{nn} = \mathbf{i}_n \cdot e \cdot \mathbf{i}_n \tag{2.50a}$$

and

$$\tau_{nn} = \overset{n}{\mathbf{T}} \cdot \mathbf{i}_n = \mathbf{i}_n \cdot \boldsymbol{\tau} \cdot \mathbf{i}_n \tag{2.50b}$$

Moreover, referring to relation (1.59) the shearing component of strain referred to the directions $\mathbf{i}_n$ and $\mathbf{i}_s$ and the shearing component of stress acting on the plane normal to the unit vector $\mathbf{i}_n$ in the direction of the unit vector $\mathbf{i}_s$ are equal to

$$e_{ns} = \mathbf{i}_n \cdot e \cdot \mathbf{i}_s$$
$$\tau_{ns} = \overset{n}{\mathbf{T}} \cdot \mathbf{i}_s = \mathbf{i}_n \cdot \boldsymbol{\tau} \cdot \mathbf{i}_s \tag{2.51}$$

As discussed previously, the three unit elongations E_{11}, E_{22} and E_{33} and the three unit shears $\tilde{a}_{12}$, $\tilde{a}_{13}$ and $\tilde{a}_{23}$ of a particle of a body referred to a rectangular system of axes x_1, x_2 and x_3 specify completely the deformation of this particle. In the case the unit elongations, the unit shears and the rotations are very small compared to unity and the rotations are not of a higher order of magnitude than the unit elongations and unit shears. Referring to relations (2.15), we see that *the nine components of strain of a particle referred to a rectangular system of axes x_1, x_2 and x_3 specify completely the deformation of this particle; that is, the deformation of a particle of a body is specified by a symmetric tensor of the second rank whose components are linearly related to the derivatives of the components of the displacement vector* $\hat{\mathbf{u}}$ *of the particle [see relations (2.16)].*

In what follows we present four examples.

Example 1 The components of strain of a particle of a body were measured as

$$e_{11} = e_{13} = e_{12} = 0$$

$$e_{23} = 0.004 \text{ rad} \qquad e_{22} = 0.008 \qquad e_{33} = 0.002$$

1. Compute the principal normal components of strain and the directions of the principal axes and show them on a sketch.

2. Compute the components of strain with respect to the axes x_1', x_2' shown in Fig. a.

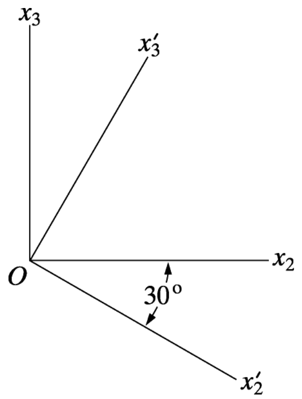


Figure a Orientation of the x_2', x_3' axes.

Solution We know the coordinates of two points on Mohr's circle, namely, $X_2(0.008, 0.004)$ and $X_3(0.002, -0.004)$ which are located on the same diameter. In Fig. b we plot these points on the e_{22}' e_{23}' plane and we draw the diameter of Mohr's circle. It is apparent that the center of the circle is at $e_{22}' = 0.005$. Moreover, the radius of the circle is equal to

$$R = \sqrt{(0.003)^2 + (0.004)^2} = 0.005$$

Thus, referring to Fig. b, the principal values of the normal components of strain are

$$(e'_{22})_{max} = 0.005 + 0.005 = 0.01$$

$$(e'_{22})_{min} = 0$$

Referring to Fig. b, the angle $2\tilde{\phi}_{11}$ is equal to

$$\tan 2\tilde{\phi}_{11} = \frac{0.004}{0.003} = \frac{4}{3}$$

Thus,

$$2\tilde{\phi}_{11} = 53.13° \qquad \text{or} \qquad \tilde{\phi}_{11} = 26.56°$$

The direction of the principal axes is shown in Fig. c.

As shown in Fig. b, point X'_2 whose coordinates are the unknown components of strain with respect to the x'_1, x'_2 axes is located 60° counterclockwise from point X_2. Thus, the angle $\angle X'_2CA$ is equal to

$$\angle X'_2CA = 180 - 60 - 53.13 = 66.87°$$

Consequently, referring to Fig. b, we have

$$e'_{23} = R \sin 66.87 = 0.005 \sin 66.87 = 0.00460 \quad \text{rad}$$
$$e'_{22} = R - CA = 0.005 - R\cos 66.87 = 0.00304$$
$$e'_{33} = R + CB = 0.005 + R\cos 66.87 = 0.00696$$

The results indicate that the right angle between two material lines which before deformation were the one in the direction of the line Ox'_2 and the other in the direction of the line Ox'_3 will decrease, due to deformation, by

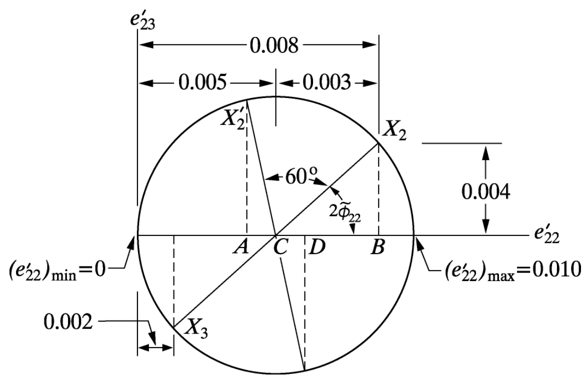


Figure b Mohr's circle.

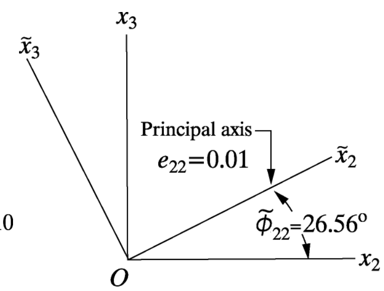


Figure c Direction of the principal axes.

$$\gamma'_{23} = 2e'_{23} = 0.0092 \text{ rad}$$

Example 2 The components of stress acting on a particle with respect to the rectangular system of axes x_j (j = 1, 2, 3) are given as

$$\begin{array}{ll} \tau_{11} = 80 \text{ MPa} & \tau_{12} = 15 \text{ MPa} \\ \tau_{22} = 40 \text{ MPa} & \tau_{13} = \tau_{32} = 0 \\ \tau_{33} = 10 \text{ MPa} & \end{array}$$

(a) Find the principal values of stress and the directions of the principal axes and show them on a sketch.

(b) Compute the components of stress with respect to the rectangular system of axes x'_1, x'_2, x'_3, which are obtained by rotating the system of axes x_1, x_2, x_3 about the x_3 axis, as shown in Fig. a.

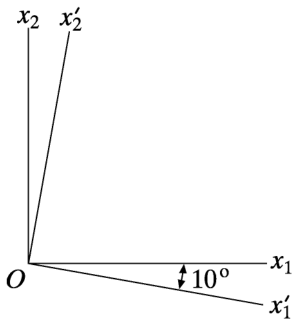


Figure a Location of the system of axes x'_1, x'_2, x'_3.

Solution

Part a

STEP 1 We make a sketch showing a plane view of the particle subjected to the given components of stress (see Fig. b).

STEP 2 We draw the $\hat{o}'_{12}$ and the $\hat{o}'_{11}$ axes of reference, as shown in Fig. c and we plot point X_1 whose coordinates are the components of stress acting on the plane normal to the x_1 axis. Tensile stress is plotted as positive; shearing stress tending to turn the element in a counterclockwise direction is plotted as positive. Thus, the coordinates of point X_1 are 80 MPa and 15 MPa. Then we plot point X_2 whose coordinates are the components of stress acting on the plane normal to x_2 axis, that is, 40 MPa and −15 MPa. Points X_1 and X_2 lie on the same diameter of Mohr's circle. The center of this circle is on the $\hat{o}'_{11}$ axis at a distance ½($\hat{o}_{11} + \hat{o}_{22}$) = 60 MPa from the origin while the radius of the circle is

$$R = [(20)^2 + (15)^2]^{½} = 25$$

STEP 3 Referring to Mohr's circle of Fig. c, we compute the maximum value $\hat{o}_1$ of the normal components of stress acting on any plane normal to the $x_1\, x_2$ plane. That is

$$\tau_1 = OC + CB = 60 + 25 = 85 \text{ MPa}$$

Moreover, the minimum value $\hat{o}_2$ of the normal component of stress acting on any plane normal to the $x_1\, x_2$ plane is equal to

$$\tau_2 = OC - CA = 60 - 25 = 35 \text{ MPa}$$

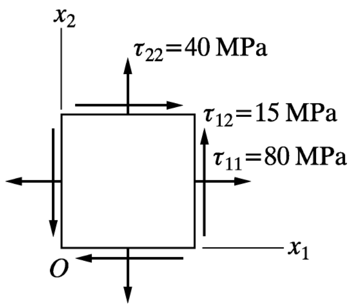
Notice that the minimum normal component of stress is not $\hat{o}_2$ but rather $\hat{o}_3 = \hat{o}_{33} = 10$ MPa.

The angle ϕ_{11} from the x_1 axis to the principle axis $\tilde{x}_1$ may be computed geometrically by referring to Fig. c. That is,

$$\tan 2\tilde{\phi}_{11} = \frac{X_1 D}{CD} = \frac{3}{4}$$

Thus,

$$2\tilde{\phi}_{11} = 36.87° \qquad \text{or} \qquad \tilde{\phi}_{11} = 18.43°$$



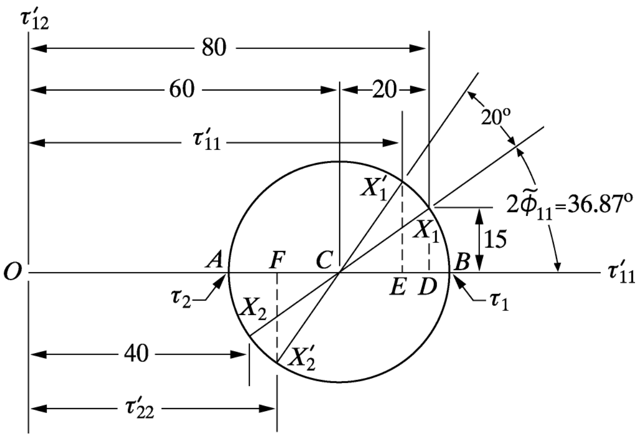


Figure b State of stress of the particle. **Figure c** Mohr's circle.

STEP 4 We locate the principal planes. One principal plane is the $x_1\, x_2$ plane. The other two are normal to x_1, x_2 plane. The plane on which the maximum normal component of stress $\hat{o}_1$ acts is located by the angle $\tilde{\phi}_{11} = 18.43°$ counterclockwise from the x_1 axis. This is so because the point on the Mohr's circle whose coordinates are $(\hat{o}_1, 0)$ is located by the angle $2\,\tilde{\phi}_{11} = 36.87°$ clockwise from point X_1. The principal planes are shown in Fig. d.

Part b

STEP 5 We locate on Mohr's circle the points X'_1 and X'_2 whose coordinates are the components of stress acting on the planes normal to the x'_1 and x'_2 axes, respectively.

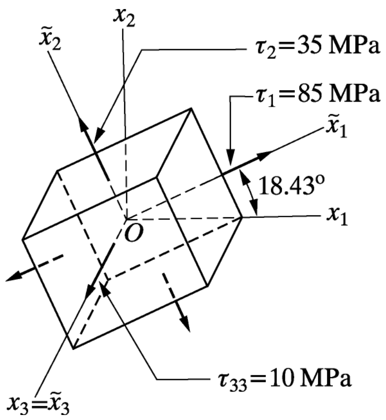


Figure d Principal planes and principal normal components of stress.

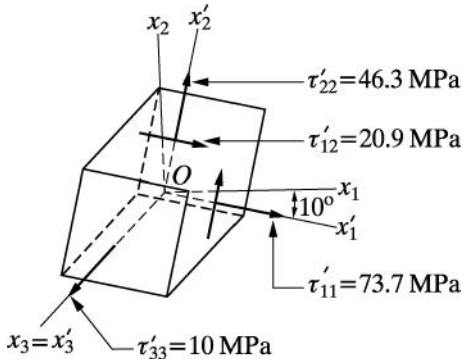


Figure e Components of stress on acting the planes normal to the x_1', x_2', x_3' axes.

Since the x_1' axis is located 10° clockwise from the x_1 axis (see Fig. a), point X_1' is located on Mohr's circle 20° counterclockwise from X_1 (see Fig. c). From geometric considerations, referring to Fig. c, we establish the components of stress acting on the planes normal to x_1' and x_2' axes as

$$\tau_{11}' = \overline{OC} + \overline{CE} = 60 + 25 \cos 56.87° = 73.7 \text{ MPa}$$

$$\tau_{22}' = \overline{OC} - \overline{CF} = 60 - 25 \cos 56.87° = 46.3 \text{ MPa}$$

$$\tau_{12}' = \overline{X_1'E} = 25 \sin 56.87° = 20.9 \text{ MPa}$$

$$\tau_{33}' = 10 \text{ MPa}$$

The components of stress acting on the planes normal to the x_1', x_2', x_3' axes are shown in Fig. e. Notice that the shearing components of stress acting on the plane normal to the x_1' axis is positive and consequently its direction must be such that it tends to turn the element in a counterclockwise direction.

Example 3 The components of stress acting on a particle of a body when referred to the system of axes specified by the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are given as

$$[\tau] = \begin{bmatrix} -60 & 10 & 10 \\ 10 & -60 & 20 \\ 10 & 20 & 100 \end{bmatrix} \text{MPa}$$

Compute the normal and shearing components of stress acting on a plane specified by the unit vector $\mathbf{i}_n = (1/\sqrt{3})\,(\mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3)$

Solution For any two conveniently chosen mutually perpendicular directions $\mathbf{i}'_2$ and $\mathbf{i}'_3$ in the plane normal to the unit vector $\mathbf{i}_n = \mathbf{i}'_1$ the components of stress $\hat{o}'_{12}$ and $\hat{o}'_{13}$ may be found by using relations (2.47a) or (2.48a). The resultant shearing stress acting on the plane normal to the unit vector $\mathbf{i}_n$ may be found from the components of shearing stress $\hat{o}'_{12}$ and $\hat{o}'_{13}$. However, this straightforward procedure is somewhat cumbersome. Instead, the components of traction acting on the plane normal to the unit vector $\mathbf{i}_n$ may be found by using relations (2.44). That is

$$\overset{n}{\mathbf{T}} = \mathbf{i}_n \cdot \boldsymbol{\tau} = \frac{1}{\sqrt{3}}(-60\ \mathbf{i}_1 + 90\ \mathbf{i}_2 + 90\ \mathbf{i}_3)$$

The traction may now be decomposed into two components - one normal and the other tangential to the plane on which it acts (see Fig. a). The normal component $\hat{o}_{nn}$ is given by

$$\tau_{nn} = \overset{n}{\mathbf{T}} \cdot \mathbf{i}_n = \frac{-60 - 90 + 90}{3} = -20\ \text{MPa}$$

Referring to Fig. a the tangential component of traction $\hat{o}_{nt}$ may be found from the following relation:

$$\tau_{nt}\mathbf{i}_t = \overset{n}{\mathbf{T}} - \tau_{nn}\mathbf{i}_n = \frac{1}{\sqrt{3}}\left(-60\mathbf{i}_1 + 90\mathbf{i}_2 + 90\mathbf{i}_3\right) + \frac{1}{\sqrt{3}}\left(20\mathbf{i}_1 - 20\mathbf{i}_2 + 20\mathbf{i}_3\right)$$
$$= \frac{1}{\sqrt{3}}\left(-40\mathbf{i}_1 + 70\mathbf{i}_2 + 110\mathbf{i}_3\right)$$

Hence

$$\tau_{nt} = \frac{\sqrt{(40)^2 + (70)^2 + (110)^2}}{\sqrt{3}} = 78.7\ \text{MPa}$$

The shearing component of stress acts in the direction specified by the unit vector $\mathbf{i}_t$ given by

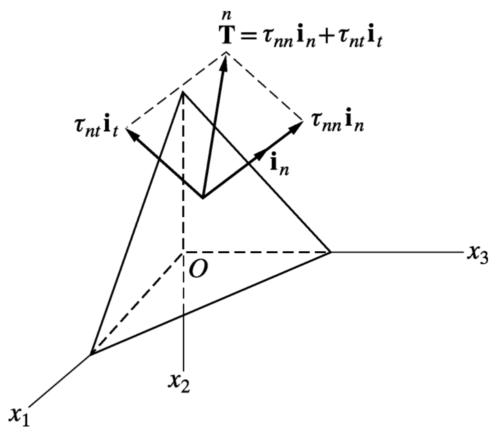


Figure a Traction and components of stress acting on the plane normal to the unit vector $\mathbf{i}_n$.

$$\mathbf{i}_t = \frac{-40\mathbf{i}_1 + 70\mathbf{i}_2 + 110\mathbf{i}_3}{\sqrt{3}(78.7)} = -0.29\mathbf{i}_1 + 0.51\mathbf{i}_2 + 0.81\mathbf{i}_3$$

Example 4 Compute the stationary values of stress and the principal directions at a particle whose components of stress with respect to the x_1, x_2, x_3 axes are

$$[\ \tau\ ] = \begin{bmatrix} -60 & 10 & 10 \\ 10 & -60 & 20 \\ 10 & 20 & 100 \end{bmatrix} \text{ MPa}$$

Solution On the basis of their definitions (1.78) to (1.80) the invariants of stress are

$$II_1 = -20 \text{ MPa} \qquad II_2 = -9{,}000 \text{ (MPa)}^2 \qquad II_3 = 384{,}000 \text{ (MPa)}^3 \tag{a}$$

Therefore, equation (1.105) becomes

$$\tau_k^3 + 20\,\tau_k^2 - 9{,}000\,\tau_k - 384{,}000 = 0 \tag{b}$$

This equation may be solved either by employing one of the standard procedures for establishing the roots of a cubic equation or by employing a numerical technique, to yield

$$\tau_1 = 103.2 \text{ MPa} \qquad \tau_2 = -70.3 \text{ MPa} \qquad \tau_3 = -52.9 \text{ MPa} \tag{c}$$

In order to establish the direction of the unit vector normal to the plane on which the stationary value of stress $\hat{o}_1$ acts, we substitute the value of $\hat{o}_1$ from the first of relations (c) into the first and second of equations (1.103). Thus,

$$\begin{aligned} (-60 - 103.2)\lambda_{11} + 10\lambda_{12} + 10\lambda_{13} &= 0 \\ 10\lambda_{11} - 163.2\lambda_{12} + 20\lambda_{13} &= 0 \end{aligned} \tag{d}$$

Solving equations (d), we obtain

$$\lambda_{12} = 1.8362\lambda_{11} \qquad \lambda_{13} = 14.4837\lambda_{11} \tag{e}$$

Substituting relations (e) into the orthogonality condition $\mathbf{i}_1 \cdot \mathbf{i}_1 = 1$ [see relation (1.25b)] or $\ddot{e}_{11}^2 + \ddot{e}_{12}^2 + \ddot{e}_{13}^2 = 1$, we get

$$\begin{aligned} \lambda_{11}^{(1)} &= 0.068335 & \phi_{11} &= 86.08^\circ \\ \lambda_{12}^{(1)} &= 0.12548 & \phi_{12} &= 82.80^\circ \\ \lambda_{13}^{(1)} &= 0.98927 & \phi_{13} &= 8.43^\circ \end{aligned}$$

It can be shown that the calculated direction cosines satisfy the last of equations (1.103) with $\tau_k = \tau_1 = 03.2$. Similarly, using the values of τ_2 and τ_3 we obtain the following sets of direction cosines corresponding to directions normal the planes on which τ_2 and τ_3 act.

$$
\begin{array}{llll}
\lambda_{21}^{(2)} = -0.6745 & \phi_{21}^{(2)} = 132.42° & \lambda_{31}^{(3)} = -0.7390 & \phi_{31}^{(3)} = 137.65° \\
\lambda_{22}^{(2)} = 0.7371 & \phi_{22}^{(2)} = 42.52° & \lambda_{32}^{(3)} = -0.6601 & \phi_{32}^{(3)} = 131.30° \\
\lambda_{23}^{(2)} = -0.0423 & \phi_{23}^{(2)} = 92.42° & \lambda_{33}^{(3)} = 0.1352 & \phi_{33}^{(3)} = 82.23°
\end{array}
$$

As a check, it can be shown that the calculated direction cosines satisfy relations (1.25).

2.10 Components of Displacement for a General Rigid Body Motion of a Particle

Within the range of validity of the theory of small deformation, the components of strain e_{ij} (i, $j = 1, 2, 3$) of a particle specify completely its deformation. Consequently, when all the components of strain of a particle vanish, the particle has not been deformed. Either it did not move at all or it was displaced as a rigid body. Therefore, referring to relations (2.16), the components of displacement of a particle of a body for a general rigid-body motion of this particle must satisfy the following relations:

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} = 0 \tag{2.52a}$$

$$e_{22} = \frac{\partial \hat{u}_2}{\partial x_2} = 0 \tag{2.52b}$$

$$e_{33} = \frac{\partial \hat{u}_3}{\partial x_3} = 0 \tag{2.52c}$$

$$e_{21} = e_{12} = \frac{1}{2}\left(\frac{\partial \hat{u}_2}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial x_2}\right) = 0 \tag{2.52d}$$

$$e_{31} = e_{13} = \frac{1}{2}\left(\frac{\partial \hat{u}_3}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial x_3}\right) = 0 \tag{2.52e}$$

$$e_{32} = e_{23} = \frac{1}{2}\left(\frac{\partial \hat{u}_2}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_2}\right) = 0 \tag{2.52f}$$

Integrating relations (2.52a) to (2.52c), we obtain

$$\hat{u}_1 = \hat{u}_1(x_2, x_3) \tag{2.53a}$$

$$\hat{u}_2 = \hat{u}_2(x_1, x_3) \tag{2.53b}$$

$$\hat{u}_3 = \hat{u}_3(x_1, x_2) \tag{2.53c}$$

Substituting relations (2.53b) and (2.53c) into relation (2.52f), we obtain

$$\frac{\partial \hat{u}_3(x_1, x_2)}{\partial x_2} = -\frac{\partial \hat{u}_2(x_1, x_3)}{\partial x_3} = f(x_1)$$

The left side of the above relation is a function of x_1 and x_2, while the right side is a function of x_1 and x_3 only. Thus, both sides must be functions of x_1 only. Integrating the above relations, we obtain

$$\hat{u}_2 = -x_3\, \mathrm{f}(x_1) + c_2\,(x_1) \tag{2.54a}$$

$$\hat{u}_3 = x_2\, \mathrm{f}(x_1) + c_3\,(x_1) \tag{2.54b}$$

Substituting relations (2.54) into (2.52d) and (2.52e), we get

$$\frac{\partial \hat{u}_1(x_2, x_3)}{\partial x_3} = -x_2 \frac{df(x_1)}{dx_1} - \frac{dc_3(x_1)}{dx_1} = \psi_2(x_2) \tag{2.55a}$$

$$\frac{\partial \hat{u}_1(x_2, x_3)}{\partial x_2} = x_3 \frac{df(x_1)}{dx_1} - \frac{dc_2(x_1)}{dx_1} = \psi_3(x_3) \tag{2.55b}$$

The left side of the above relations is a function of x_2 and x_3 only, while the right side of the first is a function of x_1 and x_2 only and of the second is a function of x_1 and x_3 only. Consequently, both sides of the first relation must be functions of x_2 only and both sides of the second relation must be functions of x_3 only. Thus, relations (2.55) are valid if

$$\frac{df(x_1)}{dx_1} = c_4 \qquad \frac{dc_3(x_1)}{dx_1} = c_5 \qquad \frac{dc_2(x_1)}{dx_1} = c_6 \tag{2.56a}$$

Consequently,

$$f(x_1) = c_4 x_1 + c_7 \tag{2.56b}$$

$$c_3(x_1) = c_5 x_1 + c_8 \tag{2.56c}$$

$$c_2(x_1) = c_6 x_1 + c_9 \tag{2.56d}$$

Where c_i ($i = 4, 5, 6, 7, 8, 9$) are constants. Substituting relations (2.56) into (2.55) and integrating, we get

$$\hat{u}_1 = -c_4 x_2 x_3 - c_5 x_3 + c_{10}(x_2) \tag{2.57a}$$

$$\hat{u}_1 = c_4 x_2 x_3 - c_6 x_2 + c_{11}(x_3) \tag{2.57b}$$

Comparing equations (2.57a) with (2.57b), we have

$$c_4 = 0 \quad c_{10}(x_2) = -c_6 x_2 + c_{12} \qquad c_{11}(x_3) = -c_5 x_3 + c_{12} \tag{2.58}$$

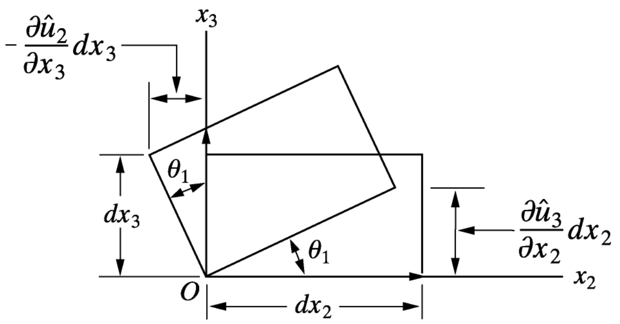


Figure 2.11 Rigid-body rotation of a particle about the x_1 axis.

Substituting relations (2.58b) into (2.57), we get

$$\hat{u}_1 = -c_6 x_2 - c_5 x_3 + c_{12} \tag{2.59a}$$

Substituting relations (2.56b, c, d) into (2.54) and using the first of relations (2.58), we have

$$\hat{u}_2 = c_6 x_1 - c_7 x_3 + c_9 \tag{2.59b}$$

$$\hat{u}_3 = c_5 x_1 + c_7 x_2 + c_8 \tag{2.59c}$$

The constants c_8, c_9 and c_{12} represent rigid-body translation of the particles of the body, while the terms involving the constants c_5, c_6 and c_7 represent rigid-body rotation of the particles of the body. For example, referring to Fig. 2.11, we see that the rigid-body rotation of the particle about the x_1 axis is equal to

$$\theta_1 \approx \tan\,\theta_1 = \frac{\partial \hat{u}_3}{\partial x_2} = -\frac{\partial \hat{u}_2}{\partial x_3} \tag{2.60a}$$

Substituting relation (2.59b) and (2.59c) into (2.60a), we get

$$\theta_1 = c_7 \tag{2.60b}$$

2.11 The Compatibility Equations

If we assume a distribution of the three components of displacement $\hat{u}_i(x_1, x_2, x_3)$ ($i = 1, 2, 3$) in a body, we can find the corresponding components of strain by substituting the assumed components of displacement in relation (2.16). However, if we assume a distribution of the six components of strain $e_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3, e_{ij} = e_{ji})$, it may not be possible to find a corresponding distribution of the three components of displacement by integrating the six partial differential equations (2.16). In this section we establish the restrictions that must be imposed on the function $e_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3, e_{ij} = e_{ji})$ in order to ensure that equations (2.16) can be integrated to give a set of continuous single-valued components of displacement $\hat{u}_i$ $(i = 1, 2, 3)$.

Consider a body in its stress-free, strain-free reference state of mechanical and thermal equilibrium at a uniform temperature T_o. Moreover, consider two particles of the body located at points P_o and Q_o which are an infinitesimal distance $d\mathbf{s}$ apart. The body deforms due to the application on it of external loads and reaches a second state of mechanical equilibrium. As shown in Fig. 2.12, after deformation the two particles under consideration move to points P and Q, respectively. We denote the displacement vector of the two particles which were located prior to deformation at point P_0 and Q_0 by $\mathbf{u}$ and $\hat{\mathbf{u}} + \mathbf{d}\hat{\mathbf{u}}$, respectively. The distance between points P and Q is equal to $\mathbf{d}s + \mathbf{d}\hat{\mathbf{u}}$. It is apparent that if the components of displacement are single-valued continuous functions of the space coordinates, all particles located on a closed curve before deformation must move to a closed curve after deformation. Moreover, if the particles of every closed curve of an undeformed body move to a closed curve after deformation, the components of displacement of the particles of the body are single-valued continuous functions of the space coordinates. That is, referring to Fig. 2.12a, a necessary and sufficient condition which ensures that the components of displacement of the particles of a body are continuous single-valued functions of the space coordinates is that the following relation is valid for the particles of every closed curve of the body:

$$\oint (d\hat{\mathbf{u}} + d\mathbf{s}) = 0 \tag{2.61a}$$

Noting that for a closed curve

$$\oint d\mathbf{s} = 0 \tag{2.61b}$$

we obtain the following necessary and sufficient condition for single-valuedness and continuity of the components of displacement of the particles of a body

$$\oint d\hat{\mathbf{u}} = 0 \tag{2.62}$$

where the integral is taken around every closed curve of the body. It can be shown that a necessary and sufficient condition to ensure the validity of relation (2.62) in a simply

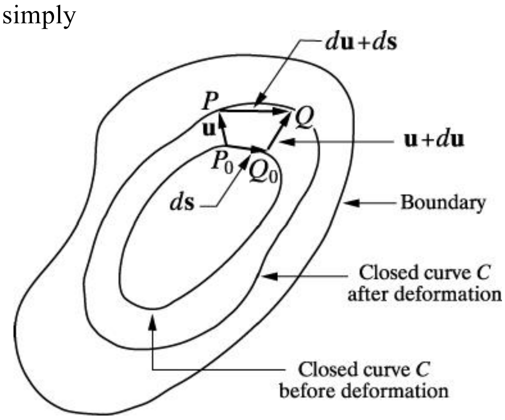


Figure 2.12 Deformation of a closed curve.

connected body (that is, a body without holes) is that the components of strain satisfy the following relations:

$$\frac{\partial^2 e_{11}}{\partial x_2^2} + \frac{\partial^2 e_{22}}{\partial x_1^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2}$$

$$\frac{\partial^2 e_{33}}{\partial x_1^2} + \frac{\partial^2 e_{11}}{\partial x_3^2} = 2\frac{\partial^2 e_{31}}{\partial x_3 \partial x_1}$$

$$\frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2} = 2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3}$$

$$\frac{\partial^2 e_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} \right) \tag{2.63}$$

$$\frac{\partial^2 e_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(\frac{\partial e_{23}}{\partial x_1} - \frac{\partial e_{31}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3} \right)$$

$$\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} - \frac{\partial e_{12}}{\partial x_3} \right)$$

Thus, for a body without holes the satisfaction of equations (2.63) by a set of six functions e_{11}, e_{22}, e_{33}, $e_{12} = e_{21}$, $e_{13} = e_{31}$, $e_{23} = e_{32}$ is a necessary and sufficient condition for ensuring that relations (2.16) can be integrated to yield a set of three single-valued continuous functions of the space coordinates. However, for a body with holes, the satisfaction of relations (2.63) by a set of six functions e_{11}, e_{22}, e_{33}, $e_{12} = e_{21}$, $e_{13} = e_{31}$, $e_{23} = e_{32}$ ensures only that when these functions are substituted into relations (2.16), they can be integrated to give a set of three functions $\hat{u}_i\,(x_1, x_2, x_3)(i = 1, 2, 3)$, which may or may not be single valued and continuous. If they are not, it is an indication that the six function e_{ij} $(i, j = 1, 2, 3)$ cannot be the components of strain of the body.

The six equations (2.63) are called the *compatibility equations*. It can be shown that they are equivalent to a system of three independent fourth order equations. However, it is more convenient to use the system of six second order equations (2.63) instead of the equivalent system of three fourth order equations.

2.12 Measurement of Strain

Any device or instrument employed in measuring the change of a specified length of a body is called an *extensiometer*. In general, inasmuch as the change of this length is very small, it is usually magnified. The most commonly used extensiometers are the *resistance wire electrical strain gages*, which are composed of a grid of fine wire filament (usually 0.001 in. diameter) cemented on the surface of the body so that the wire of the strain gage deforms as the surface of the body deforms. The operation of the strain

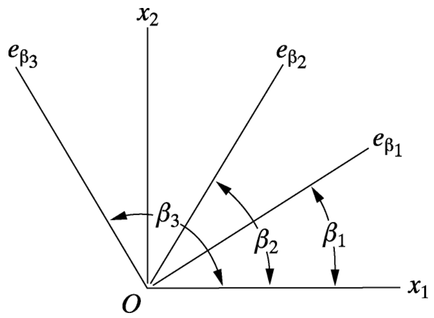


Figure 2.13 Directions along which the normal component of strain is measured.

gage is based on the principle that, as the wire elongates or shrinks, its electrical resistance changes in proportion to the change of its length. The change of the resistance of the wire is then measured accurately with a simple Wheatstone Bridge circuit, and converted into a direct measure of the normal component of strain at the point of the body at which the gage is bonded, in the direction of the gage.

Two strain gages would suffice in establishing the state of strain of a particle in a state of plane strain, if the directions of the principal components of strains are known. In general, however, the directions of the principal components of strain are not known and, consequently, the state of strain of a particle in plane strain can be specified by measuring the normal component of strain in two mutually perpendicular directions and the shearing component of strain associated with these directions. The experimental measurement of the shearing component of strain, however, is more difficult and for this reason, normal components of strain are measured in three non-collinear directions. An arrangement of three strain gages which measure the normal components of strain in three different directions is known as a *strain rosette*. The gage lines of a strain rosette are located with respect to an axis x_1 (see Fig. 2.13) by the known angles $\beta_1, \beta_2, \beta_3$ Inasmuch as a plane strain condition is assumed ($e_{13} = e_{13} = e_{33} = 0$), the measured strain components designated by $e_{\beta_1}, e_{\beta_2}, e_{\beta_3}$ are related to the strain components e_{11}, e_{12}, and e_{12} [see relations (1.116)] by the following relations:

$$
\begin{aligned}
e_{\beta_1} &= \frac{(e_{11} + e_{22})}{2} + \frac{(e_{11} - e_{22})}{2} \cos 2\beta_1 + e_{12} \sin 2\beta_1 \\
e_{\beta_2} &= \frac{(e_{11} + e_{22})}{2} + \frac{(e_{11} - e_{22})}{2} \cos 2\beta_2 + e_{12} \sin 2\beta_2 \\
e_{\beta_3} &= \frac{(e_{11} + e_{22})}{2} + \frac{(e_{11} - e_{22})}{2} \cos 2\beta_3 + e_{12} \sin 2\beta_3
\end{aligned}
\tag{2.64}
$$

This set of three simultaneous equations may be solved for the desired strain components e_{11}, e_{12}, e_{22} referred to the directions x_1 and x_2. However, in order to simplify the computations, the strain rosettes are manufactured with conveniently chosen angles $\beta_1, \beta_2, \beta_3$, the most common of which are the 45° strain rosette ($\beta_1 = 0$, $\beta_2 = 45°$, $\beta_3 = 90°$) and the 60° strain rosette ($\beta_1 = 0$, $\beta_2 = 60°$, $\beta_3 = 120°$). For the 45° strain rosette, relations (2.64) yield

$$e_{11} = e_{\beta_1} \qquad e_{22} = e_{\beta_3} \qquad e_{12} = e_{\beta_2} - \frac{1}{2}(e_{\beta_1} + e_{\beta_3}) \tag{2.65}$$

while for the 60° strain rosette, relations (2.65) reduce to

$$\begin{aligned} e_{11} &= e_{\beta_1} \qquad e_{22} = \frac{1}{3}\left[2e_{\beta_2} + 2e_{\beta_3} - e_{\beta_1} \right] \\ e_{12} &= \frac{1}{\sqrt{3}}(e_{\beta_2} - e_{\beta_3}) \end{aligned} \tag{2.66}$$

In what follows we present an example.

Example 5 The following readings were taken by three strain gages attached to a body in a state of plane strain, in the directions shown in Fig. a.

$$e_{11} = 0.002 \qquad e_{22} = 0.006 \qquad e_{\beta_2} = 0.0037$$

1. Find the principal normal components of strain and the direction of the principal axes.
2. Compute the components of strain with respect to the set of axes x_1', x_2', shown in Fig. b.

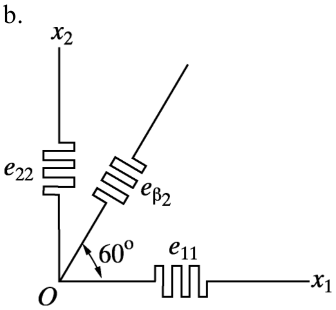


Figure a Location of the strain gages.

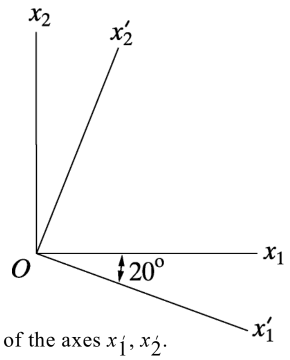


Figure b Location of the axes x_1', x_2'.

Solution Referring to relation (2.64), we have

$$\begin{aligned} e_{\beta_2} = 0.0037 &= \frac{e_{11} + e_{22}}{2} + \frac{(e_{11} - e_{22})}{2}\cos 2\beta_2 + e_{12}\sin 2\beta_2 \\ &= \frac{0.002 + 0.006}{2} + \left(\frac{0.002 - 0.006}{2}\right)(-0.5) + e_{12}\left(\frac{\sqrt{3}}{2}\right) \end{aligned}$$

or

$$e_{12} = -0.0015 \text{ rad} = -0.086°$$

The minus sign indicates that the angle $\angle x_1 O x_2$, which was a right angle prior to deformation, has increased due to deformation by approximately $\tilde{a}_{12} = 2e_{12} = 0.17°$.

Referring to Fig. c, we plot point X_1 whose coordinates are 0.002 and −0.0015 and point X_2 whose coordinates are 0.006 and 0.0015. These points lie on the same diameter of Mohr's circle. The center of this circle is on the e_{11}' axis at a distance 1/2 $(e_{11} + e_{22})$ = 0.004 from the origin. Moreover, from geometric considerations, referring to Fig. c, the radius of Mohr's circle is

$$R = \sqrt{(0.002)^2 + (0.0015)^2} = 0.0025$$

Part a

Referring to Fig. c, we see that the maximum normal component of strain is

$$e_2 = 0.004 + 0.0025 = 0.0065$$

while the minimum in the x_1x_2 plane is

$$e_1 = 0.0015$$

Using geometry, the angle $\tilde{\phi}_{11}$ between the x_1 axis to the principal axis $\tilde{x}_1$ may be computed as

$$\tan 2\tilde{\phi}_{11} = \frac{3}{4} \qquad \text{and} \qquad 2\tilde{\phi}_{11} = 36.87° \qquad \tilde{\phi}_{11} = 18.43°$$

The principal axes $\tilde{x}_1, \tilde{x}_2$ are shown in Fig. d.

Part b

Since the axis x_1' is located 20° clockwise from the axis x_1, point X_1' on Mohr's circle whose coordinates are the components of strain e_{11}' and e_{12}' with respect to the x_1' and x_2' axes is located 40° counter clockwise from point X_1. Referring to Fig. c, from geometric considerations, we obtain

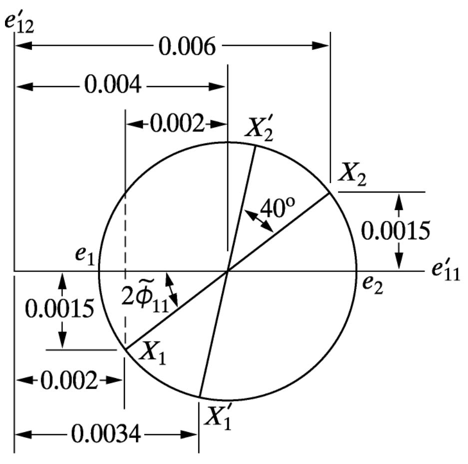


Figure c Mohr's circle.

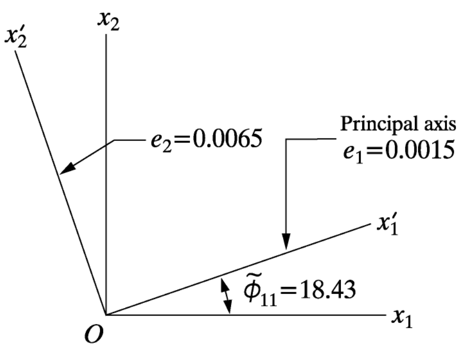


Figure d Location of the principal axes.

$$e'_{11} = 0.004 - 0.0025 \cos 76.87° = 0.0034$$
$$e'_{22} = 0.004 + 0.0025 \cos 76.87° = 0.0047$$
$$e'_{12} = -0.0025 \sin 76.87° = -0.0024$$

The minus sign in e'_{12} indicates that the angle $\angle x_1Ox_2$ increased due to the deformation by

$$\gamma'_{12} = 2e'_{12} = 0.0048 \text{ rad} = 0.275°$$

2.13 The Requirements for Equilibrium of the Particles of a Body

Consider a body initially in a stress-free, strain-free state of mechanical and thermal equlbrium at the uniform temperature T_0. Subsequently, the body is subjected to external loads and reaches a second state of mechanical but not necessarily thermal equilibrium. In this text we are interested in establishing the displacement and stress fields in the body when it is in its second state of mechanical equilibrium. Clearly the stress field must be such that every particle of the body is in equilibrium. That is, the sum of the forces of the components of stress acting on each particle of the body and their moments about any point must vanish.

In what follows, we consider two particles of the body in the second state of mechanical equilibrium. The one is located inside the volume of the body or on the portion of its surface where components of displacement are specified (see Fig. 2.14). The other is located on the portion of the surface of the body where components of tractions are specified (see Fig. 2.15). The particle shown in Fig. 2.14 is subjected to

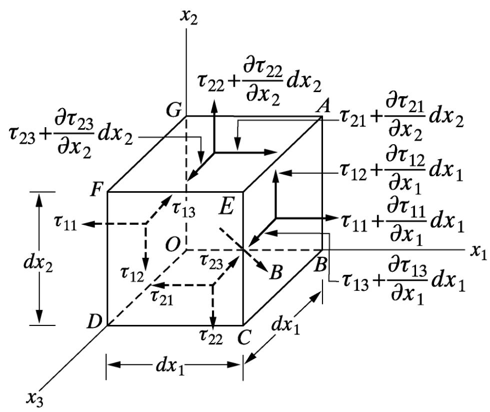


Figure 2.14 Free-body diagram of a particle located inside the volume of a body or on the portion of its surface where the components of displacement are specified.

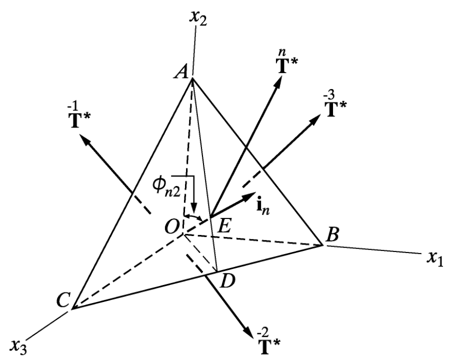


Figure 2.15 Free-body diagram of a particle located on the portion of the surface of the body where the components of traction are specified.

components of stress on its faces and distribution of body forces whose resultant acts through its mass center. The components of stress acting on any face of the particle in general vary throughout this face. Thus, referring to Fig. 2.14, $\tau_{11}(x_1, x_2, x_3)$, $\tau_{12}(x_1, x_2, x_3)$ and $\tau_{13}(x_1, x_2, x_3)$ are the average components of stress acting on face *OGFD*, while $\tau_{11}(x_1 + dx_1, x_2, x_3)$, $\tau_{12}(x_1 + dx_1, x_2, x_3)$ and $\tau_{13}(x_1 + dx_1, x_2, x_3)$ are the average components of stress acting on face *ABCE*. Inasmuch as the components of stress are differentiable functions, $\tau_{11}(x_1 + dx_1, x_2, x_3)$can be expressed in terms of its values at the neighboring point (x_1, x_2, x_3) using a Taylor series expansion. That is, disregarding the terms involving second or higher powers of dx_1, we have

$$\tau_{11}(x_1 + dx_1, x_2, x_3) = \tau_{11}(x_1, x_2, x_3) + \frac{\partial \tau_{11}}{\partial x_1} dx_1 \tag{2.67}$$

where the partial derivative of $\hat{o}_{11}$ is evaluated at point (x_1, x_2, x_{13}). Since the element is in equilibrium, the sum of the forces acting on it must vanish. Therefore, we may write

$$\begin{aligned}\sum F_1 = 0 = &- \tau_{11} dx_2 dx_3 + \left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1 \right) dx_2 dx_3 - \tau_{21} dx_1 dx_3 \\ &+ \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} dx_2 \right) dx_1 dx_3 - \tau_{31} dx_1 dx_2 \\ &+ \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3 \right) dx_1 dx_2 + B_1 dx_1 dx_2 dx_3\end{aligned} \tag{2.68}$$

Collecting terms and dividing by $dx_1\, dx_2\, dx_3$, we have

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + B_1 = 0 \tag{2.69a}$$

Similarly, we obtain

$$\sum F_2 = 0 \qquad \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + B_2 = 0 \tag{2.69b}$$

$$\sum F_3 = 0 \qquad \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} + B_3 = 0 \tag{2.69c}$$

Equations (2.69) are referred to as the *equations of equilibrium*. They must be satisfied by the components of stress acting on any three mutually perpendicular planes at every particle located inside the volume of a body.

In addition to the above, the equilibrium of an element requires that the sum of the moments of all the forces acting on it, about any point, must vanish. We may, therefore, write

$$\begin{aligned}\sum M_3 = & -\tau_{11} dx_2 dx_3 \frac{dx_2}{2} + \left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 \frac{dx_2}{2} - \left(\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1} dx_1\right) dx_2 dx_3 dx_1 \\ & + \tau_{22} dx_1 dx_3 \frac{dx_1}{2} - \left(\tau_{22} + \frac{\partial \tau_{22}}{\partial x_2} dx_2\right) dx_1 dx_3 \frac{dx_1}{2} + \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 dx_2 \\ & - \tau_{31} dx_1 dx_2 \frac{dx_2}{2} + \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3\right) dx_1 dx_2 \frac{dx_2}{2} - \left(\tau_{32} + \frac{\partial \tau_{32}}{\partial x_3} dx_3\right) dx_1 dx_2 \frac{dx_1}{2} \\ & + \tau_{32} dx_1 dx_2 \frac{dx_1}{2} - B_1 dx_1 dx_2 dx_3 \frac{dx_2}{2} + B_2 dx_1 dx_2 dx_3 \frac{dx_1}{2} = 0\end{aligned} \tag{2.70}$$

Simplifying and neglecting higher order terms, we obtain

$$\tau_{12} = \tau_{21} \tag{2.71a}$$

Similarly, we have

$$\sum M_2 = 0 \qquad \tau_{13} = \tau_{31} \tag{2.71b}$$

$$\sum M_1 = 0 \qquad \tau_{23} = \tau_{32} \tag{2.71c}$$

Therefore, we may conclude that *the components of stress of a particle are components of a symmetric tensor of the second rank.*

In Fig. 2.15 we show a particle of a body, in the form of tetrahedron $OABC$, located on its boundary. The surface ABC of the particle is a portion of area ΔA of the surface of the body where the components of traction $\overset{n}{T}{}_1^s$, $\overset{n}{T}{}_2^s$, $\overset{n}{T}{}_3^s$ are specified. From the equilibrium of the particle we have

$$\begin{aligned}\sum F_1 &= -\tau_{11}(\text{area}\,COA) - \tau_{21}(\text{area}\,COB) + \tau_{31}(\text{area}\,AOB) + \overset{n}{T}{}_1^s(\Delta A) + \frac{\Delta A \Delta h B_1}{3} = 0 \\ \sum F_2 &= -\tau_{12}(\text{area}\,COA) - \tau_{22}(\text{area}\,COB) - \tau_{32}(\text{area}\,AOB) + \overset{n}{T}{}_2^s(\Delta A) + \frac{\Delta A \Delta h B_2}{3} = 0 \\ \sum F_3 &= -\tau_{13}(\text{area}\,COA) - \tau_{23}(\text{area}\,COB) - \tau_{33}(\text{area}\,AOB) + \overset{n}{T}{}_3^s(\Delta A) + \frac{\Delta A \Delta h B_3}{3} = 0\end{aligned} \tag{2.72}$$

where referring to Fig. 2.15 Δh is the distance from point O to the plane ABC. Substituting relations (2.34) into (2.72) and dividing by ΔA, in the limit as Δh approaches zero, we obtain

$$\overset{n}{T}{}_1^s = \overset{n}{\mathbf{T}}{}^s \cdot \mathbf{i}_1 = \tau_{11}\lambda_{n1} + \tau_{21}\lambda_{n2} + \tau_{31}\lambda_{n3} \tag{2.73a}$$

$$\overset{n}{T}{}_2^s = \overset{n}{\mathbf{T}}{}^s \cdot \mathbf{i}_2 = \tau_{12}\lambda_{n1} + \tau_{22}\lambda_{n2} + \tau_{32}\lambda_{n3} \tag{2.73b}$$

$$\overset{n}{T}{}_3^s = \overset{n}{\mathbf{T}}{}^s \cdot \mathbf{i}_3 = \tau_{13}\lambda_{n1} + \tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3} \tag{2.73c}$$

These relations may be written as

$$\overset{n}{T}{}_i^s = \overset{n}{\mathbf{T}} \cdot \mathbf{i}_i = \sum_{j=1}^{3} \tau_{ji}\lambda_{nj} \qquad (i = 1, 2, 3) \tag{2.74}$$

Relations (2.73) relate the specified components of tracton acting on a particle of the surface of a body to the components of stress acting on three mutually perpendicular planes of this particle. It is clear that the actual components of stress acting on this particle must satisfy relations (2.73). In this case we say that the components of stress satisfy the given traction boundary conditions at that particle. Notice that relations (2.73) apply to any particle of the body. For a particle which is located inside the volume of a body they represent relations among the components of stress acting on three mutually perpendicular planes of the particle and the components of traction acting on the plane of the particle specified by the unit vector $\mathbf{i}_n$ (see Fig. 2.15).

2.14 Cylindrical Coordinates

Problems involving deformable bodies having circular cylindrical boundaries can be solved more conveniently by using cylindrical coordinates $x_1, r, è$. Referring to Fig. 2.16, the cylindrical coordinates of a point are related to its cartesian coordinates by the following relations:

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= r \cos \theta \\ x_3 &= r \sin \theta \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} x_1 &= x_1 \\ r &= \sqrt{x_2^2 + x_3^2} \\ \theta &= \tan^{-1} \frac{x_3}{x_2} \qquad 0 \leq \theta \leq 2\pi \end{aligned} \tag{2.76}$$

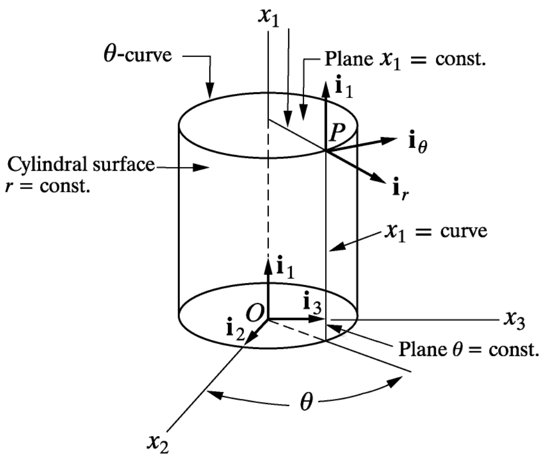


Figure 2.16 Cylindrical coordinates.

All points which have the same coordinate x_i (i = 1, 2, or 3) are located on one plane normal to axis x_i. Thus, the cartesian coordinates of a point specify three mutually perpendicular planes which intersect at that point. The cylindrical coordinates of a point specify three mutually perpendicular surfaces which intersect at that point. Referring to Fig. 2.16, we see that

1. All points which have the same coordinates r = constant = r^* are located on a cylindrical surface whose radius is r^* and whose axis is the x_1 axis.

2. All points which have the same coordinates θ = constant = θ^* are located on a plane containing the x_1 axis and forming an angle θ^* with the $x_1\,x_2$ plane.

3. All points which have the same coordinates x_1 = constant = x_1^* are located on a plane perpendicular to the x_1 axis at $x_1 = x_1^*$.

If any two of the coordinates x_1, r, è are constant, relations (2.75) or (2.76) specify a curve in space called *coordinate curve*. For every point in space we have three mutually perpendicular coordinate curves which are the intersections of the three coordinate surfaces (see Fig. 2.16). The x_1-curve corresponds to r = constant and θ = constant and is a straight line. The r-curve corresponds to x_1 = constant and θ = constant and is a straight line. The θ-curve corresponds to x_1 = constant and r = constant and is a circle (see Fig. 2.16). Consider a system of three mutually perpendicular unit vectors $\mathbf{i}_1$, $\mathbf{i}_r$, $\mathbf{i}_{è}$ which, as shown in Fig. 2.16, are tangent to the x_1, r and è coordinate curves, respectively. Referring to Fig. 2.16 the direction cosines of these unit vectors with respect to the x_1, x_2, x_3 axes are

$$\begin{array}{lll} \lambda_{11} = 1 & \lambda_{12} = 0 & \lambda_{13} = 0 \\ \lambda_{r1} = 0 & \lambda_{r2} = \cos\theta & \lambda_{r3} = \sin\theta \\ \lambda_{\theta 1} = 0 & \lambda_{\theta 2} = -\sin\theta & \lambda_{\theta 3} = \cos\theta \end{array} \tag{2.77}$$

Thus, the transformation matrix which transforms the cartesian components of a vector to cylindrical is

$$[\,\Lambda_s\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{2.78}$$

Consider vector û and denote its cartesian and cylindrical components as follows

$$\hat{\mathbf{u}} = \hat{u}_1\mathbf{i}_1 + \hat{u}_2\mathbf{i}_2 + \hat{u}_3\mathbf{i}_3 = \hat{u}_1\mathbf{i}_1 + \hat{u}_r\mathbf{i}_r + \hat{u}_\theta\mathbf{i}_\theta \tag{2.79}$$

Its cartesian components may be expressed in terms of its cylindrical ones, using relations (1.34) and (2.78). That is,

$$[\,\hat{u}\,] = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = [\,\Lambda_s\,]^T \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_r \\ \hat{u}_\theta \end{Bmatrix} = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_r\cos\theta - \hat{u}_\theta\sin\theta \\ \hat{u}_r\sin\theta + \hat{u}_\theta\cos\theta \end{Bmatrix} \tag{2.80a}$$

Moreover, its cylindrical components may be expressed in terms of its cartesian ones, using relations (1.33) and (2.78). That is,

$$\begin{Bmatrix} \hat{u}_1 \\ \hat{u}_r \\ \hat{u}_\theta \end{Bmatrix} = [\,\Lambda_s\,] \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{Bmatrix} = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \cos\theta + \hat{u}_3 \sin\theta \\ -\hat{u}_2 \sin\theta + \hat{u}_3 \cos\theta \end{Bmatrix} \tag{2.80b}$$

Using relations (2.76) and (2.75), we find that the partial derivatives of a function $F(x_1, x_2, x_3)$, in terms of cartesian coordinates, are related to its partial derivatives in terms of cylindrical coordinates, by the following relations:

$$\begin{aligned} \frac{\partial F}{\partial x_2} &= \frac{\partial F}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x_2} = \cos\theta\frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta} \\ \frac{\partial F}{\partial x_3} &= \frac{\partial F}{\partial r}\frac{\partial r}{\partial x_3} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x_3} = \sin\theta\frac{\partial F}{\partial r} + \frac{\cos\theta}{r}\frac{\partial F}{\partial \theta} \end{aligned} \tag{2.81}$$

2.15 Strain–Displacement Relations in Cylindrical Coordinates

In order to convert to cylindrical coordinates the strain–displacement relations given by relations (2.16) with respect to cartesian coordinates, we adhere to the following steps:

STEP 1 We establish the cylindrical components of strain in terms of its cartesian components by substituting the direction cosines (2.77) into the transformation relations (2.48a). That is,

$$\begin{aligned} e_{rr} &= e_{22}\cos^2\theta + e_{33}\sin^2\theta + 2\,e_{23}\sin\theta\cos\theta \\ e_{r\theta} &= -(e_{22} - e_{33})\sin\theta\cos\theta + e_{23}(\cos^2\theta - \sin^2\theta) \\ e_{r1} &= e_{21}\cos\theta + e_{31}\sin\theta \\ e_{\theta\theta} &= e_{22}\sin^2\theta + e_{33}\cos^2\theta - 2e_{23}\sin\theta\cos\theta \\ e_{\theta 1} &= -e_{21}\sin\theta + e_{31}\cos\theta \end{aligned} \tag{2.82}$$

STEP 2 We eliminate the cartesian components of strain from relations (2.82), using the strain–displacement relations (2.16).

STEP 3 In the expression for the cylindrical components of strain obtained in step 2, we convert differentiation with respect to cartesian coordinates to differentiation with respect to cylindrical coordinates using relations (2.81).

STEP 4 In the expressions for the cylindrical components of strain obtained in step 3, we replace the cartesian components of the displacement vector with its cylindrical components, using relations (2.80a), and we obtain the following strain–displacement relations in cylindrical coordinates:

$$
\begin{aligned}
e_{11} &= \frac{\partial \hat{u}_1}{\partial x_1} \\
e_{rr} &= \frac{\partial \hat{u}_r}{\partial r} \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial \hat{u}_\theta}{\partial \theta} + \frac{\hat{u}_r}{r} \\
e_{1r} &= \frac{1}{2}\left(\frac{\partial \hat{u}_r}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial r}\right) \\
e_{1\theta} &= \frac{1}{2}\left(\frac{\partial \hat{u}_\theta}{\partial x_1} + \frac{1}{r}\frac{\partial \hat{u}_1}{\partial \theta}\right) \\
e_{r\theta} &= \frac{1}{2}\left(\frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r} + \frac{1}{r}\frac{\partial \hat{u}_r}{\partial \theta}\right)
\end{aligned}
\tag{2.83}
$$

The afore described procedure is long and algebraically tedious. For this reason, the strain–displacement relations in cylindrical coordinates are often established by considering the geometry of deformation of an infinitesimal material element bounded by surfaces x_1 and $x_1 + dx_1$, r and $r + dr$ and *rè* and *rè* + *rdè*.

2.16 The Equations of Compatibility in Cylindrical Coordinates

The equations of compatibility (2.63) in cartesian coordinates can be converted to equations of compatibility in cylindrical coordinates in two steps:

1. Differentiation with respect to cartesian coordinates in equation (2.63) is converted to differentiation with respect to cylindrical coordinates using relations (2.81).

2. The cartesian components of strain are replaced with cylindrical using the transformation relation (2.48b) and the direction cosines (2.77).

After algebraically cumbersome manipulations we obtain the following equations of compatibility in cylindrical coordinates:

$$
\begin{gathered}
r\frac{\partial e_{rr}}{\partial r} - \frac{\partial^2 e_{rr}}{\partial \theta^2} - \frac{\partial}{\partial r}\left[r\left(r\frac{\partial e_{\theta\theta}}{\partial r} - 2\frac{\partial e_{r\theta}}{\partial \theta}\right)\right] = 0 \\
\frac{\partial}{\partial x_1}\left[r\frac{\partial e_{rr}}{\partial \theta} - \frac{\partial (r^2 e_{r\theta})}{\partial r}\right] + r^2\frac{\partial}{\partial r}\left\{\frac{1}{r}\left[\frac{\partial r e_{1\theta}}{\partial r} - \frac{\partial e_{1r}}{\partial \theta}\right]\right\} = 0 \\
\frac{\partial}{\partial x_1}\left[e_{rr} - \frac{\partial r e_{\theta\theta}}{\partial r} + \frac{\partial e_{r\theta}}{\partial \theta}\right] + \frac{\partial}{\partial \theta}\left[\frac{\partial r e_{1\theta}}{\partial r} - \frac{\partial e_{1r}}{\partial \theta}\right] = 0
\end{gathered}
\tag{2.84}
$$

$$
\begin{aligned}
r^2\frac{\partial^2 e_{\theta\theta}}{\partial x_1^2} + r\frac{\partial e_{11}}{\partial r} + \frac{\partial^2 e_{11}}{\partial \theta^2} - 2r\frac{\partial}{\partial x_1}\left(\frac{\partial e_{1\theta}}{\partial \theta} + e_{1r}\right) &= 0 \\
\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial e_{11}}{\partial \theta}\right) + \frac{\partial}{\partial x_1}\left[\frac{\partial e_{r\theta}}{\partial x_1} - r\frac{\partial}{\partial r}\left(\frac{e_{1\theta}}{r}\right) - \frac{1}{r}\frac{\partial e_{1r}}{\partial \theta}\right] &= 0 \\
r\frac{\partial^2 e_{rr}}{\partial x_1^2} + \frac{\partial^2 e_{\theta\theta}}{\partial r^2} - 2\frac{\partial^2 e_{1r}}{\partial r \partial x_1} &= 0
\end{aligned}
\tag{2.84}
$$

2.17 The Equations of Equilibrium in Cylindrical Coordinates

The equations of equilibrium in cylindrical coordinates may be derived by considering the equilibrium of the element shown in Fig. 2.17. For instance, setting the sum of the forces in the radial direction equal to zero, we obtain

$$
\begin{aligned}
\sum F_r = 0 = &-\left(\tau_{\theta\theta} + \frac{\partial \tau_{\theta\theta}}{\partial \theta}d\theta\right) dr\, dx_1 \sin\left(\frac{d\theta}{2}\right) - \tau_{\theta\theta}\, dr\, dx_1 \sin\left(\frac{d\theta}{2}\right) \\
&+ \left(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta}d\theta\right) dr\, dx_1 - \tau_{r\theta}\, dr\, dx_1 \\
&+ \left(\tau_{rr} + \frac{\partial \tau_{rr}}{\partial r}dr\right)(r + dr)\, d\theta\, dx_1 - \tau_{rr}\, r d\theta\, dx_1 \\
&- \tau_{1r}(r d\theta\, dr) + \left(\tau_{1r} + \frac{\partial \tau_{1r}}{\partial x_1}dx_1\right) r d\theta\, dr + B_r\, r d\theta\, dr\, dx_1
\end{aligned}
\tag{2.85}
$$

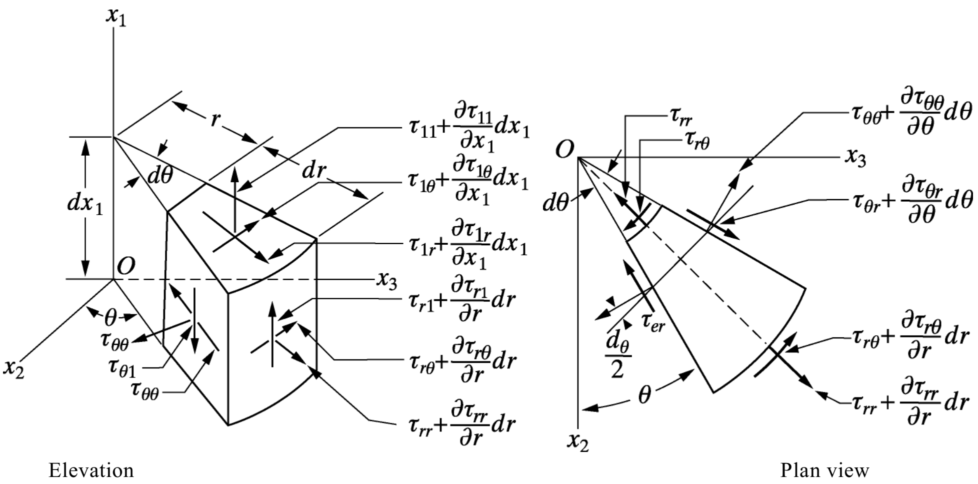


Figure 2.17 Material element in cylindrical coordinates subjected to components of stress.

Collecting terms, disregarding higher order terms and dividing by $dV = rdè\ dr\ dx_1$, equation (2.85) reduces to

$$\frac{\partial \tau_{r1}}{\partial x_1} + \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} + B_r = 0 \tag{2.86a}$$

Similarly, by setting equal to zero the sum of the forces in the tangential and the axial directions, we get

$$\frac{\partial \tau_{\theta 1}}{\partial x_1} + \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} + B_\theta = 0 \tag{2.86b}$$

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{r1}}{\partial r} + \frac{\partial \tau_{r1}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta 1}}{\partial \theta} + B_1 = 0 \tag{2.86c}$$

2.18 Problems

1. The displacement field of a body is given as

$$\hat{\mathbf{u}} = -k(2x_1 - 2x_2)\mathbf{i}_1 + k(3x_1 + 2x_2)\mathbf{i}_2$$

where k is a small number so that the magnitudes of the components of strain are within the range of validity of the theory of small deformation.

(a) Consider a material straight-line segment which prior to deformation was extending from point $P(3, 1, 0)$ to point $Q(1, 2, 0)$. Find the equation of this line after deformation. *Hint:* The line under consideration is in the x_1x_2 plane.

Ans. $(4k - 1)\hat{\imath}_1 = (2 - 6k)\hat{\imath}_2 + 50k^2 - 5$

(b) Determine the equation of the ellipse into which a circle $x_{12} + x_{22} = 1$ is deformed.

Ans. Inclined ellipse $[(2k + 1)^2 + 9k^2]\xi_1^2 + [4k^2 + (1 - 2k)^2]\xi_2^2 - k(10 - 4k)\xi_1\xi_2 = (1 - 10k^2)^2$

(c) Compute the components of the strain tensor.

Ans. $[e] = \frac{k}{2}\begin{bmatrix} -4 & 5 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) Compute the change of angle due to the deformation between two infinitesimal material elements oriented in the direction of the unit vectors $\mathbf{i}_2$ and $\mathbf{i}_n = 1/5(3\mathbf{i}_1 - 4\mathbf{i}_3)$.

Ans. $ã_{2n} = 3k$

2. The parallelepiped of Fig. 2P2 has deformed into the shape indicated by the dashed straight lines. The components of displacement of the parallelepiped are given as

$$\hat{u}_1 = A_1x_1x_2x_3 \qquad \hat{u}_2 = A_2x_1x_2x_3 \qquad \hat{u}_3 = A_3x_1x_2x_3$$

The coordinates of point E' are $\hat{\imath}_1 = 1.503$ mm, $\hat{\imath}_2 = 1.004$ mm, $\hat{\imath}_3 = 1.997$ mm.

(a) Establish the components of strain of the particle located prior to deformation at point E.

Ans.
$$[e] = \begin{bmatrix} 0.00200 & 0.00283 & -0.00025 \\ 0.00283 & 0.00400 & -0.00050 \\ -0.00025 & -0.00050 & -0.00150 \end{bmatrix}$$

(b) Determine the normal component of strain in the direction $\overrightarrow{EA}$ of the particle located prior to deformation at point E. Ans. $e_{EA} = 8(10^{-4})$

(c) Determine the change due to the deformation of the before deformation right angle $\angle AEF$. Ans. $\tilde{a}_{AEF} = 0.00298$ rad.

Warning: Do not attempt to find the change of the angle $LAEF$ by using

$$\sin \gamma_{A'E'F'} = \frac{\overrightarrow{E'A'} \times \overrightarrow{E'F'}}{|\overrightarrow{E'A'}|\ |\overrightarrow{E'F'}|}$$

because line EA deforms into a curve and the orientation after deformation of an infinitesimal segment of line EA at point E is not in the direction of Å'Á'.

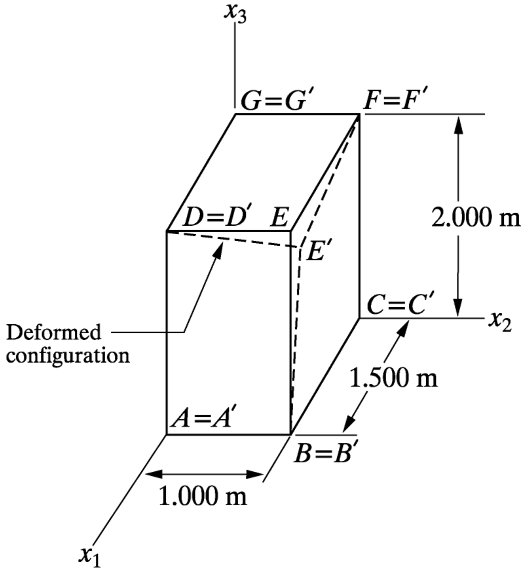


Figure 2P2

3. The components of strain at a point of a body referred to the x_1, x_2, x_3 axes are given as

$$e_{11} = -0.002 \qquad e_{22} = 0.0012 \qquad e_{12} = -0.002 \qquad e_{33} = e_{31} = e_{32} = 0$$

Using Mohr's circle:

(a) Compute the principal values of the normal components of strain and show in a sketch the direction in which they act. Ans. $\tilde{e}_1 = 0.00216$, $\tilde{e}_2 = -0.00296$, $\tilde{\phi}_{11} = 64.33$ clockwise

(b) Compute the maximum shearing component of strain and show in a sketch the axes to which it is referred. Ans. $(e'_{12})_{max} = 0.00256$, $\tilde{\tilde{\phi}}_{11} = 109.33°$ clockwise

(c) Compute the components of strain e_{11}', e_{12}' with respect to the x_1', x_2' axes shown in Fig. 2P3. Ans. $e'_{11} = -0.00293, \quad e'_{12} = 0.000385$

(d) Compute the normal component of strain in the direction $\mathbf{i}_n = 3/5\mathbf{i}_1 + 4/5\mathbf{i}_2$.

Ans. $e_{nn} = -0.00187$

(e) Compute the change of angle between the directions $\mathbf{i}_n$ and $\mathbf{i}_s = 4/5\mathbf{i}_1 - 3/5\mathbf{i}_2$ due to the deformation. Ans. $\tilde{a}_{ns} = 0.00419$ rad (increase)

4. The components of strain at a point of a body referred to the x_1, x_2, x_3 axes are

$$e_{11} = 0.0020 \qquad e_{22} = 0.0005 \qquad e_{12} = 0.0010 \qquad e_{33} = e_{31} = e_{32} = 0$$

Using Mohr's circle:

(a) Compute the principal values of normal components of strain and show in a sketch the direction in which they act. Ans. $\tilde{e}_1 = 0.0025, \ \tilde{e}_2 = 0, \ \tilde{\phi}_{11} = 26.565°$ counterclockwise

(b) Compute the maximum shearing component of strain and show in a sketch the axes to which it is referred. Ans. $(e'_{12})_{max} = 0.00125$ rad, $\quad \tilde{\tilde{\phi}}_{11} = 18.435°$ clockwise

(c) Compute the components of strain e'_{11}, e'_{12} with respect to the x_1', x_2' axes shown in Fig. 2P4. Ans. $e'_{11} = 0.00249, \ e'_{12} = -0.00015$ rad

(d) Compute the normal component of strain in the direction $\mathbf{i}_n = 3/5\mathbf{i}_1 + 4/5\mathbf{i}_2$.

Ans. $e_{nn} = 0.002$

(e) Compute the change of angle between the directions $\mathbf{i}_n$ and $\mathbf{i}_s = 4/5\mathbf{i}_1 - 3/5\mathbf{i}_2$ due to deformation. Ans. $\tilde{a}_{ns} = 0.002$ rad

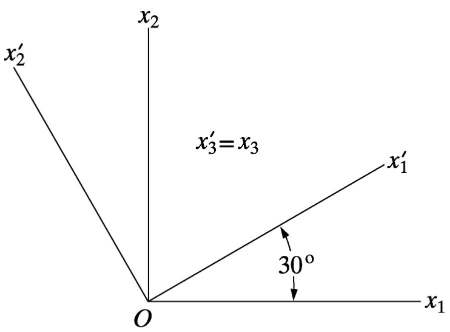


Figure 2P3

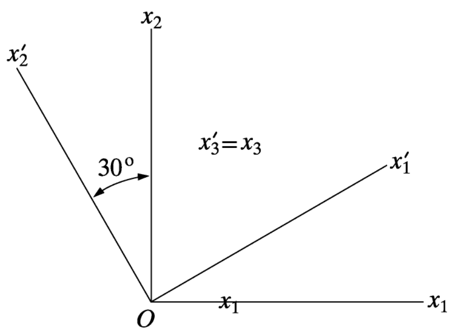


Figure 2P4

5. The components of strain at a point of a body referred to the x_1, x_2, x_3 axes are given as:

$$e_{11} = -0.002 \qquad e_{33} = -0.008 \qquad e_{13} = -0.004 \qquad e_{22} = e_{23} = e_{12} = 0$$

Using Mohr's circle:

(a) Compute the principal values of the normal components of strain and show in a sketch the direction in which they act.

Ans. $(e'_{33})_{min} = -0.01, \quad (e'_{33})_{max} = 0, \quad \tilde{\phi}_{33} = 26.565°$ counterclockwise

(b) Compute the maximum shearing component of strain and show in a sketch the axes to which it is referred. Ans. $(e'_{13})_{max.} = 0.005, \quad \tilde{\tilde{\phi}}_{33} = 71.565°$ counterclockwise

(c) Compute the components of strain e'_{11}, e'_{13} with respect to the x_1', x_3' axes shown in Fig. 2P5. Ans. $e'_{11} = -0.0098, \ e'_{13} = -0.00114$

(d) Compute the normal component of strain in the direction $\mathbf{i}_n = 3/5\mathbf{i}_1 + 4/5\mathbf{i}_3$.

Ans. $e_{nn} = -0.00968$

(e) Compute the change of angle between the directions $\mathbf{i}_n$ and $\mathbf{i}_s = 4/5\mathbf{i}_1 - 3/5\mathbf{i}_3$ due to deformation.

Ans. $\gamma_{ns} = 0.00352$ rad (decrease)

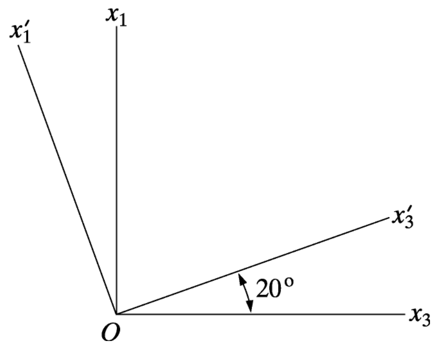


Figure 2P5

6. The following readings were taken by three gages attached to a body in a state of plane strain, in the directions shown in Fig. 2P6.

$$e_{11} = 0.002 \qquad e_{\beta_2} = -0.002 \qquad e_{\beta_3} = 0.006$$

Compute the maximum normal component of strain and show the direction in which it acts.

Ans. $e_1 = 7.2235(10^{-3})$, $\tilde{\phi}_{11} = 48.64$ clockwise

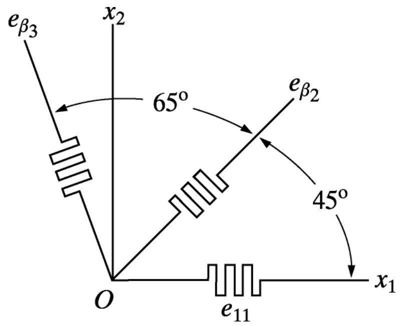


Figure 2P6

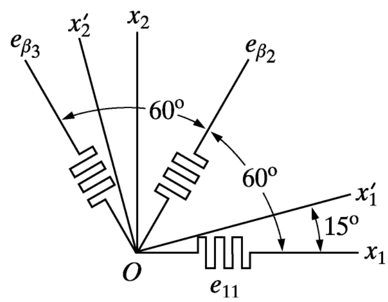


Figure 2P7

7. The following readings were taken by a 60° strain rosette attached to a structure in a state of plane strain

$$e_{\beta_1} = 0.006 = e_{11} \qquad e_{\beta_2} = 0.004 \qquad e_{\beta_3} = 0.002$$

(a) Find the principal values of the normal components of strain and the direction of the principal axes.

Ans. $\tilde{e}_1 = 0.00631$, $\tilde{e}_2 = 0.00169$, $\tilde{\phi}_{11} = 15°$ counterclockwise

(b) Compute the components of strain with respect to a set of axis $x_1'\, x_2'$ shown in Fig. 2P7.

Ans.
$$[e'] = \begin{bmatrix} 0.00631 & 0 \\ 0 & .00169 \end{bmatrix}$$

8. The following readings were taken from a 45° strain rosette attached to a structure in a state of plane strain:

$$e_{\beta_1} = 0.004 = e_{11} \qquad e_{\beta_2} = -0.004 \qquad e_{\beta_3} = 0.006$$

(a) Find the principal values of the normal components of strain and the direction of the principle axes. Ans. e_1=0.014055, e_2=−0.004055, $\tilde{\phi}_{11}$48.17° clockwise

(b) Compute the components of strain with respect to a set of axis $x_1' \, x_2'$ as shown in the Fig. 2P8.

$$\text{Ans. } [e'] = \begin{bmatrix} 0.01229 & 0.005366 & 0 \\ 0.005366 & -0.002294 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

9. The following readings were taken by three gages attached to a structure in a state of plane strain, in the directions shown in Fig. 2P9 ($e_{33} = e_{31} = e_{32} = 0$):

$$e_{11} = 0.006 \qquad e_{\beta_2} = 0.004 \qquad e_{\beta_3} = 0.002$$

(a) Find the principal values of the normal components of strain and show the directions of the principal axes. Ans. $\tilde{e}_1 = 0.00622$, $\tilde{e}_2 = 0.00130$, $\tilde{\phi}_{11} = 12.19°$ clockwise

(b) Compute the components of strain with respect to a set of axes $x_1'x_2'$ shown in Fig.2P9. Ans. $e_{11}' = 0.00576$, $e_{12}' = 0.00143$, $e_{22}' = 0.00176$

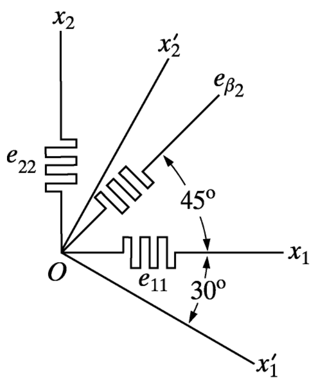


Figure 2P8

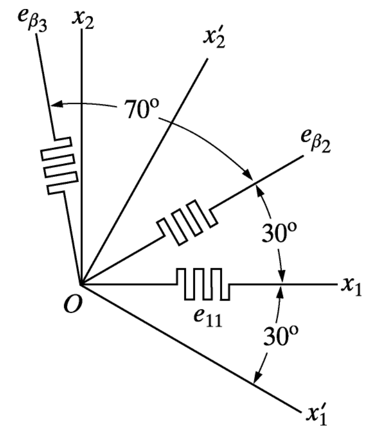


Figure 2P9

10. Consider a bar of constant circular cross section of diameter d = 40 mm subjected to an axial centroidal tensile force of P = 60 kN (see Fig. 2P10). For this loading the component of stress $\hat{o}_{11}$ acting on the particles of the bar is equal to P/A (A is the area of the cross section of the bar), while all other components of stress vanish.

(a) Compute the magnitude of the maximum shearing component of stress and show on a sketch the direction of the plane on which it acts.

(b) Compute the traction acting on the plane *mm* which is normal to the x_1, x_3 plane.

(c) Compute the components of stress acting on the plane normal to the unit vector.

$$\mathbf{i}_n = \frac{4}{5}\mathbf{i}_1 + \frac{3}{5}\mathbf{i}_2$$

Ans. (a) $(\tau'_{13})_{max} = 23.87$ MPa, $\tilde{\phi}_{33} = 45°$ counterclockwise
(b) $\overset{m}{\mathbf{T}} = 23.87\ \mathbf{i}_1$
(c) $\tau_{mn} = 30.56$ MPa, $\tau_{ns} = -22.92$ MPa

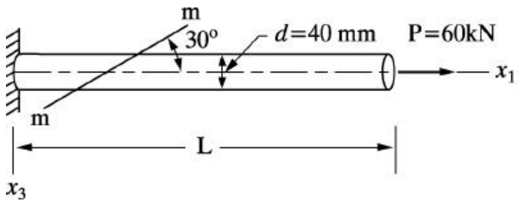


Figure 2P10

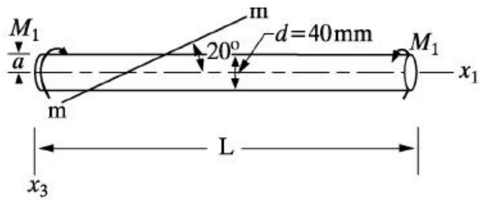


Figure 2P11

11. Consider a bar of circular cross section subjected to equal and opposite torsional moments at its ends M_1 = 2 kN·m (see Fig. 2P11). In this case, the only non-zero component of stress acting on a particle is $\hat{o}_{1\grave{e}}$ which is equal to

$$\tau_{1\theta} = \frac{M_1 r}{I_P}$$

where r is the radial coordinate of the particle and I_P is the polar moment of inertia of the cross section of the bar with respect to centroidal axes ($I_P = \pi a^4/2$).

(a) Compute the principal normal components of stress at a particle of the surface of the bar and show the direction of the planes on which they act.

Ans. $\tau_1 = -159.15$ MPa, $\tau_\theta = -159.15$ MPa, $\tilde{\phi}_{\theta\theta} = 45°$ counterclockwise

(b) Compute the components of stress acting on the plane mm. Find the traction acting on the plane mm (the normal to this plane is in the $x_1\, x_3$ plane).

Ans. $\tau'_{mt} = 121.92$ MPa, $\tau'_{mm} = 102.30$ MPa

12. to **14**. The state of plane stress at a particle of a body is shown in the Fig. 2P12.

(a) Using equations (1.116), find the normal and shearing components of stress acting on the plane mm shown in Fig. 2P12. Show the results on a sketch.

(b) Find the components of stress acting on the plane

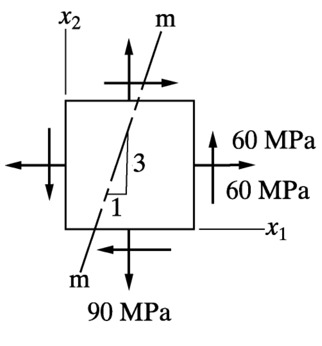


Figure 2P12

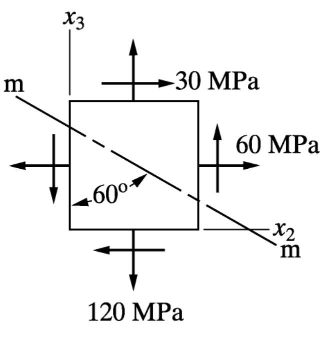


Figure 2P13

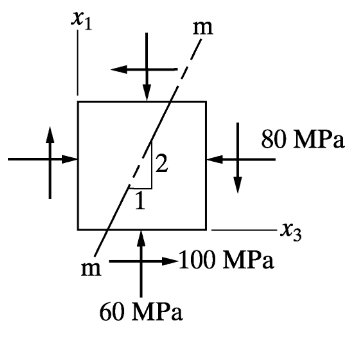


Figure 2P14

$$\mathbf{i}_n = \frac{1}{3}(\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)$$

Repeat with the state of stress shown in Figs. 2P13 and 2P14.

Ans. 12 (a) $\tau_{ms} = 39$ MPa Ans. 13 (a) $\tau_{mm} = 130.98$ MPa, $\tau_{ms} = 10.98$ MPa Ans. 14 (a) $\tau_{ms} = -68$ MPa
(b) $\tau_{ns} = 44.72$ MPa (b) $\tau_{ns} = 24.27$ MPa (b) $\tau_{ns} = 39.57$ MPa

15. to **17**. Solve problems 12 to 14 using Mohr's circle.

18. to **20**. For the state of plane stress at a particle of a body shown in Fig. 2P18, using Mohr's circle.

(a) Compute the principal normal components of stress and show on a sketch the planes on which they act.

Ans. 18 $\tau_1 = 128.1$ MPa, $\tau_2 = -28.1$ MPa, $\tilde{\phi}_{11} = 64.90°$ counterclockwise
Ans. 19 $\tau_2 = 120$ MPa, $\tau_3 = -120$ MPa, $\tilde{\phi}_{22} = 45°$ counterclockwise
Ans. 20 $\tau_3 = -4.56$ MPa, $\tau_1 = -175.44$ MPa, $\tilde{\phi}_{33} = 34.72°$ clockwise

(b) Compute the maximum shearing component of stress and the associated normal component of stress and show the planes on which they act on a sketch.

Ans. 18 $(\tau'_{12})_{max} = 78.1$ MPa, $\tilde{\tilde{\phi}}_{11} = 19.90°$ counterclockwise
Ans. 19 $(\tau'_{23})_{max} = 120$ MPa, $\tilde{\tilde{\phi}}_{22} = 0°$ counterclockwise
Ans. 20 $(\tau'_{13})_{max} = 85.44$ MPa, $\tilde{\tilde{\phi}}_{33} = 72.72°$ clockwise

(c) Compute the components of stress acting on plane mm.

Repeat with the states of stress shown in Figs. 2P19 and 2P20.

Ans. 18 $\tau_{mm} = 6.4$ MPa, $\tau_{mt} = 64.80$ MPa
Ans. 19 $\tau_{mm} = -96$ MPa, $\tau_{mt} = 72.0$ MPa
Ans. 20 $\tau_{mm} = -4.8$ MPa, $\tau_{mt} = 6.41$ MPa

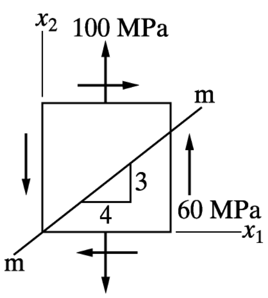


Figure 2P18

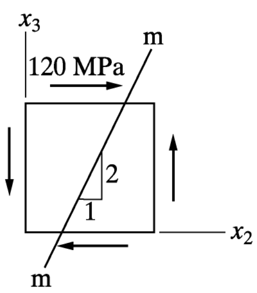


Figure 2P19

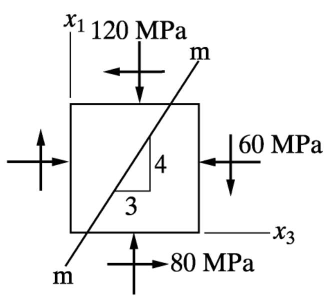


Figure 2P20

21. and 22. The tractions on planes *mm* and *nn* at a particle are shown in Fig. 2P21. They act in the $x_1\, x_2$ plane. Compute the principal normal components of stress and locate the planes on which they act. Show the results on a sketch. Repeat with the tractions shown in Fig. 2P22.

Ans. 21 (a) $\overset{n}{\mathbf{T}} = 72.79$ MPa, $\tau_{11} = 360.78$ MPa, $\tau_{12} = 81.52$ MPa, $\tau_{22} = -29.67$ MPa
(b) $\tau_1 = 377.14$ MPa, $\tau_2 = -46.01$ MPa, $\hat{\phi}_{11} = 11.33°$ counterclockwise

Ans. 22 (a) $\overset{n}{\mathbf{T}} = 207.85$ MPa, $\tau_{11} = -207.85$ MPa, $\tau_{12} = 120$ MPa, $\tau_{22} = 69.28$ MPa
(b) $\tau_1 = 114.01$ MPa, $\tau_2 = -252.59$ MPa, $\hat{\phi}_{11} = 69.55°$ counterclockwise

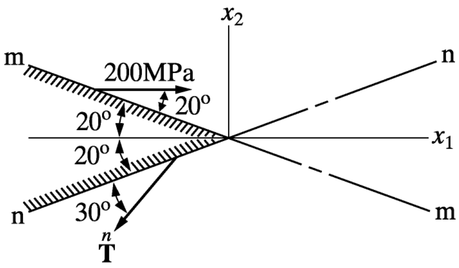


Figure 2P21

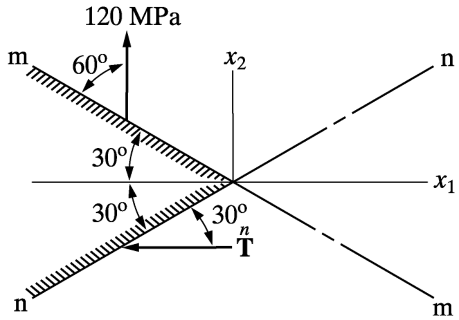


Figure 2P22

23. and 24. By considering the equilibrium of the element shown in Fig. 2P23, determine the unknown stress components acting on it. The element is in a state of plane stress

$$\tau_{13} = \tau_{23} = \tau_{33} = 0$$

Repeat with the element of Fig. 2P24.

Ans. 23 $\tau_1 = 41.44$ MPa, $\tau_2 = 226.19$ MPa
Ans. 24 $\tau_{12} = 126.96$ MPa, $\tau_{22} = 33.40$ MPa

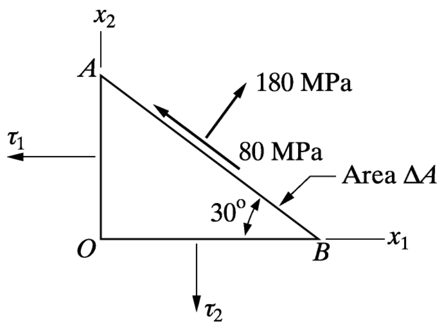


Figure 2P23

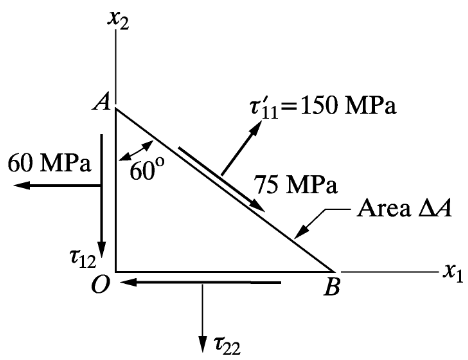


Figure 2P24

25. The state of stress at a point is

$$\begin{bmatrix} 0 & 40 & 30 \\ 40 & 0 & 0 \\ 30 & 0 & 0 \end{bmatrix} \text{MPa}$$

(a) Compute the stationary values of the normal components of stress and the directions of the planes on which they act.

Ans. $\tau_1 = 0 \qquad \tau_2 = 50$ MPa $\qquad \tau_3 = -50$ MPa

$$\lambda_{11}^{(1)} = 0 \qquad \lambda_{21}^{(2)} = \pm\frac{\sqrt{2}}{2} \qquad \lambda_{31}^{(3)} = \pm\frac{\sqrt{2}}{2}$$

$$\lambda_{12}^{(1)} = \mp\frac{3}{5} \qquad \lambda_{22}^{(2)} = \pm\frac{2\sqrt{2}}{5} \qquad \lambda_{32}^{(3)} = \mp\frac{2\sqrt{2}}{5}$$

$$\lambda_{13}^{(1)} = \pm\frac{4}{5} \qquad \lambda_{23}^{(2)} = \pm\frac{3\sqrt{2}}{10} \qquad \lambda_{33}^{(3)} = \mp\frac{3\sqrt{2}}{10}$$

(b) Compute the components of stress in the direction of the axes shown in Fig. 2P25.

$$\text{Ans.} \begin{bmatrix} 36.34 & 20 & 25.98 \\ 20 & -34.64 & -15 \\ 25.98 & -15 & 0 \end{bmatrix} \text{MPa}$$

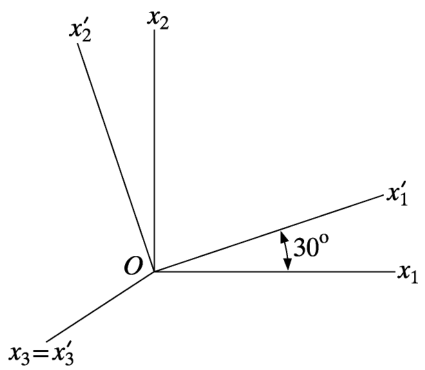


Figure 2P25

26. The components of stress with respect to a system of axes x_1, x_2, x_3 are given as

$$\begin{aligned} \tau_{11} &= 10 \text{ MPa} & \tau_{12} &= 20 \text{ MPa} \\ \tau_{22} &= -5 \text{ MPa} & \tau_{23} &= 10 \text{ MPa} \\ \tau_{33} &= 0 \text{ MPa} & \tau_{31} &= 0 \text{ MPa} \end{aligned}$$

(a) Compute the components of stress acting on the plane specified by the unit vector

$$\mathbf{i}_n = \frac{1}{3}(2\mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)$$

Ans. $\tau_{nn} = 25/3$ MPa, $\tau_{nt} = 12.02$ MPa

(b) Compute the stationary values of the normal components of stress and the direction of the principal axes.

Ans. $\tau_1 = 25.29$ MPa $\quad \tau_2 = 1.799$ MPa $\quad \tau_3 = -22.0285$ MPa

$$\lambda_{11}^{(1)} = \pm 0.7737 \qquad \lambda_{21}^{(2)} = \mp 0.3965 \qquad \lambda_{31}^{(3)} = \pm 0.4942$$

$$\lambda_{12}^{(1)} = \pm 0.5891 \qquad \lambda_{22}^{(2)} = \pm 0.1625 \qquad \lambda_{32}^{(3)} = \mp 0.7915$$

$$\lambda_{13}^{(1)} = \pm 0.2335 \qquad \lambda_{23}^{(2)} = \pm 0.9035 \qquad \lambda_{33}^{(3)} = \pm 0.3593$$

(c) Compute the magnitude of maximum shear stress and indicate the plane on which it acts.

Ans. $(\hat{\sigma}_{12})_{\max} = 23.63$ MPa

(d) Compute the components of stress referred to the system of axes x_1', x_2', x_3' shown in Fig. 2P25.

$$\text{Ans.}\quad [\tau'] = \begin{bmatrix} 23.57 & 3.50 & 5.00 \\ 3.50 & -18.57 & 8.66 \\ 5.00 & 8.66 & 0.00 \end{bmatrix}$$

27. The components of stress acting on a particle of a body with respect to a system of axes x_1, x_2, x_3 are given as

$$[\tau] = \begin{bmatrix} -30 & 20 & 10 \\ 20 & 0 & -40 \\ 10 & -40 & 60 \end{bmatrix}$$

(a) Compute the components of stress acting on the plane specified by the unit vector

Ans. $\tau_{nn} = -30$ MPa, $\tau_{nt} = 42.94$ MPa

$$\mathbf{i}_n = \frac{3}{5}\mathbf{i}_1 - \frac{4}{5}\mathbf{i}_2$$

(b) Compute the stationary values of the normal components of stress and the direction of the principal axes.

Ans. $\tau_1 = -2.087$ MPa $\quad \tau_2 = -47.91$ MPa $\quad \tau_3 = 80$ MPa

$\lambda_{11}^{(1)} = \pm 0.6252 \quad \lambda_{21}^{(2)} = \pm 0.7804 \quad \lambda_{31}^{(3)} = 0$

$\lambda_{12}^{(1)} = \pm 0.6980 \quad \lambda_{22}^{(2)} = \mp 0.5592 \quad \lambda_{32}^{(3)} = \pm 0.4472$

$\lambda_{13}^{(1)} = \pm 0.3490 \quad \lambda_{32}^{(2)} = \mp 0.2796 \quad \lambda_{33}^{(3)} = \mp 0.8944$

(c) Compute the magnitude of maximum shear stress and indicate the plane on which it acts.

Ans. $(\tau_{12})_{\max} = 22.91$ MPa

$(\tau_{13})_{\max} = 41.04$ MPa

$(\tau_{23})_{\max} = 63.96$ MPa

(d) Compute the components of stress referred to the system of axes x_1', x_2', x_3' whose transformation matrix is

$$\text{Ans.}\quad [\tau] = \begin{bmatrix} -5.18 & 22.99 & -11.34 \\ 22.99 & -24.82 & -39.64 \\ -11.34 & -39.64 & 60.00 \end{bmatrix}$$

$$[\Lambda_s] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

28. The rectangular coordinates of point P are $x_1 = 80$ mm, $x_2 = 120$ mm, $x_3 = 40$ mm.

(a) Compute the cylindrical coordinates x_1, r, è of point P.

Ans. $x_1 = 80$ mm $\quad r = 126.49 \quad$ è = 18.43o

(b) If the components of strain at point P are

$$e_{12} = e_{13} = e_{23} = e_{11} = 0, \qquad e_{22} = 0.002, \qquad e_{33} = -0.004$$

determine e_{rr}, $e_{\grave{e}\grave{e}}$ and $e_{r\grave{e}}$ at this point. Ans. $e_{rr} = 0.0014$ $e_{\grave{e}\grave{e}} = -0.0034$ $e_{r\grave{e}} = -0.0018$

29. The state of strain of a particle of a body is specified with respect to the orthogon axes x_1, x_2, x_3, by the following matrix:

$$[\varepsilon] = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix} \text{x}10^{-5}$$

Referring to Fig. 2P29 compute the change of the angles $\angle \text{AOx}_1$ due to the deformation.

Ans. $\tilde{a}_{n1} = 1.6(10^{-5})$ rad

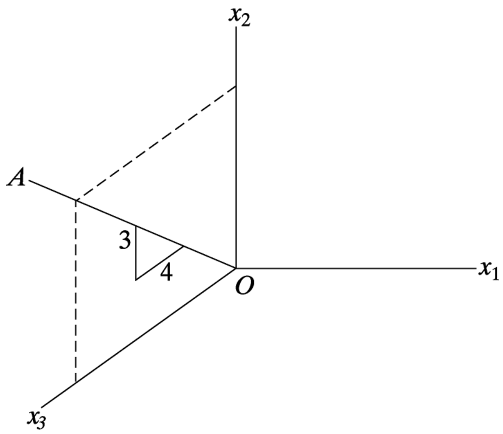


Figure 2P29

Chapter 3

Stress–Strain Relations

3.1 Introduction

In Section 2.4 we consider a body subjected to external loads and we define certain quantities (the components of strain) which specify completely the deformation of its particles. In Section 2.6 we consider a body subjected to external loads and we define certain quantities (the components of stress) which specify completely the internal forces acting on its particles. In this chapter we present relations among the components of strain and stress of a particle of a body subjected to external loads. These relations represent properties of the material from which the body is made and they are known as its *constitutive equations*. They are based on experimental observations of the macroscopic phenomenological behavior of bodies of simple geometry subjected to such loads that the distribution of the components of stress acting on their particles can be easily established. The constitutive equations of materials are classified into two groups — *time independent and time dependent.* Those belonging to the second group include first and possibly higher time derivatives of stress and/or strain, and are sensitive to the rate and history of loading. For instance, the stress–strain relations of metals at high temperatures and of certain non-metallic materials, such as plastics, ceramics and rubbers at room temperature are affected appreciably by the rate of loading.

Actually, the constitutive relations of all materials are time dependent. Time-independent constitutive relations represent an idealization which is approached rather closely by many materials when subjected to loads at ordinary temperatures. For any material there exists a range of temperature in which its constitutive relations may be considered time independent. For example, the constitutive relations of most metals are considered time independent at temperatures up to a little above room temperature and time dependent at higher temperatures. The time-dependent response of the materials is of paramount importance in the design of boilers, gas turbines or reactor shields as well as of other structures which are subjected to stress fields in an environment of high temperature.

Many features of the time-independent behavior of a body under a general loading are revealed by simple tests involving specimens of simple geometry (for example, prismatic bars of circular cross section) subjected to external forces which induce only one component of stress. The most commonly used such tests are the uniaxial tension or compression and the torsion. In these tests a specimen of constant circular cross section is subjected to external actions (axial centroidal forces or torsional moments) at its ends. These actions induce a uniform stress field in the specimen which can be specified by only one component of stress. Such a stress field is known as *uniaxial*.

3.2 The Uniaxial Tension or Compression Test Performed in an Environment of Constant Temperature

In the uniaxial tension test, a prismatic specimen (see Fig. 3.1) of circular cross section of area A_0 is placed in a tension–compression testing machine and it is subjected at its ends to equal and opposite axial centroidal tensile or compressive forces whose magnitude P is read on a dial. In Chapter 8 we show that when the resultant of the forces applied at the ends of the specimen passes through the centroid of its cross sections, in a portion of its volume which is not located very close to its ends, the normal component of stress τ_{11} and the normal component of strain e_{11} are constant† while the other components of stress vanish. That is, the particles of this portion of the specimen are subjected to the following uniaxial state of stress:

$$\tau_{11} = \frac{P}{A_o} \qquad \tau_{22} = \tau_{33} = \tau_{13} = \tau_{23} = \tau_{12} = 0 \tag{3.1a}$$

where A_0 is the area of the cross section of the specimen before deformation. The axial component of strain corresponding to the state of stress (3.1a) is equal to

$$e_{11} = \frac{u_1^L}{L_0} \tag{3.1b}$$

where u_1^L is the total elongation or shrinkage of the before deformation length L_0 (the distance AB of the specimen of Fig. 3.1). As shown in Fig. 3.1, u_1^L can be measured by means of a dial gage attached to the specimen. Moreover, e_{11} can also be measured by a strain gage (see Section 2.12). The normal component of stress τ_{11} and of strain e_{11} defined by relations (3.1) are called *conventional.* The curve obtained by plotting the conventional normal component of stress τ_{11} versus the conventional normal component of strain e_{11} is called the *conventional stress–strain diagram* in uniaxial tension or compression (see Fig. 3.2). This diagram represents a property of the material from which this specimen is made. Different stress–strain curves are obtained for different materials.

The true stress $\tau_{11}{}^T$ *and strain* $e_{11}{}^T$ are defined by

$$\tau_{11}^T = \frac{P}{A} \tag{3.2a}$$

† As a result of the forces exerted by the jaws of the tension–compression machine on the ends of a specimen, its particles which are located very close to its ends are not in a state of uniaxial stress. Moreover, the magnitude of the components of stress acting on these particles could vary from particle to particle.

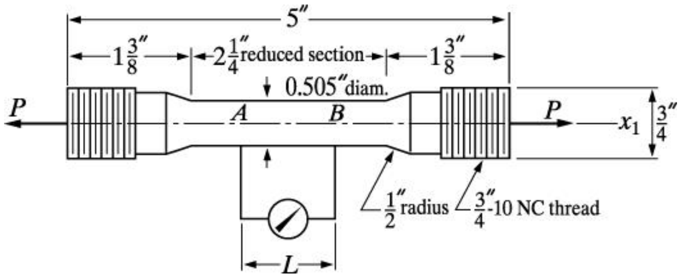


Figure 3.1 ASTM† standard metal specimen for uniaxial tension test.

$$e_{11}^{T} = \int_{L_o}^{L_F} \frac{dL}{L} = \ln \frac{L_F}{L_o} \tag{3.2b}$$

where, at any instance during the process of loading, A is the area of the cross section of the specimen; dL is the increment of the instantaneous length L of the specimen; L_F is the final deformed length of the specimen. For small values of the forces applied to the specimen, the difference between the conventional normal component of stress or strain and the true normal component of stress or strain, respectively, is negligible.

Generally, the stress–strain curve of the same material may be affected appreciably by the temperature at which the experiment is performed, and by the rate of loading (see Section 3.5.1). However, in this section *we assume that the load is applied slowly and that the experiment is performed at room temperature*.

The following observations have been made from uniaxial tension or compression tests ($\tau_{11} \neq 0, \tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$) performed in an environment of constant temperature:

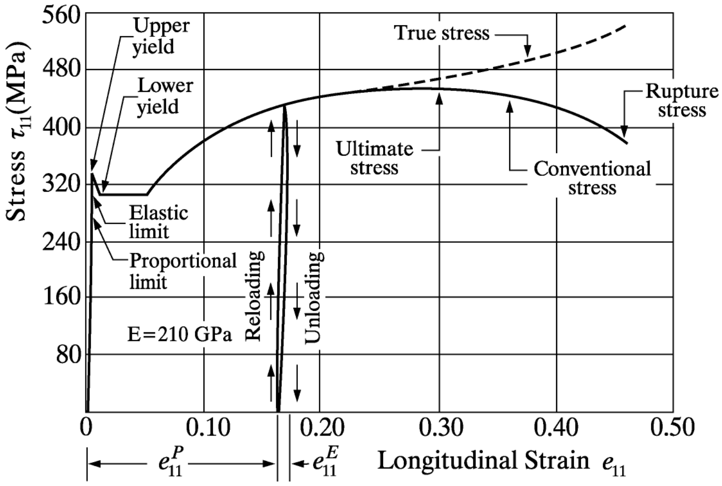


Figure 3.2 Stress–strain diagram of mild steel.

† In standard tension or compression tests, a standard specimen (see Fig. 3.1) is used, as described by specification A370-77 in the annual book of standards Vol. 01.01 of the American Society for Testing Materials (ASTM) Philadelphia, PA 19103.

1. For any material there is a range of temperature and a range of loading rate in which its response may be considered as time independent. For many materials this range is of practical interest.

2. The stress–strain curve upon unloading, of many materials of engineering interest, when subjected to small values of stress, is identical to that obtained during loading. That is, the relation between the component of stress and the corresponding component of strain is unique and energy is not lost during the process of loading and unloading. This response of a material to external forces is called *elastic.* The maximum value of stress for which a material exhibits elastic response is called *elastic limit* or *yield stress*. The response of a material above the elastic limit is called *plastic*. This response of a material is independent of the rate of loading but it depends on the path of loading.

3. For certain materials, such as metals at room temperature, subjected to values of the components of stress below a certain limiting value known as the *proportional limit*, the relation between the normal component of stress and strain is linear (see Fig. 3.2). That is,

$$\tau_{11} = Ee_{11} \tag{3.3}$$

The constant of proportionality E is known as *Young's modulus* or *modulus of elasticity*. For many materials, the proportional limit and the yield limit are very close and may be considered identical.

4. Certain elastic materials, such as rubber, have a non-linear stress–strain relation, even at very low values of stress. Such a relation involves more than one elastic constant.

5. When a specimen is subjected to uniaxial tension (or compression), it elongates (or shrinks) in the axial direction and shrinks (or elongates) in the lateral directions. For elastic materials, the ratio of the lateral to the axial component of strain of any particle of the specimen remains constant during the process of loading and unloading. That is,

$$e_{22} = -\nu_{12}e_{11} \qquad e_{33} = -\nu_{13}e_{11} \tag{3.4}$$

This phenomenon of lateral contraction is referred to as *Poisson's effect*. The effect of lateral contraction or expansion upon the cross sectional area of the specimen is small for small values of the normal component of strain e_{11}.

6. The microstructure of many engineering materials, such as metals, consists of small crystals, randomly oriented. These materials are referred to as *polycrystalline materials*. Their macroscopic behavior may be assumed independent of direction. That is, it is assumed that they have the same properties in all directions. This independence of material properties of direction is referred to as *isotropy* and materials possessing this property are called *isotropic*. Actually, isotropy is an idealized behavior which is approached rather closely by many materials. For isotropic, linearly elastic materials $\nu_{12} = \nu_{13} = \nu$ and consequently,

$$\nu e_{11} = -e_{22} = -e_{33} \tag{3.5}$$

The positive constant ν is referred to as *Poisson's ratio*. Its value is less than 0.5. It can be computed from relations (3.5) using data obtained from the uniaxial tension experiment. The component of strain e_{22} can be established by measuring the change of the diameter of the cylindrical specimen. That is,

$$e_{22} = \frac{\text{change of the diameter}}{\text{undeformed diameter}}$$

7. When the value of the component of stress acting on a particle of a specimen does not exceed the elastic limit of the material from which the specimen is made for any two different values of the component of stress [$\tau_{11}^{(1)}$ and $\tau_{11}^{(2)}$], there correspond two different values of the corresponding component of strain [$e_{11}^{(1)}$ and $e_{11}^{(2)}$] of the particle and vice versa. Moreover, an increase or a decrease of the component of stress τ_{11} acting on a particle results in an increase or a decrease, respectively, of the corresponding component of strain e_{11} of this particle and vice versa. That is,

$$(\tau_{11}^{(2)} - \tau_{11}^{(1)})(e_{11}^{(2)} - e_{11}^{(1)}) \geq 0 \tag{3.6a}$$

The equality holds only when the two values of the components of stress and strain are identical [$\tau_{11}^{(1)} = \tau_{11}^{(2)}$ and $e_{11}^{(1)} = e_{22}^{(2)}$]. Inasmuch as relation (3.6a) is valid for any two values of the components of stress and strain, it will also hold when the differences of the components of stress and of the components of strain are infinitesimal. That is,

$$d\tau_{11} de_{11} \geq 0 \tag{3.6b}$$

Condition (3.6a) or (3.6b) is referred to as the condition of *material stability*. Materials processing the property indicated by relations (3.6) are called *stable*. On the basis of the above observations, the stress–strain diagram of a specimen made from a stable elastic material subjected to a state of uniaxial tension in an environment of constant temperature cannot be like the diagrams shown in Fig. 3.3.

8. The yield stress of mild steel, that is, steel having a low carbon content, is well defined (see Fig. 3.2). For most ductile materials, however, the transition from elastic behavior to non-elastic (plastic) behavior is gradual and, consequently, difficult to establish. For these materials, *the so–called offset yield stress* is defined as the stress which produces a small but measurable permanent strain. Generally, this permanent strain is taken as 0.2%. The yield stress is taken as the stress corresponding to the intersection with the stress–strain curve of a line parallel to its straight-line portion, commencing from the value of the permanent strain (see Fig. 3.4).

9. Loading beyond the elastic limit brings the material into *the plastic range*. In this range

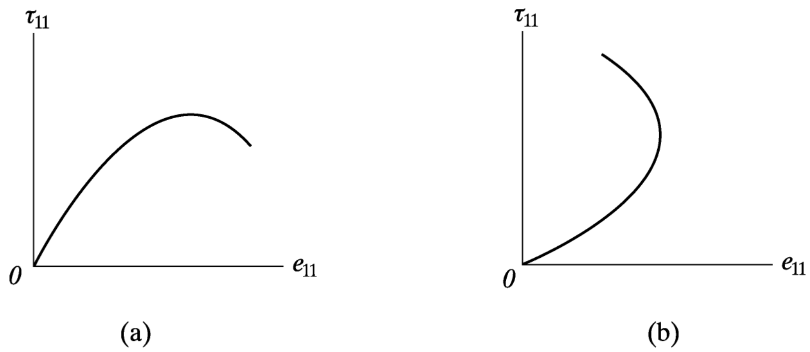


Figure 3.3 Stress–strain diagram of unstable materials.

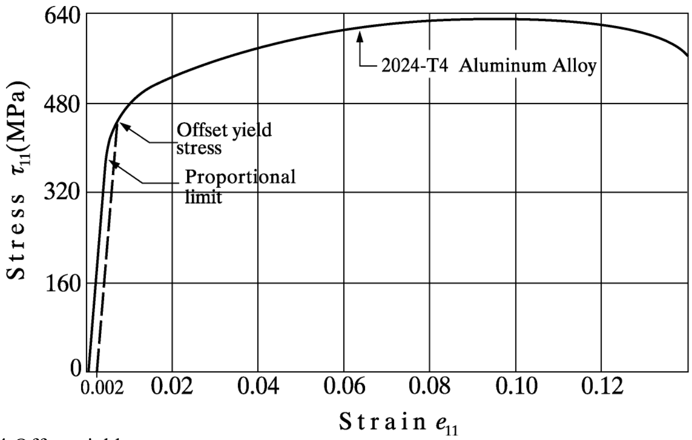


Figure 3.4 Offset yield stress.

only a portion of the deformation is elastic, whereas the remaining portion is irreversible plastic deformation. That is, at every particle of a specimen, the component of strain e_{11} is equal to the sum of an elastic part e_{11}^{E}, which is recoverable upon unloading, and a plastic part e_{11}^{P} , which is not recoverable upon unloading. Hence

$$e_{11} = e_{11}^{E} + e_{11}^{P} \tag{3.7}$$

For mild steel, when the component of stress reaches a certain critical value, referred to as the *upper yield stress*, it drops sharply to a lower value, referred to as the *lower yield stress*. Subsequently, yielding takes place at constant stress (see Fig. 3.2). However, resistance to such deformation is created within the material and increasing stress is required to produce additional strain. This final effort of the material to resist deformation is referred to as *strain hardening* (see Fig. 3.2). The flat portion of the stress–strain diagram may extend over a range of strain 10 to 40 times the value of strain at the yield stress.

10. If a specimen is stressed beyond the elastic limit and subsequently unloaded, the stress–strain curve during unloading is virtually a straight line with a slope equal to the original modulus of elasticity of the material from which the specimen is made. When all the load is removed, the specimen does not assume its before-deformation configuration. That is, it has been deformed permanently. A residual strain remains in the particles of the specimen. Upon reloading, the specimen yields at a stress approximately equal to the stress just before unloading. The stress–strain curve for reloading may differ slightly from that at unloading. A small hysteresis loop may be formed, whose area represents the energy lost during the cycle of unloading and reloading (see Fig. 3.2). The size of the hysteresis loop varies with the speed of loading — it is larger at higher rates of loading. For a metal at ordinary temperatures, the energy lost per cycle of loading and unloading is small and may be disregarded, except in cases involving many cycles. The hysteresis loop is an indication that the actual mechanical behavior of materials is not entirely time independent. The size of the hysteresis loop shown in Fig. 3.2 is greaterly exaggerated.

11. A material, such as mild steel, which fails subsequent to large, predominantly plastic

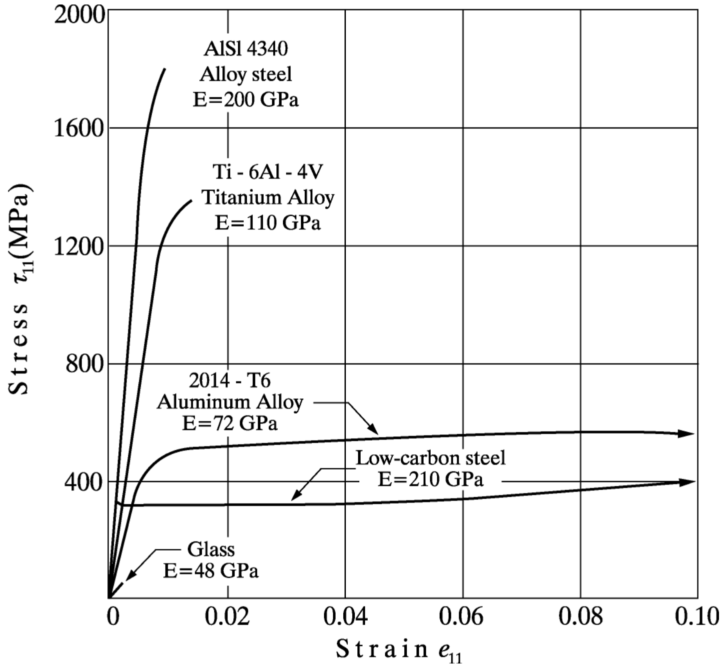
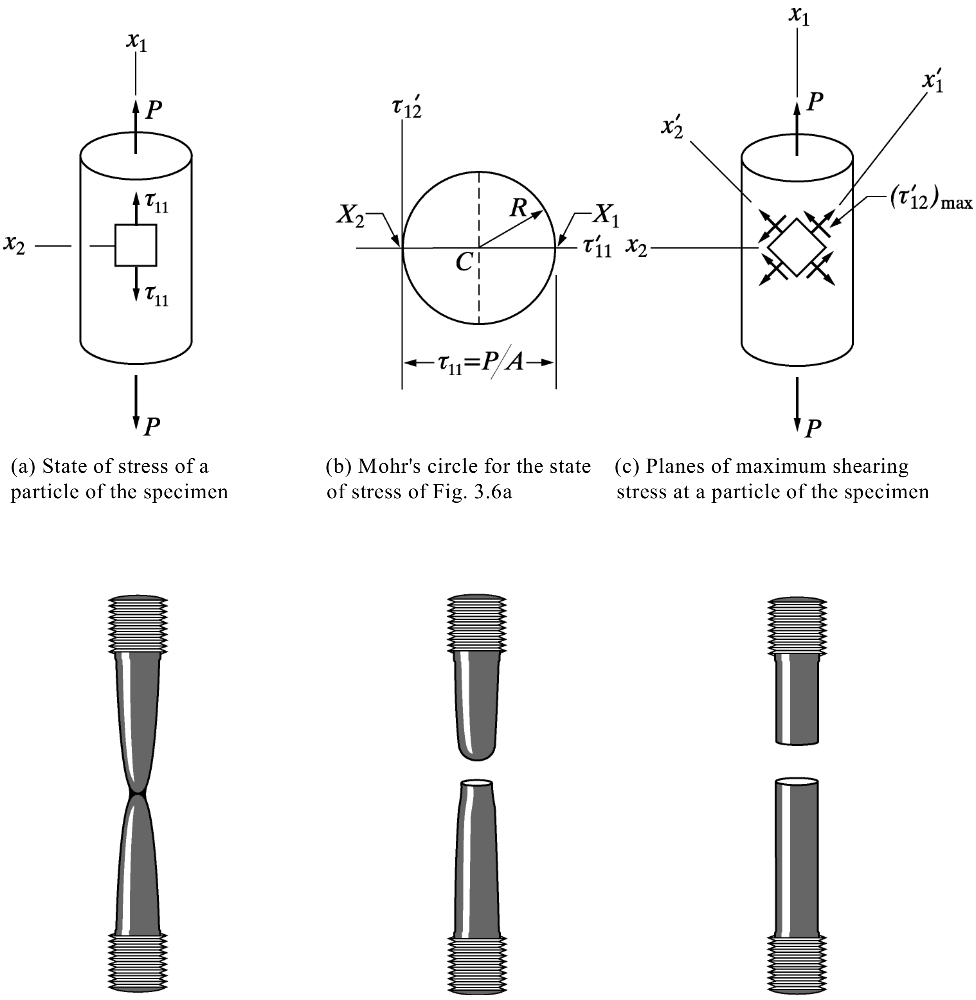


Figure 3.5 Stress–strain curve of various engineering materials.

deformation (see Fig. 3.5), is referred to as a *ductile material*, whereas a material which fails after small, predominantly elastic deformation (see Fig. 3.5), is referred to as a *brittle material*. Depending upon the environmental conditions, the rate of loading and the geometry of the specimen, the same material could behave either as a brittle or as a ductile material. Some materials, such as ceramics and glass, do not exhibit plastic deformation at ordinary temperatures. They fracture at small strains, while deforming elastically.

12. From the conventional stress–strain curve of a ductile material, it can be seen that the conventional stress and, consequently, the axial force P (since $P = \tau_{11} A_0$) applied to a specimen attain their maximum value at an appreciably lower strain than that at fracture. The maximum value of the conventional stress is called the *ultimate stress*. Actually, the true stress increases up to fracture (see Fig. 3.2). That is, strain hardening continues throughout the whole plastic range. At the maximum value of the applied force, the deformation becomes unstable and a localized rapid decrease of the cross sectional area (neck) occurs at some location of the specimen (see Fig. 3.6d). This results in a non-uniform stress and strain distribution in the neighborhood of the neck which differs from the true stress P/A and the true strain (3.2b). This phenomenon occurs in ductile materials and is referred to as *necking*. After necking has begun, somewhat lower forces are sufficient to keep the specimen elongating further until it fractures. Fracture occurs along a cone-shaped surface which is inclined to the axis of the specimen by approximately 45° (see Fig. 3.6e). Referring to Fig. 3.6c, we see that this mode of fracture indicates that the shearing stress is primarily responsible for the fracture of ductile materials.

13. When a specimen made from a brittle material is subjected to increasing equal and opposite axial tensile forces at its ends, fracture occurs at the plane normal to its axis (see



(a) State of stress of a particle of the specimen

(b) Mohr's circle for the state of stress of Fig. 3.6a

(c) Planes of maximum shearing stress at a particle of the specimen

(d) Necking of a ductile specimen

(e) Fracture of a ductile specimen

(f) Fracture of a brittle specimen

Figure 3.6 State of stress, necking and fracture of specimens subjected to equal and opposite axial centroidal tensile forces at their ends.

Fig. 3.6f). As we see in Fig. 3.6b, this is the plane of maximum tensile stress. Brittle materials exhibit little or no plastic deformation when they fracture.

14. When a metal specimen is subjected to a tensile or compressive force which stresses its particles above the elastic limit and, subsequently, it is unloaded and subjected to opposite forces, it yields at a considerably smaller stress than its initial yielding stress. This phenomenon is referred to as the *Bauschinger effect* and is illustrated in Fig. 3.7.

15. Specimens having large length to diameter ratios, when subjected to axial centroidal compressive forces on their end surfaces, become unstable and buckle when the applied forces reach a certain critical value. Moreover, short specimens subjected to high values of axial centroidal compressive forces on their end surfaces are not in a state of uniaxial

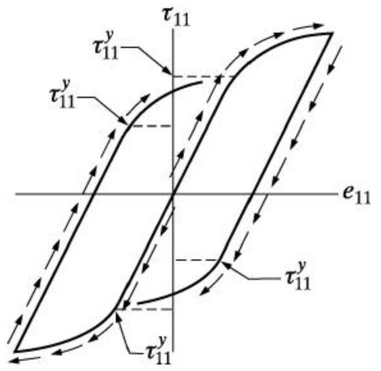


Figure 3.7 Bauschinger effect.

compression because the platens of the compression machine restrain the radial expansion of the end cross sections of the specimens.

16. Consider an element of volume $dx_1 dx_2 dx_3$ of a cylindrical specimen of constant cross sections subjected to axial tension in the x_1 direction. Referring to relations (2.6) and (2.17) the volume of this element subsequent to deformation may be approximated as $(1 + e_{11} + e_{22} + e_{33})\, dx_1\, dx_2\, dx_3$. Hence, taking into account the symmetry of the specimen ($e_{22} = e_{33}$) the change of the volume of the element due to its deformation is

$$\Delta V = (e_{11} + e_{22} + e_{33})\, dx_1\, dx_2\, dx_3 = (e_{11} + 2e_{22})\, dx_1\, dx_2\, dx_3 \tag{3.8}$$

In the plastic range, the total strain e_{11} of a particle is the sum of its elastic e_{11}^E and plastic e_{11}^P components. Therefore, substituting relation (3.7) into (3.8), we obtain

$$\Delta V = \Delta V^E + \Delta V^P = (e_{11}{}^E + 2e_{22}{}^E)\, dx_1\, dx_2\, dx_3 + (e_{11}{}^P + 2e_{22}{}^P)\, dx_1\, dx_2\, dx_3 \tag{3.9}$$

Plastic deformation can be assumed equivoluminal. Consequently, ΔV^P can be assumed to vanish. Therefore,

$$-\frac{e_{22}{}^P}{e_{11}{}^P} = \frac{1}{2} \tag{3.10}$$

3.3 Strain Energy Density and Complementary Energy Density for Elastic Materials Subjected to Uniaxial Tension or Compression in an Environment of Constant Temperature

Consider a body made from an elastic material, subjected to external forces producing a state of uniaxial tension or compression ($\tau_{11} \neq 0 \quad \tau_{12} = \tau_{13} = \tau_{23} = \tau_{22} = \tau_{33} = 0$), in an environment of constant temperature, from a stress-free, strain-free state. The *strain energy density* U_s and the *complementary energy density* U_c of a particle of this body are defined as

$$U_s(e_{11}) = \int_0^{e'_{11} = e_{11}} \tau_{11}(e'_{11})\, de'_{11} \tag{3.11}$$

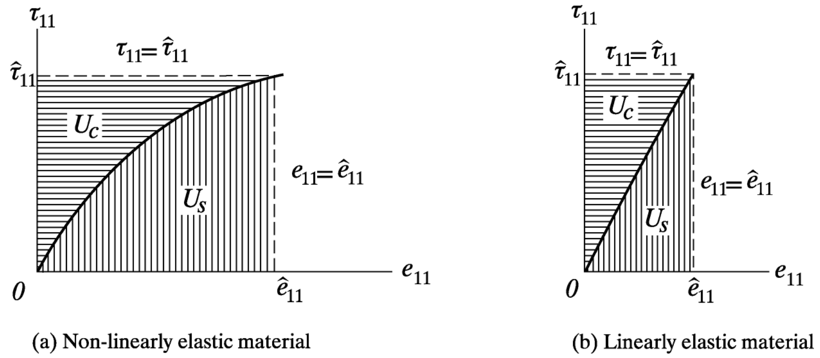


Figure 3.8 Strain energy density and complementary energy density.

$$U_c(\tau_{11}) = \int_0^{\tau'_{11} = \tau_{11}} e_{11}(\tau'_{11})\, d\tau'_{11} \tag{3.12}$$

Referring to Fig. 3.8a, from its definition (3.11), the strain energy density corresponding to any value $\hat{e}_{11}$ of the component of strain e_{11}, is the area between the stress–strain curve, the e_{11} axis and the line $e_{11} = \hat{e}_{11}$. Moreover, referring to Fig. 3.8a, from its definition (3.12), the complementary energy density for any value $\hat{\tau}_{11}$ of the component of stress τ_{11} is the area between the stress–strain curve, the τ_{11} axis and the line $\tau_{11} = \hat{\tau}_{11}$. For linearly elastic materials, referring to Fig. 3.8b, it is apparent that the strain energy density equals the complementary energy density. That is,

$$U_s(e_{11}) = U_c(\tau_{11}) \qquad \text{if} \quad \tau_{11} = Ee_{11} \tag{3.13}$$

The total strain energy $(U_s)_T$ and the total complementary energy $(U_c)_T$ of a body of volume V are equal to

$$(U_s)_T = \iiint_V U_s\, dV = \iiint_V \int_0^{e'_{11} = e_{11}} \tau_{11}(e'_{11})\, de'_{11}\, dV \tag{3.14}$$

$$(U_c)_T = \iiint_V U_c\, dV = \iiint_V \int_0^{\tau'_{11} = \tau_{11}} e_{11}(\tau'_{11})\, d\tau'_{11}\, dV \tag{3.15}$$

As stated in Section 3.2, for bodies made from an elastic material, the component of stress τ_{11} is uniquely determined from the component of strain e_{11} and vice versa. This implies that the integrands $\tau_{11}(e'_{11})de'_{11}$ and $e_{11}(\tau'_{11})d\tau'_{11}$ of relations (3.14) and (3.15) must be perfect differentials. Hence,

$$dU_s = \tau_{11}\, de_{11} \tag{3.16}$$

$$dU_c = e_{11}\, d\tau_{11} \tag{3.17}$$

Consequently,

$$\tau_{11} = \frac{dU_s}{de_{11}} \tag{3.18}$$

$$e_{11} = \frac{dU_c}{d\tau_{11}} \tag{3.19}$$

In order to establish a physical meaning of the strain energy density let us consider an infinitesimal parallelepiped of a body subjected to external forces which induce in it a state of uniaxial tension ($\tau_{11} \neq 0$, all other components of stress vanish). Referring to Fig. 3.9, this parallelepiped is subjected to two equal and opposite forces of magnitude $\tau_{11}\, dx_2\, dx_3$ acting on its faces AB and CD. We denote by $\hat{u}_1$ the component of displacement in the x_1 direction of face AB. The component of displacement in the x_1 direction of face CD is equal to

$$\hat{u}_1^{CD} = \hat{u}_1 + \left(\frac{\partial \hat{u}_1}{\partial x_1}\right) dx_1 = \hat{u}_1 + e_{11}\, dx_1 \tag{3.20}$$

When the external forces change by an infinitesimal amount, the components of displacement in the x_1 direction of faces AB and CD change by du_1 and $d(u_1 + e_{11}\, dx_1)$, respectively. The increment of work performed by the forces acting on the parallelepiped due to this additional displacement is equal to

$$d(dW) = -(\tau_{11} dx_2 dx_3) du_1 + \tau_{11} dx_2 dx_3 d(u_1 + e_{11} dx_1) = \tau_{11} de_{11} dx_1 dx_2 dx_3 = \tau_{11} de_{11} dV \tag{3.21}$$

The total work performed by the forces acting on the infinitesimal parallelipiped of Fig. 3.9, as the body deforms from its stress-free, strain-free state to a state specified by the component of strain e_{11}, is equal to

$$dW = \left(\int_0^{e'_{11}=e_{11}} \tau_{11}(e'_{11})\, de'_{11}\right) dV \tag{3.22}$$

The total work of the component of stress τ_{11} acting on all the infinitesimal parallelepipeds which make up the volume of the body is equal to

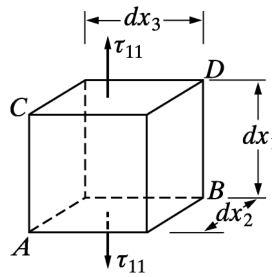
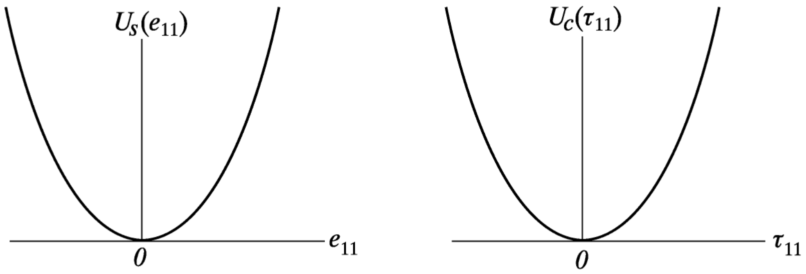


Figure 3.9 Infinitesimal element of a specimen subjected to uniaxial tension.



(a) Strain energy density (b) Complementary energy density

Figure 3.10 Strain energy density and complementary energy density.

$$W = \iiint_V \left(\int_0^{e'_{11}=e_{11}} \tau_{11}(e'_{11})\, de'_{11} \right) dV \tag{3.23}$$

Referring to relations (3.14) and (3.23) we have

$$W = \iiint_V U_s\, dV = (U_s)_T \tag{3.24}$$

where $(U_s)_T$ represents the total strain energy of the body. Since the common boundary of two adjacent elements is subjected to equal and opposite components of stress and is displaced by the same amount, the sum of the work performed by these equal and opposite components of stress vanishes. Consequently, the total work performed by the forces acting on all the elements of the body consists only of the work of the components of stress acting on the faces of the elements which are part of the surfaces of the body and the work of the body forces. Thus, W in equation (3.24) is equal to the work of the known external forces (body forces and surface tractions) acting on the body and of the reactions of its supports. If the supports of the body do not move, the work of the reactions vanishes. On the basis of the foregoing presentation, we have:

$$W_{ext.\ force} + W_{reac.} = (U_s)_T \tag{3.25}$$

Thus, we may conclude that the total strain energy of an elastic body subjected to surface tractions and body forces inducing only one component of stress τ_{11} *in an environment of constant temperature is equal to the work performed by these tractions and bodyforces in bringing it from its stress-free, strain-free reference state to its deformed state.*

Let us now examine the restrictions imposed on the strain energy density and on the complementary energy density by the condition of stability (3.6). Referring to relation (3.18), this condition may be rewritten as

$$d\tau_{11} de_{11} = \frac{d\tau_{11}}{de_{11}}(de_{11})^2 = \frac{d^2 U_s}{d^2 e_{11}}(de_{11})^2 > 0 \tag{3.26a}$$

and

$$de_{11}d\tau_{11} = \frac{de_{11}}{d\tau_{11}}(d\tau_{11})^2 = \left(\frac{d^2U_c}{d^2\tau_{11}}\right)(d\tau_{11})^2 > 0 \tag{3.26b}$$

From the above relations we see that

$$\frac{d^2U_s}{d^2e_{11}} > 0 \tag{3.27a}$$

and

$$\frac{d^2U_c}{d^2\tau_{11}} > 0 \tag{3.27b}$$

At the stress-free, strain-free state, from relations (3.18), (3.19), we have

$$\left.\frac{dU_s}{de_{11}}\right|_{e_{11}=0} = \tau_{11} = 0$$

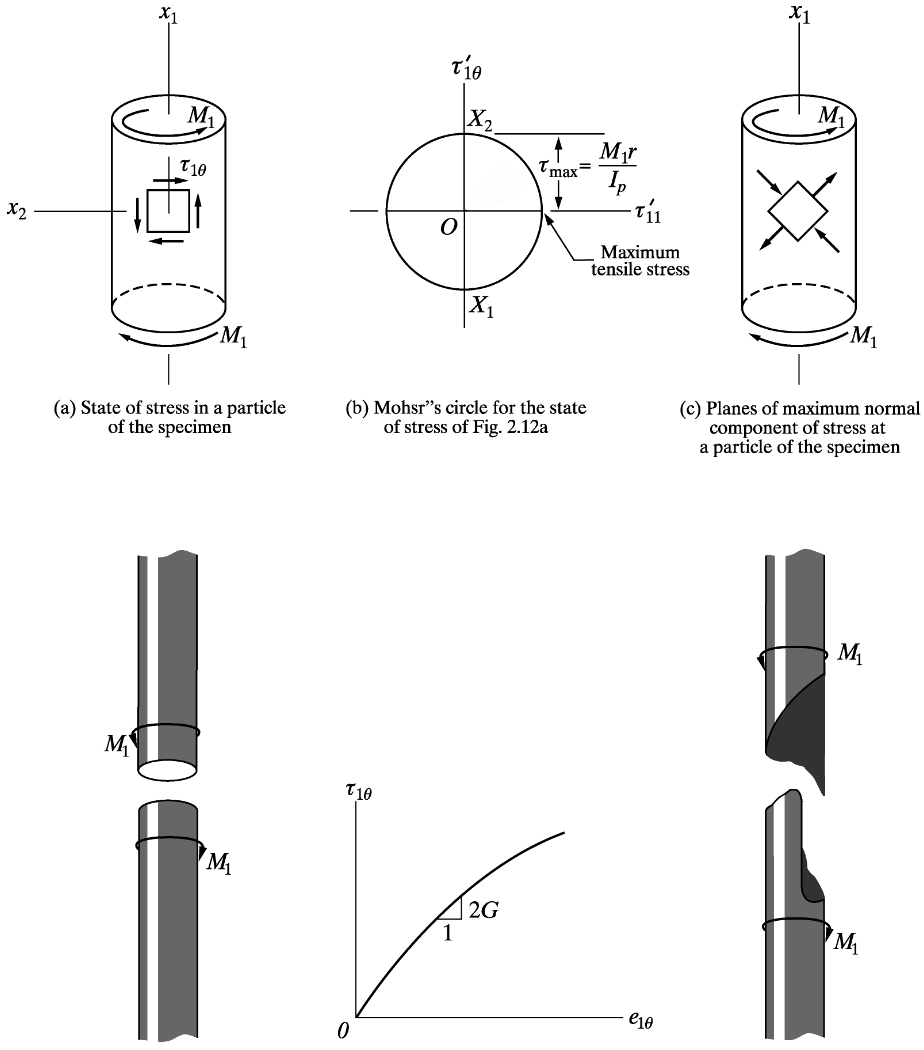
and

$$\left.\frac{dU_c}{d\tau_{11}}\right|_{\tau_{11}=0} = e_{11} = 0$$

These relations indicate that the functions $U_s(e_{11})$ and $U_c(\tau_{11})$ assume an extremum value at the stress-free, strain-free state, while relations (3.27) *indicate that this extremum is a minimum.* The functions $U_s(e_{11})$ and $U_c(\tau_{11})$ are plotted in Fig. 3.10. It is apparent that both functions are non-negative and vanish only when the variable (e_{11} or τ_{11}) vanishes. A function satisfying such conditions is referred to as *positive definite*. Therefore, we may conclude that the condition of stability (3.6) includes the requirement that the strain energy density $U_s(e_{11})$ *is a positive definite function of the component of* e_{11} *and that the complementary energy density* $U_c(\tau_{11})$ *is a positive definite function of the component of stress* τ_{11}. However, relation (3.27a) holds for any value of e_{11}, while relation (3.27b) holds for any value of τ_{11}. Thus, the condition of stability (3.6) imposes a stronger restriction on the strain energy density and the complementary energy density functions, than the requirement that they are positive definite. It does not allow for a change of the sign of the curvature (d^2U_s/d^2e_{11}) of the strain energy density versus strain curve and of the curvature $(d^2U_c/d^2\tau_{11})$ of the complementary energy density versus stress curve.

3.4 The Torsion Test

In the torsion test, a specimen of constant circular *cross section of radius r* is placed in a torsion testing machine and it is subjected at its ends to equal and opposite torsional moments M_1 (see Fig. 3.11a). In Chapter 6 we show that only a shearing component of stress $\tau_{1\theta}$ acts on the cross sections of this specimen in the direction normal to their radius. That is, referring to relations (6.14) and (k) of the example of Section 6.5 the



(a) State of stress in a particle of the specimen

(b) Mohsr"s circle for the state of stress of Fig. 2.12a

(c) Planes of maximum normal component of stress at a particle of the specimen

(d) Fracture of a specimen made from a ductile material

(e) Stress–strain diagram

(f) Fracture of a specimen made from a brittle material

Figure 3.11 The torsion test.

components of stress acting on the particles of the surface of the specimen are

$$\tau_{1\theta} = \frac{M_1 r}{I_p} \qquad \tau_{11} = \tau_{rr} = \tau_{\theta\theta} = \tau_{1r} = \tau_{r\theta} = 0 \tag{3.28}$$

where I_p is the polar moment of inertia of the cross section of the specimen about its centroid. This uniaxial state of stress is known as a state of *simple shear*. The stress–strain relation for isotropic linearly elastic materials subjected to a state of simple shear can be written as

$$\tau_{1\theta} = 2Ge_{1\theta} \tag{3.29}$$

The constant G is a mechanical property of the material from which the specimen is made, called the *shear modulus.*

The relative rotation θ_1^* of the end cross section of the specimen due to its deformation is measured for various values of the applied torsional moments. From this, the shearing component of strain $e_{1\theta}$ of the particles of the surface of the specimen is computed and the stress–strain curve is plotted (see Fig 3.11e). Referring to relations (3.28), (3.29) and (w) of the example of Section 6.5, the shearing component of strain is equal to

$$2e_{1\theta} = \frac{\tau_{1\theta}}{G} = \frac{M_1 r}{I_p G} = \frac{d\theta_1}{dx_1} r = \frac{\theta_1^* r}{L} \tag{3.30}$$

Observations similar to those made in Section 3.2 from uniaxial tension or compression tests are made from torsion tests.

When a specimen made from a ductile material is subjected to increasing equal and opposite torsional moments at its ends, it fractures on a plane normal to its axis (see Fig. 3.11d). Referring to Fig. 3.11b, we see that this is the plane of maximum shearing stress. That is, as in the case of specimens subjected to uniaxial tension, shearing stress is responsible for the fracture of specimens made from a ductile material.

When a specimen made from a brittle material is subjected to increasing equal and opposite torsional moments at its ends, it fractures at a plane inclined at 45° to its axis (see Fig. 3.11f). Referring to Fig. 3.11b and c, we see that on this plane the normal component of stress assumes its maximum value. This indicates that, as in the case of specimens subjected to uniaxial tension, tensile stress is responsible for the fracture of specimens made from a brittle material.

The strain energy density U_s and the complementary energy density U_c of a particle of a body in a state of simple shear are defined as

$$U_s(e_{1\theta}) = \int_0^{e'_{1\theta}=e_{1\theta}} \tau_{1\theta}(e'_{1\theta})\, de'_{1\theta} \tag{3.31a}$$

$$U_c(\tau_{1\theta}) = \int_0^{\tau'_{1\theta}=\tau_{1\theta}} e_{1\theta}(\tau'_{1\theta})\, d\tau'_{1\theta} \tag{3.31b}$$

The condition of stability for a particle of a body in a state of simple shear is

$$(\tau_{1\theta}^{(2)} - \tau_{1\theta}^{(1)})(e_{1\theta}^{(2)} - e_{1\theta}^{(1)}) \geq 0 \tag{3.32a}$$

or

$$d\tau_{1\theta}\, de_{1\theta} \geq 0 \tag{3.32b}$$

3.5 Effect of Pressure, Rate of Loading and Temperature on the Response of Materials Subjected to Uniaxial States of Stress

Experimental results have revealed that when a particle of a body is subjected to a hydrostatic state of stress, its volume does not change. Moreover, results of uniaxial tension tests of ductile materials† under superimposed hydrostatic pressure have revealed that their modulus of elasticity, their yield stress and their component of plastic strain are not affected by the pressure. However, the strain at fracture increases with the intensity of the pressure. That is, the material becomes more ductile. This is due to the fact that the compressive stresses produced by the pressure prevent the formation of microcracks which lead to fracture. For every material there exists a critical pressure above which the material behaves as ductile.

3.5.1 Effect of the Rate of Loading

1. The rate of loading can have a significant effect on the stress–strain curve of materials. This is particularly so at elevated temperatures††.
2. At room temperature the stress–strain curve of metals may be considered independent of the rate of loading for a certain range of rates of loading of practical interest.
3. At high rates of loading the yield stress and modulus of elasticity of ductile materials increase with the rate of loading†††. Moreover, the magnitude of the plastic components of
strain which precedes fracture of ductile materials is reduced considerably as the rate of loading is increased. That is, the material becomes less ductile. In fact at very high rates of loading materials which are ductile at ordinary rates of loading, behave as brittle. These effects are more pronounced at elevated temperature.
4. At high rates of loading it is difficult to maintain a constant temperature during a test since there is not enough time for the heat generated to dissipate into the environment. That is, at high rates of loading the process of deformation is approximately adiabatic.
5. In order to take into account the effect of the rate of loading on the stress–strain curve of a material, it must be considered as time dependent. That is, first or higher time derivatives of stress or strain must be included in its stress–strain relation.

3.5.2 Effect of Temperature

A change in the temperature of the particles of a body produces a change of their dimensions (deformation). The particles of ordinary materials expand when heated and contract when cooled. That is, the normal components of strain increase as the temperature increases and decrease as the temperature decreases, while the shearing components of strain are not affected by changes of temperature. However, some unusual materials do not obey this rule. For example, the particles of water expand when heated at temperatures above 4°C but also expand when cooled at temperatures below 4°C. A measure of the deformation of a particle due to a change of its temperature is its thermal strain tensor $[e^T]$ whose components referred to the system of axes x_1, x_2, x_3 are denoted as

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† A detailed account of experimental investigations of the effect of hydrostatic pressure on the stress–strain curves of metals can be found in
(a) Bridgman, P.W., *Studies in Large Plastic Flow and Fracture,* McGraw-Hill, New York, 1952.
(b) Pugh, H.Li D. (ed.), *Mechanical Behavior of Materials under Pressure,* Elsevier, Amsterdam, 1970.
†† The temperature of a body in the absolute scale is considered elevated if it exceeds one third of its melting temperature.
††† Morkovin, D., Sidebottom, O., The effects of nonuniform distribution of stress on the yield strength of steel, *Engineering Experimental Station University of Illinois Bulletin,* 373, Urbana, IL, 1947.

$$[e^T] = \begin{bmatrix} e_{11}^T & 0 & 0 \\ 0 & e_{22}^T & 0 \\ 0 & 0 & e_{33}^T \end{bmatrix} \tag{3.33}$$

The thermal components of strain of a particle of a body made from a general non-isotropic material are equal to

$$e_{11}^T = \alpha_1 \Delta T \qquad e_{22}^T = \alpha_2 \Delta T \qquad e_{33}^T = \alpha_3 \Delta T \tag{3.34a}$$

where α_1, α_2 and α_3 are called the *coefficients of linear thermal expansion*. The coefficients α (i = 1, 2 or 3) represent the component of thermal strain e_{ii}^T (i = 1, 2 or 3) of a particle in the direction of the x_i (i = 1, 2 or 3) axis due to an increase in temperature of 1°. The units of the coefficients of thermal expansion are the reciprocal of the units of the change of temperature ΔT. That is, in the International System of Units (SI) they are given either as strain per degree Kelvin (1/°K) or as strain per degree Celsius (1/°C), while in the English system of units are given as strain per degree Fahrenheit (1/°F). The thermal components of strain of a particle made from an isotropic material are equal to

$$e_{11}^T = e_{22}^T = e_{33}^T = \alpha \Delta T \tag{3.34b}$$

Consider an infinitesimal orthogonal parallelepiped whose edges are oriented in the direction of the x_1, x_2, x_3 axes. The parallelepiped is made from an isotropic linearly elastic material. Its dimensions in the stress-free and strain-free reference state at the uniform temperature T_o are dx_1, dx_2, dx_3. When this parallelpiped is subjected to an increase of temperature ΔT and it is free to expand, no stresses are induced in it. The dimensions of its edges change to $(1 + e_{11}^T)dx_1$, $(1 + e_{22}^T)dx_2$ and $(1 + e_{33}^T)dx_3$. If,

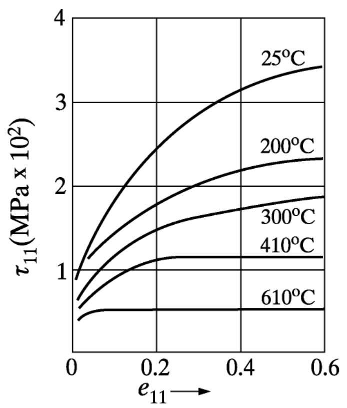


Figure 3.12 Effect of temperature on the stress–strain curve of annealed copper at a strain rate of 10-5/sec†.

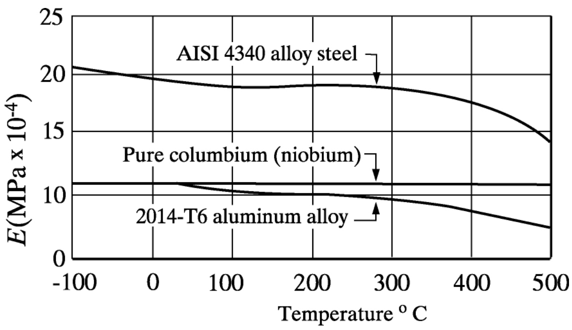


Figure 3.13 Variation of the modulus of elasticity in tension with the temperature.

† Taken from Mahtah, F.U., Johnson, W., Stater, R.A.C., *Proceedings of Institute of Mechanical Engineers,* 180, p. 285, 1965.

however, this parallelepiped is not free to expand, a state of stress will be induced in it.

Generally, all the mechanical properties of a material (*E, G,* ν or α) are affected by changes of temperature (see Fig. 3.12). However, for a range of temperature in the vicinity of room temperature, these changes are rather small for many materials of engineering interest (see Fig. 3.13).

The ductility or brittleness of materials depends on their temperature. For instance, a metal specimen which, when subjected to uniaxial tension at room temperature, fails in ductile manner, may fail in a brittle manner when subjected to uniaxial tension at a temperature substantially below room temperature. Moreover, if a metal specimen is subjected to a constant axial centroidal force at a temperature higher than its recrystallization temperature, its normal components of strain will continue to increase with time until the specimen fractures. This time-dependent deformation is known as *creep*. It is discussed in Section 3.15.

3.6 Models of Idealized Time-Independent Stress–Strain Relations for Uniaxial States of Stress

The stress–strain relations of engineering materials are highly non-linear in the plastic region. This difficulty is circumvented by using idealized stress–strain relations. The idealized stress–strain relations for uniaxial tension or compression or for simple shear most commonly used in engineering are illustrated in Fig. 3.14.

A crude but useful idealization of the stress–strain relation of materials subjected to uniaxial tension or compression or to simple shear above the elastic limit is shown in Fig. 3.14c and e. Consider a specimen made from this idealized material subjected to external forces inducing a state of uniaxial stress. When the component of stress acting on the particles of this specimen reaches its value at yielding, it will continue to deform at constant stress. The deformation of the particles of the specimen at constant stress is irreversible plastic deformation. If the specimen is unloaded before it fractures, it will deform elastically. Its stress–strain curve will be parallel to that during loading. The value of the yield stress does not depend on the number of cycles of loading and unloading. This behavior of a material is referred to as *ideally plastic*. The model shown in Fig. 3.14e is referred to as *linearly elastic — ideally plastic* and it is used when the plastic components of strain of the particles of a body are of the same order of magnitude as their elastic components. If the elastic components of strain are small as compared to the plastic components of strain, it may be possible to disregard their effect and assume rigid body behavior up to yielding, and perfectly plastic behavior subsequent to yielding (see Fig. 3.14c). This model is called *rigid — ideally plastic* and it has been used in problems involving large plastic strains such as those associated with forging, rolling, extruding and drawing.

If the effect of strain hardening cannot be entirely disregarded, two simple idealizations of the behavior of materials subjected to a uniaxial state of stress above the elastic limit are shown in Fig. 3.14d and f. In these idealizations the strain hardening is linear. When a body is subjected to a uniaxial state of stress of magnitude higher than the elastic limit, its behavior during unloading is linearly elastic with modulus of elasticity equal to that during loading. On reloading, the yield stress is raised to the value of the stress just before unloading. The initial yield stress τ_{11}^{Y} and the slope E_t of the strain hardening line of the idealized stress–strain curve may be determined by matching this curve to that of the real material (see Fig. 3.14f).

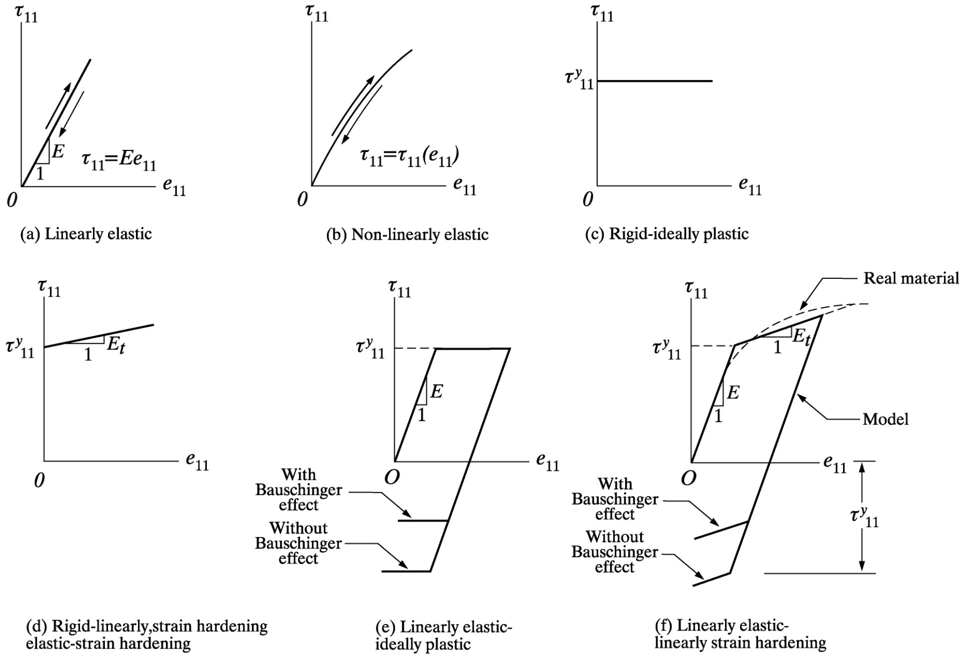


Figure 3.14 Idealized stress–strain relations for time-independent materials subjected to a uniaxial state of stress.

Example 1 Consider a prismatic bar of circular cross section and length of 120 mm. The bar is subjected to axial centroidal forces at its ends which produce an uniaxial state of tensile stress of τ_{11} = 250 MPa. The bar is made from an isotropic, linearly elastic–linearly strain hardening material (E = 200 GPa, E_1 = 100 GPa) whose stress–strain diagram is shown in Fig. a. Compute

(a) The elongation of the bar when loaded

(b) The residual elongation of the bar when unloaded

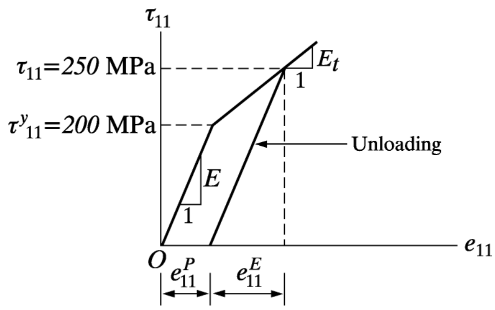


Figure a Stress–strain diagram for the material from which the bar is made.

Solution When the magnitude of the applied forces is such that the normal component of stress τ_{11} acting on the particles of the bar is equal to the yield stress $\tau_{11}^{Y} = 200$ MPa, the corresponding component of strain e_{11}^{Y} is equal to

$$e_{11}^{Y} = \frac{\tau_{11}^{Y}}{E} = \frac{200}{200(10^3)} = 0.001 \tag{a}$$

When the magnitude of the applied forces reaches the value which produces a normal component of stress of 250 MPa, the additional normal component of strain Δe_{11} of the particles of the bar is equal to

$$\Delta e_{11} = \frac{250 - 200}{100(10^3)} = 0.0005 \tag{b}$$

Thus, when the normal component of stress in the bar is $\tau_{11} = 250$ MPa, the total normal component of strain e_{11}^{T} is equal to

$$e_{11}^{T} = e_{11}^{Y} + \Delta e_{11} = 0.001 + 0.0005 = 0.0015 \tag{c}$$

The elongation of the bar ΔL is equal to

$$\Delta L = L_0 e_{11}^{T} = 120(0.0015) = 0.18 \text{ mm}$$

When the bar is unloaded, its particles will not assume their undeformed configuration. A plastic normal component of strain e_{11}^{P} will remain. Referring to Fig. a this is equal to

$$e_{11}^{P} = e_{11}^{T} - e_{11}^{E} \tag{e}$$

where

$$e_{11}^{E} = \frac{250}{200(10^3)} = 0.00125 \tag{f}$$

Substituting relations (c) and (f) into (e), we get

$$e_{11}^{P} = 0.0015 - 0.00125 = 0.00025 \tag{g}$$

Consequently, the residual elongation of the bar upon unloading is

$$(\Delta L)^{P} = L_0 e_{11}^{P} = 120(0.00025) = 0.03 \text{ mm}$$

3.7 Stress–Strain Relations for Elastic Materials Subjected to Three-Dimensional States of Stress

In this section we extend to bodies made from an elastic material and subjected to three-dimensional states of stress the findings from the uniaxial tension or compression and the torsion tests described in Sections 3.2 and 3.4. When a body made from an elastic material is subjected to loads which do not induce large components of stress at its particles, it recovers its original shape and size upon removal of the loads. Energy is not lost during the process of loading and unloading. Moreover, for a given temperature, the relations among the components of stress and strain are unique. They are independent not only of time, but also of the path of loading. That is, for any given temperature, a certain set of values of the components of stress corresponds to given values of the components of strain, notwithstanding whether this set of values of the components of stress has been reached by loading or unloading, subsequent to one or more cycles. Therefore, an elastic material is defined as one having the following properties:

1. It has a stress-free, strain-free natural state, at a uniform temperature T_0.
2. The components of stress τ_{ij} $(i, j = 1, 2, 3)$ acting on any particle of a body made from such material are single-valued functions of the components of strain $e_{mn}(m, n = 1, 2, 3)$ and the temperature T of this particle having continuous first derivatives. That is,

$$\tau_{ij} = F_{ij}\,[e_{mn},\ T] \tag{3.35a}$$

where F_{ij} are such that

$$F_{ij}\,[0,\ T_o] = 0 \tag{3.35b}$$

Moreover, relation (3.35a) can be uniquely inverted to yield the components of strain as single-valued functions of the components of stress and temperature, having continuous first derivatives. That is,

$$e_{mn} = G_{mn}\,[\tau_{ij}, T] \tag{3.36a}$$

such that

$$G_{mn}\,[0, T_o] = 0 \tag{3.36b}$$

Furthermore, if the body is subjected to surface tractions and body forces in an environment of constant temperature, the functions $F_{ij}(e_{mn}, T_0)$ and $G_{mn}(\tau_{ij}, T_0)$ satisfy the following relations:

$$\frac{\partial \tau_{ij}(e_{mn},\ T_0)}{\partial e_{mn}} = \frac{\partial \tau_{mn}(e_{ij},\ T_0)}{\partial e_{ij}} \tag{3.37a}$$

and

$$\frac{\partial e_{ij}(\tau_{mn},\ T_0)}{\partial \tau_{mn}} = \frac{\partial e_{mn}(\tau_{ij},\ T_0)}{\partial \tau_{ij}} \tag{3.37b}$$

Relations (3.37a) and (3.37b) impose restrictions on the functions $F_{ij}(e_{mn}, T_0)$ and $G_{mn}(\tau_{ij}, T_0)$, respectively. These restrictions are necessary in order to ensure that for an elastic material there exists a functional† of the components of strain called *strain energy density* and a functional of the components of stress called *complementary energy density* which are independent of the history of loading and unloading to which the body has been subjected (see Section 3.11). If the relations among the components of stress and strain are nonlinear, the sum of the components of strain of a particle $\tilde{e}_{11}$ and $\tilde{\tilde{e}}_{11}$ which are produced when the components of stress $\tilde{\tau}_{11}$ and $\tilde{\tilde{\tau}}_{11}$ act on this particle separately is unequal to the component of strain e_{11} produced when the component of stress

† A functional is an expression whose value depends on one or more functions. For example, the strain energy density is a function of the components of strain which are functions of the space coordinates.

$\tau_{11} = \tilde{\tau}_{11} + \tilde{\tilde{\tau}}_{11}$ acts on the particle. That is, the results cannot be superimposed.

3. It is assumed that when a body made from an elastic material is subjected to external forces in an environment of constant temperature, any two different sets of components of stress [$\tau_{ij}^{(2)}$ and $\tau_{ij}^{(1)}$ $(i, j = 1, 2, 3)$] acting on any one of its particles correspond to two different sets of components of strain [$e_{ij}^{(2)}$ and $e_{ij}^{(1)}$ $(i, j = 1, 2, 3)$] of this particle and vice versa. Moreover, these sets of components of stress and strain satisfy the following relation:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} (\tau_{ij}^{(2)} - \tau_{ij}^{(1)}) (e_{ij}^{(2)} - e_{ij}^{(1)}) \geq 0 \tag{3.38a}$$

The left side of relation (3.38a) is equal to zero only if $\tau_{ij}^{(2)} = \tau_{ij}^{(1)}$ and, consequently, $e_{ij}^{(2)} = e_{ij}^{(1)}$. Inasmuch as relation (3.38a) is valid for any two sets of values of the components of stress, it will also hold when the differences of the components of stress and strain are infinitesimal. Thus,

$$\sum_{i=1}^{3} \sum_{j=1}^{3} d\tau_{ij}\, de_{ij} \geq 0 \tag{3.38b}$$

Elastic materials possessing the property (3.38) are called *stable*.

An elastic material is an idealization approached by some materials. In reality, even when a body is subjected to very small forces, some permanent distortion of its microscopic structure occurs. Consequently, for any load, no matter how small, there is some permanent deformation and some energy is lost during the process of loading and unloading. When the applied forces are not very large, the permanent deformation of bodies made from certain materials within a certain range of temperature is of a smaller magnitude than we are able to observe. Moreover, the energy lost per cycle of loading and unloading is small and can be disregarded except in cases involving many cycles. For instance, in the case of a vibrating metal part, the energy lost results in the damping of the vibrations. The response of these materials is considered as *elastic*. As the applied forces increase, the permanent deformation reaches detectable magnitudes. It becomes evident that the body does not recover its original volume and shape upon unloading. The process of deformation has obviously become *irreversible*. However, the relations among the components of stress and strain do not depend upon time but upon the direction and path of loading. That is, while elastic deformation depends only upon the initial and final states of stress, the total irreversible time-independent deformation depends upon both the final state of stress and the path of loading. The time-independent behavior of a material subjected to loads beyond the maximum corresponding to elastic response is termed *plastic*. The mathematical theory concerned with this material behavior is termed *plasticity*.

3.8 Stress–Strain Relations of Linearly Elastic Materials Subjected to Three-Dimensional States of Stress

From the uniaxial tension and the torsion experiments it has been established that for certain engineering materials the relationship among the components of stress and strain is approximately linear, for values of the components of stress below the elastic limit. This result has been extended to three-dimensional states of stress by Robert Hooke and is generally referred to in the literature as Hooke's law. It states that the components of stress, in three-dimensional states of stress, are linearly related to the components of strain. In the most general form, Hooke's law for a body in an environment of constant temperature or for a body thermally isolated from its environment may be written as

$$\begin{aligned}
\tau_{11} &= C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33} + C_{1112}e_{12} + C_{1113}e_{13} + C_{1123}e_{23} \\
\tau_{22} &= C_{2211}e_{11} + C_{2222}e_{22} + C_{2233}e_{33} + C_{2212}e_{12} + C_{2213}e_{13} + C_{2223}e_{23} \\
\tau_{33} &= C_{3311}e_{11} + C_{3322}e_{22} + C_{3333}e_{33} + C_{3312}e_{12} + C_{3313}e_{13} + C_{3323}e_{23} \\
\tau_{12} &= C_{1211}e_{11} + C_{1222}e_{22} + C_{1233}e_{33} + C_{1212}e_{12} + C_{1213}e_{13} + C_{1223}e_{23} \\
\tau_{13} &= C_{1311}e_{11} + C_{1322}e_{22} + C_{1333}e_{33} + C_{1312}e_{12} + C_{1313}e_{13} + C_{1323}e_{23} \\
\tau_{23} &= C_{2311}e_{11} + C_{2322}e_{22} + C_{2333}e_{33} + C_{2312}e_{12} + C_{2313}e_{13} + C_{2323}e_{23}
\end{aligned} \tag{3.39}$$

These relations can be inverted to yield the components of strain as functions of the components of stress

$$\begin{aligned}
e_{11} &= S_{1111}\tau_{11} + S_{1122}\tau_{22} + S_{1133}\tau_{33} + S_{1112}\tau_{12} + S_{1113}\tau_{13} + S_{1123}\tau_{23} \\
e_{22} &= S_{2211}\tau_{11} + S_{2222}\tau_{22} + S_{2233}\tau_{33} + S_{2212}\tau_{12} + S_{2213}\tau_{13} + S_{2223}\tau_{23} \\
e_{33} &= S_{3311}\tau_{11} + S_{3322}\tau_{22} + S_{3333}\tau_{33} + S_{3312}\tau_{12} + S_{3313}\tau_{13} + S_{3323}\tau_{23} \\
e_{12} &= S_{1211}\tau_{11} + S_{1222}\tau_{22} + S_{1233}\tau_{33} + S_{1212}\tau_{12} + S_{1213}\tau_{13} + S_{1223}\tau_{23} \\
e_{13} &= S_{1311}\tau_{11} + S_{1322}\tau_{22} + S_{1333}\tau_{33} + S_{1312}\tau_{12} + S_{1313}\tau_{13} + S_{1323}\tau_{23} \\
e_{23} &= S_{2311}\tau_{11} + S_{2322}\tau_{22} + S_{2333}\tau_{33} + S_{2312}\tau_{12} + S_{2313}\tau_{13} + S_{2323}\tau_{23}
\end{aligned} \tag{3.40}$$

The coefficients C_{ijkl} and S_{ijkl} are referred to as the *elastic constants* and the *elastic compliances*, respectively. Generally, they are functions of the space coordinates. In this text we limit our attention to *homogeneous materials*, that is, materials whose properties do not vary from point to point. For these materials, the coefficients C_{ijkl} and S_{ijkl} are constant throughout the body. Their value depends upon the material and the direction of the axes to which they are referred. That is, generally, if the stress–strain relations are referred to another set of rectangular axes x'_1, x'_2, x'_3, the coefficients assume different values† C'_{ijkl} or S'_{ijkl}, which may be obtained from the values of the coefficients C_{ijkl} or S_{ijkl}, respectively, referred to the set of rectangular axes x_1, x_2, x_3.

Inasmuch as the stress–strain relations (3.40) must satisfy relation (3.37b), we have

$$\begin{aligned}
&S_{1122} = S_{2211} \quad && S_{1223} = S_{2312} \quad && S_{1213} = S_{1312} \quad && S_{2313} = S_{1323} \\
&S_{1112} = S_{1211} && S_{1133} = S_{3311} && S_{1113} = S_{1311} && S_{1123} = S_{2311} \\
&S_{2233} = S_{3322} && S_{2213} = S_{1322} && S_{2322} = S_{2223} && S_{2212} = S_{1222} \\
&S_{3312} = S_{1233} && S_{3313} = S_{1333} && S_{3323} = S_{2333} &&
\end{aligned} \tag{3.41}$$

or in shorthand notation

$$S_{ijkl} = S_{klij} \qquad (i, j, k, l = 1, 2, 3) \tag{3.42}$$

The physical interpretation of relations (3.41) is that the component of strain e_{ij} due to the component of stress $\tau_{mn} = \tau$, must be equal to the component of strain e_{mn}, due to the component of stress $\tau_{ij} = \tau$.

Similarly in order that the stress–strain relations (3.39) satisfy relation (3.37a), we

† See next page.

have

$$C_{ijkl} = C_{klij} \qquad (i, j, k, l = 1, 2, 3) \tag{3.43}$$

Therefore, it is apparent that the relations among the components of stress and strain of a general non-isotropic, linearly elastic material involve a maximum of 36 − 15 = 21 material constants, which may be established experimentally.

The last three of relations (3.39), when referred to the principal axes ($e_{12} = e_{13} = e_{23} = 0$) of strain, reduce to

$$\begin{aligned} \tilde{\tau}_{12} &= \tilde{C}_{1211}e_1 + \tilde{C}_{1222}e_2 + \tilde{C}_{1233}e_3 \\ \tilde{\tau}_{23} &= \tilde{C}_{2311}e_1 + \tilde{C}_{2322}e_2 + \tilde{C}_{2333}e_3 \\ \tilde{\tau}_{13} &= \tilde{C}_{1311}e_1 + \tilde{C}_{1322}e_2 + \tilde{C}_{1333}e_3 \end{aligned} \tag{3.44}$$

where e_1, e_2, e_3 are the principal components of strain. Therefore, it is apparent that the shearing components of stress $\tilde{\tau}_{12}$, $\tilde{\tau}_{23}$, $\tilde{\tau}_{13}$ corresponding to the principal axes of strain do not vanish. *That is, for a non-isotropic material, the principal axes of strain do not coincide with the principal axes of stress.*

3.9 Stress–Strain Relations for Orthotropic, Linearly Elastic Materials

Many important engineering materials possess a certain symmetry of their microscopic structure, which is reflected in their stress–strain relations. For instance, certain engineering materials, such as fiber reinforced composites, have a microstructure which

---

† The coefficients C'_{ijkl} can be established as follows:
The stress–strain relations (3.39) can be written in indicial notation as

$$\tau_{ij} = \sum_{k=1}^{3}\sum_{m=1}^{3} \hat{C}_{ijkm} e_{km}$$

where

$$\hat{C}_{ijkm} = \begin{cases} \dfrac{C_{ijkm}}{2} & \text{if } k \neq m \\ C_{ijkm} & \text{if } k = m \end{cases}$$

The 81 quantities $\hat{C}_{ijkm}$ relate the components of two tensors of the second rank, the stress and strain tensors. Consequently, as we know from tensor analysis, they are components of a tensor of the fourth rank. Moreover, if the components C_{ijkm} $(i, j, k, m = 1, 2, 3)$ of a tensor of the fourth rank are known with respect to the rectangular system of axes x_1, x_2, x_3, its components $\hat{C}'_{pqrs}$ $(p, q, r, s = 1, 2, 3)$ with respect to any other rectangular system of axes x_1', x_2', x_3' can be established using the following transformation relation:

$$\hat{C}'_{pqrs} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \lambda_{pi}\lambda_{qj}\lambda_{rk}\lambda_{sm}\hat{C}_{ijkm}$$

where λ_{ij} $(i, j = 1, 2, 3)$ are the direction cosines of the rectangular system of axes x_1', x_2', x_3', with respect to the rectangular system of axes x_1, x_2, x_3.

his symmetric with respect to three mutually perpendicular planes. Consequently, their elastic constants are also symmetric with respect to these planes. Materials having this symmetry are known as *orthotropic*. For these materials, as shown in Appendix B, the stress–strain relations (3.39) when referred to axes which are perpendicular to the planes of symmetry of their microstructure reduce to

$$\begin{aligned}
\tau_{11} &= C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33} \\
\tau_{22} &= C_{2211}e_{11} + C_{2222}e_{22} + C_{2233}e_{33} \\
\tau_{33} &= C_{3311}e_{11} + C_{3322}e_{22} + C_{3333}e_{33} \\
\tau_{12} &= C_{1212}e_{12} \\
\tau_{13} &= C_{1313}e_{13} \\
\tau_{23} &= C_{2323}e_{23}
\end{aligned} \tag{3.45}$$

Referring to relations (3.43), we find that

$$C_{1122} = C_{2211} \qquad C_{1133} = C_{3311} \qquad C_{2233} = C_{3322} \tag{3.46}$$

Thus, when the stress–strain relations for an orthotropic material are referred to axes which are perpendicular to the planes of symmetry of their microstructure, they involve only nine elastic constants. If the stress–strain relations for an orthotropic material are referred to an arbitrary set of axes, not perpendicular to the planes of orthotropic symmetry of the material, they assume the form (3.39). For instance, a normal component of stress τ'_{22} acting on a plane which is not a plane of symmetry of the material, generally, induces shearing components of strain e'_{12}, e'_{13} and e'_{23}. However, the 21 coefficients C'_{ijkl} involved in the stress–strain relations for an orthotropic material, referred to any set of axes, may be derived from its nine coefficients C_{ijkl}, referred to the axes normal to the planes of symmetry of its microstructure (see footnote, previous page).

Referring to equation (3.45), one may form the impression that for orthotropic materials the principal axes of strain coincide with the principal axes of stress. However, this is not true, because for a given state of stress the principal directions of stress may not coincide with the axes that are perpindicular to the planes of symmetry of the microstructure of the material. Consequently, in order to compute the components of strain corresponding to the principal directions of stress relations (3.40) must be employed. *Thus, in general the sheering components of strain corresponding to the principle directions of stress do not vanish* .

Example 2 Wood is an orthotropic material. Birch wood has the following stress–strain relations, relative to its orthotropic axes x_1, x_2, x_3:

$$\begin{aligned}
10^6 \times e_{11} &= 72.50\,\tau_{11} - 36.25\,\tau_{11} - 36.25\,\tau_{11} \\
10^6 \times e_{11} &= -36.25\,\tau_{11} + 942.25\,\tau_{11} - 652.5\,\tau_{11} \\
10^6 \times e_{11} &= -36.25\,\tau_{11} - 652.25\,\tau_{11} - 1450.00\,\tau_{11} \\
10^6 \times e_{11} &= 507.5\,\tau_{11} \\
10^6 \times e_{11} &= 543.8\,\tau_{11} \\
10^6 \times e_{11} &= 2175.0\,\tau_{11}
\end{aligned} \tag{a}$$

The state of stress at a particle of Birch wood acting on the planes normal to its orthotropic axes is

$$[\tau] = \begin{bmatrix} 12 & 3 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -5 \end{bmatrix} \tag{b}$$

Compute the direction of the principal axes of the normal components of stress and strain, of this particle.

Solution: The given state of stress is a quasi-plane state. Consequently, the axis x_3 is one of the three principal axes while the other two lie in the plane $x_1 x_2$. The principal axis $\tilde{x}_1$ is specified by the angle $\tilde{\phi}_{11}$ which, referring to relation (1.122), is equal to

$$\tan 2\tilde{\phi}_{11} = \frac{2\tau_{12}}{\tau_{11} - \tau_{22}} = \frac{2(3)}{12 - 4} = 0.75$$

Thus,

$$2\tilde{\phi}_{11} = 151.85^\circ \qquad \tilde{\phi}_{11} = 75.63^\circ \text{ counterclockwise}$$

The results are shown in Fig. a.

The components of strain may be computed by substituting the given components of stress (b) into the stress–strain relations (a). That is,

$$\begin{aligned} 10^6 \times e_{11} &= 72.50(12) - 36.25(4) - 36.25(-5) = 906.25 \\ 10^6 \times e_{22} &= -36.25(12) + 942.50(4) - 652.50(-5) = 6597.50 \\ 10^6 \times e_{33} &= -36.25(12) - 652.50(4) - 1450.00(-5) = 9435.00 \\ 10^6 \times e_{12} &= 507.5(3) = 1522.5 \end{aligned} \tag{c}$$

Referring to relation (1.122), we get

$$\tan 2\tilde{\phi}_{11} = \frac{2e_{12}}{e_{11} - e_{22}} = \frac{2(1522.5)}{906.25 - 6597.50} = -0.535032$$

and

$$2\tilde{\phi}_{11} = 36.87^\circ \qquad \text{or} \qquad \tilde{\phi}_{11} = 18.43^\circ \text{ counterclockwise}$$

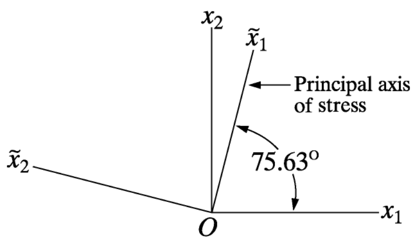


Figure a Location of the principle axes of stress.

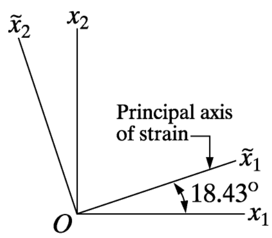


Figure b Location of the principle axes of strain.

Thus, for orthotropic linear elastic materials the principal axes for stress and strain do not coincide as they do for isotropic linearly elastic materials.

3.10 Stress–Strain Relations for Isotropic, Linearly Elastic Materials Subjected to Three-Dimensional States of Stress

To the order of accuracy of most engineering calculations, the behavior of many engineering materials under stress can be considered independent of the direction of the stress field. That is, the constants involved in the relations among the components of stress and strain of a particle of a body made from one of these materials do not change with the system of axes to which the components of stress are referred. These materials are called *isotropic*. As shown in Appendix B, for *isotropic* linearly elastic materials the stress–strain relations (3.39) are simplified to

$$
\begin{aligned}
\tau_{11} &= \frac{E}{(1+\nu)(1-2\nu)}[\,(1-\nu)e_{11} + \nu(e_{22}+e_{33})\,] \\
\tau_{22} &= \frac{E}{(1+\nu)(1-2\nu)}[\,(1-\nu)e_{22} + \nu(e_{11}+e_{33})\,] \\
\tau_{33} &= \frac{E}{(1+\nu)(1-2\nu)}[\,(1-\nu)e_{33} + \nu(e_{22}+e_{11})] \\
\tau_{12} &= 2Ge_{12} \\
\tau_{13} &= 2Ge_{13} \\
\tau_{23} &= 2Ge_{23}
\end{aligned}
\tag{3.47}
$$

Relations (3.47) can be solved for e_{ij} $(i, j = 1, 2, \text{ or } 3)$ to obtain

$$
\begin{aligned}
e_{11} &= \frac{1}{E}[\tau - \nu(\tau_{22}+\tau_{33})] \\
e_{22} &= \frac{1}{E}[\tau_{22} - \nu(\tau_{11}+\tau_{33})] \\
e_{33} &= \frac{1}{E}[\tau_{33} - \nu(\tau_{11}+\tau_{22})] \\
e_{12} &= \frac{\tau_{12}}{2G} \\
e_{13} &= \frac{\tau_{13}}{2G} \\
e_{23} &= \frac{\tau_{23}}{2G}
\end{aligned}
\tag{3.48}
$$

The material constant E is evaluated from a uniaxial tension test ($\tau_{11} \neq 0$, $\tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$), described in Section 3.2. In this case, relations (3.48) reduce to

$$\tau_{11} = Ee_{11} \qquad e_{22} = e_{33} = -\frac{\nu}{E}\tau_{11} = -\nu e_{11} \qquad e_{12} = e_{13} = e_{23} = 0$$

Therefore, E is the slope of the stress–strain curve referred to as the modulus of elasticity, and ν is Poisson's ratio.

The material constant G may be evaluated from a simple torsion test described in Section 3.4.

The material constants G, E and ν are actually related. That is, the response of an isotropic, linearly elastic material is characterized by two material constants. Physically this makes sense. It implies that the response of a material to a shearing component of stress is not independent of its response to a normal component of stress. As shown in Appendix B, the relation between the elastic constants is

$$G = \frac{E}{2(1 + \nu)} \tag{3.49}$$

Mechanical properties of some commonly used engineering materials are given in Appendix A.

3.10.1 Stress–Strain Relations for States of Plane Strain and Plane Stress

A state of plane strain is defined as one which has one component of displacement equal to zero (say $u_1 = 0$), while the other components of displacement are functions only of two space coordinates (say x_2 and x_3). In this case, referring to relations (2.16), we have

$$e_{11} = e_{12} = e_{13} = 0$$

Taking into account the above, relations (3.47) reduce to

$$\begin{aligned}
\tau_{11} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)}(e_{22} + e_{33}) = \nu(\tau_{22} + \tau_{33}) \\
\tau_{22} &= \frac{E}{(1 + \nu)(1 - \nu)}[(1 - \nu)e_{22} + \nu e_{33}] \\
\tau_{33} &= \frac{E}{(1 + \nu)(1 - 2\nu)}[(1 - \nu)e_{33} + \nu e_{22}] \\
\tau_{12} &= \tau_{13} = 0 \\
\tau_{23} &= 2Ge_{12}
\end{aligned} \tag{3.50}$$

or

$$
\begin{aligned}
e_{22} &= \frac{1+\nu}{E}[(1-\nu)\tau_{22} - \nu\tau_{33}] \\
e_{33} &= \frac{1+\nu}{E}[(1-\nu)\tau_{33} - \nu\tau_{22}] \\
e_{23} &= \frac{\tau_{23}}{2G}
\end{aligned}
\tag{3.51}
$$

A state of plane stress is defined as one which has three components of stress acting on one plane equal to zero. If we choose $\tau_{11} = \tau_{12} = \tau_{13} = 0$, relations (3.48) reduce to

$$
\begin{aligned}
e_{11} &= -\frac{\nu}{E}[\tau_{22} + \tau_{33}] \\
e_{22} &= \frac{1}{E}[\tau_{22} - \nu\tau_{33}] \\
e_{33} &= \frac{1}{E}[\tau_{33} - \nu\tau_{22}] \\
e_{32} &= \frac{\tau_{32}}{2G} \\
e_{13} &= e_{12} = 0
\end{aligned}
\tag{3.52}
$$

or

$$
\begin{aligned}
\tau_{22} &= \frac{E}{1-\nu^2}(e_{22} + \nu e_{33}) \\
\tau_{33} &= \frac{E}{1-\nu^2}(e_{33} + \nu e_{22}) \\
\tau_{23} &= 2Ge_{23}
\end{aligned}
\tag{3.53}
$$

3.11 Strain Energy Density and Complementary Energy Density of a Particle of a Body Subjected to External Forces in an Environment of Constant Temperature

Consider a symmetric tensor of the second rank $\tau_{ij}(e_{pq})(i, j, p, q = 1, 2, 3)$ whose components are functions of the components of another symmetric tensor of the second rank e_{pq} and have continuous first derivatues. A theorem of calculus states that the necessary and sufficient condition for the integral $\int_0^{e_{mn}} \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}(e'_{pq})de'_{ij}$ to exist and be independent of the path of integration is

$$
\frac{\partial \tau_{ij}}{\partial e_{mn}} = \frac{\partial \tau_{mn}}{\partial e_{ij}} \qquad (i, j, m, n = 1, 2, 3) \tag{3.54}
$$

Consequently, on the basis of the definition of an elastic material given in Section 3.7

[see relation (3.37)], it is apparent that, when a body made from an elastic material is subjected to surface tractions and/or body forces in an environment of constant temperature (deforms isothermally), the following integrals exist and are independent of the path of integration:

$$U_s(e_{mn}) = \int_0^{e_{mn}} \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}(e'_{pq})\, de'_{ij} \tag{3.55}$$

$$U_c(\tau_{mn}) = \int_0^{\tau_{mn}} \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}(\tau'_{pq})\, d\tau'_{ij} \tag{3.56}$$

The functional $U_s(e_{mn})$ is called *strain energy density*, while the functional $U_c(\tau_{mn})$ is called *complementary energy density*. Since the functions $\tau_{ij}(e_{mn})$ and $e_{ij}(\tau_{mn})$ have continuous first derivatives, on the basis of their definitions (3.55) and (3.56), the strain energy density $U_s(e_{mn})$ and the complementary energy density $U_c(\tau_{mn})$ have continuous second derivatives and vanish at the stress-free, strain-free reference state. In relations (3.55) or (3.56), integration is carried out over a curve in the nine-dimensional space of e_{pq} or τ_{pq} $(p, q = 1, 2, 3)$, respectively, connecting the origin (undeformed state) with the point $P(e_{mn})$ or $P(\tau_{mn})$ (deformed state). Inasmuch as the integrals in relations (3.55) and (3.56) are independent of the path of integration, their integrands $\sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}(e'_{pq})de'_{ij}$ and $\sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}(\tau'_{pq})d\tau'_{ij}$ must be perfect differentials. That is,

$$dU_s = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial U_s}{\partial e_{ij}} de_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\, de_{ij} \tag{3.57}$$

and

$$dU_c = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial U_c}{\partial \tau_{ij}} d\tau_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}\, d\tau_{ij} \tag{3.58}$$

Consequently,

$$\tau_{ij} = \frac{\partial U_s}{\partial e_{ij}} \qquad (i, j = 1, 2, 3) \tag{3.59}$$

and

$$e_{ij} = \frac{\partial U_c}{\partial \tau_{ij}} \qquad (i, j = 1, 2, 3) \tag{3.60}$$

In relation (3.59), $U_s(e_{mn})$ is considered a function of the nine components of strain, while in relation (3.60) $U_c(\tau_{mn})$ is considered a function of the nine components of stress. Therefore, the partial derivative of $U_s(e_{mn})$ with respect to any component of strain and the partial derivative of $U_c(\tau_{mn})$ with respect to any component of stress, respectively, imply that the other eight components of stress are considered constant during differentiation. It is apparent that for an elastic material the relations between the components of stress and strain may be established if the strain energy density U_s or the complementary energy density U_c is known. If U_s is a second degree polynomial of the components of strain, the relations between the components of stress and strain are linear. If U_s is a polynomial of higher degree than the second of the components of strain or any non-linear function, then the relations between the components of stress and strain are non-linear.

Substituting relation (3.59) and (3.60) into the condition of stability (3.38b), we get

$$\sum_{i=1}^{3}\sum_{j=1}^{3} d\left(\frac{\partial U_s}{\partial e_{ij}}\right) de_{ij} \geq 0 \qquad \sum_{i=1}^{3}\sum_{j=1}^{3} d\tau_{ij} d\left(\frac{\partial U_c}{\partial \tau_{ij}}\right) \geq 0 \tag{3.61}$$

or

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} \frac{\partial^2 U_s}{\partial e_{pq}\partial e_{ij}} de_{pq} de_{ij} \geq 0 \qquad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} \frac{\partial^2 U_c}{\partial \tau_{pq}\partial \tau_{ij}} d\tau_{pq} d\tau_{ij} \geq 0 \tag{3.62}$$

Referring to relations (3.59) and (3.60), at the stress-free, strain-free state of mechanical equilibrium, we have

$$\left.\frac{\partial U_s}{\partial e_{ij}}\right|_{e_{pq}=0} = \tau_{ij} = 0 \qquad \left.\frac{\partial U_c}{\partial \tau_{ij}}\right|_{\tau_{pq}=0} = e_{ij} = 0 \tag{3.63}$$

Relations (3.63) indicate that the functions $U_s(e_{mn})$ and $U_c(\tau_{mn})$ assume an extremum value at the stress-free, strain-free state, while relations (3.62) indicate that the extremum is a minimum. Thus, the functions $U_s(e_{mn})$ and $U_c(\tau_{mn})$ are non-negative and vanish only when the variables (e_{pq} or τ_{pq}) vanish. *Functions satisfying such a condition are referred to as positive definite. Therefore, we may conclude that the condition of stability (3.38b) implies that the strain energy density $U_s(e_{mn})$ is a positive definite function of the components of strain and that the complementary energy density $U_c(\tau_{mn})$ is a positive definite function of the components of stress.*

On the basis of the foregoing presentation, the elastic model of material behavior under isothermal conditions may be defined either as was done in Section 3.2 or as one whose components of stress are obtained from the strain energy density U_s, on the basis of relation (3.59). The strain energy density is a positive definite function of the components of strain, having continuous second derivatives and vanishing at the stress-free, strain-free state at the uniform temperature T_o. The two definitions are almost equivalent. However, the condition of stability imposes greater restrictions on the stress-strain relations of materials subjected to uniaxial tension or compression, than the requirement that the strain energy density is a positive definite function of the components of strain.

From their definition, it is apparent that the strain energy density and the complementary energy density are related. This relation may be obtained using relations (3.60), and (3.57) as follows:

$$dU_c = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial U_c}{\partial \tau_{ij}} d\tau_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}\, d\tau_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}\, d\tau_{ij} + \sum_{i=1}^{3}\sum_{j=1}^{3} de_{ij}\tau_{ij} - \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\, de_{ij}$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} d(e_{ij}\tau_{ij}) - dU_s \tag{3.64}$$

Relation (3.64) may be integrated to yield

$$U_c = \sum_{i=1}^{3}\sum_{j=1}^{3} e_{ij}\tau_{ij} - U_s \tag{3.65}$$

3.11.1 Physical Significance of the Strain Energy Density

Consider a body initially in a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, the body is subjected to external loads and reaches a second state of mechanical but not necessarily thermal equilibrium. Moreover, consider the material element (particle) *ABCDEFGO* of this body of dimensions dx_1, dx_2 and dx_3 whose free-body diagram in the second state of mechanical equilibrium is shown in Fig. 3.15. This particle is subjected to components of stress on its faces and to a distribution of body forces whose resultant acts through its mass center. The components of stress acting on any face of the particle in general vary throughout this face. Thus, referring to Fig. 3.15, $\tau_{11}(x_1, x_2, x_3)$, $\tau_{12}(x_1, x_2, x_3)$ and $\tau_{13}(x_1, x_2, x_3)$ are the average components of stress acting on face *OGFD*, while $\tau_{11}(x_1 + dx_1, x_2, x_3)$, $\tau_{12}(x_1 + dx_1, x_2, x_3)$ and $\tau_{13}(x_1 + dx_1, x_2, x_3)$ are the average components of stress acting on face *ABCE*.
We denote the components of displacement of the mass center of the particle under consideration by $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$. We assume that the average displacement of each face of the element is equal to the displacement of its centroid. Thus, if we denote by $\overset{-1}{\hat{\mathbf{u}}}$ the average component of displacement of face *OGFD*, we have

$$\overset{-1}{\hat{\mathbf{u}}} = \left(\hat{u}_1 - \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_1}dx_1\right)\mathbf{i}_1 + \left(\hat{u}_2 - \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_1}dx_1\right)\mathbf{i}_2 + \left(\hat{u}_3 - \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_1}dx_1\right)\mathbf{i}_3 \tag{3.66}$$

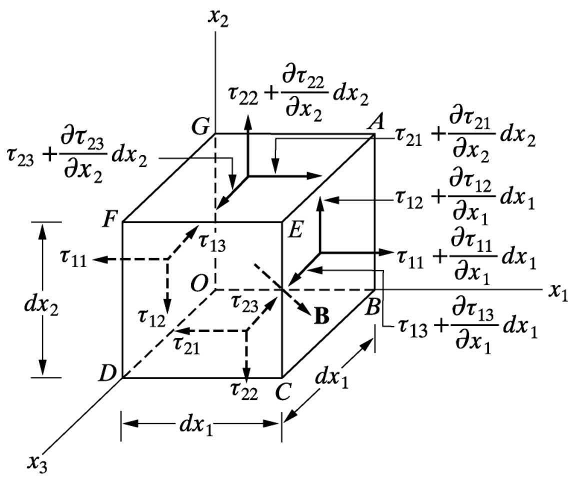


Figure 3.15 Free-body diagram of an element of the body.

Moreover, referring to Fig. 3.15, the average displacement of face *AECB* is equal to

$$\overset{+1}{\hat{\mathbf{u}}} = (\hat{u}_1 + \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_1}dx_1)\mathbf{i}_1 + (\hat{u}_2 + \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_1}dx_1)\mathbf{i}_2 + (\hat{u}_3 + \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_1}dx_1)\mathbf{i}_3 \tag{3.67}$$

Suppose that the surface traction and the distribution of body forces acting on the body change by an infinitesimal amount. The corresponding changes of the average components of displacements of the faces of the infinitesimal element *ABCEDFGO* are denoted as $d\overset{+1}{\hat{\mathbf{u}}}$, $d\overset{-1}{\hat{\mathbf{u}}}$, $d\overset{+2}{\hat{\mathbf{u}}}$, $d\overset{-2}{\hat{\mathbf{u}}}$, $d\overset{+3}{\hat{\mathbf{u}}}$ and $d\overset{-3}{\hat{\mathbf{u}}}$. The increment of work performed by the component of stress and the body force, acting on the particle *ABCDEFGO* of volume dx_1 dx_2 dx_3 as a result of this additional deformation is

$$\begin{aligned} d(dW) &= (\overset{+1}{\mathbf{T}} \cdot d\overset{+1}{\hat{\mathbf{u}}} + \overset{-1}{\mathbf{T}} \cdot d\overset{-1}{\hat{\mathbf{u}}})\, dx_2\, dx_3 + (\overset{+2}{\mathbf{T}} \cdot d\overset{+2}{\hat{\mathbf{u}}} + \overset{-2}{\mathbf{T}} \cdot d\overset{-2}{\hat{\mathbf{u}}})\, dx_1\, dx_3 \\ &\quad + (\overset{+3}{\mathbf{T}} \cdot d\overset{-3}{\hat{\mathbf{u}}} + \overset{-3}{\mathbf{T}} \cdot d\overset{-3}{\hat{\mathbf{u}}})\, dx_1\, dx_2 + \mathbf{B} \cdot d\hat{\mathbf{u}}\, dx_1\, dx_2\, dx_3 \end{aligned} \tag{3.68}$$

where the average traction on face *ODCB* or *OBAG* or *ODFG* is

$$\overset{-i}{\mathbf{T}} = \tau_{i1}\mathbf{i}_1 + \tau_{i2}\mathbf{i}_2 + \tau_{i3}\mathbf{i}_3 \qquad (i = 1, 2, 3) \tag{3.69}$$

while the average traction on face *AEFG* or *DCEF* or *AECB* is

$$\overset{i}{\mathbf{T}} = \left(\tau_{i1} + \frac{\partial \tau_{i1}}{\partial x_1}dx_1\right)\mathbf{i}_1 + \left(\tau_{i2} + \frac{\partial \tau_{i2}}{\partial x_1}dx_1\right)\mathbf{i}_2 + \left(\tau_{i3} + \frac{\partial \tau_{i3}}{\partial x_1}dx_1\right)\mathbf{i}_3 \tag{3.70}$$

$$(i = 1, 2, 3)$$

Substituting relations (3.66), (3.67), (3.69) and (3.70) into (3.68), we obtain

$$\begin{aligned} d(dW) &= (\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1}dx_1)\, d\left(\hat{u}_1 + \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_1}dx_1\right) dx_2\, dx_3 - \tau_{11}\, d\left(\hat{u}_1 - \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_1}dx_1\right) dx_2 dx_3 \\ &\quad + (\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1}dx_1)\, d\left(\hat{u}_2 + \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_1}dx_1\right) dx_2\, dx_3 - \tau_{12}\, d\left(\hat{u}_2 - \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_1}dx_1\right) dx_2 dx_3 \\ &\quad + (\tau_{13} + \frac{\partial \tau_{13}}{\partial x_1}dx_1)\, d\left(\hat{u}_3 + \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_1}dx_1\right) dx_2\, dx_3 - \tau_{13}\, d\left(\hat{u}_3 - \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_1}dx_1\right) dx_2\, dx_3 \\ &\quad + (\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2}dx_2)\, d\left(\hat{u}_1 + \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_2}dx_2\right) dx_1 dx_3 - \tau_{21}\, d\left(\hat{u}_1 - \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_2}dx_2\right) dx_1 dx_3 \\ &\quad + (\tau_{22} + \frac{\partial \tau_{22}}{\partial x_2}dx_2)\, d\left(\hat{u}_2 + \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_2}dx_2\right) dx_1 dx_3 - \tau_{22}(\hat{u}_2 - \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_2}dx_2) dx_1 dx_3 \\ &\quad + (\tau_{23} + \frac{\partial \tau_{23}}{\partial x_2}dx_2)\, d\left(\hat{u}_3 + \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_2}dx_2\right) dx_1 dx_3 - \tau_{23}\, d\left(\hat{u}_3 - \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_2}dx_2\right) dx_1 dx_3 \end{aligned}$$

$$
\begin{aligned}
&+ (\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3)\, d\left(\hat{u}_1 + \frac{\partial \hat{u}_1}{\partial x_3} dx_3 \right) dx_1\, dx_2 - \tau_{31}\, d\left(\hat{u}_1 - \frac{1}{2}\frac{\partial \hat{u}_1}{\partial x_3} dx_3 \right) dx_1\, dx_2 \\
&+ (\tau_{32} + \frac{\partial \tau_{32}}{\partial x_3} dx_3)\, d\left(\hat{u}_2 + \frac{\partial \hat{u}_2}{\partial x_3} dx_3 \right) dx_1\, dx_2 - \tau_{32}\, d\left(\hat{u}_2 - \frac{1}{2}\frac{\partial \hat{u}_2}{\partial x_3} dx_3 \right) dx_1\, dx_2 \\
&+ (\tau_{33} + \frac{\partial \tau_{33}}{\partial x_3} dx_3)\, d\left(\hat{u}_3 + \frac{\partial \hat{u}_3}{\partial x_3} dx_3 \right) dx_1\, dx_2 - \tau_{33}\, d\left(\hat{u}_3 - \frac{1}{2}\frac{\partial \hat{u}_3}{\partial x_3} dx_3 \right) dx_1\, dx_2 \\
&+ B_1\, d\hat{u}_1\, dx_1\, dx_2\, dx_3 + B_2\, d\hat{u}_2\, dx_1\, dx_2\, dx_3 + B_3\, d\hat{u}_3\, dx_1\, dx_2\, dx_3 = 0
\end{aligned} \tag{3.71}
$$

Referring to relations (2.16), we have

$$
de_{ij} = \frac{1}{2}\left[d\left(\frac{\partial \hat{u}_i}{\partial x_j} \right) + d\left(\frac{\partial \hat{u}_j}{\partial x_i} \right) \right] \tag{3.72}
$$

Using relation (3.72) in relation (3.71) and disregarding higher order terms, the following expression is obtained for the increment of work performed by the surface tractions and body forces acting on the material element *ABCDEFGO* due to the change by an infinitesimal amount of the surface tractions and body forces acting on the body under consideration:

$$
\begin{aligned}
d(dW) = {} & [\tau_{11} de_{11} + \tau_{22} de_{22} + \tau_{33} de_{33} + 2\tau_{12} de_{12} + 2\tau_{13} de_{13} + 2\tau_{23} de_{23}]\, dx_1 dx_2 dx_3 \\
& + \left[[\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + B_1] d\hat{u}_1 \right] dx_1 dx_2 dx_3 \\
& + \left[[\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + B_2] d\hat{u}_2 \right] dx_1 dx_2 dx_3 \\
& + \left[[\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} + B_3] d\hat{u}_3 \right] dx_1 dx_2 dx_3
\end{aligned} \tag{3.73}
$$

Referring to the equations of equilibrium (2.69), we see that the coefficients of du_1 du_2 du_3 in relation (3.73) vanish. Therefore, the increment of the work $d(dW)$ is given as

$$
\begin{aligned}
d(dW) &= [\ \tau_{11} de_{11} + \tau_{22} de_{22} + \tau_{33} de_{33} + 2\tau_{12} de_{12} + 2\tau_{13} de_{13} + 2\tau_{23} de_{23}\ ]\ dV \\
&= [\ \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij}\ de_{ij}\ ]\ dV
\end{aligned} \tag{3.74}
$$

This expression is valid for bodies made from any material subjected to a general process of deformation. The total work of the surface traction and body forces acting on the infinitesimal element under consideration during the processes of deformation from its stress-free, strain-free reference state to its deformed state of equilibrium characterized by the components of strain e_{mn} (x_1, x_2, x_3) is equal to

$$dW = \int_0^{e_{mn}} \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij}(e_{pq}')\, de_{ij}'\; dV \tag{3.75}$$

The sum of the work performed by the forces acting on all the elements of the body as a result of its deformation from the stress-free, strain-free state to a general deformed state characterized by the components of strain e_{mm}, is equal to

$$W = \iiint_V \left(\int_0^{e_{mn}} \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij}\left(e'_{pq}\right) de'_{ij} \right) dV \tag{3.76}$$

Since the common boundary of two adjacent elements is subjected to equal and opposite components of stress and is displaced by the same amount, the sum of the work performed by these equal and opposite components of stress vanishes. Consequently, the total work performed by the forces acting on all the elements of the body consists only of the work of the components of stress acting on the faces of the elements which are part of the surface of the body and the work of the body forces. Thus, W in relation (3.76) is equal to the work of the known external forces (body forces and surface tractions) acting on the body and of the unknown reactions at its supports. If the supports of the body do not move, the work of its reactions must vanish. On the basis of the foregoing discussion, relation (3.76) becomes

$$W_{ext\ forces} + W_{reactions} = \iiint_V \left(\int_0^{e_{mn}} \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij}\left(e'_{pq}\right) de'_{ij} \right) dV \tag{3.77}$$

Referring to relation (3.77), for an elastic body subjected to body forces and surface tractions in an environment of constant temperature, the above relation can be rewritten as

$$W_{ext\ forces} + W_{reactions} = \iiint_V U_s\, dV = (U_s)_T \tag{3.78}$$

where $(U_s)_T$ represents the total strain energy of the body. Thus, we may conclude that *the total strain energy of an elastic body subjected to surface tractions and body forces in an environment of constant temperature is equal to the work performed by these forces in bringing it from its stress-free, strain-free reference state to its deformed state.*

3.11.2 Strain Energy Density for Linearly Elastic Materials

Recall that the strain energy density of an isotropic linearly elastic body is equal to the complementary energy density. Substituting the stress–strain relations (3.47) in relation (3.55) and integrating, we obtain that for a body made from an isotropic linearly elastic material, we have

$$U_s = U_c = \frac{E\nu}{2(1+\nu)(1-2\nu)}(e_{11} + e_{22} + e_{33})^2 + G(e_{11}^2 + e_{22}^2 + e_{33}^2) + 2G(e_{12}^2 + e_{13}^2 + e_{23}^2) \tag{3.79}$$

Using the stress–strain relations (3.48), we may write the strain energy density and the complementary energy density (3.79) for a body made from an isotropic, linearly elastic material only in terms of the components of stress. That is,

$$U_s = U_c = \frac{1}{2E}(\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2) - \frac{\nu}{E}(\tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{11}\tau_{33}) + \frac{1}{2G}(\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2) \tag{3.80}$$

Using relations (3.48) relation (3.79) may be rewritten as

$$U_s = U_c = \frac{1}{2}(\tau_{11}e_{11} + \tau_{22}e_{22} + \tau_{33}e_{33} + 2\tau_{12}e_{12} + 2\tau_{13}e_{13} + 2\tau_{23}e_{23}) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}e_{ij} \tag{3.81}$$

Differentiating relation (3.81) with respect to τ_{ij}, we get

$$e_{ij} = \frac{\partial U_s}{\partial \tau_{ij}} \qquad i, j = 1, 2, 3 \tag{3.82}$$

These relations are referred to as *Castigliano's formulae.* They are valid only for bodies made from linearly elastic materials, whereas relation (3.59) and (3.60) are valid for bodies made from any elastic material. Relations (3.59) are based on the property of elastic materials that the work performed by the external forces acting on a body made from an elastic material is independent of the path of loading. In relations (3.82) the strain energy density U_s is considered a function of the nine components of stress.

In Section 3.11 we have shown that the strain energy density of stable materials is a "positive definite" function of the components of strain. Referring to equation (3.79), we see that the "positive definite" of the strain energy density function imposes the following restrictions on the magnitudes of the elastic constants:

$$G > 0 \qquad E > 0 \qquad -1 < \nu < \frac{1}{2} \tag{3.83}$$

3.12 Thermodynamic Considerations of Deformation Processes Involving Bodies Made from Elastic Materials

Consider a deformable body in an initial stress-free, strain-free state, at the uniform temperature T_o. We assume that all the particles of the body are at rest. Consequently, all the quantities that specify the state of the body (the state variables) do not change with time. The body, therefore, is in *thermodynamic equilibrium*. Imagine that we bring this body to a second state of thermodynamic equilibrium through a process of deformation. This state is characterized by the uniform temperature T, the components of strain e_{ij} (I, j = 1, 2, 3) and the components of stress τ_{ij} The number of admissible independent state variables describing a state of thermodynamic equilibrium may be established from experimental observation. *In the case of an elastic body we may choose as the independent state variables the temperature and the six components of strain.* From experimental observations, it has been established that some relations, known as *equations of state*, exist among the state variables. *The relations among the components of stress and strain are equations of state*.

During the process of deformation, the body is acted upon by surface tractions

$\hat{\mathbf{T}}^s(\mathbf{r}^s,t)$ and body forces $\mathbf{B}(\mathbf{r},t)$, where $\mathbf{r}^s$ and $\mathbf{r}$ denote the position vectors in the initial state of a particle on the surface and in the interior of the body, respectively. If the displacement vector of a particle at time t is denoted by $\hat{\mathbf{u}}(\mathbf{r}, t)$, the total work W of the external forces acting on a body due to its deformation is equal to

$$W = \int_0^{\hat{\mathbf{u}}(\mathbf{r})} dW = \int_0^{\hat{\mathbf{u}}(\mathbf{r})} \iiint_V \mathbf{B}(\mathbf{r}, t) \cdot d\hat{\mathbf{u}}(\mathbf{r}, t)\, dV + \int_0^{\hat{\mathbf{u}}(\mathbf{r})} \iint_S \mathbf{T}^s(\mathbf{r}^s, t) \cdot d\hat{\mathbf{u}}(\mathbf{r}^s, t)\, dS \tag{3.84}$$

where $\hat{\mathbf{u}}(\mathbf{r})$ is the displacement vector of a particle which in the undeformed state was located at the point whose position vector is $\mathbf{r}$.

Referring to relation (3.76) the total work of the external forces acting on a body due to its deformation may be expressed as

$$W = \iiint_V U_s^*\, dV \tag{3.85}$$

where $U^*_s(e_{pq})$ is equal to

$$U_s^* = \int_0^{e_{mn}} \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij}\, de_{ij} \tag{3.86}$$

We assume that the process of deformation is *adiabatic.* That is, throughout the process, the body is entirely separated from its surroundings by an adiabatic wall which preserves the thermodynamic equilibrium of the body when it is brought into contact with another body through this wall. In an adiabatic deformation process the temperature is a function of the components of strain. Thus, as in the case of an isothermal deformation process, in an adiabatic deformation process of an elastic body the components of strain may be taken as the independent variables. It has been verified experimentally that the work performed during adiabatic deformation of an elastic body is dependent solely upon the initial and final values of the displacement vector $\hat{\mathbf{u}}(\mathbf{r}, t)$ [or equivalently of the components of strain $e_{ij}(\mathbf{r}, t)$], and not on the particular path of loading. Moreover, it has been established experimentally that there exists an adiabatic deformation process which connects any two equilibrium states of a body. Consequently, for an adiabatic deformation process, U_s^* depends only on the final values of the components of strain, and thus $\sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}(e'_{pq})de'_{ij}$ must be a perfect differential. That is,

$$dU_s^* = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial U_s^*}{\partial e_{ij}} de_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\, de_{ij} \tag{3.87}$$

Consequently,

$$\tau_{ij} = \frac{\partial U_s^*(e_{pq})}{\partial e_{ij}} \qquad i, j = 1, 2, 3 \tag{3.88}$$

U_s^* is the strain energy density for an adiabatic deformation process given by relation (3.86). The coefficients of the monomials of the components of strain in the expression for the strain energy density for an adiabatic deformation process differ slightly from those in the expression for the strain energy density for an isothermal deformation process. That is, the elastic constants for an elastic body undergoing adiabatic deformation are slightly different than the corresponding constants for the same body undergoing isothermal deformation.

The change of the internal energy ΔE of the body is defined as the work performed by the external forces acting on the body in bringing it from one state of equilibrium to another, adiabatically. That is,

$$\Delta E^{a} = W^{a} \tag{3.89}$$

From relation (3.85) we see that for an adiabatic deformation process the total strain energy is equal to the work of the external forces acting on a body in bringing it from its undeformed to its deformed state. Consequently, for an adiabatic deformation process, the total strain energy density of a body is equal to the change of its internal energy.

For a non-adiabatic process, the difference between the change in internal energy and the work performed by the external forces acting on the body in bringing it from one state of equilibrium to another, is defined as the total heat Q, absorbed by the body from its surroundings, during this process. That is,

$$Q = \Delta E - W \tag{3.90}$$

This definition of total heat is the first law of thermodynamics, which is also referred to as the *principle of conservation of energy*. It can be shown that relation (3.90) couples the stress field with the temperature field of a body. That is, the deformation of a body affects the distribution of its temperature. *However, we assume that the effect of the deformation of the body on the distribution of its temperature is very small and can be disregarded*. On the basis of this assumption relation (3.90) involves only the temperature field and it is known as the *heat conduction equation* which is used to compute the distribution of temperature in a body. *In this book we assume that the temperature distribution has been computed and it is given*.

Consider an elastic body subjected to a distribution of surface tractions and body forces. If the body is isolated by an adiabatic wall, the work of the surface tractions and body forces is utilized in increasing the internal energy of the body [see relation (3.89)]. The temperature of the body will increase or decrease, depending upon whether the body has been subjected to compression or tension, respectively. Inasmuch as the process of deformation for a body made from an elastic material is reversible the body will revert to its undeformed shape, volume and temperature, upon unloading.

Clearly, for a body made from an elastic material enclosed in an adiabatic wall or loaded fast enough so that heat does not have time to dissipate into the environment, yet slow enough so that the kinetic energy of its particles is negligible, the relations between the components of stress and strain may be established if the strain energy density U_s^* is known. If U_s^* is a homogeneous quadratic form of the components of strain, as that given in relation (3.79), then the relations between the components of stress and strain will be linear. If U_s^* is of a higher degree than a quadratic form of the components of strain, then the relations between the components of stress and strain will be non-linear.

Let us now consider a process of isothermal deformation of an elastic body. During this process, reversible heat energy is dissipated into or absorbed from the environment, which during unloading is absorbed from or dissipated into the environment. For instance, when a bar made from an elastic material is subjected to axial tensile forces isothermally, it absorbs from the environment a considerable amount of reversible heat energy which it dissipates into the environment when the load is released. Thus, during the process of isothermal deformation of an elastic body, the work of the external forces, which we have shown to be equal to the total strain energy of the body, is not equal to the energy stored into the body.

Referring to relation (3.59), (3.55), (3.86) and (3.88), it is apparent that for isothermal and adiabatic deformation the temperature does not appear explicitly in the stress–strain relations. However, the elastic constants such as the modulus of elasticity or the Poisson ratio, established by subjecting to uniaxial tension a completely thermally isolated bar, differ slightly from those established by subjecting the same bar to uniaxial tension under isothermal conditions.

In other deformation processes, the temperature must appear explicitly in the stress–strain relations. The components of strain e_{ij} $(i, j = 1, 2, 3)$ of a particle of a body subjected to such deformation processes may be considered as the sum of two parts: a part $e_{ij}^T (i, j = 1, 2, 3)$ due to the change of temperature known as the thermal components of strain and a part $e_{ij}^s (i, j = 1, 2, 3)$ due to the components of stress acting on the particle. That is,

$$e_{ij} = e_{ij}^s + e_{ij}^T \delta_{ij} \tag{3.91}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \tag{3.92}$$

If the body is made from an isotropic, linearly elastic material, referring to relation (3.34a), the thermal components of strain e_{ij}^T $(i, j = 1, 2, 3)$ of its particles are equal to

$$e_{11}^T = e_{22}^T = e_{33}^T = \alpha \Delta T \qquad e_{12} = e_{13} = e_{23} = 0 \tag{3.93}$$

where α is the coefficient of linear thermal expansion for an isotropic, linearly elastic material (see Section 3.5.2). This coefficient is a material property which may be taken as constant for moderate changes of temperature. The coefficients of linear thermal expansion, for certain engineering materials at room temperature, are given in Appendix A. In general, at a given temperature, the coefficient of linear thermal expansion of a material changes with the magnitude and the character of stress. This change, however, is very small for states of stress within the elastic limit. The components of strain e_{ij}^s $(i, j = 1, 2, 3)$ are related to the components of stress by the stress–strain relations (3.48). Substituting relations (3.48) and (3.93) into (3.91), we obtain the following stress–strain relations for the particles of a body made from an isotropic, linearly elastic material

$$\begin{aligned} e_{11} &= \frac{1}{E}[\tau_{11} - \nu(\tau_{22} + \tau_{33})] + \alpha(T - T_0) \\ e_{22} &= \frac{1}{E}[\tau_{22} - \nu(\tau_{11} + \tau_{33})] + \alpha(T - T_0) \\ e_{33} &= \frac{1}{E}[\tau_{33} - \nu(\tau_{11} + \tau_{22})] + \alpha(T - T_0) \\ e_{12} &= \frac{\tau_{12}}{2G} \qquad e_{13} = \frac{\tau_{13}}{2G} \qquad e_{23} = \frac{\tau_{23}}{2G} \end{aligned} \tag{3.94}$$

The constants E, ν, G are established from uniaxial tension and torsion tests in an environment of constant temperature. Relations (3.94) may be solved for the components of stress to give

$$
\begin{aligned}
\tau_{11} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{11} + \nu(e_{22} + e_{33})] - \beta(T - T_0) \\
\tau_{22} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{22} + \nu(e_{11} + e_{33})] - \beta(T - T_0) \\
\tau_{33} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{33} + \nu(e_{22} + e_{11})] - \beta(T - T_0) \\
\tau_{12} &= 2Ge_{12} \\
\tau_{13} &= 2Ge_{13} \\
\tau_{23} &= 2Ge_{23}
\end{aligned}
\tag{3.95}
$$

where

$$\beta = \frac{E\alpha}{1 - 2\nu} \tag{3.96}$$

Notice that in relation (3.47) the components of strain are only due to the components of stress. Thus we can replace e_{ij} by $e^s_{ij} = e_{ij} - e^T_{ij}$ and get relation (3.95).

Consider a body made from an elastic material subjected to a general deformation process. The strain energy density of a particle of this body is defined as a function of the components of strain, which on the basis of relation (3.88), yields the stress–strain relations for that material from which the body is made. On the basis of this definition for a body made from an isotropic, linearly elastic material, we have

$$
\begin{aligned}
\hat{U}_s = {} & \frac{1}{2}\frac{E\nu}{(1+\nu)(1-2\nu)}(e_{11} + e_{22} + e_{33})^2 + G(e_{11}^2 + e_{22}^2 + e_{33}^2) \\
& + 2G(e_{12}^2 + e_{23}^2 + e_{13}^2) - \frac{E}{1-2\nu}\alpha(T - T_0)[e_{11} + e_{22} + e_{33}]
\end{aligned}
\tag{3.97}
$$

In a general non-isothermal and non-adiabatic deformation process the components of strain and the temperature are the independent variables. It can be shown, that the total strain energy obtained from the strain energy density defined above for a body made from an elastic material undergoing a general non-isothermal and non-adiabatic deformation process is not equal to the work of the external forces.

3.13 Linear Response of Bodies Made from Linearly Elastic Materials

Consider a body made from a linearly elastic material, subjected to loads of such magnitude that the deformation of its particles is within the range of validity of the assumption of small deformation. Thus, the relations between the measure of the deformation (the components of strain) of a particle of the body and its components of displacement are linear [see relations (2.16)]. Moreover, the change of the dimensions of any portion of the body due to its deformation is considered negligible compared to its original dimensions. Thus, the loads acting on the body and the components of stress acting on its particles may be considered independent of the change of its dimensions due to its deformation. This indicates that the components of stress acting on a particle of the body are related to the magnitude of the loads acting on it by linear relations. That is, when the loads acting on the body are doubled, the magnitude of the components of stress acting on its particles is also doubled.

On the basis of the foregoing discussion and taking into account that the relations between the components of stress and strain of the particles of the body are linear, we may conclude that the relations between the external loads acting on the body and the components of stress, the components of strain or the components of displacement are linear. That is, the effects are linearly related to the causes. In this case we say that the *response of the body is linear.*

A direct consequence of the linear response of a body is that the principle of superposition is valid for this body. That is, its response due to a number of simultaneously applied loads is equal to the sum of its responses due to the application of each of these loads separately.

3.14 Time-Dependent Stress–Strain Relations

In the Section 3.6, we present idealized models of material behavior. The constitutive relations of these models are time independent. When a thermally isolated body made from an elastic material is subjected to external forces, all the work performed by these forces is stored in the body as elastic energy. Upon unloading, this energy is used to restore the body to its undeformed configuration. Consider a thermally isolated body made of an elastoplastic material in an unstressed unstrained state of thermodynamic equilibrium at the uniform temperature T_0. When the body is subjected to external forces inducing plastic components of strain, only part of the work performed by these forces is stored in the body as elastic energy. The remaining is converted to heat and raises the temperature of the body. Moreover, upon removal of the external forces, the body does not revert to its undeformed configuration and its temperature remains higher than that at its undeformed stress-free, strain-free state. The total amount of energy used to raise the temperature of a particle depends not only upon the final state of stress at this particle, but also upon the way this particle was stressed (that is, upon the path of loading in the six-dimensional stress space). However, the total amount of energy used to raise the temperature of a particle is not dependent upon the time history of stress acting on this particle. For example, the total amount of energy used to raise the temperature of a particle depends on the number of cycles of loading and unloading and on the magnitude of the components of stress acting on the particle during each cycle. It does not depend on the rate of loading and unloading or on the time it remained under load.

A model of time-dependent behavior employed in describing the response to stress of many fluids is the *viscous fluid.* When this model is subjected to external forces, inducing a state of stress with zero hydrostatic stress, all the work performed is dissipated into the environment as heat. Moreover, when the applied forces reach a constant value, this model continues to deform at a constant rate. If the rate of straining of the model is linearly related to the magnitude of the applied stress, the model is called a *Newtonian fluid.*

The response of real materials is time dependent. However, for some materials, under certain environmental conditions, as, for instance, relatively low temperature, the amount of energy dissipated due to viscous (time-dependent) deformation is very small compared to the energy stored in the material or to the energy dissipated due to plastic deformation and can be disregarded. Consequently, the response of theses materials may be represented by that of the elastic or elastoplastic models provided that they are not subjected to many cycles of loading and unloading. The fact that the amplitude of vibrations of a bar made from a material such as steel, which under static loading is considered elastic, decreases with time (it is damped), even when the bar vibrates in a vacuum, is a clear indication that the behavior of real materials is not entirely time independent.

For low values of the applied stress, at relatively low temperature, the response of many metals is only slightly dependent on the rate and the path of loading and it can be described to a considerable degree of accuracy by that of the elastic model. For higher values of the applied stress, even at relatively low temperatures, the response of these metals is highly dependent on the path of loading (plastic deformation).

Under certain environmental conditions, the response to external forces of many materials of technological interest (i.e., metals at relatively high temperatures or polymers and ceramics at room temperature) cannot be represented adequately by the response of the elastic or the elastoplastic models. Moreover, at relatively low temperatures the response of many fluids cannot be adequately represented by that of the viscous fluid model. Under the forementioned environmental conditions, the response of these materials can be represented adequately only by the response of models which are a combination of the elastic and the viscous models or of the elastic, the plastic and the viscous models. These models of material behavior are referred to as *viscoelastic* or *elastoviscoplastic,* respectively.

When a body made from a viscoelastic or an elastoviscoplastic material is subjected to external forces, only part of the work performed by these forces is stored in the body; the remaining is dissipated into the environment as irreversible heat.

3.15 The Creep and the Relaxation Tests

Some characteristics of the behavior of bodies made of materials having time-dependent constitutive relations are revealed by the creep and the relaxation tests. In this section we describe these two tests.

In the creep test, a specimen is subjected to a state of constant uniaxial stress in an environment of constant temperature and the corresponding component of strain is measured at a number of time intervals. For metals, a prismatic specimen of circular cross section is used subjected to a constant tensile force in an environment of constant temperature. The specimen is in a state of uniaxial tension. In Fig. 3.16 we plot the normal component of strain e_{11} of the specimen as a function of time. As shown in Fig. 3.16, the specimen exhibits an instantaneous (time-independent) deformation (denoted as *OA*) and then continues to deform. This time-dependent deformation of the specimen at constant load is called *creep*. Part of the instantaneous initial deformation of the specimen is reversible (elastic) and the rest is irreversible (plastic). The time rate of change of strain is high immediately after the application of the load. This time-dependent deformation is called *transient creep*. However, the time rate of strain

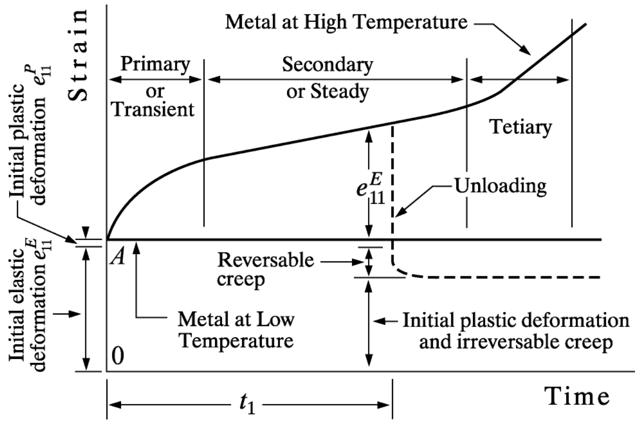


Figure 3.16 Creep curve of metal.

decreases with time until a constant rate of strain is achieved. This time-dependent deformation is called *steady-state creep* and is analogous to viscous flow. If the magnitude of the applied constant force is sufficiently high, the velocity of straining will ultimately start to increase and fracture will occur. This time-dependent deformation is referred to as *tertiary creep*. Upon unloading, at time $t = t_1$, the initial elastic deformation is recovered instantaneously, while a portion of the time-dependent deformation is recovered subsequently, at a progressively decreasing rate, until the residual strain decreases to a permanent value representing the sum of the instantaneous initial plastic deformation and the irrecoverable time-dependent deformation. In metals, creep deformation is predominantly irrecoverable deformation. That is, upon unloading, the specimen does not revert to its original size and shape.

In some non-metallic materials, such as plastics and rubber, creep may involve predominantly recoverable elastic deformation. For instance, when a piece of natural rubber (vulcanized without filters) is subjected to constant shearing stress (see Fig. 3.17), it exhibits some instantaneous deformation and subsequently depending on the temperature, it may continue to deform with time (see Fig. 3.17). In this case, if the load is suddenly released, the rubber immediately recovers only a portion of its deformation. A residual deformation remains, reducing with time, to a limit of small permanent deformation. The magnitude of this residual permanent deformation is dependent upon the temperature, the magnitude of the applied load and the duration of its application.

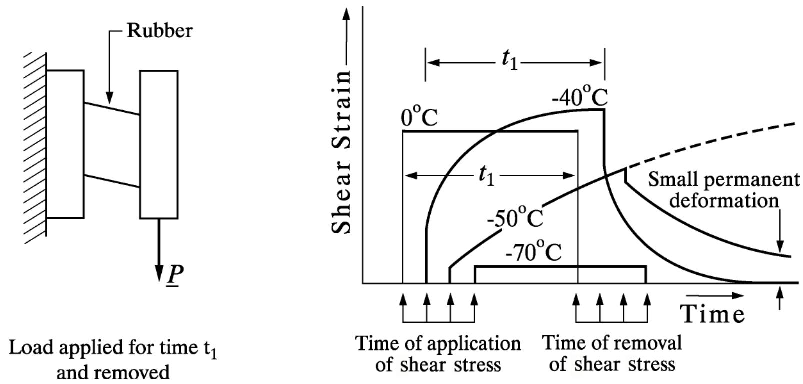


Figure 3.17 Retarded elasticity of vulcanized rubber.

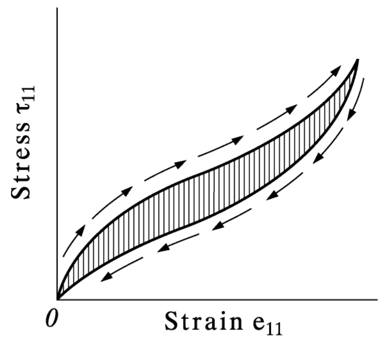


Figure 3.18 Stress–strain diagram for a material exhibiting retarded elasticity.

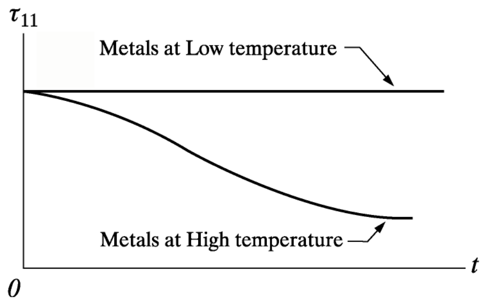


Figure 3.19 Stress relaxation under constant strain.

This phenomenon of time-dependent elastic recovery is referred to in literature as *retarded elasticity* and is more pronounced when the rubber is subjected to higher strains or to longer periods of stretching. In general, this retarded elastic effect and the residual permanent deformation is small at room temperature, and increases at lower temperatures (see Fig. 3.17). Some of the work performed by the applied forces during both recoverable and irrecoverable time-dependent deformation is converted into irreversible heat and dissipated into the surroundings. Materials exhibiting only recoverable, time-dependent deformation, like those exhibiting elastic time-independent deformation, "remember" their unstressed–unstrained reference state. However, their stress–strain curve during loading differs from that at unloading — the area between the two curves represents the energy lost (see cross-hatched area in Fig. 3.18).

In the relaxation test, the stress required to maintain constant the corresponding component of strain of a specimen subjected to a state of uniaxial stress in an environment of constant temperature is measured at a number of time intervals. For metals, a prismatic specimen of circular cross section is used subjected to the axial centroidal force required to maintain its elongation constant in an environment of constant temperature. The applied force is measured at a number of time intervals. For a metal specimen at a relatively high temperature or a specimen made from some non-metallic material at room temperature, this force decreases continuously with time (see Fig. 3.19). This phenomenon is referred to as *relaxation*.

3.16 Problems

1. A circular cylindrical specimen of 20-mm diameter has been subjected to uniaxial tension of 50 kN. The change of length of a before-deformation 80-mm length was measured by an extensiometer as 0.08 mm, while the change of its diameter was measured as 0.006 mm. The specimen is made from a homogeneous, isotropic, linearly elastic material. Compute the true stress at a point of the specimen, the modulus of elasticity and the Poisson ratio of the material from which the specimen is made.

Ans. $E = 159.15$ GPa, $\nu = 0.3$

2. Consider a structure made from an isotropic, linearly elastic material whose modulus of elasticity and shear modulus are $E = 210$ GPa and $G = 75$ GPa, respectively. If the components of strain of a particle of this structure are given as

$$[e] = \begin{bmatrix} 0.0002 & 0 & -0.0001 \\ 0 & 0 & -0.0003 \\ -0.0001 & -0.0003 & 0.0005 \end{bmatrix}$$

compute the components of stress acting on this particle.

Ans. $[\tau] = \begin{bmatrix} 240 & 0 & -15 \\ 0 & 210 & -45 \\ -15 & -45 & 285 \end{bmatrix}$ MPa

3. The Poisson ratio of an isotropic, linearly elastic material is 1/3 and its modulus of elasticity is 210 GPa. Determine the components of strain of a particle of a body made from this material, when subjected to the following state of stress:

$$\begin{bmatrix} 70 & 140 & 0 \\ 140 & 40 & 280 \\ 0 & 280 & 140 \end{bmatrix} \text{MPa}$$

Ans. $[e] = \begin{bmatrix} 0.476 & 8.889 & 0 \\ 8.889 & -1.429 & 17.778 \\ 0 & 17.778 & 4.921 \end{bmatrix} \text{x } 10^{-4}$

4. Strain gages were placed at various points on the surface of a plate made from a homogeneous, isotropic, linearly elastic material with $\nu = 1/3$, $E = 210$ GPa. The x_1, x_2 axes are in the plane of the plate. The gages were placed in such a way as to measure at each point the principal strains e_1 and e_2. The values of these strains are given below:

	Point A	**Point B**	**Point C**	**Point D**
e_1	0.0004	0.0006	−0.0002	−0.0009
e_2	−0.0002	0.0003	−0.0006	0.0002

Compute the principal stresses at these points.

(*Hint*: On any point of the surface the stresses τ_{33}, τ_{31} and τ_{32} vanish.)

Ans. Point A $\tau_1 = 78.75$ MPa, $\tau_2 = -15.75$ MPa Point C $\tau_1 = -94.5$ MPa, $\tau_2 = -157.5$ MPa
Point B $\tau_1 = 70.88$ MPa, $\tau_2 = 118.13$ MPa Point D $\tau_1 = -196.88$ MPa, $\tau_2 = -23.63$ MPa

5. A bar of circular cross sections and undeformed diameter of 8 mm is subjected to the following state of uniaxial stress:

$$\tau_{11} = 140 \text{ MPa} \quad \tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$$

Compute the change of the diameter of the bar due to its deformation. The bar is made from a homogeneous, isotropic, linearly elastic material with $E = 210$ GPa and $\nu = 1/3$.

Ans. $\Delta d = -0.00177$ mm

6. An incompressible material is defined as one that cannot experience any volume change. If such a material is linearly elastic, what is its Poisson ratio? Ans. $\nu = 0.5$

7. The state of stress at a point of a steel member (E = 210 GPa, ν = 1/3) is given by

$$\tau_{11} = 120 \text{ MPa} \quad \tau_{22} = -40 \text{ MPa} \quad \tau_{12} = -60 \text{ MPa} \quad \tau_{23} = \tau_{13} = \tau_{33} = 0$$

Determine the principal normal components of strain and their directions.

Ans. $e_1 = 0.7619(10^{-3})$ $e_2 = -0.5079\,(10^{-3})$ $e_3 = -0.1270\,(10^{-3})$ $\tilde{\phi}_{11} = 18.435^o$ clockwise

8. The components of displacement in a body made from a homogeneous, isotropic, linearly elastic material have been found to have the following form:

$$u_1 = Ax_1^2 + Bx_2^2, \qquad u_2 = Cx_1x_2 \qquad u_3 = 0$$

Disregarding the effect of body forces, determine the relation between the parameters A, B and C, so that the components of stress obtained from the given components of displacement satisfy the equilibrium equations.

. Ans. $4(1 - \nu)A + 2(1 - 2\nu)B + C = 0$

9. Wood is an orthotropic material. Douglas fir has the following strain–stress relations relative to orthtropic axes x_1, x_2, x_3:

$$\begin{aligned}
(10^6)e_{11} &= 87.0\tau_{11} - 34.8\tau_{22} - 43.5\tau_{33} \\
(10^6)e_{22} &= -34.8\tau_{11} + 1{,}305.0\tau_{22} - 609.0\tau_{33} \\
(10^6)e_{33} &= -43.5\tau_{11} - 609.0\tau_{22} + 1{,}740.0\tau_{33} \\
(10^6)e_{12} &= 696.0\tau_{12}\ , \quad (10^6)e_{13} = 290.0\tau_{13} \\
(10^6)e_{23} &= 3{,}045.0\tau_{23}
\end{aligned}$$

where the components of stress are given in MPa. The x_1 axis is parallel to the grain, the x_2 axis is in the radial direction of the tree and the x_3 axis is tangent to the growth rings of the tree. At a particle of a Douglas fir log subjected to external forces, the components of stress are τ_{11} = 8 MPa, τ_{22} = 2.4 MPa, $\tau_{33} = -2.8$ MPa, τ_{12} = 1.0 MPa and $\tau_{13} = \tau_{23}$ = 0.

(a) Determine the orientation of the principal axes of stress at that particle.

Ans. $\tilde{\phi}_{11} = 9.825°$ counterclockwise

(b) Determine the components of strain of that particle with respect to the axes x_1, $_2$, x_3x

Ans. $e_{11} = 734.28\,(10^{-6})$ $e_{22} = 4558.8\,(10^{-6})$ $e_{33} = -6681.6\,(10^{-6})$ $e_{12} = 696.0\,(10^{-6})$

(c) Determine the orientation of the principal axes of strain at that particle.

Ans. $\tilde{\phi}_{11} = 80.0°$ counterclockwise

10. The bar of Fig. 3P10 has a square cross section (160 mm x 160 mm) and is 400 mm long. The bar is made from an isotropic, linearly elastic material (E = 70 GPa, ν = 0.33). The components of stress τ_{11} = 228 MPa and τ_{22} are uniformly distributed as shown in

Fig. 3P10.

(a) Determine the magnitude of τ_{22}, so that the dimension b = 160 mm does not change under the load. Ans. τ_{22} = 154.4 MPa

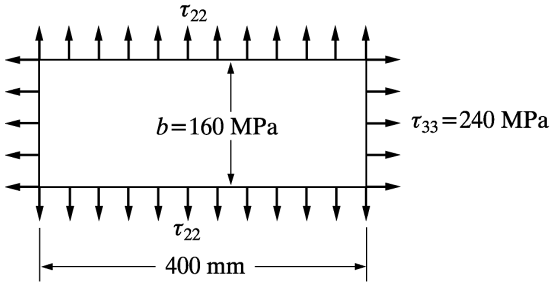
(b) Determine the amount by which the length L = 400 mm of the bar changes due to its deformation. Ans. ΔL = 0.65 mm

(c) Determine the change in the cross-sectional area of the bar.

Ans. ΔA = 35.84 mm^2

11. A prismatic bar of circular cross section of 20-mm diameter is subjected to uniaxial tension τ_{11} = 250 MPa. The bar is made from an isotropic, linearly elastic, strain hardening material, whose stress–strain curve is shown in Fig. 3P11 (E_1 = 200 GPa, E_2 = 70 GPa, ν = 1/3). Compute the change of diameter of the bar due to its deformation. Compute the change of the diameter of the bar when the load is removed.

[*Hint*: $e_{11}^P = -2e_{22}^P$.] Ans. $\Delta d = -0.01297$ mm $(\Delta d)_{\text{residual}} = -0.00464$ mm



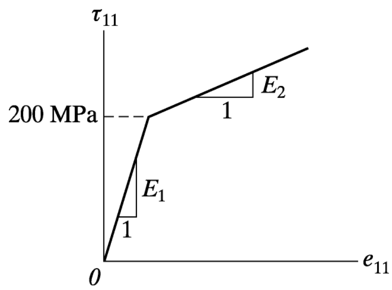


Figure 3P10 **Figure 3P11**

12. A circular bar of undeformed diameter 20 mm is subjected to a state of uniaxial stress τ_{11} = 120 MPa, $\tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$. Compute the change of its diameter as a result of the deformation. E = 200 GPa, ν = 1/3. Ans. Δd = − 0.004 mm

13. The coefficient of thermal expansion, the modulus of elasticity and the Poisson ratio of an isotropic, linearly elastic material are $\alpha = 10^{-5}/°C$, E = 200 GPa, ν = 0.3. Compute the stress field of a body made from this material when its temperature increases by 20°C while every point of its surface is completely restrained from moving. (*Hint* : Assume $u_1 = u_2 = u_3 = 0$.)

$$\text{Ans.} \quad [\tau] = -100\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\text{MPa}$$

14. Express the strain energy density of an orthotropic linearly elastic medium in an environment of constant temperature, as a function of the components of strain referred to the set of axes which is normal to the planes of symmetry of the material.

15. Consider a cylinder whose wall thickness is small compared to its mean radius. The cylinder is formed from an inner aluminum cylinder of outer radius R and thickness t_A and from an outer steel cylinder of inner radius R and thickness t_S bound rigidly together. The dimensions t_A and t_S are very small compared to R. The supports of the cylinder can restrain its ends only from moving in the direction of its axis. That is, they can exert only axial component of traction. An inner pressure p is applied to the cylinder and moreover its temperature increases by ΔT^{o}C. Since the wall of the cylinder is free to move in the radial direction it can be shown that

$$\tau_{1r} = \tau_{1\theta} = 0$$

Moreover, it can be shown that the component of stress $\tau_{r\theta}$ vanishes. Furthermore, since the thickness of the cylinder is very small compared to its radius the radial component of stress acting on its particles is negligible compared to the tangential component of stress acting on them. Thus, there are only two components of stress acting on the particles of this cylinder, namely, τ_{11} and $\tau_{\theta\theta}$. We denote the tangential components of stress acting on the particles of the steel cylinder by $\tau^{S}_{\theta\theta}$ and those acting on the particles of the aluminum cylinder by $\tau^{A}_{\theta\theta}$. Since the aluminum and steel cylinders are very thin, these components of stress $\tau^{S}_{\theta\theta}$ and $\tau^{A}_{\theta\theta}$ vary negligibly in the radial direction. Furthermore, since the loads and the geometry of the cylinder do not vary in the axial direction, the components of stress τ_{11} and $\tau_{\theta\theta}$ must be constant.

Compute the stress distribution acting on the particles of the cylinder. Evaluate the results for $t_S = t_A = t = 8$ mm, $\nu_S = \nu_A = 1/3$, $E_S = 200$GPa, $E_A = 70$GPa, $\alpha = 2(10^{-5})/^{o}$C, $R = 70$ mm, $p = 20$ MPa, $\Delta T = 30^{o}$C.

Ans. $\tau^{A}_{\theta\theta} = 22.03$ MPa, $\tau^{S}_{\theta\theta} = 152.96$ MPa.

Chapter 4

Yield and Failure Criteria

4.1 Yield Criteria for Materials Subjected to Triaxial States of Stress in an Environment of Constant Temperature

When a body is subjected to external forces (body forces and surface tractions), its particles are in general subjected to components of stress as a result of which they deform. When the values of the external forces are small, the body and its particles assume their undeformed configuration upon removal of the external forces. That is, the deformation of the body is reversible. No energy is lost during the process of loading and unloading. In this case we say that the deformation of the body and its particles is *elastic*. For sufficiently high magnitudes of the external forces, the components of strain of some particles of the body may include an irreversible part. We say that at these particles *plastic deformation* or *yielding* has occurred. The components of strain e_{ij} of these particles are equal to the sum of an elastic part e_{ij}^{E} and a plastic part e_{ij}^{P}. That is,

$$e_{ij} = e_{ij}^{E} + e_{ij}^{P} \tag{4.1}$$

In this section we focus our attention to two models of idealized time-independent material behavior under general three-dimensional states of stress. These models are the isotropic, rigid-ideally plastic and the isotropic, linearly elastic-ideally plastic. In the special case of a uniaxial state of stress, these models reduce to the corresponding models described in Section 3.6 (see Fig. 3.14c and e, respectively). For these models of idealized time-independent material behavior, we present criteria for defining the limiting combinations of values of the components of stress acting on a particle of a body, for which its deformation is elastic.

It is assumed that plastic components of strain are produced at a particle of a body subjected to a general three-dimensional state of stress, when the components of stress acting on it satisfy a certain relation referred to as the *yield criterion* or the *yield condition*. The yield criterion for a particle of a body made from a material which exhibits strain-hardening (see Section 3.6, Fig. 3.14d and f) depends on the path of loading and unloading to which this particle has been subjected since the last annealing† of the body. Consequently, it is a function of the components of stress and of the plastic

† Annealing is the process of heating a metal to an appropriate temperature for a specified time and subsequently cooling it at an appropriate slow rate. This process can restore a metal to its stress-free, strain-free state of mechanical and thermal equilibrium.

components of strain of the particle. The yield criterion for a particle of a body made from an ideally plastic material does not depend on the path of loading and unloading to which this particle has been subjected. That is, it is not a function of the plastic components of strain. Thus, the yield criterion for the particles of a body made from an ideally plastic material has the following form:

$$f\,(\tau_{11},\ \tau_{12},\ \tau_{13},\ \tau_{22},\ \tau_{23},\ \tau_{33}) = 0 \tag{4.2}$$

where the *yield function* $f(\tau_{nm})$ has first derivatives which are sectionally continuous. *The sign of the yield function is chosen so that an increment of the components of stress does not change the plastic components of strain at a particle of a body, if the components of stress at this particle satisfy the relation* $f(\tau_{nm}) < 0$. On the basis of this convention, it is apparent that a state of stress satisfying the relation $f(\tau_{nm}) > 0$, cannot be realized at a particle of a body made from an ideally plastic material. If the components of stress satisfy the relation $f(\tau_{nm}) = 0$ at a particle of a body made from an ideally plastic material, an increment $d\tau_{ij}$ $(i, j = 1, 2, 3)$ of the components of stress may or may not produce changes in the components of plastic strain at that particle. That is, if for an increment $d\tau_{ij}$ $(i, j = 1, 2, 3)$ of the components of stress, the increment of the yield function is negative $\left(df = \sum_{q=1}^{3} \sum_{p=1}^{3} \frac{\partial f}{\partial \tau_{pq}} d\tau_{pq} < 0 \right)$, the new value of this function will be negative and, thus, the forementioned increment $d\tau_{ij}$ $(i, j = 1, 2, 3)$ does not produce any change in the components of plastic strain of the particle. Thus, for any increment of the components of stress $d\tau_{ij}$ $(i, j = 1, 2, 3)$, the plastic components of strain of a particle do not change if

(a) $f(\tau_{nm}) < 0$

$$\text{(b)}\quad f(\tau_{nm}) = 0 \quad \text{and} \quad df = \sum_{p=1}^{3}\sum_{q=1}^{3} \frac{\partial f}{\partial \tau_{pq}} d\tau_{pq} < 0 \quad \text{(unloading)} \tag{4.3}$$

whereas, for any increment of the components of stress $d\tau_{ij}$ $(i, j = 1, 2, 3)$, the plastic components of strain of a particle change if

$$f(\tau_{nm}) = 0 \quad \text{and} \quad df = \sum_{p=1}^{3}\sum_{q=1}^{3} \frac{\partial f}{\partial \tau_{pq}} d\tau_{pq} = 0 \tag{4.4}$$

When a body made from an elastic-ideally plastic material is subjected to increasing external forces from its unstressed–unstrained state, it deforms elastically until for a certain value of the external forces the components of stress acting on one or more particles of the body reach values satisfying the yield criterion (4.2). Then, an increase of the external forces acting on the body could produce plastic components of strain at these particles. Thus, for sufficiently high values of the external forces, there will be one or more parts of the body whose particles are subjected to components of stress which satisfy relation (4.2). These parts of the body are referred to as the *plastic regions*. Generally, the plastic regions of a body are surrounded by elastic regions. The boundaries between the elastic and plastic regions are referred to as the *elastoplastic boundaries*. If

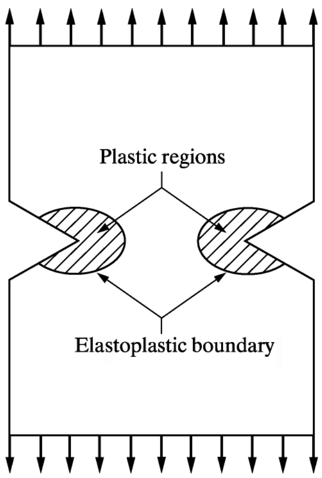


Figure 4.1 Symmetrically notched plate subjected to tensile forces.

the elastic regions of a body made from an ideally plastic material, can restrain the deformation of its plastic regions, the magnitude of the deformation of the particles of the plastic regions could be within the range of validity of the theory of small deformation. *As the external forces increase, the size of the plastic regions increases until they extend to a sufficiently large* portion of the volume of the body that the elastic regions are not able to restrain the deformation of the plastic regions. The body then deforms without any further increase of the external forces. In this case, we say that the body collapses plastically. For instance, consider the symmetrically notched plate shown in Fig. 4.1 in an environment of constant temperature subjected on its end surfaces to a uniform gradually increasing distribution of the axial component of traction from its unstressed and unstrained reference state. Assume that the plate is made from an elastic-ideally plastic material. For small values of the external forces, the distribution of the components of stress throughout the plate is elastic and can be established using one of the formulations presented in Section 5.2. For such values of the external forces, a high-stress concentration occurs at the root of the notches. Thus, as the external forces increase, the components of stress at a number of particles in the neighborhood of the root of the notches satisfy the yield criterion (4.2) and plastic deformation occurs at these particles. Consequently, two plastic regions are formed in the plate, as shown in Fig. 4.1. In general, the location and geometry of the elastoplastic boundary are unknown. In many cases, the geometry of the elastoplastic boundary is complex and can only be established using numerical methods. Once the elastoplastic boundary corresponding to a given value of the applied forces is established, the components of stress and displacement of every particle of the elastic and the plastic regions can be computed. In the elastic regions, the stress–strain relations (3.47) for an isotropic, linearly elastic material are used. In the plastic regions, appropriate relations between the increments of the component of stress and strain are employed. When the external forces are increased by a small amount, the location and the geometry of the elastoplastic boundary and the distribution of the components of stress and displacement in the elastic and plastic regions will change and must be recomputed. The location of the elastoplastic boundary and the distribution of the components of stress and displacement depend on the number of cycles of loading and unloading to which the plate has been subjected and on the extent of plastic deformation in each cycle. The plate does not deform extensively until, for some value of the external forces, the plastic regions extend across its width. Then the plate deforms to failure while the external forces remain constant. We say that the plate has *collapsed plastically*.

Consider a body made from an elastic-ideally plastic material subjected to a sufficiently high loading so that components of plastic strain have been produced at some of its particles, and suppose that it is subsequently unloaded. As the body is unloaded, the changes (decrements) of the components of strain of each of its particles are elastic. That is, the components of plastic strain do not change during unloading. When the body has been unloaded, the components of plastic strain of its particles do not vanish and, consequently, the plastic regions of the body inhibit its elastic regions from reverting to their unstrained state. Thus, generally, a *residual stress* field exists in the body after the loading has been removed.

If the material from which the body is made is isotropic, the values of the yield function f must be independent of the axes to which the components of stress are referred. That is, the value of the yield function f does not change if the components of stress are referred to the x_1, x_2, x_3 axes or to another set of axes. Hence, the yield function for isotropic, ideally plastic, materials can be expressed in terms of the three principal components of stress and thus, the yield criterion assumes the following form:

$$f(\tau_1, \tau_2, \tau_3) = 0 \tag{4.5}$$

In the sequel we limit our attention only to isotropic materials.

It has been verified experimentally[†] *that to a reasonable degree of approximation, hydrostatic pressure does not affect the yield condition or the components of plastic strain*. The form of the yield function may be restricted further by assuming that *the yield condition*

in tension is identical to the yield condition in compression. This implies that *the material does not exhibit a Bauschinger effect*. On the basis of this assumption, we have

$$f(\tau_{11}, \tau_{12}, \tau_{13}, \tau_{22}, \tau_{23}, \tau_{33}) = f(-\tau_{11}, -\tau_{12}, -\tau_{13}, -\tau_{22}, -\tau_{23}, -\tau_{33}) \tag{4.6}$$

In a three-dimensional[††] space where the principal components of stress τ_1, τ_2, τ_3 are taken as the cartesian coordinates (principal stress space) (see Fig. 4.2), the state of stress of a particle of a body during a program of loading may be represented by a curve. The yield criterion (4.2) may be represented by a closed surface for which it can be shown that *it must be convex* and for an ideally plastic material it is a cylindrical surface (see Fig. 4.2). For isotropic materials, the axis of the yield surface must pass through the origin of the τ_1, τ_2, τ_3 axes. Moreover, since it is assumed that hydrostatic pressure does not produce yielding, it can be shown that the axis of the yield surface is in the direction of the following unit vector:

$$\mathbf{i}_n = \frac{1}{\sqrt{3}}(\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3) \tag{4.7}$$

[†] Bridgmen, P.W., The compressibility of thirty metals as a function of pressure and temperature, *Proceedings of the American Academy of Arts and Science,* 58, p. 163-242, 1923.

[††] In general, the state of stress at a particle of a body may be represented by a point in the six-dimensional space of the components of stress. However, for an isotropic material, we prefer to think in terms of the three-dimensional prinicpal stress space.

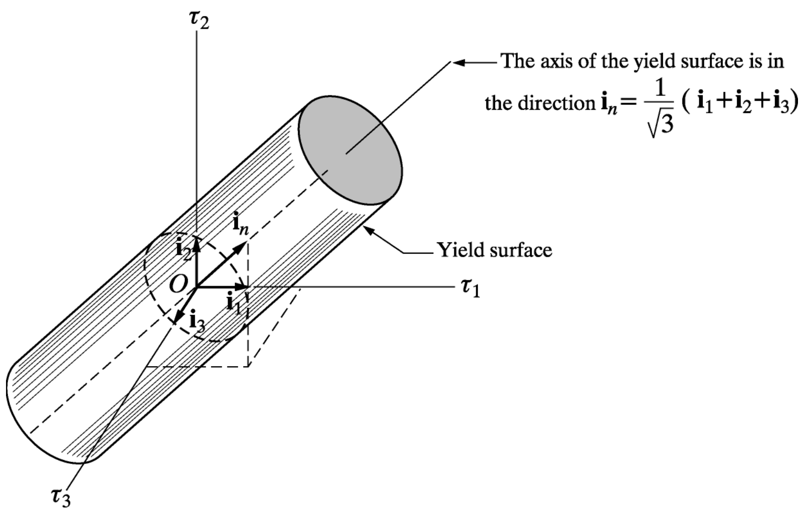


Figure 4.2 Graphical representation of the yield criterion for isotropic, ideally plastic material.

The geometry of the cross section of the yield surface depends on the prescribed yield criterion (4.2). The state of stress at any particle of a body made from an ideally plastic material may be represented by a point, either on or inside the yield surface but not outside the yield surface.

4.2 The Von Mises Yield Criterion

A widely employed form of the yield criterion for metals due to Von Mises†, states that yielding occurs at a particle of a body when the components of stress acting on it, satisfy the following relation:

$$\frac{1}{6}[(\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2] = k^2 \tag{4.8a}$$

or

$$\frac{1}{6}[(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2)] = k^2 \tag{4.8b}$$

where k^2 is a constant to be determined from the uniaxial tension experiment††. Denoting the yield stress in uniaxial tension by τ^Y, the Von Mises yield criterion (4.8) for uniaxial tension reduces to

$$\frac{(\tau^Y)^2}{3} = k^2 \tag{4.9}$$

† Von Mises, R., *Mechanik der Festen Koerper im Plastischen Deformablen Zustand* Goettinger Nachr., Math-Phys. K1, p. 582-592, 1913.

†† The constant k^2 may also be established from an experiment in which the specimen is subjected to a state of pure shear $\tau_{11} = \tau_{rr} = \tau_{\theta\theta} = \tau_{1r} = \tau_{r\theta} = 0$, $\tau_{1\theta} \neq 0$. In this case, denoting by $\tau_{1\theta}^Y$ the yield stress in pure shear, the Von Mises yield criterion (4.8b) reduces to $\left(\tau_{1\theta}^Y\right)^2 = k^2$.

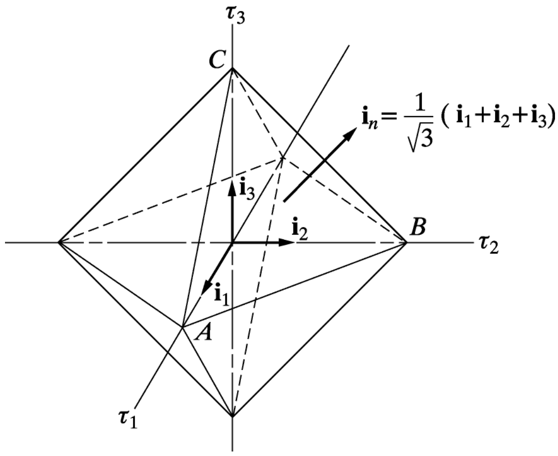


Figure 4.3 Octahedron.

It can be shown that the shearing stress acting on the faces of an octahedron whose faces form equal angles with each of the principal axes of stress (see Fig. 4.3) is equal to

$$\tau_{nt} = \frac{1}{3}[(\tau_1 - \tau_2)^2 + (\tau_1 - \tau_3)^2 + (\tau_2 - \tau_3)^2]^{\frac{1}{2}} \tag{4.10}$$

In the case of uniaxial tension ($\tau_1 \neq 0$, $\tau_2 = \tau_3 = 0$), from relation (4.10), we obtain that the total shearing stress acting on the face of the octahedron is equal to

$$\tau_{nt} = \frac{\sqrt{2}}{3}\tau_1 \tag{4.11}$$

Referring to relations (4.10) and (4.11), it can be seen that the Von Mises yield criterion is obtained if it is postulated that, *yielding occurs at a particle subjected to a triaxial state of stress, when the value of the total shearing stress acting on a face of the octahedron described above is equal to the value of the shearing stress acting on a face of this octahedron when the particle yields while subjected to uniaxial tension* ($\tau_{11} = \tau^Y$). For this reason, the Von Mises yield criterion is also known as the *octahedral yield criterion*. A general state of stress of a particle may be regarded as the superposition of two parts. The one part causes only change of its volume and it is called *dilatational* while the other part causes only change of its shape (distortion) and its is called *distortional*. The dilatational part of the state of stress of a particle (see Fig. 4.4b) is a hydrostatic state of stress given as

$$[\tau] = \begin{bmatrix} \bar{\tau} & 0 & 0 \\ 0 & \bar{\tau} & 0 \\ 0 & 0 & \bar{\tau} \end{bmatrix} \tag{4.12}$$

where $\bar{\tau}$ is referred to as the *mean stress* and it is given as

$$\bar{\tau} = \frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) \tag{4.13}$$

Adding the first three of relations (3.47), we obtain:

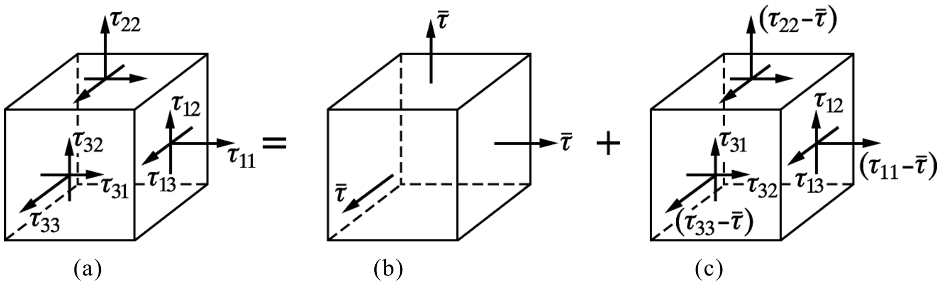


Figure 4.4 Dilatational and distortional part of the states of stress of a particle.

$$\bar{\tau} = \frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) = 3K\frac{(e_{11} + e_{22} + e_{33})}{3} = 3K\bar{e} \tag{4.14}$$

where

$$K = \frac{E}{3(1 + \nu)} \tag{4.15}$$

$\bar{e}$ is called the *mean strain* and K is called the *bulk modulus*.

The distortional part of the state of stress of a particle (see Fig. 4.4c) is equal to

$$[\tau_{distor}] = \begin{bmatrix} (\tau_{11} - \bar{\tau}) & \tau_{12} & \tau_{13} \\ \tau_{21} & (\tau_{22} - \bar{\tau}) & \tau_{23} \\ \tau_{31} & \tau_{32} & (\tau_{33} - \bar{\tau}) \end{bmatrix} \tag{4.16}$$

The strain energy density of a particle corresponding to the dilatational part of its state of stress is known as *dilatational* and we denote it by U_V. Substituting relation (4.12) into (3.80), we get

$$U_V = \frac{2}{2E}(1 - 2\nu)\bar{\tau}^2 = \frac{(1 - 2\nu)}{6}E(\tau_{11} + \tau_{22} + \tau_{33})^2 \tag{4.17}$$

The strain energy density of a particle corresponding to the distortional part of its state of stress is called *distortional* and we denote it by U_d. It can be established by substituting relation (4.16) into (3.82) or using the following relation:

$$U_d = U_s - U_V \tag{4.18}$$

Substituting relation (3.82) and (4.17) in the above, we get

$$U_d = \frac{(1 + \nu)}{6E}\left[(\tau_{11} - \tau_{22})^2 + (\tau_{11} - \tau_{33})^2 + (\tau_{22} - \tau_{33})^2 + 6(\tau_{12}^{\,2} + \tau_{13}^{\,2} + \tau_{23}^{\,2})\right] \tag{4.19}$$

Referring to relation (4.19) the distortional part of its strain energy density when the particle yields while subjected to a state of uniaxial tension or compression, is equal to

$$U_d\left(\begin{array}{l}\text{At yielding in a state}\\ \text{of uniaxial stress}\end{array}\right) = \frac{1+\nu}{3E}(\tau^{\nu})^2 \tag{4.20}$$

Referring to relation (4.18) and (4.20) it can be seen that the Von Mises criterion (4.7) is obtained if it is postulated that yielding occurs at a particle subjected to a general triaxial state of stress when the value of the corresponding distortional strain energy density is equal to the distortional strain density energy density when the particle yields while subjected to a state of uniaxial stress. For this reason the Von Mises yield criterion is also known as the *distortional energy density yield criterion*.

4.3 The Tresca Yield Criterion

Another well-known yield criterion was proposed by Tresca[†]. It postulates that yielding occurs when the maximum shearing stress at a particle of a body in a general, triaxial state of stress attains a value equal to the maximum shearing stress at yielding in uniaxial tension. That is why this criterion is also known as the *maximum shearing stress yield criterion*. In uniaxial tension ($\tau_{11} \neq 0$, $\tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$) the maximum shearing stress is $\tau_{11}/2$. In a general, triaxial state of stress if the principal components of stress are $\tau_1 > \tau_2 > \tau_3$, the maximum shearing stress is $(\tau_1 - \tau_3)/2$ [see relation (1.143)]. Thus, the Tresca yield criterion can be written as

$$f = (\tau_1 - \tau_3)^2 - (\tau^Y)^2 = 0 \tag{4.21}$$

This simple form of the Tresca yield criterion can be used only in problems wherein the ordering of the magnitude of the principal components of stress is known. If the ordering of the magnitude of the principal components of stress is not known, the Tresca yield criterion must be written in the following general form:

$$f = [(\tau_1 - \tau_2)^2 - (\tau^Y)^2][(\tau_2 - \tau_3)^2 - (\tau^Y)^2][(\tau_3 - \tau_1)^2 - (\tau^Y)^2] = 0 \tag{4.22}$$

In the principal stress space, the Tresca yield criterion is an infinitely long cylindrical surface of hexagonal cross section inscribed in the Von Mises circular, cylindrical surface (see Fig. 4.5). Thus, the Tresca yield criterion satisfies the requirement of convexity of the yield surface.

4.4 Comparison of the Von Mises and the Tresca Yield Criteria

The Von Mises yield criterion is non-linear, whereas the Tresca yield criterion is piecewise linear. However, if the ordering of the magnitudes of the principal components of stress is not known, the Tresca yield surface involves singularities (edges and corners) and is difficult to handle.

The difference in the predictions of the two criteria is not appreciable. This may be illustrated by considering a state of plane stress at a particle of a body specified by

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[†] Tresca, H., *Memoire sur 1 'Eccoulement des Corps Solides,* Mem. Pres. Par Div. Sav., 18, p. 733-799, 1868.

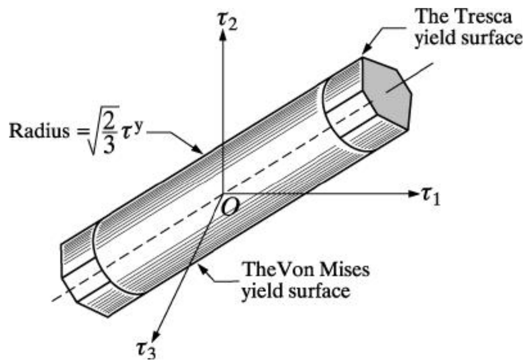


Figure 4.5 The Von Mises and the Tresca yield surfaces.

$$\tau = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.23}$$

Such a state of stress may be realized by subjecting a thin, circular, cylindrical tube to axial tension combined with internal pressure. For this case, the Von Mises yield criterion (4.8a) reduces to

$$\tau_1^2 - \tau_1\tau_2 + \tau_2^2 = (\tau^Y)^2 \tag{4.24}$$

This is an ellipse in the τ_1, τ_2 plane, with its major and minor axes inclined 45° to the τ_1, and τ_2 axes (see Fig. 4.6).

The Tresca yield criterion depends on the ordering of the magnitudes of τ_1 and τ_2. If $\tau_1 > \tau_2 > 0$, the Tresca yield criterion reduces to

$$\tau_1 = \tau^Y \tag{4.25a}$$

If $\tau_2 > \tau_1 > 0$ it reduces to

$$\tau_2 = \tau^Y \tag{4.25b}$$

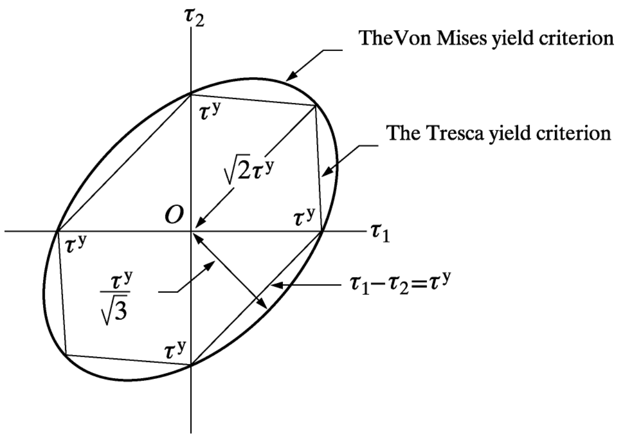


Figure 4.6 Comparison of the Von Mises and the Tresca yield criteria.

if $\tau_1 > 0 > \tau_2$ it reduces to

$$\tau_1 - \tau_2 = \tau^Y \tag{4.25c}$$

As can be seen from Fig. 4.6, the greatest divergence in the predictions of the two criteria occurs at $\tau_1 = -\tau_2$. For this case, the Von Mises yield criterion gives $\tau^M = \tau^Y/\sqrt{3}$, while the Tresca yield criterion gives $\tau_1{}^T = \tau^Y/2$. Thus, the maximum discrepancy $(\tau_1^M - \tau_1^T)/\tau_1^M$ in the predictions of the two criteria is approximately 15%.

In what follows, we present an example.

Example 1 The state of stress of a particle of a body, made from an isotropic linearly elastic-plastic material, is specified by two parameters τ and τ_{33} and it is given as

$$[\tau] = \begin{bmatrix} 0 & \frac{2\tau}{3} & 0 \\ \frac{2\tau}{3} & \tau & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix} \tag{a}$$

Determine and plot in the τ, τ_{33} space the yield surface corresponding to this state of stress. The yield stress in uniaxial tension or compression for the material from which the body is made is $\tau^Y = 200$ MPa. Assume that the material from which the body is made, obeys (a) the Von Mises yield criterion, (b) the Tresca yield criterion.

Solution

Part a

Substituting the values of the components of stress (a) in the Von Mises yield condition (4.8b), we get

$$\frac{1}{2}\left[(\tau_{33} - \tau)^2 + \tau_{33}^2 + \tau^2 + 6\left(\frac{2\tau}{3}\right)^2\right] = 40{,}000$$

or

$$3\tau_{33}^2 - 3\tau\tau_{33} + 7\tau^2 = 120{,}000 \tag{b}$$

This is the equation of an ellipse whose one axis, as shown in Fig. a, is assumed to be inclined by an angle α with respect to the axis τ. Referring to relation (1.36b) the coordinates of any point, on the τ, τ_{33} plane, referred to the axes τ, τ_{33} are related to its coordinates, referred to the axes x_1, x_2 (see Fig. a) by the following relations:

$$\begin{aligned} \tau &= x_1 \cos\alpha - x_2 \sin\alpha \\ \tau_{33} &= x_1 \sin\alpha - x_2 \cos\alpha \end{aligned} \tag{c}$$

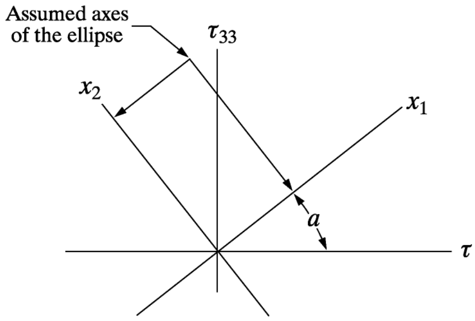


Figure a Assumed axes of the ellipse.

Substituting relations (c) into (b), we obtain

$$3(x_1 \sin\alpha + x_2 \cos\alpha)^2 - 3(x_1 \cos\alpha - x_2 \sin\alpha)(x_1 \sin\alpha + x_2 \cos\alpha) + 7(x_1 \cos\alpha - x_2 \sin\alpha)^2 = 120{,}000 \quad \text{(d)}$$

or

$$x_1^2(3\sin^2\alpha - 3\cos\alpha\sin\alpha + 7\cos^2\alpha) + x_1x_2(-8\sin\alpha\cos\alpha - 3\cos^2\alpha + 3\sin^2\alpha) + x_2^2(3\cos^2\alpha + 3\sin\alpha\cos\alpha + 7\sin^2\alpha) = 120{,}000 \quad \text{(e)}$$

In order that relation (e) assumes the form

$$\frac{x_1^2}{\alpha^2} + \frac{x_2^2}{\beta^2} = 1 \quad \text{(f)}$$

the coefficient of $x_1\, x_2$ must vanish. Thus,

$$-8\cos\alpha\sin\alpha - 3\cos^2\alpha + 3\sin^2\alpha = 0$$

or

$$-\cos 2\alpha - \frac{4}{3}\sin 2\alpha = 0$$

and

$$\tan 2\alpha = -\frac{3}{4} \qquad \text{and} \qquad 2\alpha = -36.87^\circ \qquad \alpha = -18.43^\circ$$

Comparing relations (e) and (f), we get

$$\alpha^2 = \frac{120{,}000}{3\sin^2\alpha - 3\cos\alpha\sin\alpha + 7\cos^2\alpha} = \frac{120{,}000}{7.498} = 16{,}004$$

$$b^2 = \frac{120{,}000}{3\cos^2\alpha + 3\sin\alpha\cos\alpha + 7\sin^2\alpha} = \frac{120{,}000}{2.50177} = 47{,}966$$

and

$$a = 126.51 \qquad b = 219.01$$

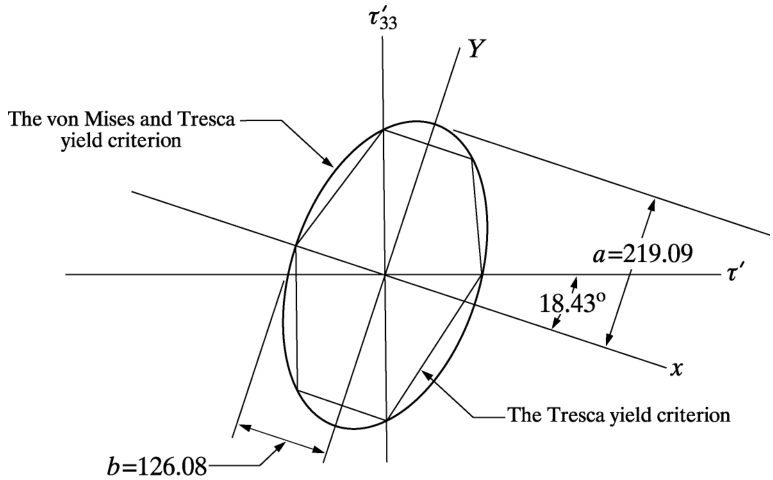


Figure b The Von Mises and Tresca yield criteria in the τ, τ_{33} space.

Thus, the Von Mises yield condition is an ellipse (see Fig. b), whose minor axis is inclined to the axis τ by 18.43° clockwise. The equation of this ellipse with respect to the axes x_1 and x_2 is

$$\frac{x_1^2}{(126.51)^2} + \frac{x_2^2}{(219.01)^2} = 1 \tag{g}$$

Part b

In Fig. c we plot Mohr's circle for the state of stress (a) using its two points $X_1\left(0, \frac{2\tau}{3}\right)$ and $X_2\left(\tau, -\frac{2\tau}{3}\right)$ Referring to fig. C, we have

$$\bar{OC} = \frac{\tau}{2}$$

$$R = \sqrt{\left(\frac{\tau}{2}\right)^2 + \left(\frac{\tau}{2}\right)^2} = \frac{5\tau}{6}$$

$$\tau_1 = \frac{\tau}{2} + R = \frac{4}{3}\tau \qquad \tau_2 = \frac{\tau}{2} - R = -\frac{\tau}{3} \tag{h}$$

Referring to relations (4.22) and (h), we find that, on the basis of the Tresca yield criterion, yielding occurs when the values of τ and τ_{33} satisfy one of the following relations:

$$(\tau_1 - \tau_{33})^2 - (\tau^y)^2 = 0 \qquad \text{or} \qquad 4\tau - 3\tau_{33} = \pm 600\,\text{MPa}$$
$$(\tau_2 - \tau_{33})^2 - (\tau^y)^2 = 0 \qquad \text{or} \qquad -\tau - 3\tau_{33} = \pm 600\,\text{MPa}$$
$$(\tau_1 - \tau_2)^2 - (\tau^y)^2 = 0 \qquad \text{or} \qquad \tau = \pm 120\,\text{MPa}$$

The results are plotted in Fig. b.

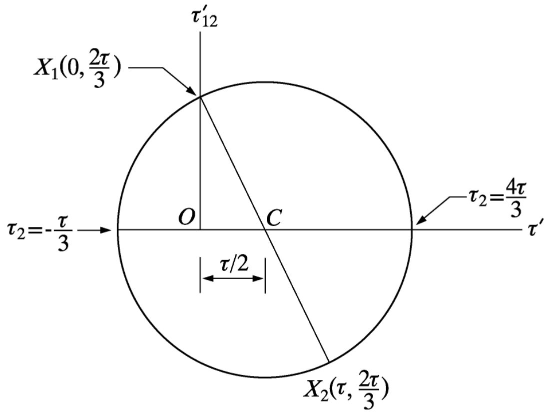


Figure c Mohr's circle for the state of stress (a).

4.5 Failure of Structures — Factor of Safety for Design

We define a structure as a man-made object, such as a building, a machine, a vehicle, an aircraft, or a ship, which supports or transmits loads. A structure is designed to perform a certain function. The design of a structure entails

1. The determination of the loads (external forces, change of temperature, etc) to which it is expected to be subjected during its lifetime.

2. The selection of the dimensions and material of its members as well as the details of their connections which ensure that the structure is able to perform its function when subjected to the maximum value of the anticipated loads.

The capacity of a structure to support or transmit loads without failing to perform its function is known as *its strength*. Due to the many uncertainties involved in the design of a structure, the strength for which it is designed is greater than that required. The ratio of the design strength to the required strength is known as the *factor of safety* (F.S.). That is,

$$\text{factor of safety} = \text{F.S.} = \frac{\text{design strength}}{\text{required strength}} > 1 \qquad (4.26)$$

The following are some of the criteria which must be taken into account in determining the F.S., which should be used for a structure of a particular type of structures (as, for example, buildings or machines):

1. The type (i.e., static, dynamic, repeated) and magnitude of the loads to which it is anticipated that the structure will be subjected during its lifetime, as well as the expected accuracy of their estimation. A load-inducing strain rate greater than 10^{-4}/sec is considered dynamic.

2. The anticipated quality of construction.

3. The quality of available materials for construction.

4. The anticipated damage from environmental conditions such as corrosion of metal parts. **5**. The nature of the anticipated failure. Gradual failures give time to reinforce or modify

the structure and prevent its failure.

6. The consequences of failure. Greater factor of safety is used if the consequences of

failure are catastrophic.
7. The cost of increasing the factor of safety. In an airplane, a small increase of the factor of safety may result in a prohibitive increase of cost of operation because the weight of the airplane increases as the F.S. increases and the load which can be transported decreases.
8. The effect of the approximations involved in the theory used to analyze the structure.
9. The determination of the factor of safety for a structure or for a particular type of structures (like buildings or bridges), a difficult task which is usually performed by groups of experienced engineers who write the design specifications† or codes which are used by designers.

The F.S. is applied in one of the following ways:
1.The loads for which the structure is designed (design loads) are obtained by multiplying by the F.S. the maximum value of the loads to which the structure is anticipated that it will be subjected during its lifetime. That is,

$$\text{design loads} = (\text{F.S.})\ (\text{maximum anticipated loads}) \tag{4.27}$$

The dimensions of the members of the structure are then chosen so that the structure is just able to perform its function when subjected to the design loads.
2. The yield stress or the ultimate stress of the material from which the structure is made is
divided by the F.S. and an allowable or working stress is obtained. That is,

$$\text{allowable yield stress} = \frac{\text{actual yield stress}}{\text{F.S.}} \tag{4.28}$$

or

$$\text{allowable ultimate stress} = \frac{\text{actual ultimate stress}}{\text{F.S.}} \tag{4.29}$$

The actual yield stress is replaced by the allowable one in the yield criteria described in Sections 4.2 and 4.3, to obtain *allowable yield criteria*. Moreover, the actual ultimate stress is replaced by the allowable one in the fracture criteria described later in this section, to obtain *allowable fracture criteria*. The members of the structure are then chosen so that when the structure is subjected to the maximum values of the loads to which it is anticipated to be exposed during its lifetime, the values of the components of stress acting on its particles are less or just equal to those satisfying the allowable yield of fracture criterion.

In establishing the required strength of a structure, all possible ways (modes) that a structure can fail to perform its faction must be taken into account. The following are the most commonly encountered modes of failure:

---

† Examples of design specifications are
1. The specifications of the AISC (American Institute of Steel Construction) for the design and erection of structural steel for buildings.
2. The specifications for reinforced concrete of ACI (American Concrete Institute)
3. The standard specifications for highway bridges of AASHO (American Association of State Highway Officials).
4. The specifications for stress–grade lumber and its fastenings of the National Forest Products Association.

1. Failure of structures due to excessive elastic or viscoelastic deformation of one or more of their members

In order to prevent this type of failure, the structure is designed so that when it is subjected to the design loads, the displacements of its particles do not exceed the maximum permissible displacement.

2. Failure of structures made from a ductile material due to initiation of yielding

It is postulated that structures made from ductile materials fail as soon as the components of stress acting on one of their particles satisfy the yield criterion for the material from which they are made. In order to prevent this type of failure, structures are designed so that when they are subjected to the maximum values of the loads to which it is anticipated that they will be exposed during their lifetime, the components of stress acting on their particles reach values which satisfy the "allowable" yielding criterion in one or more of their particles. When the Von Mises yield criterion is used, the failure theory is known as the *maximum distortion energy theory of failure* or the *octahedral shearing stress theory of failure*. When the Tresca yield criterion is used, the theory is known as the *maximum shearing stress theory of failure*.

When a structure is designed on the basis of this failure mode, we say that the *structure is designed for strength on the basis of elastic design.*

3. Failure of structures made from a ductile material due to plastic collapse

It is postulated that a structure made from a ductile material fails when the loads acting on it reach values (called the collapse loads of the structure) for which the structure would have collapsed, if it was made from an isotropic, linearly elastic–ideally plastic material. That is, in some of its members a sufficient number of sufficiently large plastic regions would have been formed so that the structure would deform to collapse (see Chapter 16). In order to prevent this type of failure, the structure is designed so that the design loads are smaller or equal to the collapse loads.

When a structure is designed on the basis of this failure mode, we say that the *structure is designed for strength on the basis of plastic design.*

4. Failure of structures due to elastic or plastic instability of one or more of their members (see Chapter 18)

In order to prevent this type of failure, structures are designed so that when they are subjected to their design loads, one of their members or a group of their members does not reach a state of instability and buckle (see Chapter 18).

5. Failure of structures by fracture†

† For further reading see

Barsom, J.M., Rolfe S.T., *Frature and Fatigue Control in Structures,* 2nd edition, Prentice-Hall, Englewood Cliffs, NJ, 1987.

Broek, D., *Elementary Engineering Fracture Mechanics,* 4th edition, Matinus Nijhoff, London, 1985.

Ewalds H.L., Wanhill R.J.H. *Fracture Mechanics* Edward Arnold, London 1986.

Knott, J.F., *Fundamentals of Fracture Mechanics,* J. Wiley & Sons, New York, 1973.

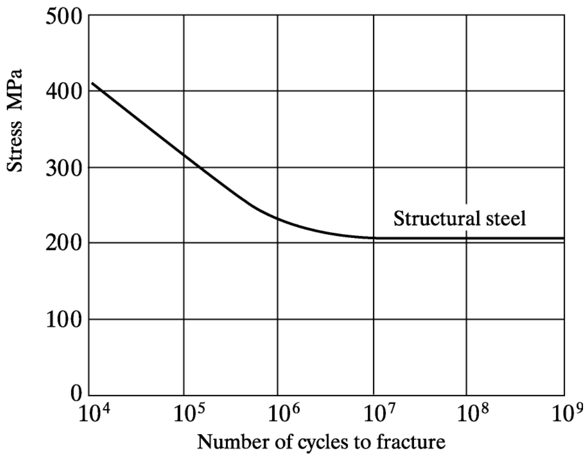


Figure 4.7 S-N diagram for structural steel.

Some structures resist the applied loads satisfactorily until one or more of their members break rather suddenly with little or no evidence of plastic deformation. We distinguish the following three different mechanisms of failure by fracture:

(a) *Sudden fracture of structures made from brittle materials*

When a specimen made from a brittle material, such as high carbon steel, gray cast iron, glass, or ceramics, is subjected to a state of uniaxial tension, it fails with little or no visual evidence of plastic deformation (see Section 3.2) as soon as the maximum normal component of stress acting on its particles reaches its ultimate value. When a structure made from a brittle material is subjected to a state of triaxial stress, it is postulated that it fails as soon as the components of stress acting on one of its particles satisfy the fracture criterion for the material from which it is made. In Sections 4.6 and 4.7, we describe two criteria for brittle fracture. The ultimate stress in uniaxial tension or compression is a parameter of these criteria, as the yield stress in uniaxial tension or compression is a parameter of the yield criteria.

(b) *Brittle fracture of structures made from ductile materials*

Structures made from ductile materials can fracture as if they were made from brittle materials. Such brittle fractures are caused by pre-existing flaws (such as cracks, small notches, or scratches) producing stress concentrations in their neighborhoods involving high tensile hydrostatic stresses. As discussed in Section 4.1 the effect of hydrostatic stress on the yield criterion is very small. Consequently, when a ductile material is subjected to states of stress involving high tensile hydrostatic stress, it may break before the components of stress acting on one of its particles satisfy the appropriate yield criterion. That is, it may fracture with little or no visual evidence of yielding. Low temperatures and dynamic loads decrease the ability of the material to resist brittle fracture. The phenomenon of brittle fracture of ductile materials is a broad subject which requires knowledge of material science including fracture processes and crack initiation and propagation, and consequently is beyond the scope of this book.

(c) *Fatigue fracture of structures*

When a member made from a ductile metal is subjected to repeated cycles of stress, minute cracks are formed at one or more of its points and the member may fracture after N cycles, provided that the maximum stress in each cycle is equal to or greater than a value known as the *fatigue strength of the metal* corresponding to N number of loading cycles. This type of failure of a member of a structure is called *fatigue failure* or f*atigue fracture*. The number of load cycles to which the member is subjected when fracture occurs is called *the fatigue life* of the member. The fatigue life of a member is affected by many factors such as the type of loading (uniaxial, bending, tension), the frequency of the loading cycles, the amplitude of the load, the size of the member, the presence of flaws, or the temperature during testing. Each of these factors can have considerable effect on the fatigue life of a member.

We distinguish two types of fatigue fracture — low cycle (10^4 cycles or less) and high cycle (usually above 10^6 cycles). When a member made from a ductile material fractures while undergoing low cycle fatigue, large plastic strains occur in the neighborhood of the fracture. When a member made from a ductile material fractures while undergoing high cycle fatigue, little or no visual plastic deformation is apparent.

The curve of Fig. 4.7 has been plotted from results of tests on a number of specimens having the same geometry and made from the same material (structural steel). Every specimen was subjected to a loading which was the same completely reversing function of time of constant amplitude. The curve gives the fatigue strength of the structural steel specimen as a function of the number of cycles required to break it. It is called the *S-N (stress–number) diagram for the metal*.

The S-N diagrams for metals are usually 50% probability of failure curves. That is, if a large number of specimens having the same geometry and made from the same material were subjected to the same loading cycles, the numbers of cycles at which they will fracture would have appreciable scatter. However, approximately 50% of the specimens are expected to fracture at a smaller number of cycles than the number indicated in the S-N diagram for the amplitude of the loading to which the specimens were subjected.

In the next two subsections we present two criteria for fracture of members made from brittle materials.

In what follows we present an example.

Example 2 A bar of circular cross section of radius R is made from a ductile, isotropic, linearly elastic material with yield stress in uniaxial tension or compression $\tau_{11}^{Y} = 240$ MPa. The bar is subjected to two equal and opposite bending moments $M_2 =$ 12 kN·m and to two equal and opposite torsional moments $M_1 = 16$ kN·m at its ends (see Fig. a). Determine the minimum diameter of the bar for safe design assuming that the bar fails as soon as yielding occurs at one of its particles. Use the maximum shearing stress

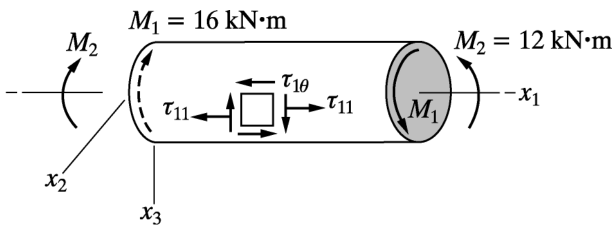


Figure a Bar subjected to torsional and bending moments.

and the maximum distortion energy theories with a factor of safety of three.

Solution The maximum tensile or compressive stress acting on the particles of the bar due to the bending moment alone occurs at $x_3 = \pm R$ and it is given by relation (9.12b) as

$$\tau_{11} = \pm \frac{M_2 R}{I_2} \tag{a}$$

where I_2 is the moment of inertia of the cross section of the bar about its x_2 axis.

The maximum shearing stress due to the torsional moment alone occurs on the particles of the lateral surface of the bar and it is equal to

$$\tau_{1\theta} = \frac{M_1 R}{I_p} = \frac{M_1 R}{2I_2} \tag{b}$$

where I_p is the polar moment of inertia for a circular cross section which referring to relation (C.10) of Appendix C is equal to

$$I_p = 2I_2 \tag{c}$$

On the basis of the foregoing presentation, the particles of the bar at $x_3 = R$ will be subjected to the maximum tensile and shearing stresses while the particles of the bar at $x_3 = -R$ will be subjected to the maximum compressive and shearing stresses. That is, yielding will start at these particles, simultaneously. The state of stress of a particle at $x_3 = R$ is shown in Fig. b. Mohr's circle for this state of stress is shown in Fig. c. Referring to fig. c, we see that

$$(\tau_{1\theta})_{\max} = \text{radius of Mohr's circle} = \frac{R}{2I_2}\sqrt{M_1^2 + M_2^2} \tag{d}$$

According to the Tresca yield criterion (see Section 4.3), yielding occurs at a particle of a body when the maximum shearing stress acting on it is equal to the maximum shearing stress at yielding in uniaxial tension; that is, $\tau_{11}^Y/2$. Thus the radius of the bar for safe design with a factor of safety of three must satisfy the following relation:

$$\frac{\tau_{11}^Y}{2(\text{F.S.})} = \frac{0.240}{6} = 0.04 \geq (\tau_{i\theta})_{\max} = \frac{2R}{2I_2}\sqrt{M_1^2 + M_2^2} = \frac{2R(10^3)}{\pi R^4}\sqrt{12^2 + 16^2} = \frac{20(10^3)}{\pi R^3} \tag{e}$$

or

$$R^3 \geq \frac{40(10^3)}{\pi(0.04)} = 318.30\,(10^3)\ \text{mm}^3 \qquad \text{and} \qquad R \geq 68.28\ \text{mm} \tag{f}$$

Thus, according to the maximum shearing stress failure criterion the minimum radius of the bar required for safe design is 68.28 mm.

According to the Von Mises yield criterion, yielding occurs at a particle of a body when the components of stress acting on it satisfy relation (4.8b). For the state of stress of Fig. b, this relation gives

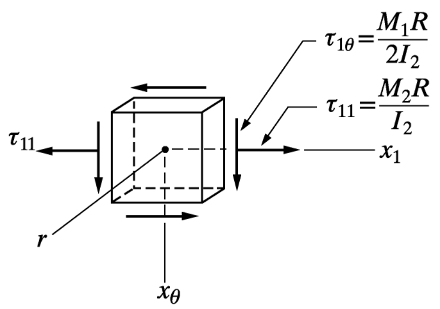


Figure b State of stress at $x_3 = R$.

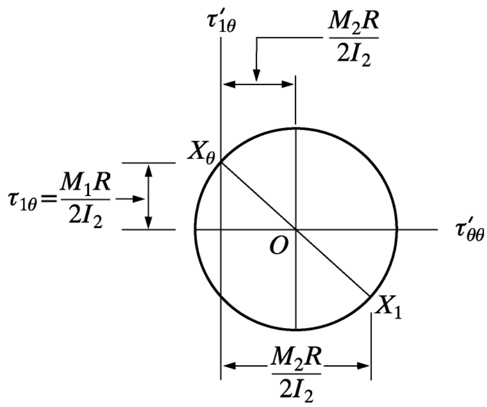


Figure c Mohr's circle for the state of stress at $x_3 = R$.

$$\frac{1}{2}(2\tau_{11}^2 + 6\tau_{1\theta}^2) = \frac{R^2}{I_2^2}(M_2^2 + 0.75M_1^2) = (\tau_{11}^Y)^2 \tag{g}$$

Consequently, for safe design of the bar with a factor of safety of 3, its radius must be such that

$$\begin{aligned}\left(\frac{\tau_{11}^Y}{\text{F.S.}}\right)^2 &= \left(\frac{0.240}{3}\right)^2 = (0.08)^2 \geq \frac{16R^2}{(\pi R^4)^2}(M_2^2 + 0.75M_1^2)\\ &= \frac{16(10^6)}{\pi^2 R^6}\left[12^2 + 0.75(16^2)\right] = \frac{544.70(10^6)}{R^6}\end{aligned} \tag{h}$$

or

$$R^6 \geq 851.10(10^8) \qquad \text{and} \qquad R \geq 66.32 \text{ mm} \tag{i}$$

Thus, according to the maximum distortion energy failure theory, the minimum radius of the bar for safe design is 66.32 mm. Referring to results (f) and (i), we see that the minimum radius of the bar for safe design based on the maximum shearing stress theory is about 3% bigger than that based on the maximum distortion energy theory.

4.6 The Maximum Normal Component of Stress Criterion for Fracture of Bodies Made from a Brittle, Isotropic, Linearly Elastic Material†

This criterion is based on the assumption that a body made from a brittle material fractures when the maximum absolute value of the normal component of stress acting at one of its particles is equal to the ultimate stress in uniaxial tension of the material from which the body is made. This implies that the response of the material to uniaxial

† This criterion is attributed to Rankine (1820–1872), a British scientist and educator.

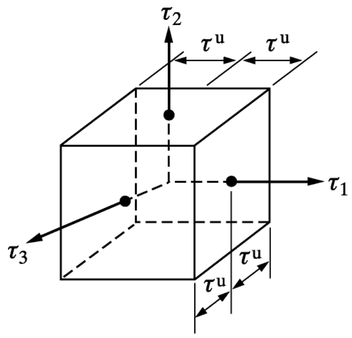


Figure 4.8 The maximum normal stress fracture criterion surface.

compression is the same as its response to uniaxial tension. As shown in Fig 4.8, in the principal stress space the maximum stress criterion is the surface of a cube with edges equal to twice the ultimate stress in uniaxial tension. Experimental results indicate that this criterion gives good results provided that a tensile principal stress exists at the particles of the body.

Example 3 A bar of circular cross section is made from a brittle, isotropic, linearly elastic material with ultimate stress in uniaxial tension or compression $\tau_{11}^{Y} = 360\ \text{MPa}$. The bar is subjected to two equal and opposite bending moments $M_2 = 12$ kN·m and to two equal and opposite torsional moments $M_1 = 16$ kN·m at its ends (see Fig. a). Determine the minimum diameter of the bar for safe design using the maximum normal component of stress criterion for fracture with a factor of safety of three.

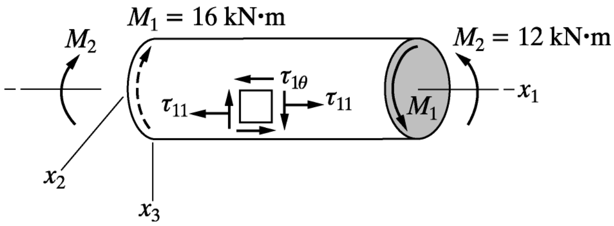


Figure a Bar subjected to torsional and bending moments.

Solution As discussed in the example of the previous section, the particles of the bar at $x_3 = \pm R$ are subjected to the maximum normal component of stress due to the bending moment and to the maximum shearing component of stress due to the torsional moment. The state of stress of these particles is shown in Fig. b of the example of the previous section, while Mohr's circle for this state of stress is shown in Fig. c of the example of the previous section. Referring to fig. c,we see that the maximum absolute value of the normal component of stress acting on the particles of the bar at $x_3 = \pm R$ is equal to

$$\left|(\tau_{\theta\theta})_{\max}\right| = \frac{M_2 R}{2I_2} + \frac{R}{2I_2}\sqrt{M_1^2 + M_2^2} = \frac{R}{2I_2}\left(M_2 + \sqrt{M_1^2 + M_2^2}\right) \qquad \text{(a)}$$

According to the maximum normal component of stress fracture criterion, in order to avoid fracture of the bar, its radius must satisfy the following relation:

$$\left|\frac{\tau_{11}^{u}}{\text{F.S.}}\right| = \frac{0.360}{3} = 0.12 \geq \frac{2R}{\pi R^4}\left(M_2 + \sqrt{M_1^2 + M_2^2}\right)$$
$$= \frac{2(10^3)}{\pi R^3}\left(12 + \sqrt{12^2+16^2}\right) = \frac{20.37(10^3)}{R^3} \tag{b}$$

or

$$R \geq 55.37 \text{ mm} \tag{c}$$

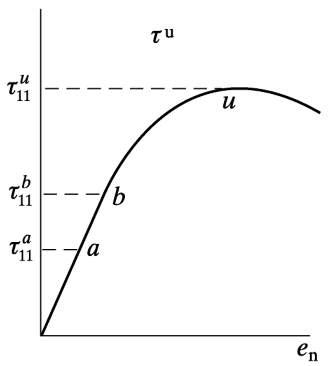
Thus, according to the maximum normal component of stress criterion for fracture, the minimum radius of the bar for safe design is 55.37 mm.

4.7 The Mohr's Fracture Criterion for Brittle Materials Subjected to States of Plane Stress

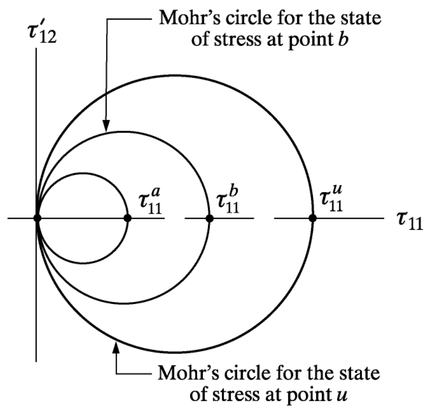
The "maximum normal stress" fracture criterion is valid for materials having the same properties in uniaxial tension and in uniaxial compression. There are brittle materials, however, as, for example, concrete, cast iron and soils whose properties in uniaxial tension are different than in uniaxial compression. A fracture criterion for such materials subjected to states of plane stress has been proposed by the German engineer Otto Mohr. In order to establish this criterion for a particular material, several different tests must be performed on it.

Consider a uniaxial tension test performed on a specimen made from a material whose fracture criterion for states of plane stress we are interested to establish. In Fig. 4.9a we show the stress–strain curve for this material and in Fig. 4.9b we plot Mohr's circles for the states of stress corresponding to points *a*, *b* and *u* of this stress–strain curve. It is clear that, when the magnitude of the applied force is equal to $P_1^{(a)} = \tau_{11}^{(a)}A$ or $P_1^{(b)} = \tau_{11}^{(b)}A$, the specimen under consideration is safe. The specimen fails when the applied force reaches the value $P_1^{(u)} = \tau_{11}^{(u)}A$. From Fig. 4.9b we see that Mohr's circles for safe states of stress lie inside the Mohr's circle for the ultimate state of stress.

Consider a body made from a brittle material whose ultimate stresses in uniaxial

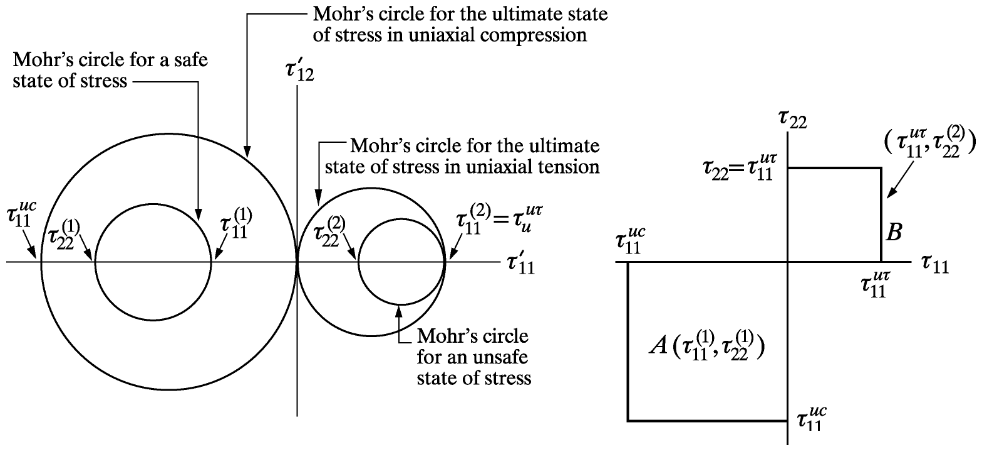


(a) Stress–strain curve in uniaxial tension



(b) Mohr's circles

Figure 4.9 Mohr's circles for state of stress at the particles of a specimen subjected to uniaxial tension.

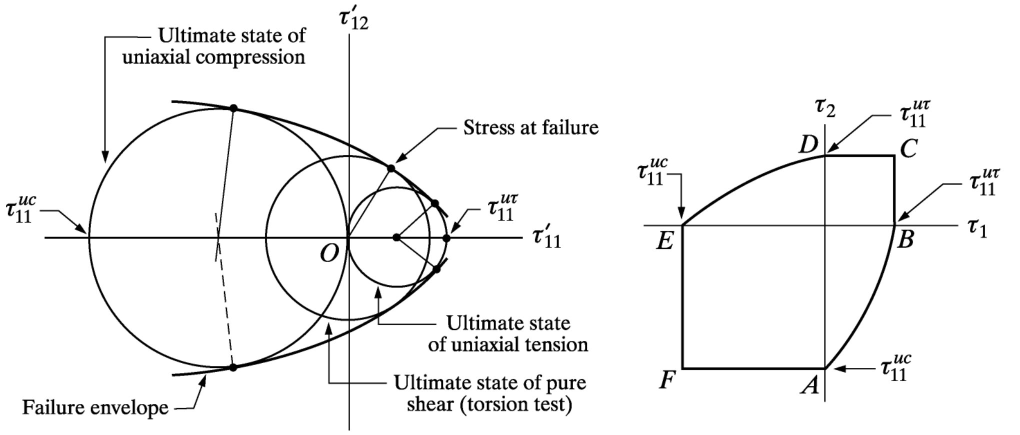


(a) Mohr's circles for a safe and an unsafe states of stress

(b) Fracture criterion for plane stress state with $\tau_1 > 0$ and $\tau_2 > 0$ or $\tau_1 < 0$ and $\tau_2 < 0$.

Figure 4.10 Safe and unsafe states of stress for a body in a state of plane stress whose principal stresses are either both tensile or both compressive.

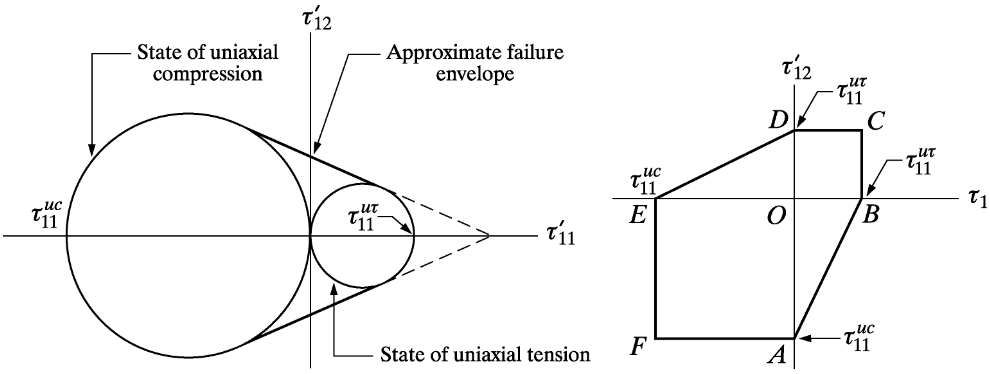
tension and compression are known. In Fig. 4.10a we plot Mohr's circles for these ultimate states of stress. Suppose that the body is subjected to loads which induce states of plane stress whose principal stresses τ_1 and τ_2 are either both tensile or both compressive. Mohr postulated that *the body under consideration is safe, when Mohr's circles for the states of stress of all its particles lie entirely within Mohr's circles for the ultimate state of stress in uniaxial tension or compression. A state of plane stress whose principal stresses are either both tensile or bothcompressive is unsafe when its maximum principal stress is equal to the ultimate stress in uniaxial tension or compression.* In Fig. 4.10a, we plot Mohr's circles for a safe and two unsafe plane states of stress. In Fig. 4.10b we plot, in the principal stress space, Mohr's fracture criterion for the body under consideration. Points A and B in this figure represent the safe and the unsafe states of stress, respectively, whose Mohr's circles are shown in Fig. 4.10a.



(a) Mohr's circles and fracture envelope

(b) Fracture criterion for states of plane stress

Figure 4.11 Mohr's fracture envelope and fracture criterion for states of plane stress.



(a) Mohr's circles and approximate fracture envelope

(b) Approximate fracture criterion for states of plane stress

Figure 4.12 Mohr's fracture envelope and approximate fracture criterion for states of plane stress.

For materials for which Mohr's postulate is assumed to be valid, we perform a number of different tests and from each test we draw Mohr's circle corresponding to the state of ultimate stress. For example, we can perform a uniaxial compression test and a torsion (pure shear) test in addition to the uniaxial tension test. In Fig 4.11a we plot Mohr's circles for the states of ultimate stress of these three tests as well as the envelope of these Mohr's circles. Mohr's fracture criterion states that a body in a state of plane stress made from an isotropic material is safe if Mohr's circles for the states of stress of its particles are located entirely within the area bounded by the envelope of Mohr's circles representing the state of ultimate stress of several tests (see Fig 4.11a). In Fig. 4.11b, we plot in the principal stress space Mohr's fracture criterion for states of plane stress.

If the only test results available for a material are from uniaxial tension and compression tests as shown in Fig 4.12a, Mohr's envelope may be approximated by straight lines tangent to the two Mohr's circles representing the ultimate state of stress in uniaxial tension and compression. The corresponding fracture criterion in the principal stress space is shown in Fig. 4.12b. Thus, Mohr's fracture criterion can be stated as follows:

> "A body in a state of plane stress fractures as soon as the loads acting on it reach such a value that the principal normal components of stress acting on one or more of its particles are coordinates of a point on line *ABCDEFA* of the fracture criterion of Fig. 4.11b or 4.12b."

For materials whose properties in tension are the same as their properties in compression, the geometry of Mohr's fracture criterion is similar to that of the Tresca yield criterion.

The fracture criterion of many cohesive materials such as concrete, soil or rock, depends on hydrostatic stress. An increase in the hydrostatic compressive stress increases the ability of the material to resist the applied stresses without fracturing. It is assumed that the addition of hydrostatic pressure does not fracture bodies made from such materials.

Example 4 A bar of circular cross section of radius R = 60 mm is made from a brittle, isotropic, linearly elastic material with ultimate stress in tension of 360 MPa and in compression of 720 MPa. The bar is fixed at its one end and is subjected to a bending moment $M_2 = 12$ kN·m and a torsional moment M_1 at its unsupported end. Using Mohr's failure criterion with a factor of safety of 2, determine the maximum allowable torsional moment M_1 to which the bar can be subjected.

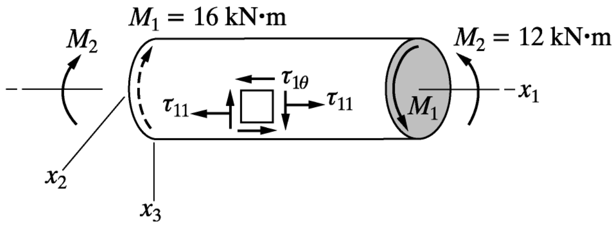


Figure a Geometry and loading of the bar.

Solution The particles of the bar at $x_3 = \pm R$ are subjected to the maximum absolute value of the normal component of stress as well as to the maximum value of the shearing stress. The state of stress of a particle at $r = R$ is shown in Fig. b. Mohr's circle for this state of stress is shown in Fig. c; referring to this figure, we see that

$$R_m = \sqrt{\left(\frac{\tau_{11}}{2}\right)^2 + \tau_{1\theta}^2} \tag{a}$$

$$\tilde{\tau}_1 = (\tau_{11})_{\max} = \frac{\tau_{11}}{2} + R_m > 0 \tag{b}$$

$$\tilde{\tau}_2 = (\tau_{11})_{\min} = \frac{\tau_{11}}{2} - R_m < 0 \tag{c}$$

where R_m is the radius of Mohr's circle. Since $R_m > \tau_{11}/2$, the minimum principal stress $\tilde{\tau}_2$ is compressive. Thus, referring to Fig. 4.12b, we see that the bar will fail when the value of the torsional moment is such that $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are the coordinates of a point of line AB. The equation of line AB is

$$\tilde{\tau}_2 = \frac{\tau_{11}^{UC}}{\tau_{11}^{UT}}\tilde{\tau}_1 - \frac{\tau_{11}^{UC}}{\text{F.S.}} = \frac{0.720}{0.360}\tilde{\tau}_1 - \frac{0.720}{2} = 2\tilde{\tau}_1 - 0.360 \tag{d}$$

Substituting relations (b) and (c) into (d), we obtain

$$\frac{\tau_{11} - 2R_m}{2} = \tau_{11} + 2R_m - 0.360 \qquad \text{or} \qquad R_m = 0.120 - \frac{\tau_{11}}{6} \tag{e}$$

The normal component of stress τ_{11} is obtained using relation (9.12b). That is,

$$\tau_{11} = \frac{M_2 R}{I_2} = \frac{12(1000)(4)}{\pi(60^3)} = 0.0707355 \text{ MPa} \tag{f}$$

Substituting relation (f) into (e), we get

$$R_m = 0.120 - \frac{0.0707355}{6} = 0.10821 \tag{g}$$

Substituting results (f) and (g) into relation (a), we have

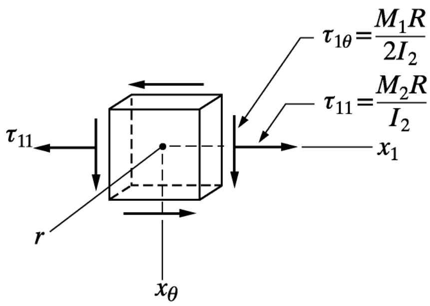


Figure b State of stress of a particle of the bar at $x_1 = R$.

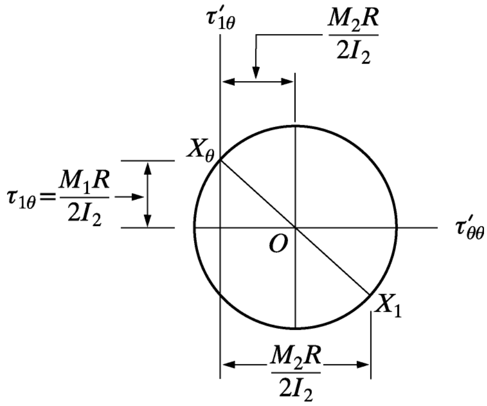


Figure c Mohr's circle for the state stress of a particle of the bar at $x_1 = R$.

$$(0.10821)^2 = \left(\frac{0.0707355}{2}\right)^2 + (\tau_{1\theta})^2_{\text{allowable}}$$

or

$$(\tau_{1\theta})_{\text{allowable}} = 0.102267 \frac{\text{kN}}{\text{mm}^2}$$

The allowable torsional moment is equal to

$$(M_1)_{\text{allowable}} = \frac{(\tau_{1\theta})_{\text{allowable}} I_p}{R} = \frac{0.102267\pi R^3}{2} = \frac{0.102267(60)^3\pi}{2}$$
$$= 34.695 \ \text{kN}\cdot\text{mm} = 34.695 \ \text{kN}\cdot\text{m}$$

4.8 Problems

1. The state of stress of a particle of a body made from an isotropic, linearly elastic–ideally plastic material with a yield stress $\tau^Y = 200$ MPa is given as

$$[\tau] = \begin{bmatrix} -60 & 40 & 0 \\ 40 & 80 & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix} \text{MPa}$$

Compute the value of τ_{33} required for yielding if the material obeys (a) the Tresca yield condition (b) the Von Mises yielding condition.

Ans. (a) $\tau_{33} = 129.38$ MPa or -109.38 MPa
(b) $\tau_{33} = 153.18$ MPa or -133.18 MPa

2. The state of stress of a particle of a body made from an isotropic linearly elastic–plastic material, is specified by two parameters τ_{11} and τ and it is given as

$$[\tau] = \begin{bmatrix} \tau_{11} & 0 & 0 \\ 0 & 0 & -4\tau \\ 0 & -4\tau & 8\tau \end{bmatrix}$$

Determine and plot in the τ_{11}, τ space the yield surface corresponding to this state of stress. The yield stress in uniaxial tension or compression for the material from which the body is made is $\tau^Y = 200$ Mpa. Assume that the material from which the body is made, obeys

(a) The Von Mises yield criterion Ans. $\tau_{11}^2 - 8\tau\tau_{11} + 112\tau^2 = 40{,}000$

(b) The Tresca yield criterion

Ans. The yield criterion is an exagon in the τ_{11}, τ plane, consisting of segments of the following six straight lines: $\tau = \pm 17.68\text{MPa}$ $\tau_{11} - 9.66\tau = \pm 200$ MPa $\tau_{11} + 1.66\tau = \pm 200$ MPa

3. Consider a particle of a body subjected to external forces and assume that its state of stress is specified by only one parameter τ and it is equal to

$$\text{(a)}\quad [\tau] = \begin{bmatrix} \tau & -\frac{3\tau}{4} & 0 \\ -\frac{3\tau}{4} & -\tau & 0 \\ 0 & 0 & \tau \end{bmatrix} \text{MPa} \qquad \text{(b)}\quad [\tau] = \begin{bmatrix} 0 & \frac{2\tau}{3} & 0 \\ \frac{2\tau}{3} & \tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{MPa}$$

Determine the values of τ for yielding if the body is made from steel ($\tau^Y = 200$ MPa) obeying (a) the Von Mises yield criterion, (b) the Tresca yield criterion.

Ans. (aa) $\tau = \pm 83.86$ MPa, (ab) $\tau = \pm 80$ MPa

(ba) $\tau = \pm 130.93$ MPa, (bb) $\tau = \pm 120$ MPa

4. For an isotropic, linearly elastic–ideally plastic material obeying the Von Mises yield condition, determine the values of τ for yielding for the following state of stress:

$$[\tau] = \begin{bmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & -\tau \end{bmatrix} \text{(a)} \quad [\tau] = \begin{bmatrix} \tau & 0 & 0 \\ 0 & -\tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(b)} \quad [\tau] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & \tau \end{bmatrix} \text{(c)} \quad [\tau] = \begin{bmatrix} \tau & \tau & 0 \\ \tau & \tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{(d)}$$

Ans. (a) $\tau = \pm\tau^y/2$ (c) $\tau = \pm\tau^y/2$

(b) $\tau = \pm\tau^y/\sqrt{3}$ (d) $\tau = \pm\tau^y/2$

5. The state of stress of a particle of a body made from an isotropic linearly elastic-plastic material, is specified by two parameters τ_{33} and τ and it is given as

$$[\tau] = \begin{bmatrix} 4\tau & 2\tau & 0 \\ 2\tau & \tau & 0 \\ 0 & 0 & \tau_{33} \end{bmatrix}$$

Determine and plot in the τ_{33}, τ space the yield surface corresponding to this state of stress. The yield stress in uniaxial tension or compression for the material from which the body is made is $\tau^Y = 200$ Mpa. Assume that the material from which the body is made obeys

(a) The Von Mises yield criterion

Ans. $\tau_{13}^2 - 5\tau\tau_{33} + 25\tau^2 = 40{,}000$

(b) The Tresca yield criterion

Ans. The yield criterion is an exagon in the τ , τ_{33} plane, consisting of segments of the following six straight lines: $\tau = \pm 40\text{MPa}$ $\quad 5\tau - \tau_{33} = \pm 200$ MPa $\quad \tau_{33} = \pm 200$ MPa

6. Consider a prismatic bar of circular cross section of radius R made from an isotropic, linearly elastic material of modulus of elasticity E and shear modulus G. When this bar is subjected to equal and opposite torsional moments of magnitude M_1 at its ends, its stress field is given as

$$[\tau] = \begin{bmatrix} 0 & 0 & \tau_{1\theta} \\ 0 & 0 & 0 \\ \tau_{\theta 1} & 0 & 0 \end{bmatrix}$$

where, referring to Fig. 3.11a, we have

$$\tau_{1\theta} = \frac{M_1 r}{I_p}$$

When the bar under consideration is subjected to equal and opposite axial centroidal tensile forces of magnitude P_1 at its ends, its stress field is given as

$$[\tau] = \begin{bmatrix} P_1/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the combination of P_1 and M_1 which is required for incipient yielding.

(a) Use the Von Mises yield criterion. Ans. $R^2 P_1^2 + 12M_1^2 = \pi^2 R^6 (\tau^Y)^2$

(b) Use the Tresca yield criterion. Ans. The Tresca yield criterion is a quadra-lateral defined by segments of the ellipse $R^2 P_1^2 + 16M_1^2 = \pi^2 R^6 (\tau^Y)^2$ and of the parabolas $\dfrac{4\, M_1^2}{(\pi R^2 \tau^Y)^2} - \pi R^2 \tau^Y = \pm P_1$

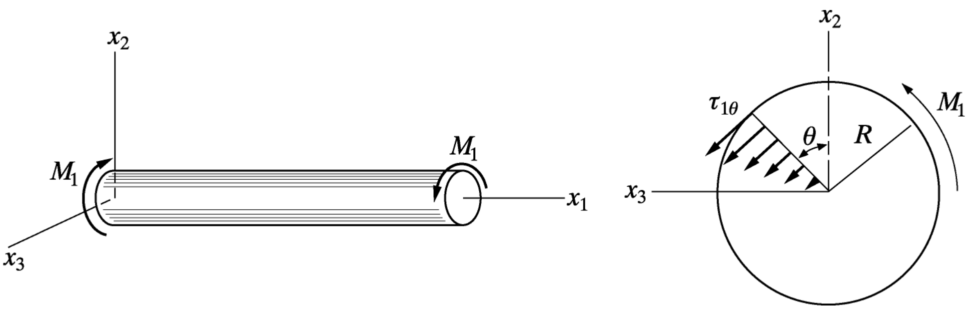
7. An 80-pmm diameter rod is made from an isotropic, linearly elastic–ideally plastic material which obeys the Von Mises yield criterion with $\tau^Y = 250$ MPa. The rod is subjected to equal and opposite torsional moments of magnitude $M_1 = 10$ kN·m at its ends, and subsequently to equal and opposite axial centroidal forces of magnitude P_1. Compute the magnitude of the axial forces P_1 for which some particles of the rod begin to yield. (*Hint*: $\tau_{1\theta} = M_1 r/I_p \quad \tau_{11} = P_1/A$) Ans. $P_1 = 910.6$ kN

8. and **9.** A particle of a body begins to yield when subjected to the state of plane stress shown in Fig. 4P8. The body is made from steel (E = 200 GPa, τ^Y = 250 MPa). Determine the factor of safety used in designing the body, assuming that the body fails as soon as yielding begins at the particle under consideration on the basis of the Von Mises yield criterion. Repeat with the particle subjected to the state of stress shown in Fig. 4P9.

Ans. 8 F.S. = 1.05
Ans. 9 F.S. = 1.44

10. Using the maximum distortional energy failure criterion with a factor of safety of three, determine the minimum safe width of the cantilever beam subjected to a concentrated force as shown in Fig. 4P10. The beam is made from a ductile, isotropic, linearly elastic material with a yield stress in tension and compression of τ^Y = 240 MPa. (*Hint*: $\tau_{11} = M_2 x_3 / I_2$).

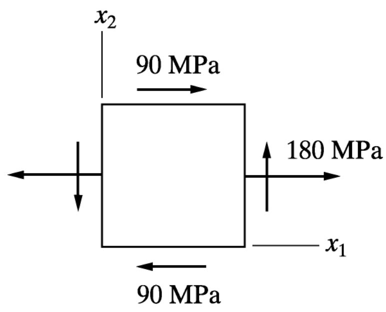
Ans. b = 24 mm



(a) Bar subjected to equal and opposite torsional moments at its ends

(b) Stress distribution on the cross section of the bar due to the torsional moments

Figure 4P6



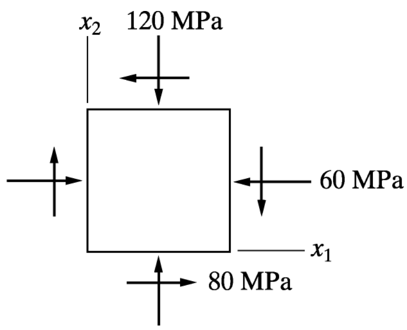


Figure 4P8

Figure 4P9

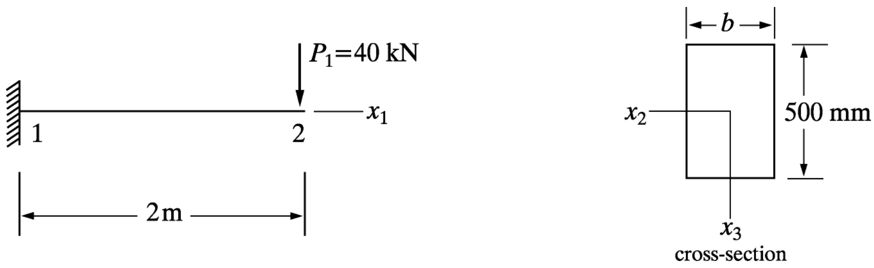


Figure 4P10

11. A body made from a ductile, isotropic, linearly elastic material with a yield stress in pure shear of 80 MPa fails according to the maximum distortion energy failure criterion, as soon as the state of stress of one of the particles is equal to

$$\tau = m \begin{bmatrix} 2 & -2 & 2 \\ -2 & 4 & 3 \\ 2 & 3 & 6 \end{bmatrix} \text{MPa}$$

Using a factor of safety of two, compute the allowable value of m.

Ans. $m^2 = 76.19$

12. A body is subjected to loads of such magnitude that yielding has just begun at one of its particles. The state of stress of this particle is

$$\tau = \begin{bmatrix} 12 & 32 & 24 \\ 32 & 72 & 44 \\ 24 & 44 & 120 \end{bmatrix} \text{MPa}$$

The body is made from a ductile, isotropic, linearly elastic material with yield stress in uniaxial tension of 320 MPa. Determine the F.S. used in the design of the body assuming (a) the maximum distortional energy failure criterion and (b) the maximum shearing stress failure criterion.

Ans. (a) F.S. = 2.298 (b) F.S. = 2.027

13. A thin-walled cylinder of radius R = 120 mm and thickness t = 10 mm is made from a brittle, isotropic, linearly elastic material, whose ultimate uniaxial tensile and compressive stresses are τ_{11}^{UT} = 360 MPa and τ_{11}^{UC} = 720 MPa, respectively. Using a F.S. of two, compute the maximum allowable axial centroidal force which can be applied to the cylinder when it is already subjected to a torsional moment of 24 kN·m.
Employ Mohr's fracture criterion. (*Hint*: $\tau_{1\theta}=M_1 r/I_p \quad \tau_{11}=P_1/A$)

Ans. Max. allow. tensile axial force = 1,245.34 kN
Max. allow. compr. axial force = 2,545.96 kN

14. A solid cylinder must withstand a maximum torsional moment of 24 kN·m together with a bending moment of 12 kN·m. The cylinder is to be made from a brittle, isotropic, linearly elastic material with ultimate stress in tension and compression of 320 MPa and 800 MPa, respectively. Using Mohr's failure criterion with a F.S. of two, determine the minimum allowable radius of the cylinder. (*Hint*: $\tau_{11} = 2M_2 x_3 / I_p \quad \tau_{1\theta} = M_1 r / I_p$.)

Ans. $r_{\text{allow.}} = 50.76$ mm

15. A solid cylinder must withstand a maximum torsional moment of 24 kN·m together with a bending moment of 12 kN·m. The cylinder is to be made from a brittle, isotropic, linearly elastic material with ultimate stress in tension and compression of 320 MPa. Using the maximum normal component of stress fracture criterion with a F.S. of two, determine the minimum allowable radius of the cylinder.

(*Hint*: $\tau_{11} = 2M_2 x_3 / I_p \quad \tau_{1\theta} = M_1 r / I_p$.) Ans. $r_{\text{allow.}} = 53.65$mm

16. Prove that the total shearing stress acting on a face of the octahedron (shown in Fig. 4.3) is given by relation (4.10).

Chapter 5

Formulation and Solution of Boundary Value Problems Using the Linear Theory of Elasticity

5.1 Introduction

The mathematical formulation of a time-independent problem (equilibrium or steady state) using the continuum model (see Section 2.1) is referred to as a *boundary value problem*. It involves the determination of a scalar, a vector or a tensor function of the space coordinates which satisfies a differential equation in a certain region called the *domain of the problem* as well as appropriate specified conditions on the boundary of the domain called the *boundary conditions*.

In this book we consider solid bodies initially in a reference stress-free, strain-free state of mechanical† and thermal†† equilibrium at a uniform temperature T_o. In this state the bodies are not subjected to external loads and heat does not flow in or out of them. Subsequently, the bodies are subjected to specified external loads described in Section 2.2, as a result of which they deform and reach a second state of mechanical, but not necessarily thermal, equilibrium. We formulate and solve boundary value problems for computing the displacement and stress fields of solid bodies subjected to external loads, using two theories: the linear theory of elasticity and the theories of mechanics of materials.

The linear theory of elasticity can be employed to formulate boundary value problems for computing the displacement, and stress fields of bodies of any geometry subjected to any loading. However, only a few such problems have been solved exactly. The rest are solved approximately using one of the modern numerical methods suitable for programming their solution on an electronic computer. The finite elements method is the most popular of these methods. Examples of boundary value problems formulated on the basis of the linear theory of elasticity are solved exactly in this and the next two chapters. They involve bodies of simple geometry supported in an idealized convenient way and subjected to external loads which induce states of stress having some vanishing components.

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† When a body is in a state of mechanical equilibrium, its particles do not accelerate. That is, the sum of the forces acting on any portion of the body and the sum of their moments about any point vanish.

†† When a body is in a state of thermal, equilibrium heat does not flow in or out of it. That is, the temperature of all its particles is the same.

The linear theory of elasticity is based on the following assumptions:

1. The response of a body may be approximated by that of the continuum model (see Section 2.1).
2. The deformation of the particles of a body is in the range of validity of the assumptions of small deformation (see Sections 2.4 and 2.5).
3. The effect of the deformation of a body on the change of its temperature is negligible.
4. The relations among the components of stress and strain are linear.

The theories of mechanics of materials are approximate. They can be employed to formulate boundary value problems for computing the displacement and stress fields in solid bodies of certain geometries (see Section 8.1). Most of these problems can be solved exactly by hand calculation if the geometry and loading of the body are simple or with the aid of a computer if the geometry and loading of the body are complex.

In Chapters 8 to 18 we formulate and solve boundary value problems for computing the displacement and stress fields of solid bodies subjected to external loads, using the theories of mechanics of materials.

5.2 Boundary Value Problems for Computing the Displacement and Stress Fields of Solid Bodies on the Basis of the Assumption of Small Deformation

Consider a solid body initially in a reference stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. Subsequently, the body is subjected to specified external loads, as a result of which it deforms and reaches a second state of mechanical, but not necessarily thermal, equilibrium. These loads could include one or more of the following:

1. Specified components of force applied to every particle of the body due to its presence in a force field as, for example, the gravitational field of the earth (see Section 2.2). They are called *components of the specific body force* and are given in units of force per unit volume of the body. We denote the specific body force by

$$\mathbf{B}(x_1, x_2, x_3) = B_1(x_1, x_2, x_3)\mathbf{i}_1 + B_2(x_1, x_2, x_3)\mathbf{i}_2 + B_3(x_1, x_2, x_3)\mathbf{i}_3$$

2. Specified components *of surface traction* applied on some particles of the surface of the body due to its contact with other bodies. They are given in units of force per unit

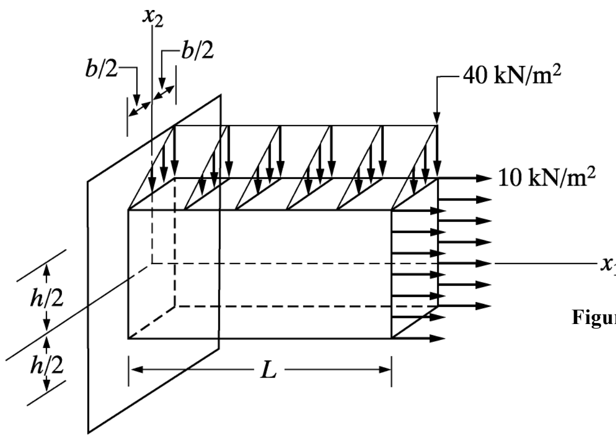

Figure 5.1 Body fixed on a portion of its surface and subjected to traction on another.

area of the surface of the body. Consider a point of the surface of the body whose coordinates are x_1^s, x_2^s, x_3^s and the unit vector normal to the surface of the body at this point is $\mathbf{i_n}$. We denote the surface traction at this point by

$$\overset{n}{\mathbf{T}}{}^s(x_1^s, x_2^s, x_3^s) = \overset{n}{T}{}_1^s(x_1^s, x_2^s, x_3^s)\mathbf{i}_1 + \overset{n}{T}{}_2^s(x_1^s, x_2^s, x_3^s)\mathbf{i}_2 + \overset{n}{T}{}_3^s(x_1^s, x_2^s, x_3^s)\mathbf{i}_3$$

3. Specified components of displacement of the other particles of the surface of the body. The points of the surface of the body where components of displacement are specified are called its *supports*. The tractions exerted by the supports of a body on its particles are not known. They are called the *reacting tractions* of the body and are determined as part of the solution of the problem. This is accomplished by substituting the computed components of stress into the traction–stress relations (2.73) and evaluating the resulting expressions at the points of the surface of the body where components of displacement have been specified.

4. Specified temperature $T(x_1^s, x_2^s, x_3^s)$ of some points of the surface of the body.

5. Specified temperature gradient $\partial T/\partial x_n$ at the other points of the surface of the body. x_n is the coordinate of a point measured in the direction normal to the surface of the body at that point.

For a properly formulated problem, four quantities must be specified at each point of the surface of a body, one from each of the following pairs of quantities:

$$\begin{array}{ccc} \hat{u}_1^s & \text{or} & \overset{n}{T}{}_1^s \\ \hat{u}_2^s & \text{or} & \overset{n}{T}{}_2^s \\ \hat{u}_3^s & \text{or} & \overset{n}{T}{}_3^s \\ T & \text{or} & \dfrac{\partial T}{\partial x_n} \end{array} \tag{5.1}$$

The quantities which are specified at the points of the surface of a body are called its *boundary conditions*. When the components of displacement and the temperature are specified at a point of the surface of a body, the boundary conditions at that point are called *essential*. When the components of traction and the temperature gradient are specified at a point of the surface of a body, the boundary conditions at that point are called *natural*. It is possible to have mixed boundary conditions at some points of the surface of a body. That is, one component of displacement could be specified (say u_1) and two components of traction ($\overset{n}{T}_2$ and $\overset{n}{T}_3$) or two components of displacement could be specified (say u_1 and u_2) and one component of traction $\overset{n}{T}_3$. As an example, consider the body of Fig. 5.1. Its boundary conditions are

$$\hat{u}_i(0, x_2, x_3) = 0 \qquad (i = 1,2,3)$$

$$\begin{aligned} &\overset{1}{T}{}_1^s(L, x_2, x_3) = 10\ \frac{\text{kN}}{\text{m}^2} && \overset{-2}{T}{}_i^s\left(x_1, -\frac{h}{2}, x_3\right) = 0 \quad (i = 1,2,3) \\ &\overset{1}{T}{}_i^s(L, x_2, x_3) = 0 \quad (i = 2,3) && \overset{\pm 3}{T}{}_i^s\left(x_1, x_2, \pm\frac{b}{2}\right) = 0 \quad (i = 1,2,3) \end{aligned} \tag{5.2}$$

Notice that the distribution of the component of traction $\overset{n}{T}{}_i^s (i = 1, 2, 3)$ on the surface of a body in equilibrium cannot be specified throughout its whole surface independently of the distribution of the components of the specific body force $B_i (i = 1, 2, 3)$. These two distributions must satisfy the conditions imposed by the equilibrium of the body as a whole. This requirement does not arise if the components of displacement $\hat{u}_i (i = 1, 2, 3)$ are specified at some points of the surface of the body. In this case, at the points where the component of displacement $\hat{u}_i$ is specified, the solution of the problem yields

$$\overset{2}{T}{}_i^s \left(x_1, \frac{h}{2}, x_3\right) = 0 \qquad (i = 1, 3)$$

$$\overset{2}{T}{}_2^s \left(x_1, \frac{h}{2}, x_3\right) = \frac{40\ x_3}{b} - 20$$

components of traction $\overset{n}{T}_i$ satisfying the equilibrium of the particle on which they act as well as the equilibrium of the body as a whole.

If the temperature T is not maintained uniform throughout its volume, the body does not reach a state of thermal equilibrium because heat will be flowing in or out of it. If during a process of deformation the temperature distribution on the surface of the body remains constant ($T = T_o$), the temperature distribution inside the volume of the body may change slightly and heat may be absorbed from or emitted to the environment. However, when the body reaches the second state of equilibrium, the temperature distribution inside its volume becomes uniform again and equal to the temperature T_o of its surface. Consequently, heat does not flow in or out of the body. When a body made from an elastic material is subjected to external forces in an environment of constant temperature, the amount of heat absorbed from or emitted to the environment during loading is equal to the amount of heat emitted to or absorbed from the environment during unloading (reversible heat).

We are interested in establishing the displacement, stress and temperature fields in the body under consideration. In general these quantities must satisfy the following laws:

1. The principle of conservation of mass. That is, the mass of any portion of a body does not change during a process of deformation. Denoting by dm the mass of a particle of a body on the basis of this principle, we have

$$dm = \rho_o\, dV_o = \rho\, dV \tag{5.3}$$

where

dV_o = the volume of the particle in the undeformed state.
dV = the volume of the particle in the deformed state.
ρ_o = the density of the particle in the undeformed state.
ρ = the density of the particle in the deformed state.

For deformation within the range of validity of the assumption of small deformation we have

$$dV \approx dV_o$$

Consequently for the boundary value problems under consideration, the principle of conservation of mass reduces to

$$\rho \approx \rho_o \tag{5.4}$$

That is, the change of the density of a particle of the body due to its deformation is negligible compared to its density before deformation.

2. The sum of the forces acting on any particle of a body in equilibrium that must vanish. In Section 2.13 we show that on the basis of this law the stress field of the body must satisfy the following requirements:

(a) The three equations of equilibrium (2.69) at every point inside the volume of the body. They can be written as

$$\begin{aligned} \sum_{j=1}^{3} \frac{\partial \tau_{1j}}{\partial x_j} + B_1 &= 0 \\ \sum_{j=1}^{3} \frac{\partial \tau_{2j}}{\partial x_j} + B_2 &= 0 \\ \sum_{j=1}^{3} \frac{\partial \tau_{3j}}{\partial x_j} + B_3 &= 0 \end{aligned} \tag{5.5}$$

(b) When substituted into the traction–stress relations, (2.73) must give components of traction which are equal to the specified components of traction when evaluated at the points of the surface of the body where components of traction are specified.

3. The sum of the moments, about any point, of the forces acting on any portion of a body in equilibrium that must vanish. In Section 2.13 we show that this law is satisfied if the stress tensor is symmetric. That is,

$$\tau_{ij} = \tau_{ji} \qquad i, j = 1, 2, 3 \tag{5.6}$$

4. The six strain–displacement relations. That is,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i, j = 1, 2, 3 \tag{5.7}$$

From relations (5.7) we see that the strain tensor is symmetric

$$e_{ij} = e_{ji} \qquad i, j = 1, 2, 3 \tag{5.8}$$

5. The relations between the components of stress and strain for the material from which the body is made.

6. The first law of thermodynamics, also known as the principle of conservation of energy. This law may be expressed as

$$\mathrm{Q} = (\Delta \mathrm{E}) - \mathrm{W} \tag{5.9}$$

where

Q = the heat absorbed by the portion of the body under consideration, from its environment during the process of deformation.

ΔE = the change of the internal energy of the portion of the body under consideration due to its deformation.

W = the work done by the external forces (surface tractions and body forces) acting on the portion of the body under consideration in moving from their position in the undeformed configuration of the body to their position in its deformed configuration.

It can be shown that relation (5.9) couples the temperature field with the strain field of a body. Thus, the deformation of a body affects the distribution of its temperature. However, *we assume that the effect of the deformation of a body on the distribution of its temperature is very small and can be neglected*. On the basis of this assumption relation (5.9) involves only the temperature field and it is known as *the heat conduction equation.* Consequently, the boundary value problem for computing the temperature, the displacement and the stress fields of a body is divided into the following two parts:

1. A boundary value problem which involves the determination of the temperature field of the body (known as *the heat conduction problem*) which satisfies the heat conduction equation at every point inside the volume of the body and the specified boundary conditions at the points of the surface of the body. For a correctly formulated problem either the temperature or its gradient $\left(\partial T/\partial x_n\right)$ must be specified at every point of the surface of the body.

2. A boundary value problem which involves the determination of the displacement and stress fields of the body using the temperature field established by solving the heat conduction problem.

In this book we assume that the temperature field of the bodies which we are considering has been established and it is known.

On the basis of the foregoing presentation it is clear that the boundary value problems for computing the components of displacement strain and stress fields in a solid body involve the following 15 unknown quantities:

$$u_1,\ u_2,\ u_3,\ e_{11},\ e_{22},\ e_{33},\ e_{12},\ e_{13},\ e_{23},\ \tau_{11},\ \tau_{22},\ \tau_{33},\ \tau_{12},\ \tau_{13},\ \tau_{23}$$

Moreover, referring to relations (5.5) and (5.7) and taking into account that the six components of stress are related to the six components of strain by six stress–strain relations, we find that 15 equations are available for the boundary value problems under consideration.

The boundary value problems for computing the displacement strain and stress fields, of bodies made from a homogeneous, isotropic, linearly elastic material are known as the boundary value problems of the linear theory of elasticity and are usually formulated either in terms of the components of displacement or in terms of the components of stress. We present these formulations in Sections 5.2.1 and 5.2.2, respectively.

As discussed in Section 3.13 the relations between the external loads acting on a body made from a linearly elastic material and the components of stress, the components of strain or the components of displacement are linear when its deformation is in the range of validity of the assumption of small deformation. That is, the effects are linearly related to the causes. In this case we say that the *response of the body is linear. A direct consequence of the linear response of a body is that the principle of superposition is valid for this body.* That is, its response due to a number of simultaneously applied loads is equal to the sum of its responses due to the application of each one of these loads separately.

The linear theory of elasticity is based on the following four assumptions:

1. The bodies which are considered are made from a continuum (see Section 2.1).
2. The magnitude of the external loads acting on the bodies which we are considering is such that their deformation is within the range of validity of the assumption of small deformation (see Section 2.4). As a result of this assumption, the following approximations can be made:

(a) The deformation of a particle of a body is completely specified by its components of strain which are related to first derivatives of the components of displacement $\hat{u}_1(x_1, x_2, x_3)$, $\hat{u}_2(x_1, x_2, x_3)$ and $\hat{u}_3(x_1, x_2, x_3)$ of the particle by the linear relations (2.16).

(b) The change of length, area or volume of a segment of a body due to its deformation is negligible compared to its undeformed length, area or volume, respectively. Consequently, when we consider the equilibrium of a portion of a body (finite of infinitesimal), we do not take into account the change of its dimensions due to its deformation. That is, when we draw its free-body diagram we use its undeformed configuration.

3. The relations among the components of stress and strain are linear and independent of time and of the history of loading and unloading. In this book we limit our attention to bodies made from isotropic, linearly elastic materials. Their stress–strain relations are (3.94) and (3.95).
4. The effect of deformation of a body on the distribution of its temperature can be neglected. As a result of this assumption the boundary value problems for computing the displacement stress and temperature fields in a body are divided into two uncoupled problems: one involving the computation of only the temperature field and the other involving the computation of only the displacement and the stress fields.

It can be shown that the boundary value problems formulated in this section have a solution[†], which moreover, is unique[††] provided that the specified components of displacement prevent the body from moving as a rigid body. If the specified components of displacement do not prevent the body from moving as a rigid body the solution is only unique to within a rigid-body motion.

5.2.1 Strong Form–Displacement Formulation

The strong form–displacement formulation of the boundary value problems for computing the components of displacement and stress of the particles of solid bodies made from linearly elastic materials when they are in the second state of mechanical equilibrium described in Section 5.2 can be formulated.

Establish the components of displacement $\hat{u}_i(x_1, x_2, x_3)(i = 1, 2, 3)$ which have the following attributes:

1. They are bounded, single-valued functions of the space coordinates having continuous

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† Rigorous but lengthy proofs of the existence of a solution is presented in the following publications: Korn, A., Uber die losung des grund problems des elastizitatsttheorie, *Mathematisch Annalen,* 75, p. 497-544, 1914, Lichtenstein, L., Uber die erste randwertaufgabe der elastizitatsttheorie, *Mathematisch Zeitschrift,* 20, p. 21-28, 1924.

†† A proof that the solution is unique is presented in the following publication: Kirchhoff, G., *Vorlesunger uber Mathematisch Phys. Mechan.* 3rd edition, Leipzig, 1882.

first derivatives.

2. They satisfy the specified essential (displacement) boundary conditions at the points of the surface of the body where such conditions are specified. If we assume that the components of displacement $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ of the particles of the portion $S-S_t$ of the surface of the body are specified, the boundary conditions at this portion are

$$\{\hat{u}\} = \begin{Bmatrix} \hat{u}_1(x_1, x_2, x_3) \\ \hat{u}_2(x_1, x_2, x_3) \\ \hat{u}_3(x_1, x_2, x_3) \end{Bmatrix} = \begin{Bmatrix} \hat{u}_1^s(x_1^s, x_2^s, x_3^s) \\ \hat{u}_2^s(x_1^s, x_2^s, x_3^s) \\ \hat{u}_3^s(x_1^s, x_2^s, x_3^s) \end{Bmatrix} \text{ on } S - S_t \tag{5.10}$$

where $\hat{u}_1^s$, $\hat{u}_2^s$, $\hat{u}_3^s$ are the specified components of displacement and x_1^s, x_2^s, x_3^s are the coordinates of the points on the surface of the body.

3. When substituted into the strain–displacement relations, components of strain which on the basis of the stress–strain relations for the material from which the body is made give components of stress which satisfy the conditions for equilibrium of every particle of the body. That is,

(a) They satisfy the specified natural (traction) boundary conditions at the points of the surface of the body where such conditions are specified. In Section 2.13 we show that the components of surface traction acting on a particle located at a point of the surface of the body are related to the components of stress acting on this particle by relations (2.73). If we assume that the components of traction acting on the particles of the portion S_t of the surface S of the body are specified, the boundary conditions at this portion are

$$\{\overset{n}{T}\} = \begin{Bmatrix} \tau_{11}\lambda_{n1} + \tau_{21}\lambda_{n2} + \tau_{31}\lambda_{n3} \\ \tau_{12}\lambda_{n1} + \tau_{22}\lambda_{n2} + \tau_{32}\lambda_{n3} \\ \tau_{13}\lambda_{n1} + \tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3} \end{Bmatrix} = \begin{Bmatrix} \overset{n}{T}{}_1^s(x_1^s, x_2^s, x_3^s) \\ \overset{n}{T}{}_2^s(x_1^s, x_2^s, x_3^s) \\ \overset{n}{T}{}_3^s(x_1^s, x_2^s, x_3^s) \end{Bmatrix} \text{ on } S_t \tag{5.11}$$

where $\overset{n}{T}{}_1^s$, $\overset{n}{T}{}_2^s$, $\overset{n}{T}{}_3^s$ are the specified components of traction acting at a point (x_1^s, x_2^s, x_3^s) of the portion S_t of the surface of the body. We denote by $\mathbf{i}_n = \lambda_{n1}\mathbf{i}_1 + \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ the unit vector outward normal to the surface at that point. The satisfaction of relation (5.11) ensures that the components of stress acting on the particles of the portion S_t of the surface of the body where components of traction are specified are in equilibrium with the specified components of traction.

(b) They satisfy the equations of equilibrium (5.5) at every point inside the volume of the body. This ensures that the sum of the forces acting on each particle which is located inside the volume of the body is equal to zero. For bodies made from isotropic, linearly elastic materials the equations of equilibrium (5.5) can be written in terms of the components of displacement $\hat{u}_i(x_1, x_2, x_3)$ ($i = 1, 2, 3$). That is, substituting the strain–displacement relations (2.16) into the stress–strain relations (3.47) and the resulting expressions into the equations of equilibrium (5.5), we obtain the following three displacement equations of equilibrium:

$$\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2} + \frac{1}{1-2\nu}\frac{\partial II_1}{\partial x_i} + \frac{2(1-\nu)}{E}B_i = 0 \quad (i = 1,2,3) \tag{5.12}$$

where II_1 is the first invariant of strain. That is, referring to relation (1.78), we have

$$II_1 = e_{11} + e_{22} + e_{33} \tag{5.13}$$

The reacting tractions of a body are computed by substituting the calculated components of stress into the traction–stress relations (2.73) and evaluating the resulting expressions at the points of the surface of the body where components of displacement are specified.

5.2.2 Strong Form–Stress Formulation

The strong form–stress formulation of the boundary value problem for computing the displacement and stress fields of solid bodies made from linearly elastic materials, when they are in the second state of mechanical equilibrium described in Section 5.2, can be formulated as shown below.

Establish the components of stress $\tau_{ij}(x_1, x_2, x_3)$ $(i, j = 1, 2, 3)$ which have the following attributes:

1. They are bounded single-valued continuous functions of the space coordinates.
2. They are symmetric $(\tau_{ij} = \tau_{ji})$.
3. They satisfy the equations of equilibrium (5.5) at every point inside the volume of the body.
4. They satisfy the natural (traction) boundary conditions at every point of the surface of the body where such conditions are specified.
5. On the basis of the stress–strain relations for the material from which the body is made, they give components of strain which when substituted into the strain–displacement relations (5.7) the resulting relations can be integrated to give components of displacement which are single-valued continuous functions of the space coordinates. A necessary condition, which ensures that the components of strain satisfy this requirement is that they satisfy the compatibility equations (2.63) at every point of the body. For simply connected (without holes) bodies the satisfaction of the equations of compatibility is a necessary and sufficient condition. For multiply connected (with holes) bodies the satisfaction of the equations of compatibility ensures only that the strain–displacement relations can be integrated. It does not ensure that the components of displacement will be continuous single-valued functions. This can be ensured by requiring that the components of strain satisfy additional relations (see Section 6.6).
6. The components of displacement obtained from the components of stress satisfy the essential (displacement) boundary conditions at the points of the surface of the body where such conditions are specified.

The reactions of the body are computed as discribed in the previous subsection.

5.3 The Principle of Saint Venant

The exact solution of static boundary value problems involving the determination of the components of displacement and stress of bodies made from linearly elastic materials

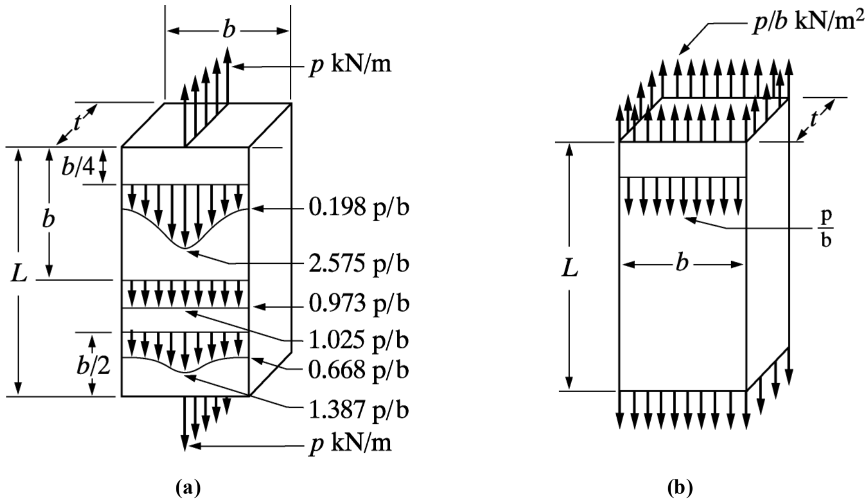

Figure 5.2 Bar subjected to statically equivalent distributions of external forces on its end surfaces and corresponding stress distribution† near the end surface.†

is usually mathematically complex. The solution of certain boundary value problems could be simplified by allowing the components of stress to satisfy convenient statically equivalent traction boundary conditions instead of the actual ones over one or more small portions of the surface of the body. The license for such modification of the boundary conditions of a body was set forth by Barré de Saint Venant (1797–1886), a French mathematician, in the form of the following principle:

> "If some distribution of traction acting on a small portion of the surface of a body is replaced by a different distribution of traction acting on the same portion of the surface of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the traction are essentially the same, provided that the two distributions of traction are statically equivalent."

"Statically equivalent" distributions of traction have the same resultant force and the same moment about any chosen point.

On the basis of the principle of Saint Venant the distribution of the components of stress in the bar loaded in the two different ways shown in Fig. 5.2 is the same at points sufficiently distant from the region where the load is applied provided that t and b are small compared to L. The results shown in Fig. 5.2 have been obtained on the basis of the linear theory of elasticity. As can be seen from Fig. 5.2, at sections sufficiently removed from the end surfaces of the bar where the load is applied, the distribution of the components of stress due to a line load approaches the distribution of the components of stress due to an equivalent uniformly distributed load. However, at sections close to the end surfaces of the bar, the distribution of the components of stress due to the line load is non-uniform. Similarly, the distribution of the components of stress in the beam loaded in the three different ways shown in Fig. 5.3 is the same at a sufficient distance from the region where the load is applied, provided that b is small compared to the length of the beam.

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† These results are adopted from Timoskenko, S. and Goodier, J.N., *Theory of Elasticity*, Mc-Graw Hill, New York, 1951, p. 52.

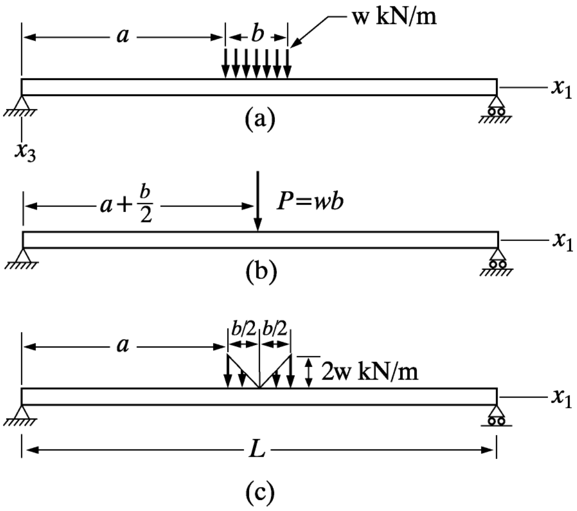


Figure 5.3 Statically equivalent distribution of forces acting on a beam.

The applicability of the principle of Saint Venant is limited to static problems. In dynamic problems, a disturbance may be propagated at distances far removed from the region of its application. Moreover, in static problems, the principle of Saint Venant is valid only if there are paths available through which the internal forces may balance themselves†. This may be illustrated by considering the very long truss shown in Fig. 5.4, loaded on its one end by a self-equilibrated force system. On the basis of the principle of Saint Venant, the effect of this load should vanish a short distance from the face on which it is applied. However, as it can be seen in Fig. 5.4 some members of truss far removed from the point of application of the external forces are stressed appreciably.

In certain boundary value problems we do not know the distribution of the components of traction acting on one or more small portions of the surface of the body. Instead, we know their resultant force and moment. Thus, in the solution of such problems we can only ensure that the components of stress give the known resultant force

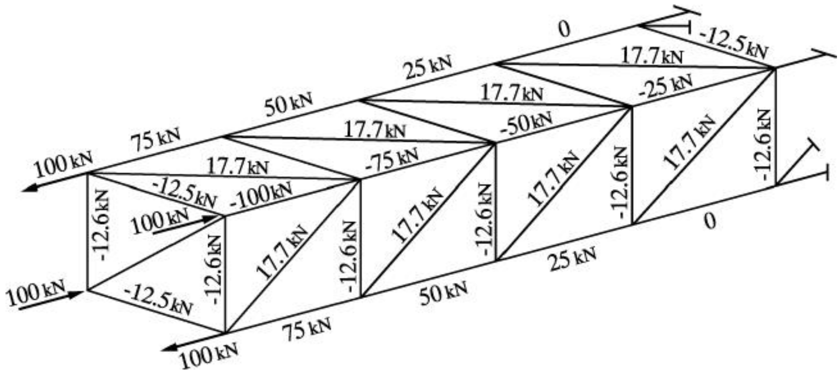


Figure 5.4 Truss loaded with self-equilibrating load system.

† Hoff, N.J., The applicability of Saint Venant's principle to airplane structures, *Journal of Aeronautical Science,* 12, p.445-460, 1945.

and moment when evaluated on each of these small portions of the surface of the body. In accordance with the principle of Saint Venant the difference between the computed and actual stress distributions will be negligible at points sufficiently removed from the forementioned small portions of the surface of the body.

5.4 Methods for Finding Exact Solutions for Boundary Value Problems in the Linear Theory of Elasticity

The exact solution for very few boundary value problems formulated on the basis of the linear theory of elasticity has been found. These problems involve bodies having a simple geometry which are subjected to simple distributions of the external loads and are supported in a convenient way. Solutions of boundary value problems involving bodies of complex geometry subjected to any distribution of external loads can be obtained with the aid of a computer using a numerical method such as the finite element method.

Some of the methods for obtaining exact solutions of boundary value problems formulated on the basis of the linear theory of elasticity are

1. The inverse method
2. The semi-inverse method
3. The method of potentials

In the inverse method, a stress field which satisfies the equilibrium equations and the compatibility conditions is usually selected and the corresponding displacement field is determined. Subsequently, the boundary conditions are found which correspond to these stress and displacement fields. This approach is of very limited use.

In the semi-inverse method, certain assumptions are made as to the form of the components of stress or of the components of displacement, on the basis of physical intuition or experimental evidence. As a result of these assumptions, the number of unknown quantities decreases and, moreover, the governing equations are reduced to a set of simpler differential equations from which the solution may be established by direct mathematical methods. In the next section we establish the displacement and stress fields of prismatic bodies made from a homogeneous, isotropic, linearly elastic material using the displacement or the stress formulation, presented in Section 5.2.1 or 5.2.2, in conjunction with the semi-inverse method.

In the method of potentials, the number of the unknown quantities is reduced by expressing two or more of them in terms of derivatives of one function called the *potential*. In Chapters 6 and 7, we establish the displacement and stress fields of prismatic bodies made from a homogeneous, isotropic, linearly elastic material using the stress formulation presented in Section 5.2.2 in conjunction with the method of potentials.

5.5 Solution of Boundary Value Problems for Computing the Displacement and Stress Fields of Prismatic Bodies Made fom Homogeneous, Isotropic, Linearly Elastic Materials

In this section we formulate three boundary value problems on the basis of the linear theory of elasticity and we solve them using the semi-inverse method. All three problems involve prismatic bodies, that is, bodies obtained by translating a surface in the direction normal to its plane without rotating it (see Fig. 5.5). The line described by the centroid

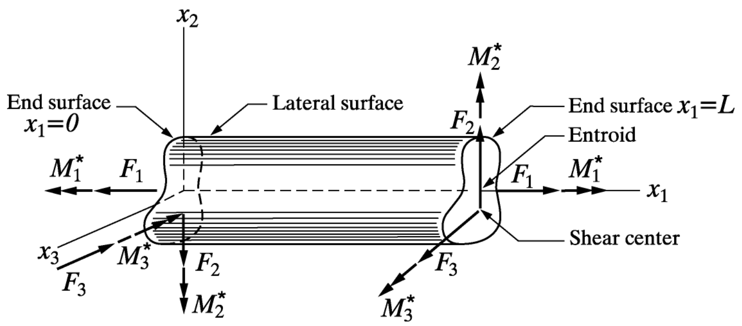


Figure 5.5 Prismatic body subjected to positive resultant forces and moments at its end.

of the plane surface as it translates is called the axis of the prismatic body. Moreover, the surface cut by an imaginary plane perpendicular to the axis of a prismatic body is called *its cross section*. We choose the axis of a prismatic body as the x_1 axis.

In practice, we often do not know the distribution of the components of traction on the end surfaces of a prismatic body. Instead we usually know the magnitudes and directions of concentrated forces and moments which are statically equivalent to the actual distribution of traction. However, in accordance with the principle of Saint Venant (see Section 5.3) if the cross-sectional dimensions of a prismatic body are small compared to its length, the difference in the components of stress acting on a particle of the body which is sufficiently removed from its end surfaces due to two statically equivalent distributions of traction on these surfaces is negligible. For such a body we only require that at its end surfaces the stress field gives the known concentrated forces and moments. In such an eventuality, we realize that the computed stress field may differ substantially from the actual at particles of the body which are close to its end surfaces.

We consider the tractions acting on an end surface of a prismatic body as being statically equivalent to one or more of the following concentrated forces and moments, known as the *resultant forces and moments:*

1. A concentrated force applied at the centroid of the end surface acting in the direction of the x_1 axis. It is called *internal axial centroidal force* and we denote it by F_1.

2. A transverse force acting in the plane of the end surface whose line of action passes through a point known as the shear center† of the end surface (see Section 9.5). It is called *shearing force*. We denote its components in the directions of the x_2 and x_3 axis by F_2 and F_3, respectively (see Fig. 5.5).

3. A moment about an axis parallel to the x_1 axis through the shear center of the end surface. It is called *torsional moment* and we denote it by M_1^* (see Fig. 5.5).

4. A moment whose vector acts in the plane of the end surface. It is called *bending moment*. We denote its components about the x_2 and about the x_3 axes by M_2^* and by M_3^*, respectively.

We consider as positive the resultant forces and moments acting on a positive†† end surface or cross section of a prismatic body if their sense coincides with the positive sense of its axes x_1, x_2, x_3 (see Fig. 5.5). Furthermore, we consider as positive the components

† The shear center of an end surface or of a cross section of a prismatic body is a point on the plane of this end surface or of this cross section which has the property that the prismatic body does not twist ($\theta_1 = 0$) when subjected to transverse forces whose line of action lies in a plane which contains the shear centers of its end surfaces and of its cross sections.

†† We call an end surface or a cross section of member positive or negative if the unit vector normal to it is directed along the positive or negative x_1 axis, respectively.

of the resultant forces and moments acting on a negative end surface or cross section of a prismatic body if their sense coincides with the negative sense of its axes x_1, x_2, x_3 (see Fig. 5.5). Thus, a tensile axial force is considered positive, while a compressive axial force is considered negative. Keeping in mind this sign convention, referring to the traction–stress relations (2.73) and noting that for the end surfaces of a prismatic body $\mathbf{i}_n = \pm\mathbf{i}_1$ $(\lambda_{n2} = \lambda_{n3} = 0,\ \lambda_{n1} = \pm 1)$ the resultant forces and moments are related to the components stress by the following relations:

$$\begin{aligned}
F_1 &= \mp\iint_A \overset{\mp 1}{T}{}_1^s dA = \iint_A \tau_{11}\, dA \\
F_2 &= \mp\iint_A \overset{\mp 1}{T}{}_2^s\, dA = \iint_A \tau_{12}\, dA \\
F_3 &= \mp\iint_A \overset{\mp 1}{T}{}_3^s\, dA = \iint_A \tau_{13}\, dA \\
M_1^* &= \mp\iint_A \left[\overset{\mp 1}{T}{}_3^s (x_2 - e_2) - \overset{\mp 1}{T}{}_2^s (x_3 - e_3)\right] dA = \iint_A \left[\tau_{13}(x_2 - e_2) - \tau_{12}(x_3 - e_3)\right] dA \\
M_2^* &= \mp\iint_A \overset{\mp 1}{T}{}_1^s x_3\, dA = \iint_A \tau_{11} x_3\, dA \\
M_3^* &= \mp\iint_A \overset{\mp 1}{T}{}_1^s x_2\, dA = -\iint_A \tau_{11} x_2\, dA
\end{aligned} \tag{5.14}$$

where e_2 and e_3 are the x_2 and x_3 coordinates, respectively, of the shear center of a cross section of the prismatic body.

When we use the displacement formulation in conjunction with the semi-inverse method for the solution of boundary value problems, we adhere to the following steps:

STEP 1 We assume the form and/or part of the displacement field which satisfies the displacement boundary conditions at the points of the surface of the body where components of displacement are specified.

STEP 2 We substitute the assumed displacement field into the strain–displacement relations (2.16) to obtain the components of strain.

STEP 3 We substitute the components of strain established in step 2 into the stress–strain relations (3.94) for an isotropic, linearly elastic material to obtain the components of stress.

STEP 4 We substitute the components of stress established in step 3 into the equations of equilibrium (2.69) and we establish the conditions which must be imposed on the assumed components of displacement in order to satisfy these equations. If this cannot be done , the assumed components of displacement cannot be modified to become the actual and we must return to step 1 and start with a new set of assumed components of displacement.

STEP 5 We substitute the components of stress established in step 3 and modified in step 4 into the traction–stress relations (2.73) and we establish the additional conditions which must be imposed on the assumed components of displacement in order to satisfy the traction boundary conditions at the points of the surface of the body where components of traction have been specified. If this cannot be done, the assumed components of displacement cannot be modified to become the actual and we must return to step 1 and start with a new set of assumed components of displacement.

When the components of stress obtained from the assumed components of displacement satisfy the equations of equilibrium and the traction boundary conditions, the assumed components of displacement as modified in steps 4 and 5 are the actual.

When we use the stress formulation of a boundary value problem we adhere to the following steps:

STEP 1 We assume some components of stress and the form of the others.

STEP 2 We substitute the assumed components of stress into the equations of equilibrium and we establish the conditions which must be imposed on them in order to make them satisfy these equations. If this cannot be done, the assumed components of stress cannot be modified to become the actual and we must return to step 1 and start with a new set of assumed components of stress.

STEP 3 We substitute the assumed components of stress as modified in step 2 into the traction–stress relations (2.73) and we establish the additional conditions which must be imposed on them in order to satisfy the traction boundary conditions at the points of the surface of the body where components of traction have been specified. If this cannot be done the assumed components of stress cannot be modified to become the actual and we must return to step1 and start with a new set of assumed components of stress.

STEP 4 We substitute the components of stress assumed in step 1 as modified in steps 2 and 3 into the stress–strain relations (3.95) to obtain the components of strain.

STEP 5 We substitute the components of strain established in step 4 into the equations of compatibility (2.63) and we establish the conditions which must be imposed on the assumed components of stress in order to satisfy these equations. If the assumed components of stress cannot be made to satisfy the equations of compatibility, we must return to step 1 and start with a new set of assumed components of stress.

STEP 6 We substitute the components of strain obtained in step 4 as modified in step 5 into the strain–displacement relations and we integrate them to obtain the components of displacement. If the body is simply connected, the components of displacement, thus obtained, are single-valued continuous functions of the space coordinates. If the body is multiply connected, the components of displacement, thus obtained, may or may not be single-valued continuous functions of the space coordinates. If they are not, the assumed components of stress are not the actual. In such an eventuality we must return to step 1 and start with a new set of assumed components of stress.

STEP 7 We check to see whether the components of displacement obtained in step 6 can be made to satisfy the displacement boundary conditions at the points of the surface of the body where components of displacement are specified. If the components of displacement cannot be made to satisfy the displacement boundary conditions, we must return to step 1 and start with a new set of assumed components of stress. If the components of displacement satisfy the displacement boundary conditions, the assumed components of stress as modified in steps 2, 3, 5 and 7 are the actual.

In what follows we formulate on the basis of the linear theory of elasticity and solve the following boundary value problems:

Problem 1. We establish the components of displacement and stress in a prismatic body with traction-free lateral surfaces subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to a distribution of traction which is statically equivalent to an axial centroidal force. The cross sections of the body are simply or multiply connected (have holes). We use the stress formulation.

Problem 2. We establish the components of displacement and stress in a prismatic body with traction-free lateral surfaces subjected to a uniform increase of temperature with its end surfaces ($x_1 = 0$ and $x_1 = L$) restrained from moving in the axial direction but free to move in the transverse directions. The cross sections of the body could be simply or multiply (have holes) connected . We use the displacement formulation.

Problem 3. We establish the components of translation (deflection) and the components of stress in a prismatic beam of simply or multiply connected cross sections subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to a distribution of traction which is statically equivalent to a bending moment. We use the stress formulation.

Example 1 Consider a prismatic body of arbitrary (simply or multiply connected) cross section made from a homogeneous (E = constant), isotropic, linearly elastic material. The dimensions of the cross sections of the body are small compared to its length. The body is initially in a reference stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. The body reaches a second state of mechanical and thermal equilibrium at the uniform temperature T_o due to the application on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) of a distribution of traction (see Fig. a) which is statically equivalent to a given concentrated axial centroidal force F_1. Assume that the body forces are negligible. On the basis of the principle of Saint Venant (see Section 5.3) all distributions of traction acting on the end surfaces of the body which are statically equivalent to the given axial centroidal force produce essentially the same results on parts of the body which are sufficiently removed from its end surfaces. For instance, the results are valid for a relatively long prismatic body fixed at its one end ($x_1 = 0$) into a rigid wall and subjected to a distribution of traction at its unsupported end ($x_1 = L$) which is statically equivalent to the given axial centroidal force. In this case, on the boundary $x_1 = 0$, the distribution of traction is such that the components of displacement $\hat{u}_i (i = 1, 2, 3)$ vanish, while the components $\overset{-1}{T}_2$ and $\overset{-1}{T}_3$ of traction do not vanish. However, the components F_2 and F_3 of the resultant force over the total cross-section vanish. Consequently, on the basis of the principle of Saint Venant for a particle sufficiently distant from the fixed end, the effect of the components of traction $\overset{-1}{T}_2$ and $\overset{-1}{T}_3$ acting on its fixed end is negligible. If, however, the cross-sectional dimensions of the body are not small, compared to its length, then the constraint at $x_1 = 0$ will affect the distribution of the components of stress throughout the length of the body and, therefore, the results will

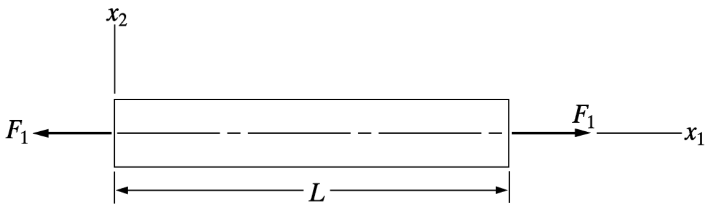


Figure a Prismatic body subjected to a uniformly distributed axial component of traction on its end surfaces.

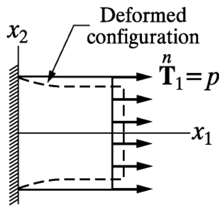


Figure b Prismatic body fixed at one end.

not be valid. In this case, the problem must be formulated as a mixed boundary value problem. That is, we must take into account that on the end surface $x_1 = 0$ of the body, the components of displacement, are specified ($\hat{u}_1 = \hat{u}_2 = \hat{u}_3 = 0$). As yet, this problem has not been solved analytically, on the basis of the theory of elasticity. However, it can be solved numerically using the method of finite elements. On the basis of the foregoing discussion, referring to relation (5.14), the boundary conditions for the body under consideration are

On its cylindrical surfaces

$$\overset{n}{\mathbf{T}} = 0 \tag{a}$$

On its end surfaces $x_1 = 0$ and $x_1 = L$

$$\begin{aligned}
F_1 &= \iint_A \tau_{11}\, dA = \text{given} \\
F_2 &= \iint_A \tau_{12}\, dA = 0 \\
F_3 &= \iint_A \tau_{13}\, dA = 0 \\
M_1^* &= \iint_A \left[\tau_{13}(x_2 - e_2) - \tau_{12}(x_3 - e_3)\right] dA = 0 \\
M_2^* &= \iint_A \tau_{11} x_3\, dA = 0 \\
M_3^* &= -\iint_A \tau_{11} x_2\, dA = 0
\end{aligned} \tag{b}$$

Notice that the above boundary conditions do not restrain the body from moving as a rigid body. Thus, we anticipate that when we use the stress formulation we will obtain expressions for the components of displacement of the particles of this body which include an unspecified rigid-body motion of the body. We will eliminate this rigid-body motion by assuming that it is possible to restrain the body from moving as a rigid-body without restraining the displacement of the particles of its end surfaces.

Solution

Stress formulation

STEP 1 From physical intuition we assume the following stress distribution in the body:

$$\tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0 \qquad \tau_{11} = C = \text{constant} \tag{c}$$

STEPS 2 and 3 It is apparent that the stress distribution (c) satisfies the equations of equilibrium (2.69). Moreover, on the lateral surfaces $[\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3 \ (\lambda_{n1} = 0)]$ of the body the traction–stress relations (2.73) reduce to

$$\begin{aligned} \overset{n}{T}{}_1^s &= \tau_{21}\lambda_{n2} + \tau_{31}\lambda_{n3} \\ \overset{n}{T}{}_2^s &= \tau_{22}\lambda_{n2} + \tau_{32}\lambda_{n3} \\ \overset{n}{T}{}_3^s &= \tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3} \end{aligned} \tag{d}$$

By inspection we see that the assumed solution (c) satisfies the boundaries conditions $(\overset{n}{T}{}_1^s = \overset{n}{T}{}_2^s = \overset{n}{T}{}_3^s = 0)$ on the lateral surfaces of the body. Moreover, on the end boundaries, $x_1 = 0$ and $x_1 = L$ ($\mathbf{i}_n = \pm \mathrm{i}_1$, $\lambda_{n1} = \pm 1$, $\lambda_{n2} = \lambda_{n3} = 0$), of the body substituting relation (c) into (b), we have

$$F_1 = \iint_A \tau_{11}\, dA = CA \qquad F_2 = F_3 = 0 \qquad M_1^* = M_2^* = M_3^* = 0 \tag{e}$$

Thus, the boundary conditions (b) are satisfied by the assumed solution (c) provided that

$$\tau_{11} = C = \frac{F_1}{A} \tag{f}$$

Notice that if the body under consideration was subjected to a specified distribution of the axial component of traction $(\overset{n}{\mathbf{T}} = \overset{n}{T}_1\mathbf{i}_1)$ on its lateral surface, the assumed solution (c) cannot give this specified distribution of traction, when substituted into relations (d). Consequently, the assumed solution (c) is not the actual solution of the boundary value problem for computing the components of displacement and stress of a prismatic body subjected to a distribution of the axial component of traction on its lateral surfaces. Moreover, notice that if a body with traction free lateral surfaces has variable cross sections $(\lambda_{n1} \neq 0)$, the assumed solution (c) does not satisfy the first of relations (2.75) on its lateral surface. Consequently, the assumed solution (c) with (f) is not the actual solution of the boundary value problem for computing the components of displacement and stress of bodies of variable cross sections subjected to a uniform distribution of the axial component of traction on each of its end surfaces ($x_1 = 0$ and $x_1 = L$). However, it can be shown that the assumed solution (c) is a good approximation to the actual distribution of the components of stress in prismatic bodies subjected to a distribution of the axial component of traction on their lateral surfaces as well as in tapered bodies, provided that the angle of taper is small.

STEP 4 Substituting relations (c) and result (f) into the stress–strain relations (2.50), we obtain

$$e_{11} = \frac{F_1}{EA} \qquad e_{22} = -\frac{\nu F_1}{EA} = e_{33} \qquad e_{12} = e_{13} = e_{23} = 0 \tag{g}$$

STEP 5 By inspection, it can be seen that the components of strain, (g), satisfy the equations of compatibility (2.63). For a simply connected body (no holes) the satisfaction of the equations of compatibility is necessary and sufficient to ensure that the strain–displacement relations (5.7) can be integrated to yield single-valued continuous components of displacement. For a multiply connected body the satisfaction of the equations of compatibility is necessary and sufficient to ensure that the strain–displacement relations (5.7) can be integrated. However, the resulting components of displacement may or may not be single-valued continuous functions of the space coordinates. If they are not, this indicates that the assumed components of stress with relation (f) are not the solution of the problem. If the components of displacement obtained by integrating the strain–displacement relations (5.7) are single-valued continuous functions of the space coordinates and moreover satisfy the essential (displacement) boundary conditions, then the assumed components of stress are the solution of the problem.

STEP 6 We substitute the components of strain (g) into the strain–displacement relations (5.7) and we integrate the resulting relations to obtain the components of displacement of the particles of the body. That is,

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} = \frac{F_1}{AE} \qquad \hat{u}_1 = \frac{x_1 F_1}{AE} + g_1(x_2, x_3) \tag{h}$$

$$e_{22} = \frac{\partial \hat{u}_2}{\partial x_2} = -\frac{\nu F_1}{AE} \qquad \hat{u}_2 = -\frac{\nu x_2 F_1}{AE} + g_2(x_1, x_3) \tag{i}$$

$$e_{33} = \frac{\partial \hat{u}_3}{\partial x_3} = -\frac{\nu F_1}{AE} \qquad \hat{u}_3 = -\frac{\nu x_3 F_1}{AE} + g_3(x_1, x_2) \tag{j}$$

Substituting relations (i) and (j) into the expression, for the shearing component of strain, $e_{23} = 0$, we obtain

$$\frac{\partial g_3(x_1, x_2)}{\partial x_2} = -\frac{\partial g_2(x_1, x_3)}{\partial x_3} = f(x_1) \tag{k}$$

The left side of the above relation is a function of x_1 and x_2 while the right side is a function of x_1 and x_3 only. Thus, both sides must be functions of x_1 only. Integrating relations (k), we obtain

$$\begin{aligned} g_2 &= -x_3\, \mathrm{f}(x_1) + c_2(x_1) \\ g_3 &= x_2\, \mathrm{f}(x_1) + c_3(x_1) \end{aligned} \tag{l}$$

Substituting relations (l) into $e_{21} = 0$ and $e_{13} = 0$, we get

$$\frac{\partial g_1(x_2, x_3)}{\partial x_3} = -x_2\frac{df(x_1)}{dx_1} - \frac{dc_3(x_1)}{dx_1} = \psi_2(x_2)$$

$$\frac{\partial g_1(x_2, x_3)}{\partial x_2} = x_3\frac{df(x_1)}{dx_1} - \frac{dc_2(x_1)}{dx_1} = \psi_3(x_3) \tag{m}$$

The left sides of the above relations are functions of x_2 and x_3 while the right-hand side of the first is a function of x_1 and x_2 and of the second is a function of x_1 and x_3. Consequently, both sides of the first relation must be functions of x_2 only and both sides of the second must be functions of x_3 only. Relations (m) are valid if

$$\frac{df(x_1)}{dx_1} = c_4 \qquad \frac{dc_3(x_1)}{dx_1} = c_5 \qquad \frac{dc_2(x_1)}{dx_1} = c_6 \tag{n}$$

Consequently,

$$f(x_1) = c_4x_1 + c_7 \qquad c_3(x_1) = c_5x_1 + c_8 \qquad c_2(x_1) = c_6x_1 + c_9 \tag{o}$$

Where $c_i(i = 4, 5, 6, 7, 8, 9)$ are constants. Substituting relations (n) into (m) and integrating, we get

$$g_1 = -c_4x_2x_3 - c_5x_3 + c_{10}(x_2)$$

$$g_1 = c_4x_2x_3 - c_6x_2 + c_{11}(x_3) \tag{p}$$

Thus,

$$c_{10}(x_2) = -c_6x_2 + c_{12} \qquad c_{11}(x_3) = -c_5x_3 + c_{12} \qquad c_4 = 0 \tag{q}$$

Substituting relations (q) into (p), we obtain

$$g_1 = -c_5x_3 - c_6x_2 + c_{12} \tag{r}$$

Substituting relations (q) into (o) and the resulting expressions into (l), we have

$$g_2 = -c_7x_3 + c_6x_1 + c_9$$

$$g_3 = c_7x_2 + c_5x_1 + c_8 \tag{s}$$

Substituting relations (r) and (s) into (h), (i) and (j), we get

$$\hat{u}_1\,(x_1, x_2, x_3) = \frac{x_1F_1}{AE} - c_5x_3 - c_6x_2 + c_{12}$$

$$\hat{u}_2\,(x_1, x_2, x_3) = -\frac{vx_2F_1}{AE} - c_7x_3 + c_6x_1 + c_9 \tag{t}$$

$$\hat{u}_3\,(x_1, x_2, x_3) = -\frac{vx_3F_1}{AE} + c_7x_2 + c_5x_1 + c_8$$

Referring to relations (2.59b) and (2.59c) we see that the terms invoIing the constants c_5,

$c_6, c_7, c_8, c_9, c_{12}$ represent unspecified rigid-body motion of the body. Thus, if we restrain the body from moving as a rigid body, these constants vanish and relations (t) reduce to

$$\hat{u}_1 = \frac{x_1 F_1}{AE} \qquad \hat{u}_2 = -\frac{\nu x_2 F_1}{AE} \qquad \hat{u}_3 = -\frac{\nu x_3 F_1}{AE} \tag{u}$$

Strictly speaking the body must be restained from moving as a rigid body in a way that its end surfaces are not inhibited from deforming freely. However, on the basis of the principle of Saint Venant, the solution is valid for particles which are not very close to the end surfaces of the body no matter how the end surfaces of the body are supported or how the tractions applied on them are distributed provided that

1. The distribution of traction on each end surface is statically equivalent to an axial centroidal force of magnitude F_1.

2. The dimensions of the cross sections of the body are small compared to its length.

STEP 7 Referring to relations (u), we see that the components of displacement obtained by integrating the strain–displacement relations (5.7) are single-valued continuous functions of the space coordinates. Consequently, the assumed distribution of the components of stress (c) with (f) is the solution of the boundary value problem under consideration.

Example 2 Consider a prismatic body of arbitrary simply or multiply connected cross-sections made from a homogeneous, isotropic, linear elastic material (E = 200 GPa; ν = 0.3; $\alpha = 10^{-5}$/°C). When this body is in its reference stress-free, strain-free equilibrium state at the uniform temperature T_o = 10°C, it fits exactly between two rigid walls (see Fig. a). Subsequently, the temperature of the environment of the body is increased to 30°C and the body reaches a second state of mechanical and thermal equilibrium. As the temperature of the body increases its length cannot change but the dimensions of its cross sections can. That is, it is assumed that the two rigid walls do not restrain the particles of the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body from moving freely in directions normal to its axis. Assuming that the body forces are negligible, compute the stress and displacement fields of the body.

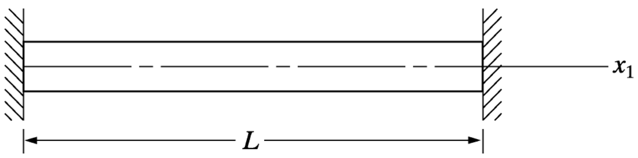


Figure a Prismatic body with its end surfaces restrained from moving in the axial direction.

Solution From physical intuition we assume that the components of stress and displacement of the body are not functions of the tangential coordinate, Thus, it is convenient to use cylindrical coordinates. In these coordinates, the traction boundary conditions become

$$\begin{aligned} \overset{n}{T}_1 &= \lambda_{n1}\tau_{11} + \lambda_{nr}\tau_{1r} + \lambda_{n\theta}\tau_{1\theta} = 0 \\ \overset{n}{T}_r &= \lambda_{n1}\tau_{r1} + \lambda_{nr}\tau_{rr} + \lambda_{n\theta}\tau_{r\theta} = 0 \end{aligned} \tag{a}$$

$$\overset{n}{T}_{\theta} = \lambda_{n1}\tau_{\theta 1} + \lambda_{nr}\tau_{\theta r} + \lambda_{n\theta}\tau_{\theta\theta} = 0 \tag{a}$$

The boundary conditions for the body in cylindrical coordinates are

$$\begin{aligned}
&\overset{n}{T}_{i} = 0 \ (i = 1, r, \theta) \text{ on the lateral boundaries of the body} \\
&\hat{u}_1 (0, r, \theta) = 0 \qquad \hat{u}_1 (L, r, \theta) = 0 \\
&\overset{-1}{T}_{r} (0, r, \theta) = 0 \qquad \overset{1}{T}_{r} (L, r, \theta) = 0 \\
&\overset{-1}{T}_{\theta} (0, r, \theta) = 0 \qquad \overset{1}{T}_{\theta} (L, r, \theta) = 0
\end{aligned} \tag{b}$$

Displacement formulation

STEP 1 We assume that the components of displacement $\hat{u}_1$ and $\hat{u}_\theta$ vanish while the component of displacement $\hat{u}_r$ is function only of the radial coordinate r. That is,

$$\hat{u}_1 = \hat{u}_\theta = 0 \qquad \hat{u}_r = \hat{u}_r(r) \tag{c}$$

This displacement field satisfies the displacement boundary conditions of the body, at $x_1 = 0$ and $x_1 = L$ [see relations (b)].

STEP 2 Substituting the assumed displacement field (d) into the strain–displacement relations (2.83) we obtain

$$e_{rr} = \frac{d\hat{u}_r}{dr} \qquad e_{\theta\theta} = \frac{\hat{u}_r}{r} \qquad e_{11} = e_{1\theta} = e_{1r} = e_{r\theta} = 0 \tag{d}$$

STEP 3 Substituting relations (d) into the stress–strain relations (3.94), we have

$$\begin{aligned}
\tau_{11} &= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{d\hat{u}_r}{dr} + \frac{\hat{u}_r}{r}\right) - \beta(T - T_0) \\
\tau_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{d\hat{u}_r}{dr} + \nu\frac{\hat{u}_r}{r}\right] - \beta(T - T_0) \\
\tau_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\frac{d\hat{u}_r}{dr} + (1-\nu)\frac{\hat{u}_r}{r}\right] - \beta(T - T_0)
\end{aligned} \tag{e}$$

$$\tau_{r\theta} = \tau_{1\theta} = \tau_{r\theta} = 0$$

STEP 4 Substituting relations (e) into the equations of equilibrium (2.86), we get

$$\frac{d^2\hat{u}_r}{dr^2} + \frac{1}{r}\frac{d\hat{u}_r}{dr} - \frac{\hat{u}_r}{r^2} = 0$$

This equation can be written as

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(r\hat{u}_r)}{dr}\right) = 0 \tag{f}$$

Integrating equation (f) twice, we have

$$\hat{u}_r = \frac{A_1 r}{2} + \frac{A_2}{r} \tag{g}$$

Substituting relations (g) into the strain–displacement relations (d), we get

$$e_{rr} = \frac{A_1}{2} - \frac{A_2}{r^2} \qquad e_{\theta\theta} = \frac{A_1}{2} + \frac{A_2}{r^2} \qquad e_{r\theta} = e_{1\theta} = e_{1r} = e_{11} = 0 \tag{h}$$

Substituting relation (h) into the stress–strain relation (e), we obtain

$$\begin{aligned}
\tau_{11} &= \frac{E\nu A_1}{(1+\nu)(1-2\nu)} - \beta(T - T_0) \\
\tau_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{A_1}{2} - \frac{A_2}{r^2} + \frac{2\nu A_2}{r^2}\right) - \beta(T - T_0) \\
\tau_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{A_1}{2} + \frac{A_2}{r^2} - \frac{2\nu A_2}{r^2}\right) - \beta(T - T_0) \\
\tau_{r\theta} &= \tau_{1\theta} = \tau_{1r} = 0
\end{aligned} \tag{i}$$

STEP 5 The stress field (i) must satisfy the traction boundary conditions (b) at the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body. Noting that the unit vectors normal at the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body are $\mathbf{i}_n = \mp\mathbf{i}_1$ ($\lambda_{nr} = \lambda_{n\theta} = 0,\ \lambda_{n1} = \mp 1$) referring to relations (a) and (b) and using the last of relations (i), we have

$$\begin{aligned}
\overset{-1}{T}_r(0, r, \theta) &= 0 = -\tau_{1r} = 0 \\
\overset{-1}{T}_\theta(0, r, \theta) &= 0 = -\tau_{1\theta} = 0 \\
\overset{1}{T}_r(L, r, \theta) &= 0 = \quad \tau_{1r} = 0 \\
\overset{1}{T}_\theta(L, r, \theta) &= 0 = \quad \tau_{1\theta} = 0
\end{aligned} \tag{j}$$

Thus, the stress field (i) satisfies the traction boundary conditions at the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body without imposing any restriction on the value of the constants A_1 and A_2.

STEP 6 The stress field (i) must satisfy the boundary conditions (b) on the lateral surfaces of the body. Noting that the unit vector normal to the lateral surfaces of the body is $\mathbf{i}_n = \lambda_{nr}\mathbf{i}_r + \lambda_{n\theta}\mathbf{i}_\theta$, ($\lambda_{n1} = 0$) referring to relations (a) and (b) and using the last of relations (i), we get

$$\begin{aligned} \overset{n}{T}_1 &= 0 = \tau_{r1}\lambda_{nr} + \tau_{\theta 1}\lambda_{n\theta} = 0 \\ \overset{n}{T}_r &= 0 = \tau_{rr}\lambda_{nr} + \tau_{\theta r}\lambda_{n\theta} = \tau_{rr}\lambda_{nr} \\ \overset{n}{T}_\theta &= 0 = \tau_{r\theta}\lambda_{nr} + \tau_{\theta\theta}\lambda_{n\theta} = \tau_{\theta\theta}\lambda_{n\theta} \end{aligned} \tag{k}$$

Thus, the assumed solution (b) with (g) gives components of stress which satisfy the boundary conditions on the lateral surfaces of the prismatic body under consideration if on these surfaces, we have

$$\tau_{rr} = 0 \qquad \tau_{\theta\theta} = 0 \tag{l}$$

Substituting the second and third of relations (i) into (l), we obtain

$$\begin{aligned} \tau_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{A_1}{2} - \frac{A_2}{r^2} + \frac{2\nu A_2}{r^2}\right)_{\substack{\text{on the lateral}\\ \text{surfaces}}} - \beta(T - T_0) = 0 \\ \tau_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{A_1}{2} + \frac{A_2}{r^2} - \frac{2\nu A_2}{r^2}\right)_{\substack{\text{on the lateral}\\ \text{surfaces}}} - \beta(T - T_0) = 0 \end{aligned} \tag{m}$$

Subtracting one of the above relations from the other, we get

$$\left[\frac{EA_2}{(1+\nu)r^2}\right]_{\substack{\text{on the lateral}\\ \text{surfaces}}} = 0 \tag{n}$$

Thus,

$$A_2 = 0 \tag{o}$$

Referring to relations (i) and taking into account relation (o) we see that the components of stress are constant throughout the body. Consequently, the components of stress τ_{rr} and $\tau_{\theta\theta}$ must be zero at every particle of the body since they vanish at the particles of its lateral surfaces. With this in mind, we conclude that the components of stress (i) with (o) satisfy the equations of equilibrium (2.86). Using relations (o) and (3.96) from any one of relations (m), we obtain

$$A_1 = \frac{2\beta(T - T_0)(1+\nu)(1-2\nu)}{E} = 2\alpha(T - T_0)(1+\nu) \tag{p}$$

Substituting relations (o) and (p) into (g) and (i) taking into account relations (b), we have

$$\hat{u}_r = \alpha(T - T_0)(1 + \nu)r = 10^{-5}(20)(1.3)r = 2.6(10^{-4})r$$

$$\hat{u}_\theta = \hat{u}_1 = 0$$

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{r\theta} = \tau_{1r} = 0 \tag{q}$$

$$\tau_{11} = -E\,\alpha(T - T_0) = -200{,}000(10^{-5})(20)$$

$$= -40 \text{ MPa} = 40 \text{ MPa compression}$$

It is clear that the components of displacement and stress (q) satisfy the equations of equilibrium and the boundary conditions (b) of the body. Consequently, they are the actual components of displacement and stress of the body.

Example 3 Consider a prismatic beam of arbitrary cross sections (simply or multiply connected) made from a homogeneous, isotropic, linearly elastic material. The beam is initially in a reference stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. The beam reaches a second state of mechanical and thermal equilibrium, at the uniform temperature T_o, due to the application on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) of a distribution of traction which is statically equivalent to specified components of moment of magnitude M_2^* and M_3^* (see Fig. a). Since the distribution of the components of traction on the end surfaces of the beam is not given, we can only ensure that the calculated components of stress satisfy the following [see relations (5.14)] boundary conditions at the surfaces of the beam:

On its lateral surfaces

$$\overset{n}{\mathbf{T}} = 0 \tag{a}$$

On its end surfaces $x_1 = 0$ and $x_1 = L$

$$F_1 = \iint_A \tau_{11} dA = 0 \tag{b}$$

$$F_i = \iint_A \tau_{1i}\, dA = 0 \qquad (i = 2, 3) \tag{c}$$

$$M_1 = \iint_A [\,\tau_{13}(x_2 - e_2) - \tau_{12}(x_3 - e_3)\,]dA = 0 \tag{d}$$

$$M_2 = \iint_A \tau_{11}\, x_3\, dA = M_2^* = \text{given} \tag{e}$$

$$M_3 = -\iint_A \tau_{11}\, x_2\, dA = M_3^* = \text{given} \tag{f}$$

In the above relations, the convention has been made that M_3 is positive if it produces tensile stress on the portion of the cross section having a negative x_2 coordinate. Moreover, M_2 is positive if it produces tensile stress on the portion of the cross section having a positive x_3 coordinate. On the basis of this convention, the components of the external moment applied on the surface $x_1 = 0$ (or $x_1 = L$) are positive when they are represented by a vector acting in the direction of the negative (or positive) axes x_1, x_2, and x_3 (see Fig. 5.5). When the loading of the beam consists of only end bending moments, we say that the beam is subjected to *pure bending*.

The principle of Saint Venant (see Section 5.3) applies to the end surfaces of the beam under consideration because the dimensions of their cross section are small compared to their length. Consequently, on the basis of this principle all distributions of traction acting on the end surfaces of the beam under consideration which are statically equivalent to the given bending moment produce essentially the same results on parts of the beam which are sufficiently removed from its end surfaces. For instance, the results are valid at particles sufficiently removed from the fixed end of a prismatic beam fixed at its end $x_1 = 0$ into a rigid wall and subjected to a distribution of traction at its unsupported end $x_1 = L$ which is statically equivalent to the given bending moment. In this case, on the boundary $x_1 = 0$, the distribution of traction is such that the components of displacement $\hat{u}_i (i = 1, 2, 3)$ vanish, while the components $\overset{-1}{T}_2$ and $\overset{-1}{T}_3$ of traction do not vanish. However, the resultant components of force F_2 and F_3 of these components of traction vanish. For a particle sufficiently distant from the fixed end of the beam, the effect of these components of traction is negligible. If, however, the cross-sectional dimensions of the beam are not small, compared to its length, then the constraint at $x_1 = 0$ will affect the distribution of the components of stress throughout the length of the beam and, therefore, the results will not be valid for any particle of the beam. In this case and in case one is interested in establishing the exact distribution of the components of stress in the neighborhood of the fixed support of a long cantilever beam, the problem must be formulated as a mixed boundary value problem. That is, one must take into account that on the end surface $x_1 = 0$ of the beam, the components of displacement are specified ($\hat{u}_1 = \hat{u}_2 = \hat{u}_3 = 0$). This problem has not been solved analytically, on the basis of the linear theory of elasticity. However, it can be solved numerically using the method of finite elements.

Notice that the specified boundary conditions (a) to (f) do not restrain the beam from moving as a rigid body. Thus, we anticipate that the calculated components of displacement of the particles of the beam will include an unspecified rigid-body motion. We will eliminate this rigid motion by assuming that it is possible to restrain the beam from moving as a rigid body without inhibiting the displacement of the particles of its end surfaces.

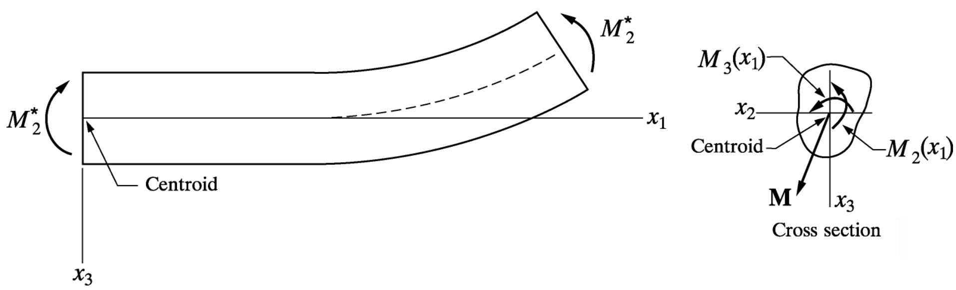


Figure a Prismatic body in pure bending.

Compute the stress and displacement fields of the beam under consideration.

Solution We solve the boundary value problem† described above using the stress formulation (see Section 5.2.2) in conjuction with the semi-inverse method (see Section 5.4). That is, we start by assuming a solution of the following form:

$$\tau_{22} = \tau_{33} = \tau_{12} = \tau_{23} = \tau_{31} = 0 \qquad \tau_{11} = \tau_{11}(x_2, x_3) \tag{g}$$

Thus, the components of the resultant moment of the tractions acting on any cross section of the beam must be equal to

$$M_2(x_1) = M_2^* \qquad M_3(x_1) = M_3^* \tag{h}$$

By inspection, we see that the components of traction acting on the lateral surfaces of the beam obtained by substituting the components of stress (g) into relations (2.73) are equal to zero. Thus, solution (g) satisfies the boundary conditions on the lateral surfaces of the beam. Moreover, the components of stress (g) satisfy the equations of equilibrium (2.69). Substituting relations (g) into the stress–strain relations (3.48), we obtain

$$e_{12} = e_{13} = e_{23} = 0 \qquad e_{33} = e_{22} = -\frac{\nu}{E}\tau_{11}(x_2, x_3) \qquad e_{11} = \frac{\tau_{11}(x_2, x_3)}{E} \tag{i}$$

Substituting relations (i) into the compatibility equations (2.63), we find that three of these equations are satisfied automatically, while the other three reduce to

$$\frac{\partial^2 \tau_{11}}{\partial x_2^{\,2}} = \frac{\partial^2 \tau_{11}}{\partial x_3^{\,2}} = \frac{\partial^2 \tau_{11}}{\partial x_2 \partial x_3} = 0$$

The solution of these equations is

$$\tau_{11} = \tau_{11}(x_2, x_3) = Ax_2 + Bx_3 + C \tag{j}$$

Thus, the assumed distribution of the components of stress (g) with (j) satisfies:

1. The equations of equilibrium occur at every particle inside the volume of the beam.
2. The traction-free boundary condition exists on the lateral surfaces of the beam.
3. When substituted in the stress–strain relations, (3.48) gives components of strain which satisfy the equations of compatibility at every point of the beam. For a simply connected body (no holes) the satisfaction of the equations of compatibility by the components of strain is necessary and sufficient to ensure that the strain–displacement relations (5.7) can be integrated to yield single-valued continuous components of displacement. For a multiply connected body the satisfaction of the equations of compatibility is necessary and sufficient to ensure that the strain–displacement relations (5.7) can be integrated. However, the resulting components of displacement may or may not be single-valued continuous functions of the space coordinates. If they are not, this indicates that the assumed components of stress are not the solution of the problem. If the components of displacement obtained by integrating the strain–displacement relations

---

† This problem was solved by Saint Venant in 1856.

(5.7) are single-valued continuous functions of the space coordinates and, moreover, satisfy the essential (displacement) boundary conditions of the beam, then the assumed components of stress are the solution of the problem.

Substituting relations (g) and (j) into the relations (i), we obtain

$$
\begin{aligned}
e_{11} &= \frac{A}{E}x_2 + \frac{B}{E}x_3 + \frac{C}{E} \\
e_{22} &= -\nu e_{11} = e_{33} \qquad e_{12} = e_{13} = e_{23} = 0
\end{aligned}
\tag{k}
$$

Substituting the components of strain (k) into the strain–displacement relations (5.7) and integrating the resulting relations, we get

$$
\begin{aligned}
\hat{u}_1(x_1, x_2, x_3) &= \frac{Ax_1x_2}{E} + \frac{Bx_1x_3}{E} + \frac{Cx_1}{E} + m_1x_2 + m_2x_3 + p_1 \\
\hat{u}_2(x_1, x_2, x_3) &= -\frac{Ax_1^2}{2E} + \frac{\nu A}{2E}(x_3^2 - x_2^2) - \frac{\nu B}{E}x_2x_3 - \frac{\nu Cx_2}{E} - m_1x_1 + nx_3 + p_2 \\
\hat{u}_3(x_1, x_2, x_3) &= -\frac{Bx_1^2}{2E} + \frac{\nu B}{2E}(x_2^2 - x_3^2) - \frac{\nu A}{E}x_2x_3 - \frac{\nu Cx_3}{E} - m_2x_1 - nx_2 + p_3
\end{aligned}
\tag{l}
$$

where A B C m_1, m_2, n, p_1, p_2 and p_3 are constants. Referring to relations (2.59), we see that the following part of the displacement field (l) represents only rigid-body motion:

$$
\begin{aligned}
\hat{u}_1 &= m_1x_2 + m_2x_3 + p_1 \\
\hat{u}_2 &= -m_1x_1 + nx_3 + p_2 \\
\hat{u}_3 &= m_2x_1 - nx_2 + p_3
\end{aligned}
\tag{m}
$$

Thus, the constants m_1, m_2, n, p_1, p_2 and p_3 in relation (l) vanish if the beam is supported in a way that it cannot move as a rigid body. However, in doing so the end surfaces ($x_1 = 0$ and $x_1 = L$) of the beam should not be restrained from deforming freely. Disregarding the rigid-body motion of the beam, relations (l) reduce to

$$
\begin{aligned}
\hat{u}_1(x_1, x_2, x_3) &= \frac{Ax_1x_2}{E} + \frac{Bx_1x_3}{E} + \frac{Cx_1}{E} \\
\hat{u}_2(x_1, x_2, x_3) &= -\frac{Ax_1^2}{2E} + \frac{\nu A}{2E}(x_3^2 - x_2^2) - \frac{\nu B}{E}x_2x_3 - \frac{\nu Cx_2}{E} \\
\hat{u}_3(x_1, x_2, x_3) &= -\frac{Bx_1^2}{2E} + \frac{\nu B}{2E}(x_2^2 - x_3^2) - \frac{\nu A}{E}x_2x_3 - \frac{\nu Cx_3}{E}
\end{aligned}
\tag{n}
$$

From relations (n) we see that the components of displacement of the beam obtained by integrating the strain–displacement relation (5.7) are single-valued continuous functions of the space coordinates. Consequently, the assumed distribution of the components of stress (g) with (j) is the actual distribution of the components of stress of the beam under consideration, provided that it satisfies the boundary conditions at the end surfaces ($x_1 = 0$ and $x_1 = L$) of the beam, that is, relations (b) to (f). By inspection we see that conditions

(c) and (d) are automatically satisfied. Noting that x_2 and x_3 are centroidal axes, we have

$$\iint_A x_2\, dA = 0 \qquad \iint_A x_3\, dA = 0 \tag{o}$$

Substituting relation (j) into relation (b) and using relations (o), we obtain

$$C = 0 \tag{p}$$

Moreover, substituting relations (j) into relations (e) and (f), we get

$$M_2 = I_{23}A + I_{22}B \tag{q}$$

$$-M_3 = I_{33}A + I_{23}B \tag{r}$$

Where I_{22} and I_{33} are the moments of inertia of the cross sections of the beam about the x_2 and x_3 axis, respectively, I_{23} is the product of inertia of the cross sections of the beam with respect to x_2 and x_3 axes (see Section C.2 of Appendix C). Solving relations (q) and (r) for A and B, we obtain

$$A = -\frac{M_3 I_{22} + M_2 I_{23}}{I_{33}I_{22} - I_{23}^2} \qquad B = \frac{M_2 I_{33} + M_3 I_{23}}{I_{22}I_{33} - I_{23}^2} \tag{s}$$

Substituting relations (p) and (s) into (j) and (n) and referring to relations (g), the solution of the problem under consideration is

$$\tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$$

$$\tau_{11} = -\frac{M_3 I_{22} + M_2 I_{23}}{I_{33}I_{22} - I_{23}^2}x_2 + \frac{M_2 I_{33} + M_3 I_{23}}{I_{33}I_{22} - I_{23}^2}x_3$$

$$\hat{u}_1(x_1, x_2, x_3) = \frac{1}{E\,(I_{22}I_{33} - I_{23}^2)}\Big[-(M_3 I_{22} + M_2 I_{23})x_1 x_2 + (M_2 I_{33} + M_3 I_{23})x_1 x_3\Big]$$

$$\hat{u}_2(x_1, x_2, x_3) = \frac{1}{2E\,(I_{22}I_{33} - I_{23}^2)}\Big[-(M_3 I_{22} + M_2 I_{23})[-x_1^2 + \nu(x_3^2 - x_2^2)] - 2\nu(M_2 I_{33} + M_3 I_{23})x_2 x_3\Big] \tag{t}$$

$$\hat{u}_3(x_1, x_2, x_3) = \frac{1}{2E\,(I_{22}I_{33} - I_{23}^2)}\Big[(M_2 I_{33} - M_3 I_{23})[-x_1^2 + \nu(x_2^2 - x_3^2)] + 2\nu(M_3 I_{33} + M_2 I_{23})x_2 x_3\Big]$$

If the length of the beam is considerably larger than its other dimensions, solution (t) represents a good approximation of the components of displacement and stress of particles located sufficiently far from the end surfaces ($x_1 = 0$ and $x_1 = L$) of the beam even when the end surfaces of the beam are restrained from deforming freely. For instance, solution (t) is valid for the particles of a relatively long beam fixed into a rigid wall at $x_1 = 0$ and subjected to a bending moment at $x_1 = L$ provided that they are not located very close to the rigid wall.

Referring to the expression for $u_1(x_1, x_2, x_3)$ in relations (t), we see that for a given value of x_1 the expression for $\hat{u}_1$ is the equation of a plane. *That is, plane sections ($\mathbf{x_1}$ = constant) which prior to deformation were perpendicular to the axis of the beam remain plane subsequent to deformation.*

The displacement of the centroidal axis of a beam is referred to as the *deflection* of the beam, while the displaced centroidal axis of a beam is called its *elastic curve*. The deflection of the beam may be obtained from relations (t) by setting $x_3 = x_2 = 0$. That is,

$$\begin{aligned} \hat{u}_1(x_1, 0, 0) &= u_1(x_1) = 0 \\ \hat{u}_2(x_1, 0, 0) &= u_2(x_1) = -\frac{Ax_1^2}{2E} \\ \hat{u}_3(x_1, 0, 0) &= u_3(x_1) = -\frac{Bx_1^2}{2E} \end{aligned} \tag{u}$$

Consider a beam subjected to a moment $\mathbf{M}^* = M^*_2\mathbf{i}_2$ ($M^*_3 = 0$) on its end surface $x_1 = L$ and an equal and opposite moment on its end surface $x_1 = 0$ and assume that the axes x_2, x_3 are principal centroidal. For this loading referring to relations (h) relations (s) reduce to

$$A = 0 \qquad\qquad B = \frac{M_2}{I_2} \tag{v}$$

Referring to relations (u) and using the above results, the deflection of the beam is

$$\begin{aligned} u_2 &= 0 \\ u_3 &= -\frac{Bx_1^{\,2}}{2E} = -\frac{M_2 x_1^{\,2}}{2EI_2} \end{aligned} \tag{w}$$

The radius of curvature ρ_{13} of the elastic curve of the beam under consideration is equal to [see relation (9.25a)].

$$\frac{1}{\rho_{13}} = \frac{\dfrac{d^2u_3}{d^2x_1}}{\left[1 + \left(\dfrac{du_3}{dx_1}\right)^2\right]^{3/2}} \tag{x}$$

When the assumption of small deformation is valid, $(du_3/dx_1)^2$ is negligible compared to unity; consequently, relation (x) reduces to

$$\frac{1}{\rho_{13}} \approx \frac{d^2u_3}{d^2x_1} \tag{y}$$

Differentiating the second of relations (w) twice with respect to x_1 and using relation (y), we obtain

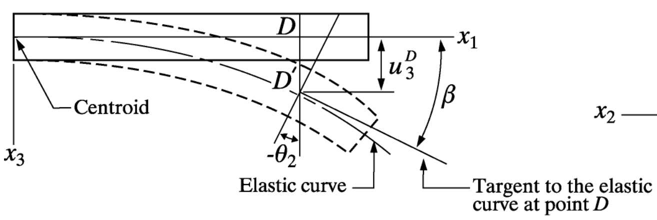


Figure b Elastic curve of a prismatic body (beam) subjected on each of its end surfaces to a moment M_2.

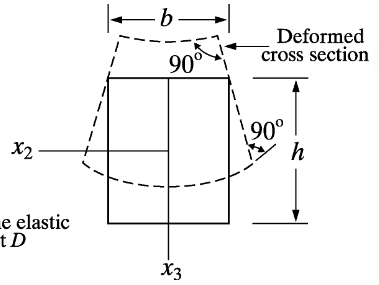


Figure c Deformation of rectangular cross section of a beam in pure bending.

$$\frac{1}{\rho_{13}} = \frac{d^2u_3}{dx_1^{\,2}} = -\frac{M_2}{EI_2} = \text{constant} \tag{z}$$

That is, the elastic curve of the beam under consideration is a circle in the $x_1\,x_3$ plane. The slope of the elastic curve of the beam is equal to

$$\frac{du_3}{dx_1} = -\frac{M_2 x_1}{EI_2} \tag{za}$$

Moreover, referring to Fig. b and the third of relations (t), the angle of rotation $-\theta_2$ of the cross section of the beam is given as

$$-\theta_2 \approx -\tan(\theta_2) = \frac{u_1}{x_3} = \frac{M_2 x_1}{EI_2} \tag{zb}$$

Comparing relations (za) and (zb), we see that the slope of any point of the elastic curve of the beam under consideration is equal in magnitude to the angle of rotation θ_2 of the cross section of the beam at that point. *Therefore, planes normal to the axis* of the beam before deformation remain plane and normal to its elastic curve, subsequent to deformation.

Consider a beam of rectangular cross section subjected to a moment $\mathbf{M}^* = M^*_2\mathbf{i}_2$ ($M^*_3 = 0$) at its end surface $x_1 = L$ and to an equal and opposite moment at its end surface $x_1 = 0$. For this loading, $A = 0$. Referring to relations (t) the deformed configuration of the edges, $x_2 = \pm b/2$ of the cross section of the beam located at $x_1 = c$ is given by

$$\begin{aligned} \hat{u}_2(c, \pm\frac{b}{2}, x_3) &= \pm\frac{\nu}{2E}Bbx_3 \\ \hat{u}_3(c, \pm\frac{b}{2}, x_3) &= -\frac{B}{2E}\left(c^2 - \nu\frac{b^2}{4} + \nu x_3^{\,2}\right) \end{aligned} \tag{zc}$$

If the applied moments M^*_2 is positive, B is positive and the sides of the cross section of the beam deform, as shown in Fig. c.

Referring to relation (t) the deformation of edges $x_3 = \pm h/2$ of the cross section of the beam located at $x_1 = c$ is given by

$$u_2\,(c, x_2, \pm\frac{h}{2}) = \pm\frac{\nu}{2E}Bhx_2 \tag{zd}$$

$$u_3\,(c,\, x_2,\, \pm\frac{h}{2}) \;=\; -\frac{B}{2E}(c^2 \;-\; \nu x_2^{\,2} \;+\; \nu\frac{h^2}{4}) \qquad \text{(zd)}$$

The top and bottom sides of the deformed cross section are second degree parabolas, tangent to the lines perpendicular to the deformed sides $x_2 = \pm\, b/2$. This is due to the fact that the shear stress is zero at the particles located at the corners of the cross section of the beam. Consequently, these particles of the beam do not distort.

5.6 Problems

1. to **4.** Write the boundary conditions for the body (parallelepiped) shown in Fig. 5P1. Determine the components of stress acting on the particle at point A on the plane $\mathbf{i}_n = \frac{1}{\sqrt{3}}(\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)$. Repeat with the bodies of Fig. 5P2 to 5P4.

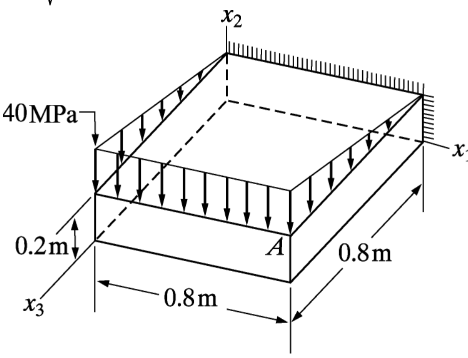


Figure 5P1

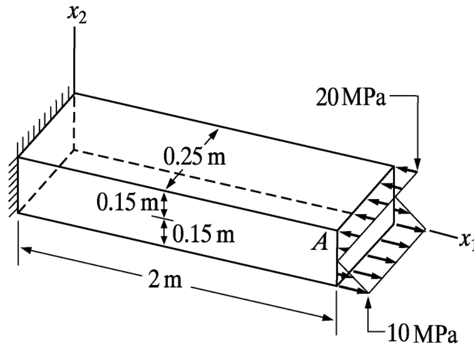


Figure 5P2

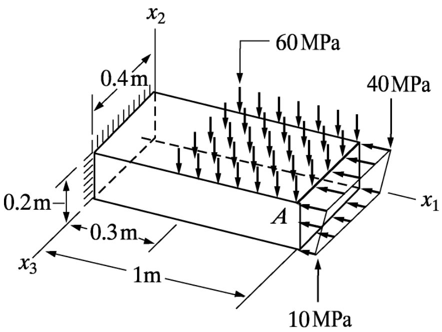


Figure 5P3

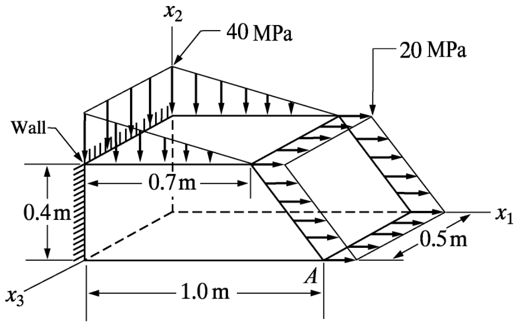


Figure 5P4

Ans. 1 At pt. A $\quad \tau_{nn} = -\frac{40}{3}$ MPa $\quad \tau_{nt} = 18.856$ MPa

Ans. 2 At pt. A $\quad \tau_{nn} = -\frac{20}{3}$ MPa $\quad \tau_{nt} = 9.428$ MPa

Ans. 3 At pt. A $\quad \tau_{nn} = -33.33$ MPa $\quad \tau_{ns} = 24.94$ MPa

Ans. 4 $\hat{u}_i(0, x_2, x_3) = 0 \quad i = 1, 2, 3 \qquad \overset{-2}{T}{}^s_i(x_1, 0, x_3) = 0 \quad i = 1, 2, 3 \qquad \overset{2}{T}{}^s_i(x_1, 0.4, x_3) = 0 \quad i = 1, 3$

$\overset{2}{T}{}^s_i(x_1, 0.4, x_3) = \frac{400x_1}{7} - 40 \qquad \overset{-3}{T}{}^s_i(x_1, x_2, 0) = 0 \quad i = 1, 2, 3 \qquad \overset{n}{T}{}^s_1(\frac{-3x_2}{4} + 1, x_2, x_3) = 20 \text{ MPa}$

Ans .4 $\overset{3}{T}{}_i^s(x_1, x_2, 0.5) = 0 \quad i = 1, 2, 3 \qquad \overset{n}{T}{}_i^s\left(\frac{-3x_2}{4} + 1, x_2, x_3\right) = 0 \quad i = 2, 3$ At *pt A* $\tau_{11} = \frac{25}{3}$ MPa

$\tau_{nt} = 11.785$ MPa

5. The components of displacement of the particles of the three-dimensional body shown in Fig. 5P5 are

$$u_1 = Ax_1 \qquad u_2 = Ax_2 \qquad u_3 = Ax_3$$

Find the traction acting on the surface of the body and the distribution of the specific body force.

Ans
$$\overset{-1}{T}_1(0, x_2, x_3) = -\frac{EA}{1-2\nu} \qquad \overset{1}{T}_1(a, x_2, x_3) = \frac{EA}{1-2\nu} \qquad \overset{\pm 1}{T}_2 = \overset{\pm 1}{T}_3 = 0$$
$$\overset{-2}{T}_2(x_1, 0, x_3) = -\frac{EA}{1-2\nu} \qquad \overset{2}{T}_2(x_1, b, x_3) = \frac{EA}{1-2\nu} \qquad \overset{\pm 2}{T}_1 = \overset{\pm 2}{T}_3 = 0$$
$$\overset{-3}{T}_3(x_1, x_2, 0) = -\frac{EA}{1-2\nu} \qquad \overset{3}{T}_3(x_1, x_2, c) = \frac{EA}{1-2\nu} \qquad \overset{\pm 3}{T}_1 = \overset{\pm 3}{T}_2 = 0$$
$$B_1 = B_2 = B_3 = 0$$

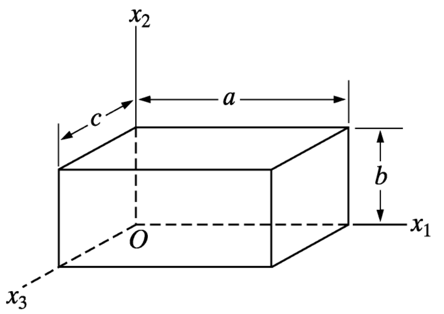


Figure 5P5

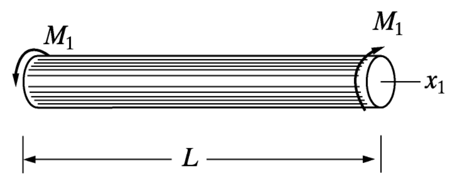


Figure 5P6

6. Consider a bar of constant circular cross section shown in Fig. 5P6. The bar is made from a homogeneous, isotropic, linearly elastic material and is supported and loaded in a way that the components of displacement of its particles are

$$u_1 = 0 \qquad u_2 = -\alpha x_1 x_3 \qquad u_3 = \alpha x_1 x_2 \tag{a}$$

where α is a constant. Disregarding the body forces, find

(a) If the equations of equilibrium are satisfied by this displacement field

(b) The value of α so that the components of displacement of its particles are the actual components of displacement of the bar Ans. $\alpha = M_1^*/GI_p$

7. Consider the cantilever beam of constant rectangular cross section of width b and depth h shown in Fig. 5P7. On the basis of theories of mechanics of materials, the components of stress acting on the particles of this beam, disregarding the effect of its weight, are equal to

$$\tau_{11} = -\frac{12P_3(L - x_1)x_3}{bh^3} \qquad \tau_{13} = \frac{3P_3(h^2 - 4x_3^2)}{2bh^3} \tag{a}$$

$$\tau_{22} = \tau_{33} = \tau_{12} = \tau_{23} = 0$$

(a) Check to see whether, the distribution of the components of stress (a) satisfies the equations of equilibrium (2.69) and the compatibility equations (2.63) at the particles of the beam.

(b) Establish the boundary conditions for the beam at $x_3 = \pm h/2$ and at $x_1 = L$, corresponding to the components of stress given by relations (a). Do they satisfy the given boundary conditions at $x_1 = 0$ and at $x_1 = L$?

Ans. $\overset{\pm 3}{T}_i(x_1, x_2, \pm\frac{h}{2}) = 0 \quad i = 2, 3 \qquad \overset{\pm 3}{T}_1(x_1, x_2, \pm\frac{h}{2}) = 0$

$\overset{\pm 1}{T}_1(L, x_2, x_3) = 0 \qquad \overset{\pm 1}{T}_2(L, x_2, x_3) = 0 \qquad \overset{\pm 1}{T}_3(L, x_2, x_3) = \frac{3P_3(h^2 - 4x_3^2)}{2bh^3}$

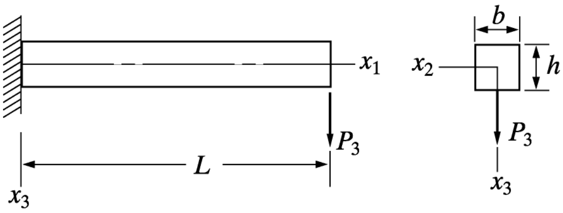


Figure 5P7

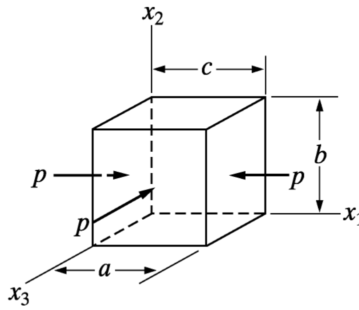


Figure 5P8

8. The particles of the surfaces $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ of the rectangular parallelepiped whose dimensions are shown in Fig. 5P8 are restrained from moving in the x_1, x_2 and x_3 directions, respectively, while they are free to move in any other direction. The parallelepiped is subjected to a uniform pressure, p(kN/m^2) on its other surfaces. Disregarding the body forces find the displacement and stress fields of the parallelepiped. Assume

$$\tau_{11} = \tau_{22} = \tau_{33} = C = \text{constant} \qquad \tau_{12} = \tau_{13} = \tau_{23} = 0$$

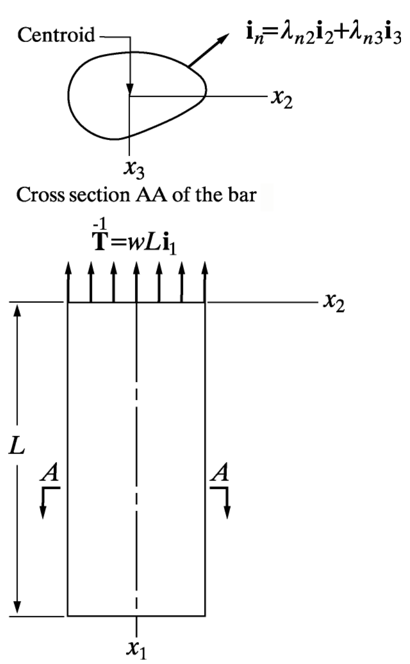
Ans. $C = -p \qquad u_i = -\frac{(1-2\nu)px_i}{E} \qquad i = 1, 2, 3$

9. Consider a bar of constant cross section, length L (m) and specific weight w (N/m^3). The bar is made from a homogeneous isotropic, linearly elastic material of modulus of elasticity E (N/m^3) and Poisson's ratio ν. As shown in Fig. 5P9 the bar is suspended at $x_1 = 0$ in a way that the traction $\overset{-1}{\mathbf{T}}$ on the surface $x_1 = 0$ is uniform and equal to

$$\left.\overset{-1}{\mathbf{T}}\right|_{x_1=0} = wL\mathbf{i}_1$$

The remaining surface of the bar is traction free. Compute the stress and displacement fields of the bar resulting from its weight.

Notice that in reality it is not possible to hold the bar in a way that a uniform traction at $x_1 = 0$ is induced. However, if the length of the bar is considerably greater than its other dimensions on the basis of the Saint Venant principle, the state of stress at points sufficiently removed from the boundary $x_1 = 0$ is not affected by the way the traction at $x_1 = 0$ is distributed.



Ans. $\tau_{11} = w(L - x_1) \qquad \tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$

$$u_1 = -\frac{w(L - x_1)^2}{2E} - \frac{\nu w(x_2^2 + x_3^2)}{2E} + \frac{wL^2}{2E}$$

$$u_2 = -\frac{\nu w(L - x_1)x_2}{E} \qquad u_3 = -\frac{\nu w(L - x_1)x_3}{E}$$

Figure 5P9

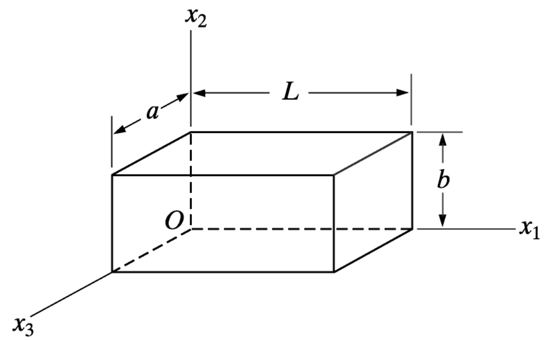


Figure 5P10

10. The body shown in Fig. 5P10 is fixed at $x_1 = L$ and it is subjected to body forces and surface tractions which are not shown in the figure. The components of displacement of the particles of the body are equal to

$$\hat{u}_1 = \frac{k}{2}(x_1 - L)^2 \qquad \hat{u}_2 = kx_1(L - x_1) \qquad \hat{u}_3 = -kx_3(x_1 - L)$$

where

$$k = \frac{(1 + \nu)(1 - 2\nu)}{E}c$$

E = modulus of elasticity $\qquad \nu$ = Poisson's ratio $\qquad c$ = small number in kN/m^3

(a) Find the distribution of the specific body force acting on the particles of the body. Ans. $2B_1 = -c(1 - 2\nu) = -B_2, \; B_3 = 0$

(b) Find the distribution of traction acting on the surface of the body.

Ans. at $x_1 = 0 \quad \overset{-1}{T}_1 = c(1 - 2\nu)L \quad \overset{-1}{T}_2 = -\dfrac{c(1 - 2\nu)L}{2} \quad \overset{-1}{T}_3 = -\dfrac{c(1 - 2\nu)x_3}{2}$

at $x_1 = L \quad \hat{u}_1 = \hat{u}_2 = \hat{u}_3 = 0$ at $x_2 = 0$ and $x_2 = b \quad \overset{\pm 2}{T}_1 = \pm\dfrac{c}{2}(1 - 2\nu)(\dfrac{L}{2} - x_1) \quad \overset{\pm 2}{T}_2 = \overset{\pm 2}{T}_3 = 0$

at $x_3 = 0 \quad \overset{-3}{T}_1 = \overset{-3}{T}_2 = 0 \quad \overset{-3}{T}_3 = c(1 - 2\nu)(x_1 - L)$ at $x_3 = a \quad \overset{3}{T}_1 = -\dfrac{c(1 - 2\nu)a}{2} \quad \overset{3}{T}_2 = 0 \quad \overset{3}{T}_3 = -c(1 - 2\nu)(x_1 - L)$

11. A body made from a homogeneous, isotropic, linearly elastic material [E = 200 GPa, ν = 1/3, α = 10^{-5}/°C] is subjected to a uniform increase in temperature ΔT while every point of its surface is restrained from moving. Disregard the effect of the body forces. Find its stress and displacement fields. Ans. $\tau_{11} = \tau_{22} = \tau_{33} = -6\Delta T$ MPa

12. The particles of the lateral surface of a prismatic body of length L = 2 m are restrained from moving in the plane normal to its axis. Moreover, the particles of its end surface $x_1 = 0$ are restrained from moving in the x_1 direction while the particles of its end surface at $x_1 = 2$m are free to move in any direction. Write the boundary conditions for the body. Determine the stress and displacement fields in the body when it is subjected to a uniform increase of temperature $\Delta T = 20$ °C. The body is made from an isotropic, linearly elastic material [E = 200 GPa, ν = 1/3, $\alpha = 10^{-5}$/°C] [*Hint:* Assume the displacement field $\hat{u}_1 = \hat{u}_1(x_1) \quad \hat{u}_2 = \hat{u}_3 = 0$].

Ans. $\tau_{11} = 0 \quad \tau_{rr} = -\dfrac{E\alpha\Delta T}{(1-\nu)} = \tau_{\theta\theta} \quad u_1 = \dfrac{\alpha\Delta T(1+\nu)x_1}{(1-\nu)} \quad \tau_{1r} = \tau_{1\theta} = \tau_{r\theta} = 0$

13. Consider the beam subjected at $x_2 = \pm c$ to the traction shown in Fig. 5P13. On its surface $x_1 = L$ the beam is subjected to a distribution of traction $\overset{1}{T}_3$ which is statically equivalent to a force P_3 = 40 kN. On its surface $x_1 = 0$ the beam is subjected to a distribution of traction $\overset{-1}{\mathbf{T}} = \overset{-1}{T}_1\mathbf{i}_1 + \overset{-1}{T}_3\mathbf{i}_3$ which is statically equivalent to a force P_3 = −40 kN and a moment M_2 = −80 kN·m. The beam is made from an isotropic, linearly elastic material (E = 200 GPa, ν = 1/3). Using L = 2m, c = 20 mm, h = 80 mm, compute the stress field in the beam. Disregard the effect of weight of the beam. Assume that

$$\tau_{11} = A(L - x_1)x_3 \qquad \tau_{13} = B(h^2 - x_3^2)$$

$$\tau_{22} = C(L - x_1)x_3 \qquad \tau_{12} = \tau_{23} = \tau_{33} = 0$$

Ans. $A = -\dfrac{P_3}{I_2} \quad B = \dfrac{P_3}{2I_2} \quad C = -\dfrac{\nu P_3}{I_2}$

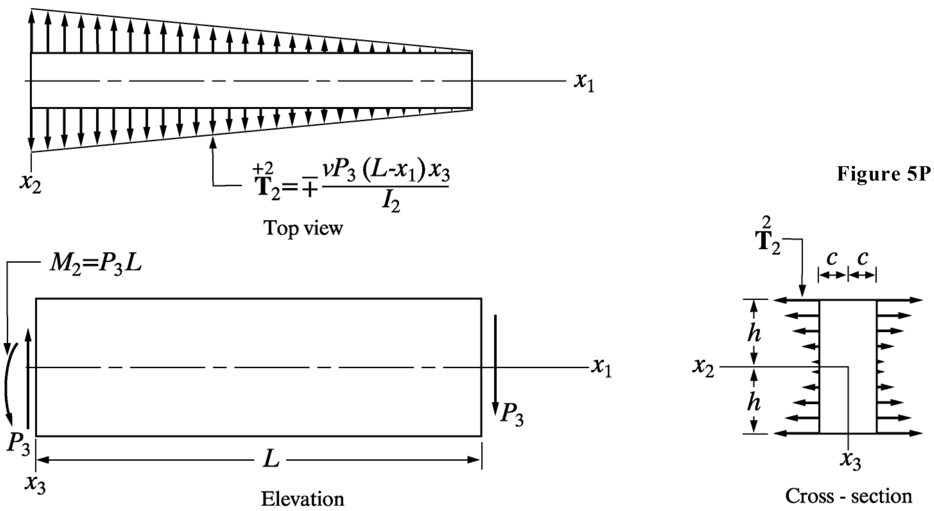


Figure 5P13

References for further reading

1. Fung, Y.C., *Foundations of Solid Mechanics,* Prentice-Hall, Englewood Cliffs, NJ, 1965.
2. Malvern, L.E., *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, NJ, 1969.
3. Timoshenko, S., Goodiar, J.N., *Theory of Elasticity,* McGraw-Hill, New York, 1970.
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Chapter 6

Prismatic Bodies Subjected to Torsional Moments at Their Ends

6.1 Description of the Boundary Value Problem for Computing the Displacement and Stress Fields in Prismatic Bodies Subjected to Torsional Moments at Their Ends

In this chapter we formulate and solve the boundary problem for computing the displacement and stress fields of prismatic bodies having cross sections of arbitrary geometry (solid or with holes)(see Fig. 6.1) made from isotropic, linearly elastic materials. The dimensions of the cross sections of these bodies are small as compared to their length. We call such bodies *prismatic bodies*. We choose the x_1 axis to be any axis parallel to their axis. The bodies are initially in a reference, stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. Subsequently, they reach a second state of mechanical equilibrium due to the application on each of their end surfaces ($x_1 = 0$ and $x_1 = L$) of a distribution of external tractions which is statically equivalent to a given concentrated torsional moment of magnitude M_1. We assume that the distribution of the external tractions is applied in such a way that it does not restrain the end surfaces of the bodies from warping. In practice we often do not know the distribution of traction acting on the end surfaces of prismatic bodies. However, we do know the magnitude of their resultant force and moment. On the basis of the principle of Saint Venant (see Section 5.3) all distributions of traction, acting on the end surfaces of the prismatic bodies under consideration, which are statically equivalent, produce essentially the same distribution of components of stress on parts of the bodies which are sufficiently removed from their end surfaces. Thus, we can only ensure that the stress field of such bodies yields a distribution of traction on each one of their end surfaces which is statically equivalent to a given torsional moment of magnitude M_1. On the

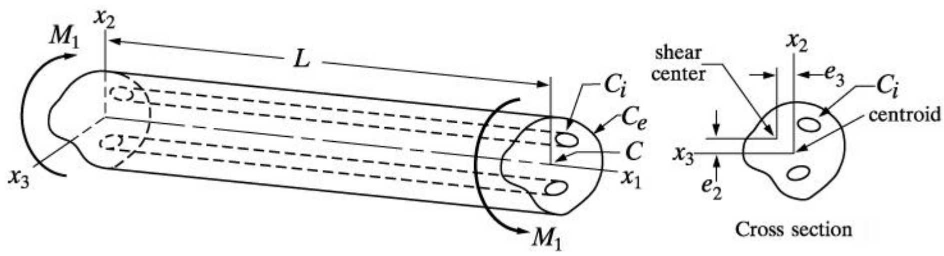


Figure 6.1 Prismatic body of arbitrary cross section subjected to torsional moments at its ends.

basis of the above presentation, referring to relations (5.14), the boundary conditions for the prismatic bodies under consideration are

On their lateral surfaces

$$\overset{n}{\mathbf{T}} = 0 \tag{6.1}$$

On their end surfaces ($x_1 = 0$ *and* $x_1 = L$)

$$F_1 = \iint_A \tau_{11}\, dA = 0 \tag{6.2a}$$

$$F_2 = \iint_A \tau_{12}\, dA = 0 \tag{6.2b}$$

$$F_3 = \iint_A \tau_{13}\, dA = 0 \tag{6.2c}$$

$$M_1 = \iint_A \left[(x_2 - e_2)\tau_{13} - (x_3 - e_3)\tau_{12}\right] dA = \iint_A (x_2\tau_{31} - x_3\tau_{12})\, dA = \text{given} \tag{6.2d}$$

$$M_2 = \iint_A x_3\tau_{11}\, dA = 0 \tag{6.2e}$$

$$M_3 = -\iint_A x_2\tau_{11}\, dA = 0 \tag{6.2f}$$

where e_2 and e_3 are the x_2 and x_3 coordinates, respectively, of the shear center of the cross sections of the prismatic bodies. Notice that since F_2 and F_3 vanish the moment of the components of tractions $\overset{n}{\mathbf{T}}_2$ and $\overset{n}{\mathbf{T}}_3$ about any axis parallel to the x_1 axis must be equal to M_1. These boundary conditions do not restrain a body from moving as a rigid-body. Thus, we anticipate that the components of displacement of a body obtained from the solution of the boundary value problem under consideration will include an unspecified rigid-body motion of this body. We will eliminate this rigid-body motion by assuming that it is possible to restrain the body from moving as a rigid-body without inhibiting the warping of its end surfaces.

When a prismatic body of non-circular (hollow or solid) cross sections is subjected to equal and opposite torsional moments at its ends, its cross sections warp. Practically, it is not possible to apply a distribution of traction on the end surfaces of a prismatic body of non-circular cross sections without restraining their warping. From physical intuition, we may deduce that when the warping of a cross section is restrained, a distribution of normal component of stress must act on it. For bodies of thin-walled, open cross sections the effect of this distribution of stress is not negligible. Moreover, although this distribution of normal component of stress is statically equivalent to zero force and moment, its magnitude on cross sections away from the one whose warping is restrained, does not reduce as quickly as one would expect that it would, on the basis of the principle of Saint Venant. For this reason in Section 9.11 we investigate the effect of restraining the warping of a cross section of prismatic bodies having thin-walled open cross sections.

6.2 Relations among the Coordinates of a Point Located on a Curved Boundary of a Plane Surface

Before proceeding with the formulation and the solution of the boundary value problem under consideration in this chapter, in what follows we derive certain relations among the coordinates of a point located on one of the lateral surfaces of a prismatic body. Referring to Fig. 6.2a consider two points P_o and P_o' located on a closed external curve C_e or internal curved C_i in the x_2x_3 plane whose coordinates are x_2, x_3 and $x_2 + \Delta x_2$, $x_3 + \Delta x_3$, respectively. These points may also be specified by the coordinates x_s and $x_s + \Delta x_s$, respectively; where x_s is measured along the external curve C_e counterclockwise from a reference point A_e or along the internal curved C_i clockwise from a reference point A_i. That is, for the points of curve C_e or C_i, we have

$$x_2 = x_2(x_s) \qquad\qquad x_3 = x_3(x_s) \tag{6.3}$$

Denoting by $\mathbf{i}_n = \lambda_{n2}\,\mathbf{i}_2 + \lambda_{n3}\,\mathbf{i}_3 = \cos\theta\,\mathbf{i}_2 + \sin\theta\,\mathbf{i}_3$ the unit vector outward normal to the curve C_e or C_i at point P_o and referring to Fig. 6.2a, from geometric considerations, we obtain

$$\frac{dx_2}{dx_s} = \lim_{\Delta x_s \to 0}\frac{\Delta x_2}{\Delta x_s} = -\sin\theta = -\lambda_{n3} \tag{6.4}$$

$$\frac{dx_3}{dx_s} = \lim_{\Delta x_s \to 0}\frac{\Delta x_3}{\Delta x_s} = \cos\theta = \lambda_{n2} \tag{6.5}$$

Referring to Fig. 6.2b, consider another point P_o'' located on the normal to curve C_e or C_i at point $P_o(x_2, x_3)$. Denoting by Δx_n the distance of point P_o'' from point P_o, from geometric considerations, referring to Fig. 6.2b, we have

$$\frac{dx_2}{dx_n} = \lim_{\Delta x_n \to 0}\frac{\Delta x_2}{\Delta x_n} = \cos\theta = \lambda_{n2} \tag{6.6}$$

$$\frac{dx_3}{dx_n} = \lim_{\Delta x_n \to 0}\frac{\Delta x_3}{\Delta x_n} = \sin\theta = \lambda_{n3} \tag{6.7}$$

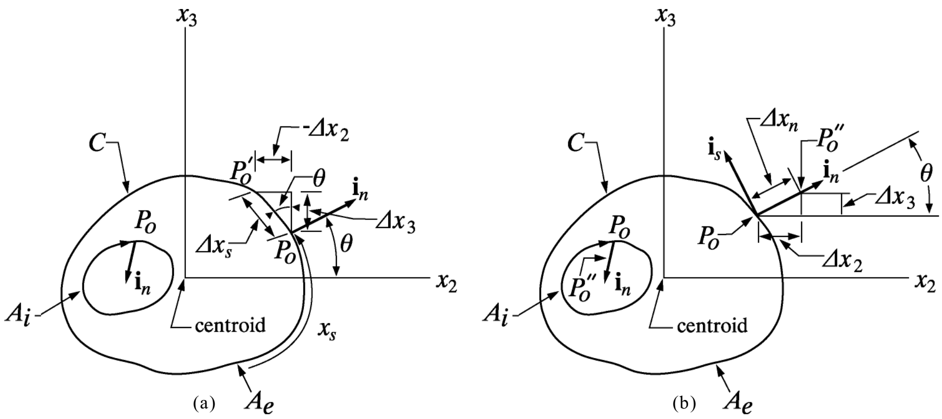


Figure 6.2 Cross section of a prismatic body.

From equations (6.4) to (6.7), we obtain

$$\frac{dx_2}{dx_s} = -\frac{dx_3}{dx_n} \tag{6.8}$$

$$\frac{dx_3}{dx_s} = \frac{dx_2}{dx_n} \tag{6.9}$$

6.3 Formulation of the Torsion Problem for Prismatic Bodies of Arbitrary Cross Section on the Basis of the Linear Theory of Elasticity

In this section we use the semi-inverse method in conjunction with the method of potentials to formulate and solve the boundary value problem for computing the components of displacement and stress of prismatic bodies of arbitrary cross section subjected to equal and opposite torsional moments at their ends.

Inasmuch as, neither the geometry of the bodies nor their loading vary with x_1, the components of stress and the component of displacement u_1 must be only functions of x_2 and x_3. This implies that

$$e_{11}(x_1, x_2, x_3) = 0 \tag{6.10}$$

Moreover, at the end surfaces $x_1 = 0$ and $x_1 = L$ the normal component of the resultant force and the resultant moments about the x_2 and x_3 axis must vanish. This suggests the assumption that

$$\tau_{11}(x_1, x_2, x_3) = 0 \tag{6.11}$$

Substituting relations (6.10) and (6.11) into the first of the stress–strain relations (3.48), for a body made from an isotropic, linearly elastic material, we get

$$\tau_{22} = -\tau_{33} \tag{6.12}$$

Taking into account that on the lateral surfaces of the bodies under consideration the traction $\overset{n}{\mathbf{T}}$ vanishes and that the unit vector outward normal to them is $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ ($\lambda_{n1} = 0$), the traction–stress relations (2.73) reduce to

$$\begin{aligned} \overset{n}{\mathbf{T}}{}_1^s &= \tau_{21}\lambda_{n2} + \tau_{31}\lambda_{n3} = 0 \\ \overset{n}{\mathbf{T}}{}_2^s &= \tau_{22}\lambda_{n2} + \tau_{23}\lambda_{n3} = 0 \\ \overset{n}{\mathbf{T}}{}_3^s &= \tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3} = 0 \end{aligned} \tag{6.13}$$

From the last two of relations (6.13) using relation (6.12) we obtain that τ_{22}, τ_{33}, τ_{23} must be equal to zero on the lateral surfaces of the bodies. Thus, we assume a solution of the following form:

$$\tau_{23} = \tau_{11} = \tau_{22} = \tau_{33} = 0 \qquad \tau_{12} = \tau_{12}(x_2, x_3) \qquad \tau_{13} = \tau_{13}(x_2, x_3) \tag{6.14}$$

Substituting relations (6.14) into the equations of equilibrium (2.69), we get

$$\frac{\partial \tau_{21}}{\partial x_2} = \frac{\partial(-\tau_{31})}{\partial x_3} \tag{6.15}$$

The above relation indicates that a function $\phi\ (x_2, x_3)$ exists referred to as *the stress function*, such that

$$\tau_{13} = -\frac{\partial \phi}{\partial x_2} \qquad \tau_{12} = \frac{\partial \phi}{\partial x_3} \tag{6.16}$$

The equations of equilibrium are therefore satisfied at every particle of the body if the components of stress are obtained from a stress function by relations (6.14) with (6.16). Thus, we have reduced the problem of finding the components of stress τ_{12} and τ_{13} acting on the particles of a prismatic body subjected to equal and opposite moments at its ends, to that of establishing the single-valued differentiable function $\phi(x_2, x_3)$ which when substituted into relation (6.16), gives components of stress which with relations (6.14) have the following attributes:

1. May satisfy the boundary conditions of the body.

2. When substituted into the stress–strain relations (3.48), they give components of strain which when substituted into the strain–displacement relations (2.16), the resulting expressions can be integrated to give single-valued continuous components of displacement.

Referring to relations (6.14) the boundary conditions (6.13) on the lateral surfaces of a prismatic body reduce to

$$\begin{aligned} \overset{n}{T}_1 &= \tau_{12}\lambda_{n2} + \tau_{13}\lambda_{n3} = 0 \\ \overset{n}{T}_2 &= 0 \\ \overset{n}{T}_3 &= 0 \end{aligned} \tag{6.17}$$

Substituting relations (6.16), (6.4) and (6.5) into the first of relations (6.17), we get

$$\left(\frac{\partial \phi}{\partial x_3}\right)\left(\frac{dx_3}{dx_s}\right) + \left(\frac{\partial \phi}{\partial x_2}\right)\left(\frac{dx_2}{dx_s}\right) = \frac{d\phi}{dx_s} = 0$$

Therefore the assumed solution (6.14) with (6.16) satisfies the boundary conditions on the traction-free lateral surfaces of the body if

$$\phi = \begin{cases} \phi_{C_e} = \text{constant on the curve } C_e \\ \\ \phi_{C_i} = \text{constant on the curves } C_i \quad (i = 1, 2, ..., N) \end{cases} \tag{6.18}$$

where C_e is the closed external curve bounding the cross sections of the body while C_i is the closed curve bounding the i^{th} ($i = 1, 2, ..., N$) hole of the cross sections of the body.

In what follows we establish the restrictions which must be imposed on the function $\phi(x_2, x_3)$ in order to ensure that the assumed stress field (6.14) with (6.16) satisfies the boundary conditions (6.2) on the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body. Inasmuch as τ_{11} is zero, it is apparent that conditions (6.2a), (6.2e) and (6.2f) are satisfied without imposing any additional restrictions on the function $\phi(x_2, x_3)$. Moreover, substituting relations (6.16) into relations (6.2b) and (6.2c) and using Green's theorem of the plane† (6.19a) and relations (6.20a) and (6.20b), (see footnote) for a prismatic body with N holes, we get

$$\begin{aligned} F_2 &= \iint_A \tau_{12}\, dA = \iint_A \left(\frac{\partial \phi}{\partial x_3} \right) dx_2 dx_3 = -\oint_{C_e} \phi\, dx_2 - \sum_{i=1}^{N} \oint_{C_i} \phi\, dx_2 \\ &= -\phi_{C_e} \oint_{C_e} dx_2 - \sum_{i=1}^{N} \phi_{C_i} \oint_{C_i} dx_2 = 0 \\ F_3 &= \iint_A \tau_{13}\, dA = -\iint_A \left(\frac{\partial \phi}{\partial x_2} \right) dx_2 dx_3 = -\oint_{C_e} \phi\, dx_3 - \sum_{i=1}^{N} \oint_{C_i} \phi dx_3 \\ &= -\phi_{C_e} \oint_{C_e} dx_3 - \sum_{i=1}^{N} \phi_{C_i} \oint_{C_i} dx_3 = 0 \end{aligned} \tag{6.21}$$

where A is the area of the cross section of the body. The line intergrals in relations (6.21) are evaluated counterclockwise on the curve C_e and clockwise on the curves C_i (i = 1, 2, ..., N). Substituting relations (6.16) into relation (6.2d), we obtain

$$M_1 = \iint_A (x_2 \tau_{31} - x_3 \tau_{12})\, dA = -\iint_A \left[x_2 \frac{\partial \phi}{\partial x_2} + x_3 \frac{\partial \phi}{\partial x_3} \right] dx_2 dx_3$$

---

† Green's theorem of the plane states:
If $M(x_2, x_3)$ and $N(x_2, x_3)$, $\partial M/\partial x_2$ and $\partial M/\partial x_3$ are finite continuous functions of x_2 and x_3 on a plane surface of area A bounded by an external closed curve C_e and N internal closed curves C_i(i = 1, 2, ..., N) then

$$\iint_A \left(\frac{\partial M}{\partial x_3} - \frac{\partial N}{\partial x_2} \right) dx_2 dx_3 = -\oint_{C_e} (M dx_2 + N dx_3) - \sum_{i=1}^{N} \oint_{C_i} (M dx_2 + N dx_3) \tag{6.19a}$$

Using relations (6.4) and (6.5), relation (6.19a) can be rewritten as

$$\iint_A \left(\frac{\partial M}{\partial x_3} - \frac{\partial N}{\partial x_2} \right) dx_2 dx_3 = \oint_{C_e} (M\lambda_{n3} - N\lambda_{n2}) dx_s + \sum_{i=1}^{N} \oint_{C_i} (M\lambda_{n3} - N\lambda_{n2}) dx_s \tag{6.19b}$$

In the above relations the line integrals are evaluated counterclockwise along the closed curve C_e and clockwise along the closed curves C_i. Notice that if $N = 0$ and $M = 1$ for a plane surface without holes, relation (6.19a) reduces to

$$\oint_{C_e} dx_2 = 0 \tag{6.20a}$$

Similarly, if $M = 0$ and $N = 1$ for a plane surface without holes relation (6.19a) reduces to

$$\oint_{C_e} dx_3 = 0 \tag{6.20b}$$

Moreover, if $M = x_3$ and $N = -x_2$ for simply connected regions relation (6.19a) gives

$$-\oint_{C_e} (x_3 dx_2 - x_2 dx_3) = 2 \iint_{A_e} dx_2 dx_3 = 2A_e$$

and

$$\oint_{C_i} (x_3 dx_2 - x_2 dx_3) = 2A_i \tag{6.20c}$$

$$= \iint_A \left[2\phi - \frac{\partial(x_2\phi)}{\partial x_2} - \frac{\partial(x_3\phi)}{\partial x_3} \right] dx_2 dx_3 \tag{6.22}$$

Using Green's theorem of the plane (6.19a) with $M = -\phi x_3$ and $N = \phi x_2$ and relations (6.20c) and (6.20d), relation (6.22) can be rewritten as

$$\begin{aligned} M_1 &= 2\iint_A \phi\, dA + \phi_{C_e} \oint_{C_e} (x_3 dx_2 - x_2 dx_3) - \sum_{i=1}^{N} \phi_{C_i} \oint_{C_i} (x_3 dx_2 - x_2 dx_3) \\ &= 2\iint_A \phi\, dA - 2\phi_{C_e} A_e + 2\sum_{i=1}^{N} \phi_{C_i} A_i \end{aligned} \tag{6.23}$$

where A is the area of the cross section and A_e is the area enclosed by the curve C_e, while A_i is the area enclosed by the curve C_i. Thus,

$$A = A_e - \sum_{i=1}^{N} A_i \tag{6.24}$$

On the basis of the foregoing presentation the components of stress (6.14) with (6.16) satisfy the equations of equilibrium at all points inside the volume of the prismatic body under consideration and the boundary conditions (6.1) and (6.2) on its surfaces provided that the stress function $\phi(x_2, x_3)$ has the following attributes:

1. It is constant on the lateral surfaces of the body.
2. It satisfies relation (6.23).

In what follows we establish the additional restrictions which must be imposed on the stress function $\phi(x_2, x_3)$ in order to ensure that the components of strain obtained from the components of stress (6.14) with (6.16) satisfy the equations of compatibility (2.63). If the body under consideration is simply connected (does not have holes), the satisfaction of the equations of compatibility is necessary and sufficient to ensure that the strain–displacement relations (2.16) can be integrated to give single-value continuous components of displacement. If the body is multiply connected (has holes), the satisfaction of the equations of compatibility is necessary and sufficient to ensure only that when the components of strain are substituted into the strain–displacement relations, the resulting relations can be integrated. However, the resulting components of displacement may or may not be single-valued, continuous functions of the space coordinates. In order to ensure that the components of displacement are single-valued, continuous functions of the space coordinates, restrictions must be imposed on the stress function $\phi(x_2, x_3)$ in addition to those imposed by the equations of compatibility. We establish these restrictions in Section 6.6.

Substituting relations (6.14) with (6.16) into the stress–strain relations (3.48) for a body made from an isotropic, linearly elastic material , we obtain

$$\begin{aligned} e_{11} &= e_{22} = e_{33} = e_{23} = 0 \\ e_{12} &= \frac{\tau_{12}}{2G} = \frac{1}{2G}\frac{\partial\phi}{\partial x_3} \\ e_{13} &= \frac{\tau_{13}}{2G} = -\frac{1}{2G}\frac{\partial\phi}{\partial x_2} \end{aligned} \tag{6.25}$$

Notice that since the normal components of strain vanish, the length of the longitudinal fibers of the body does not change.

Substituting relations (6.25) into the equations of compatibility (2.63) and noting that $e_{12} = e_{12}(x_2, x_3)$ and $e_{13} = e_{13}(x_2, x_3)$, we find that the first four of the equations of compatibility (2.63) are satisfied without imposing any additional restriction on the function $\phi(x_2, x_3)$ while the last two of these equations reduce to

$$\begin{aligned} \frac{\partial}{\partial x_2}\left(-\frac{\partial e_{13}}{\partial x_2} + \frac{\partial e_{12}}{\partial x_3}\right) &= 0 \\ \frac{\partial}{\partial x_3}\left(-\frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{13}}{\partial x_2}\right) &= 0 \end{aligned} \tag{6.26}$$

Consequently,

$$\begin{aligned} \frac{\partial e_{13}}{\partial x_2} - \frac{\partial e_{12}}{\partial x_3} &= f_2(x_3) \\ -\frac{\partial e_{12}}{\partial x_3} + \frac{\partial e_{13}}{\partial x_2} &= f_3(x_2) \end{aligned} \tag{6.27}$$

Thus,

$$f_2(x_3) = f_3(x_2) = \alpha = \text{constant}$$

Hence, relations (6.27) can be rewritten as

$$\frac{\partial e_{13}}{\partial x_2} - \frac{\partial e_{12}}{\partial x_3} = \alpha \tag{6.28}$$

Substituting relation (6.25) into (6.28), we obtain

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -2G\alpha \tag{6.29}$$

On the basis of the foregoing presentation the components of stress (6.14) with (6.16) satisfy the equations of equilibrium at all points inside the volume of the prismatic body under consideration as well as the boundary conditions on all its surfaces. Moreover, on the basis of the stress–strain relations (3.48) the components of stress (6.14) with (6.16) give components of strain which satisfy the equations of compatibility (2.63) at every point of the body, provided that the function $\phi(x_2, x_3)$ has the following attributes

1. It satisfies the differential equation (6.29) at every point of the cross sections of the body.
2. It is constant on the curves C_e and C_i ($i = 1, 2, ..., N$) bounding the cross sections of the body.

In order to obtain a differential equation which does not involve an unknown constant, like α, we introduced the function $\psi(x_2, x_3)$ defined by

$$\phi(x_2, x_3) = \alpha G\psi(x_2, x_3) \tag{6.30}$$

Substituting relations (6.30) into (6.29), we have

$$\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} = -2 \tag{6.31}$$

Moreover, substituting relation (6.30) into (6.23), we get

$$M_1 = \alpha G R_C = \alpha R_T \tag{6.32}$$

where

$$R_C = 2\left[\iint_A \psi dA - \psi_{C_e} A_e + \sum_{i=1}^{N} \psi_{C_i} A_i\right] \tag{6.33}$$

A_e is the area enclosed by the curve C_e bounding the cross sections of the body, while A_i is the area enclosed by the curve C_i bounding the i^{th} (i = 1, 2, ..., N) hole of its cross sections. R_T is called the *torsional rigidity* and R_C is called the *torsional constant*. The latter depends only on the geometry of the cross sections of the body and can be computed once the function $\psi(x_2, x_3)$ is established† using relation (6.33). Furthermore, substituting relations (6.30) into (6.16) and (6.25), we obtain

$$\tau_{12} = \alpha G\frac{\partial \psi}{\partial x_3} \qquad \tau_{13} = -\alpha G\frac{\partial \psi}{\partial x_2} \tag{6.35}$$

and

$$e_{11} = e_{22} = e_{33} = e_{23} = 0 \qquad e_{12} = \frac{\alpha}{2}\frac{\partial \psi}{\partial x_3} \qquad e_{13} = -\frac{\alpha}{2}\frac{\partial \psi}{\partial x_2} \tag{6.36}$$

Thus, we have converted the boundary value problem for computing the components of displacement and stress in a prismatic simply connected (without holes) body subjected to a distribution of shearing components of traction on each of its end surfaces which is statically equivalent to a given torsional moment, to the simpler boundary value problem of finding the function $\psi(x_2, x_3)$ which has the following attributes:

1. It satisfies the differential equation (6.31) at every point inside of the cross sections of the body.
2. It is constant on the curves C_e and C_i (i = 1, 2, ..., N) bounding the cross sections of the body.

On the basis of the above presentation it is clear that the function $\psi(x_2, x_3)$ depends

† Saint Venant presented the following empirical formula for finding R_C for simply connected cross sections of any geometry

$$R_C = \frac{0.25A^4}{I_p} \tag{6.34}$$

where A is the area of the cross section and I_p is its centroidal polar moment of inertia. This formula gives acceptable results except for bodies whose cross sections have one dimension considerably larger than the other.

only on the geometry of the cross sections of the body under consideration. Once the function $\psi(x_2, x_3)$ is established, it can be substituted into relation (6.33) to give the torsional constant which in turn can be substituted into relation (6.32) to yield the constant α. The components of stress are obtained from relations (6.34).

In what follows we substitute relations (6.35) into the strain–displacement relations (2.16) and integrate the resulting equations to obtain the components of displacement $u_2(x_2, x_3)$ and $u_3(x_2, x_3)$ of the body. That is,

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} = 0 \quad \text{therefore} \quad \hat{u}_1 = \hat{u}_1(x_2, x_3) \tag{6.37a}$$

$$e_{22} = \frac{\partial \hat{u}_2}{\partial x_2} = 0 \quad \text{therefore} \quad \hat{u}_2 = \hat{u}_2(x_1, x_3) \tag{6.37b}$$

$$e_{33} = \frac{\partial \hat{u}_3}{\partial x_3} = 0 \quad \text{therefore} \quad \hat{u}_3 = \hat{u}_3(x_1, x_2) \tag{6.37c}$$

$$e_{23} = \frac{1}{2}\left[\frac{\partial \hat{u}_2}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_2}\right] = 0 \quad \text{therefore} \quad -\frac{\partial \hat{u}_2(x_1, x_3)}{\partial x_3} = \frac{\partial \hat{u}_3(x_1, x_2)}{\partial x_2} = f(x_1) \tag{6.37d}$$

Integrating, relations (6.37d), we get

$$\hat{u}_3 = x_2 f(x_1) + g(x_1) \tag{6.38a}$$

$$\hat{u}_2 = -x_3 f(x_1) + h(x_1) \tag{6.38b}$$

From the strain–displacement relation for e_{13} and e_{12}, using relations (6.36) we have

$$\begin{aligned} \frac{\partial \hat{u}_3}{\partial x_1} &= 2e_{13} - \frac{\partial \hat{u}_1}{\partial x_3} = \alpha\frac{\partial \psi}{\partial x_2} - \frac{\partial \hat{u}_1}{\partial x_3} \\ \frac{\partial \hat{u}_2}{\partial x_1} &= 2e_{12} - \frac{\partial \hat{u}_1}{\partial x_2} = \alpha\frac{\partial \psi}{\partial x_3} - \frac{\partial \hat{u}_1}{\partial x_2} \end{aligned} \tag{6.39}$$

Differentiating the above relations with respect to x_1, we obtain

$$\begin{aligned} \frac{\partial^2 \hat{u}_3}{\partial x_1^2} &= -\alpha\frac{\partial^2 \psi}{\partial x_1 \partial x_2} - \frac{\partial^2 \hat{u}_1}{\partial x_1 \partial x_3} \\ \frac{\partial^2 \hat{u}_2}{\partial x_1^2} &= \alpha\frac{\partial^2 \psi}{\partial x_1 \partial x_3} - \frac{\partial^2 \hat{u}_1}{\partial x_1 \partial x_2} \end{aligned} \tag{6.40}$$

However, referring to relation (6.37a), we see that u_1 is not a function of x_1 and, moreover, ψ is not a function of x_1. Therefore, from relations (6.40), we have

$$\frac{\partial^2 \hat{u}_2}{\partial x_1^2} = 0 \qquad \frac{\partial^2 \hat{u}_3}{\partial x_1^2} = 0$$

Substituting relations (6.38) into the above, we get

$$-x_3 \frac{d^2 f}{dx_1^2} + \frac{d^2 h}{dx_1^2} = 0$$

$$x_2 \frac{d^2 f}{dx_1^2} + \frac{d^2 g}{dx_1^2} = 0$$

These equations are satisfied if

$$\frac{d^2 f}{dx_1^2} = 0 \qquad \frac{d^2 g}{dx_1^2} = 0 \qquad \frac{d^2 h}{dx_1^2} = 0$$

Integrating the above relations, we have

$$f(x_1) = a + mx_1$$

$$g(x_1) = b + cx_1$$

$$h(x_1) = d + ex_1$$

Substituting the above relations into (6.38), we obtain

$$\begin{aligned} \hat{u}_2 &= -mx_3x_1 + ex_1 - ax_3 + d \\ \hat{u}_3 &= mx_1x_2 + cx_1 + ax_2 + b \end{aligned} \tag{6.41}$$

where x_1, x_2 and x_3 are any set of orthogonal axes having the x_1 axis parallel to the axis of the body. Referring the relations (2.59) we see that the last two terms on the right-hand side of relations (6.41) represent rigid-body motion. That is, as expected, the components of displacement (6.41) include an unspecified rigid-body motion of the body. We can eliminate this rigid-body motion by restraining the body from moving as a rigid-body. However, this must be done in a way that the warping of the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body is not inhibited. Otherwise normal components of stress will be induced on the cross sections of the body and the length of its longitudinal fibers will change. In practice one or more cross sections of a body are usually restrained from warping. In Section 9.11 we investigate the effect of restraining the warping of a cross section on the stress field in prismatic bodies having thin-walled, open cross sections when subjected to equal and opposite torsional moments at their ends. Eliminating the terms which represent rigid-body motion, relations (6.41) may be rewritten as

$$\begin{aligned} \hat{u}_2(x_1, x_3) &= -mx_1(x_3 - e_3) \\ \hat{u}_3(x_1, x_2) &= mx_1(x_2 - e_2) \end{aligned} \tag{6.42}$$

From relation (6.42) we see that the components of displacement $\hat{u}_2$ and $\hat{u}_3$ of the point on the plane of a cross section (not necessarily on the cross section) whose coordinates are $x_2 = e_2$ and $x_3 = e_3$ vanish. That is, the particles of the cross section rotate about this point. We call this point the *center of twist* of the cross section of the body. Its location depends on the geometry of the cross section of the body. It can be shown that if a cross section has an axis of symmetry, its center of twist is located on the axis of symmetry. Thus, if a cross section has two axes of symmetry, its center of twist is the point of intersection of the two axes of symmetry. Consequently, it coincides with the centroid of the cross section. Similarly if the cross section of a body has a center of symmetry, its center of twist coincides with the center of symmetry which is also the centroid of the cross section. In Section 13.16 we show that *the center of twist of a cross section is identical to its shear center*. This is a point on the plane of each cross section of a body (not necessarily on the cross section) which has the property that when the line of action of an external transverse force acting on the body lies on a plane which contains the shear center of its cross sections, the body does not twist (see Section 9.7).

In our presentation in this chapter we consider bodies which we assume are supported and loaded in a way that all their cross sections are free to warp. Moreover, we assume that the supports restrain the bodies from moving as rigid bodies. In general these assumptions cannot be easily realized in practice. For this reason in Section 9.11 we investigate the effect of restraining the warping of a cross section of certain bodies on the state of stress of their particles and on the angle of twist of their cross sections.

Using relations (2.16), relation (6.28) may be rewritten as

$$\frac{1}{2}\left(\frac{\partial^2 \hat{u}_3}{\partial x_2 \partial x_1} - \frac{\partial^2 \hat{u}_2}{\partial x_3 \partial x_1}\right) = \alpha \tag{6.43}$$

Substituting relations (6.42) in the above, we get

$$\alpha = m$$

Substituting the above relation into (6.42) we obtain

$$\begin{aligned} \hat{u}_2(x_1, x_3) &= -\alpha x_1(x_3 - e_3) \\ \hat{u}_3(x_1, x_2) &= \alpha x_1(x_2 - e_2) \end{aligned} \tag{6.44}$$

The component of displacement $\hat{u}_1(x_2, x_3)$ is obtained by substituting the function $\psi(x_2, x_3)$ into the following relations established from the fourth and fifth of the strain-displacement relations (2.16) and using relations (6.36) and (6.44). That is,

$$\begin{aligned} \frac{\partial \hat{u}_1}{\partial x_2} &= 2e_{12} - \frac{\partial \hat{u}_2}{\partial x_1} = \alpha \frac{\partial \psi}{\partial x_3} + \alpha x_3 - \alpha e_3 \\ \frac{\partial \hat{u}_1}{\partial x_3} &= 2e_{13} - \frac{\partial \hat{u}_3}{\partial x_1} = -\alpha \frac{\partial \psi}{\partial x_2} - \alpha x_2 + \alpha e_2 \end{aligned} \tag{6.45}$$

When the function $\psi(x_2, x_3)$ corresponding to the given geometry of the cross section of a prismatic body is established, it can be substituted into relation (6.33) to get the torsional constant R_C which can in turn be used in a relation (6.32) to obtain α. The function ψ and the constant α can then be substituted into relations (6.45) and the resulting expressions can be integrated to give the component of displacement $\hat{u}_1(x_2, x_3)$.

6.4 Interpretation of the Results of the Torsion Problem

Referring to Fig. 6.3 the direction cosines of the unit vectors in the radial $\mathbf{i}_r$ and in the transverse $\mathbf{i}_\theta$ directions, with respect to the *axes* x_2^* *and* x_3^*, whose origin is the center of twist of the cross section of a body, are

$$\lambda_{r2} = \cos\theta^* \qquad \lambda_{r3} = \sin\theta^* \qquad \lambda_{\theta 2} = -\sin\theta^* \qquad \lambda_{\theta 3} = \cos\theta^*$$

Referring to relation (2.80b) and using the above relations and (6.44) the radial and transverse components of displacement of a particle located at a distance r^* from the center of twist of a cross section of a prismatic body, may be established as

$$\begin{aligned} u_r &= u_2\lambda_{r2} + u_3\lambda_{r3} = u_2\cos\theta^* + u_3\sin\theta^* \\ &= -\alpha x_1(x_3 - e_3)\cos\theta^* - (x_2 - e_2)\sin\theta^* \\ &= -\alpha x_1(r^*\cos\theta^*\sin\theta^* - r^*\cos\theta^*\sin\theta^*) = 0 \\ u_\theta &= u_2\lambda_{\theta 2} + u_3\lambda_{\theta 3} = -u_2\sin\theta^* + u_3\cos\theta^* \\ &= \alpha x_1(x_3 - e_3)\sin\theta^* + (x_2 - e_2)\cos\theta^* \\ &= \alpha x_1(r^*\sin^2\theta^* + r^*\cos^2\theta^*) = \alpha x_1 r^* \end{aligned} \tag{6.46}$$

Thus, when a prismatic body is subjected to equal and opposite torsional moments at its ends, *the radial component of displacement of its particles, with respect to the center of twist, vanishes.*

Consider a prismatic body in its stress-free, strain-free state (undeformed state) of mechanical and thermal equilibrium at the uniform temperature T_o. Moreover, consider a particle, which when the body is in its undeformed state is located at a point $P(x_1, x_2, x_3)$ of the cross section at x_1 of this body. When the body is subjected to equal and opposite torsional moments at its ends, in a way that all its cross sections are free to warp, the particle under consideration moves to point $P'(\xi_1, \xi_2, \xi_3)$. We denote by $P''(x_1, \xi_2, \xi_3)$ the projection of point P' on the cross section at x_1. Referring to relations (2.4) and (6.44), we have

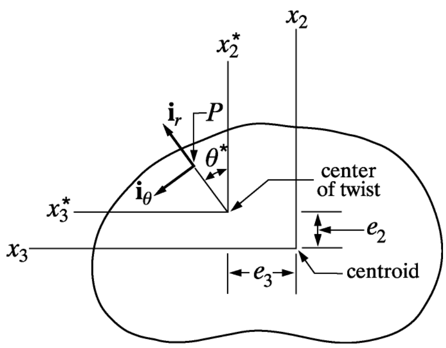


Figure 6.3 Radial and transverse directions with respect to the center of twist of a cross section.

$$\xi_2 - e_2 = (x_2 - e_2) + u_2 = (x_2 - e_2) - \alpha x_1(x_3 - e_3)$$
$$\xi_3 - e_3 = (x_3 - e_3) + u_3 = (x_3 - e_3) + \alpha x_1(x_2 - e_2)$$

Solving the above relations for x_2 and x_3, we obtain

$$x_2 - e_2 = \frac{(\xi_2 - e_2) + \alpha x_1(\xi_3 - e_3)}{1 + \alpha^2 x_1^2}$$
$$x_3 - e_3 = \frac{(\xi_3 - e_3) - \alpha x_1(\xi_2 - e_2)}{1 + \alpha^2 x_1^2} \tag{6.47}$$

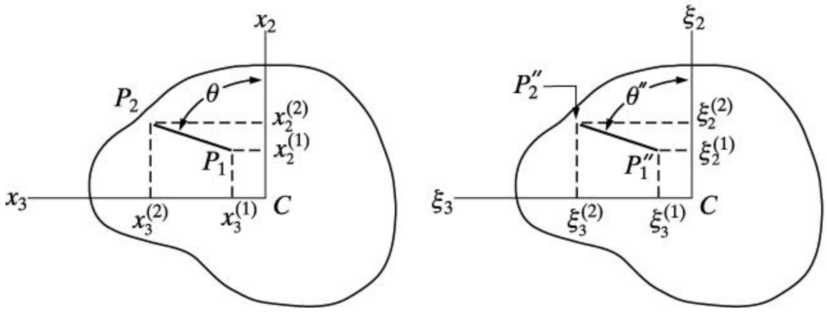
Consider a segment of a material straight line which in the undeformed state is located on the cross section at x_1 of a prismatic body and extends from point $P_1(x_1, x_2^{(1)}, x_3^{(1)})$ to point $P_2(x_1, x_2^{(2)}, x_3^{(2)})$. The equation of this line can be written in the following form:

$$x_3 - e_3 = A(x_2 - e_2) + B \tag{6.48a}$$

When the prismatic body under consideration is subjected to equal and opposite torsional moments at its end, in a way that the warping of its cross sections is not inhibited, the before deformation material straight line of a cross section deforms into a three-dimensional curve. The end particles of this line which in the unformed state were located at points $P_1(x_1, x_2^{(1)}, x_3^{(1)})$ and $P_2(x_1, x_2^{(2)}, x_3^{(2)})$ moved to points $P_1'(\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)})$ and $P_2'(\xi_1^{(2)}, \xi_2^{(2)}, \xi_3^{(2)})$, respectively. In Fig. 6.4 we denote the projection of these points, on the cross section at x_1 by $P_1''(x_1, \xi_2^{(1)}, \xi_3^{(1)})$ and $P_2''(x_1, \xi_2^{(2)}, \xi_3^{(2)})$, respectively. The equation of the projection, on the cross section at x_1, of the deformed line under consideration may be obtained by substituting relations (6.47) in (6.48a). That is,

$$\xi_3 - e_3 = \left(\frac{A + \alpha x_1}{1 - A\alpha x_1}\right)(\xi_2 - e_2) + \left(\frac{1 + \alpha^2 x_1^2}{1 - A\alpha x_1}\right)B \tag{6.48b}$$

Thus, a material straight line which, in the undeformed state, is located on a cross section at x_1 of a prismatic body, deforms into a curve when the body is subjected to equal and



(a) Undeformed straight line (b) Projection of the deformed line

Figure 6.4 Deformed configuration of a material straight line which in the undeformed state is located on a cross section of a prismatic body subjected to equal and opposite torsional moments at its ends.

opposite torsional moments at its ends without restraining the warping of its cross sections. However, as can be seen from relation (6.48b) the projection of this curve on a plane normal to the axis of the body is a straight line. Referring to Fig. 6.4, we have

$$\tan\theta = \frac{x_3^{(2)} - x_3^{(1)}}{x_2^{(2)} - x_2^{(1)}} \tag{6.49a}$$

and

$$\tan\theta'' = \frac{\xi_3^{(2)} - \xi_3^{(1)}}{\xi_2^{(2)} - \xi_2^{(1)}} \tag{6.49b}$$

From relations (6.48a), we have

$$\begin{aligned} x_3^{(2)} - e_3 &= A(x_2^{(2)} - e_2) + B \\ x_3^{(1)} - e_3 &= A(x_2^{(1)} - e_2) + B \end{aligned} \tag{6.50a}$$

Moreover, from relation (6.48b) we get

$$\begin{aligned} \xi_3^{(2)} - e_3 &= \left(\frac{A + \alpha x_1}{1 - A\alpha x_1}\right)(\xi_2^{(2)} - e_2) + \left(\frac{1 + \alpha^2 x_1^2}{1 - A\alpha x_1}\right)B \\ \xi_3^{(1)} - e_3 &= \left(\frac{A + \alpha x_1}{1 - A\alpha x_1}\right)(\xi_2^{(1)} - e_2) + \left(\frac{1 + \alpha^2 x_1^2}{1 - A\alpha x_1}\right)B \end{aligned} \tag{6.50b}$$

Substituting relations (6.50a) into (6.49a) and relations (6.50b) into (6.49b), we obtain

$$\tan\theta = A \tag{6.51a}$$

and

$$\tan\theta'' = \frac{A + \alpha x_1}{1 - A\alpha x_1} \tag{6.51b}$$

We denote by θ_1 the rotation about the x_1 axis of the line under consideration due to the deformation of the body. Thus,

$$\theta'' = \theta + \theta_1$$

Consequently, from trigonometric considerations, we have

$$\tan\theta'' = \tan(\theta + \theta_1) = \frac{\tan\theta + \tan\theta_1}{1 - \tan\theta\tan\theta_1}$$

Substituting relations (6.51) into the above, we get

$$\frac{A + \alpha x_1}{1 - A\alpha x_1} = \frac{A + \tan\theta_1}{1 - A\tan\theta_1}$$

or

$$\theta_1 \approx \tan\theta_1 = \alpha x_1 \tag{6.52}$$

On the basis of the foregoing presentation we may arrive at the following conclusions: *When a prismatic body is subjected to equal and opposite torsional moments at its ends in a way that no one of its cross sections is restrained from warping, a material straight line located on one of its cross sections before deformation in general deforms into a three- dimensional curve. However, the projection of this curve on a cross section of the body is a straight line which rotated by an angle θ_1. As can be seen from relation (6.52) all material straight lines located on the same cross section rotate by the same angle θ which is proportional to the axial coordinate x . It is called the angle of twist of the cross section. Referring to relation (6.52) we see that the constant α, introduced in relation (6.27) as a constant of integration, is the angle of twist per unit length.* That is, differentiating relation (6.52) and using relation (6.32), we have

$$\alpha = \frac{d\theta_1}{dx_1} = \frac{M_1}{R_T} = \frac{M_1}{GR_C} \tag{6.53}$$

R_T is *the torsional rigidity* and R_C is *the torsional constant*. The latter depends only on the geometry of the cross section of the body.

6.5 Computation of the Stress and Displ acement Fields of B odies of So lid Elliptical and Circular Cross Section Subjected to Equal and Opposite Torsional Moments at Their Ends

In this section we present an example

Example 1 Consider a prismatic body of solid elliptical cross section, whose geometry is shown in Fig. a, subjected to equal and opposite torsional moments at its ends of magnitude M_1. We choose the x_2 and x_3 axes to be the principal centroidal axes of the elliptical cross section of the body. With respect to these axes the equation of the boundary of the cross sections of the body is given by

$$\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1 \tag{a}$$

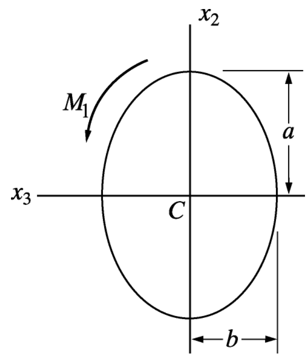


Figure a Elliptical cross section of a prismatic body.

Solution We are interested in establishing, to within a constant, the function $\psi(x_2, x_3)$ which satisfies relation (6.31) at every point of the body. That is,

$$\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} = -2 \tag{b}$$

and moreover, is a constant on the closed curve C_e (the perimeter of the cross sections of the body). Inasmuch as the components of stress are derivatives of the function ψ we can choose the constant value ψ equal to zero on the closed curve C_e. Thus, we assume that ψ is given by

$$\psi = B\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) \tag{c}$$

It is apparent that ψ vanishes on the closed curve C_e. Substituting relation (c) into (b), we obtain

$$B = -\frac{a^2 b^2}{a^2 + b^2} \tag{d}$$

Substituting relation (d) into (c), we get

$$\psi = -\frac{a^2 b^2}{a^2 + b^2}\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) \tag{e}$$

The torsional constant R_C may be computed by substituting relation (e) into (6.33). That is,

$$R_C = -\frac{2a^2 b^2}{a^2 + b^2}\iint_A \left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) dA = -\frac{2a^2 b^2}{a^2 + b^2}\left(\frac{I_3}{a^2} + \frac{I_2}{b^2} - A\right) \tag{f}$$

where I_3, and I_2 are the moments of inertia about the x_3 and x_2 axes, respectively, of the elliptical cross section of the body; A is the area of the cross section of the body. Referring to the table on the inside of the back cover of the book, these quantities are given by

$$I_2 = \frac{\pi a b^3}{4} \qquad I_3 = \frac{\pi a^3 b}{4} \qquad A = \pi a b \tag{g}$$

Substituting relations (g) into (f), we obtain the following expression for the torsional constant of the body:

$$R_C = \frac{\pi a^3 b^3}{a^2 + b^2} \tag{h}$$

Substituting relation (h) into (6.53), we get

$$\alpha = \frac{M_1}{G R_C} = \frac{M_1(a^2 + b^2)}{G \pi a^3 b^3} \tag{i}$$

Notice that using relations (g) the empirical formula (6.34) presented by Saint Venant gives

$$R_C = 0.1\pi^2\left(\frac{\pi a^3 b^3}{a^2 + b^2}\right) = 0.987\left(\frac{\pi a^3 b^3}{a^2 + b^2}\right) \tag{j}$$

Comparing relations (h) and (j), we see that they differ by 1.3%.
The components of stress may be computed by substituting relation (e) into (6.35) and using relation (i). That is,

$$\begin{aligned} \tau_{12} &= \alpha G \frac{\partial \psi}{\partial x_3} = -\frac{2M_1 x_3}{\pi a b^3} \\ \tau_{13} &= -\alpha G \frac{\partial \psi}{\partial x_2} = \frac{2M_1 x_2}{\pi a^3 b} \end{aligned} \tag{k}$$

Consider a point $A(x_2, x_3)$ located r distance from the centroid C of the cross section. Referring to Fig. b and to relations (k), the resultant shear stress at that point is given by

$$\tau_{1s} = \sqrt{\tau_{12}^2 + \tau_{13}^2} = \frac{2M_1}{\pi ab}\left(\frac{x_3^2}{b^4} + \frac{x_2^2}{a^4}\right)^{\frac{1}{2}} = \frac{2M_1 r}{\pi ab}\left(\frac{\sin^2\beta}{b^4} + \frac{\cos^2\beta}{a^4}\right)^{\frac{1}{2}}$$

It can be seen that τ_{1s} is proportional to r. Moreover, it can be shown that the maximum shearing stress occurs at $x_2 = 0$ $x_3 = b$ ($b = 90^\circ$). Referring to Fig. b and using relations (k) the direction of the resultant shearing stress is given by

$$\tan\gamma = \frac{|\tau_{12}|}{|\tau_{13}|} = \frac{a^2 x_3}{b^2 x_2} = \frac{a^2}{b^2}\tan\beta \tag{l}$$

It is apparent, that the direction of the resultant shearing stress acting on the particles of any centroidal line CB (β= constant) is constant. At point B, the resultant shearing stress must be tangent to the elliptical boundary of the cross section of the body. Consequently, the direction of the resultant shearing stress at any point of line CB, must be parallel to the tangent to the elliptical boundary at point B.

The components of displacement $\hat{u}_2$ and $\hat{u}_3$ are obtained by substituting relation (i)

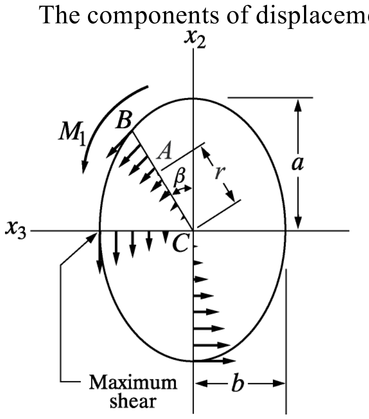


Figure b Distribution of shearing stress in a body, of elliptical cross section, subjected to equal and opposite torsional.

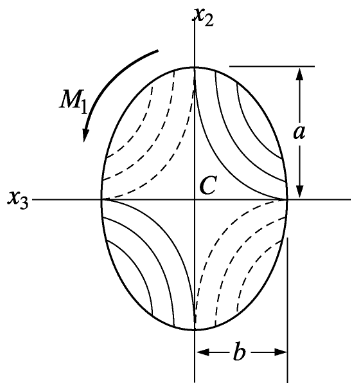


Figure c Contour lines of $\hat{u}_1$ for a prismatic body of elliptical cross section subjected, to equal and opposite torsional moments at its ends.

into relations (6.44). An elliptical cross section has two axes of symmetry; consequently, its center of twist coincides with its centroid (see Section 9.7) and, the constants e_2 and e_3 in relations (6.44) are zero. Thus,

$$\hat{u}_2 = -\alpha x_1 x_3 = -\frac{M_1}{R_T} x_1 x_3 = -\frac{M_1(b^2+a^2)x_1x_3}{G\pi b^3 a^3}$$
$$\hat{u}_3 = \alpha x_1 x_2 = \frac{M_1}{R_T} x_1 x_2 = \frac{M_1(b^2+a^2)x_1x_2}{G\pi a^3 b^3} \tag{m}$$

The displacement component $\hat{u}_1$ may be obtained by substituting relations (e) and (i) into (6.45) and integrating the resulting relations. That is,

$$\frac{\partial \hat{u}_1}{\partial x_2} = \alpha \frac{\partial \psi}{\partial x_3} + \alpha x_3 = \frac{M_1(b^2-a^2)x_3}{G\pi b^3 a^3}$$
$$\frac{\partial \hat{u}_1}{\partial x_3} = -\alpha \frac{\partial \psi}{\partial x_2} - \alpha x_2 = \frac{M_1(b^2-a^2)x_2}{G\pi a^3 b^3} \tag{n}$$

Integrating relations (n), we get

$$\hat{u}_1 = \frac{M_1(b^2-a^2)x_2x_3}{G\pi b^3 a^3} + f_2(x_3)$$
$$\hat{u}_1 = \frac{M_1(b^2-a^2)x_2x_3}{G\pi a^3 b^3} + f_3(x_2) \tag{o}$$

Equating the left-hand side of the above relations, we obtain

$$f_2(x_3) = f_3(x_2) = \text{constant} \tag{p}$$

Therefore, if we restrain the prismatic body from moving as a rigid-body by setting $\hat{u}_1 = 0$ at $x_2 = x_3 = 0$, we get

$$\hat{u}_1 = \frac{M_1(b^2-a^2)x_2x_3}{G\pi a^3 b^3} \tag{q}$$

As it can be seen from relation (q) for a prismatic body with an elliptical cross section, plane sections normal to the axis of the body before deformation do not remain plane after deformation. The contour lines of $\hat{u}_1$ are the hyperbolas plotted in Fig. c. When the body is twisted by end torsional moments acting in the direction indicated in Fig. c, the dotted lines denote the concave portion of the cross sections, whereas the solid lines denote its convex portion.

Prismatic bodies of circular cross sections subjected to equal and opposite torsional moments at their ends

The results obtained for a prismatic body of elliptical cross sections reduce to those for a prismatic body with circular cross sections of radius R by setting $a = b = R$. For this case relations (k) give

$$\tau_{12} = -\frac{2M_1 x_3}{\pi R^4} = -\frac{M_1 x_3}{I_p} \qquad \tau_{13} = \frac{M_1 x_2}{I_p} \tag{r}$$

where I_p is the polar moment of inertia of the circular cross section with respect to its centroid given by

$$I_p = \frac{\pi R^4}{2} \tag{s}$$

As has been deduced previously the direction of the resultant shearing stress is the same at all points of a radial line. Moreover, the resultant shearing stress at points of the boundary of the cross section acts in the direction of the tangent to the boundary. Thus, the resultant shearing stress acting on a particle of a cross section of a prismatic body of circular cross section must act in the direction normal to the radius passing through this particle. Consequently, for such a body the radial component τ_{1r} of shearing stress must vanish while the tangential component $\tau_{1\theta}$ of shearing stress is given by

$$\tau_{1\theta} = (\tau_{12}^2 + \tau_{13}^2)^{\frac{1}{2}} = \frac{M_1}{I_p}(x_2^2 + x_3^2)^{\frac{1}{2}} = \frac{M_1 r}{I_p} \tag{t}$$

The torsional rigidity of a prismatic body of circular cross section is obtained by substituting $a = b = R$ in relation (h). That is,

$$R_T = GR_C = \frac{G\pi R^4}{2} = GI_p \tag{u}$$

The components of displacement of the particles of a prismatic body of circular cross sections may be obtained by substituting $a = b = R$ into relations (q) and (m). That is,

$$\hat{u}_1 = 0 \qquad \hat{u}_2 = -\frac{M_1}{GI_p}x_1 x_3 \qquad \hat{u}_3 = \frac{M_1}{GI_p}x_1 x_2 \tag{v}$$

Therefore, when a prismatic body of circular cross sections is subjected to equal and opposite torsional mo ments at its ends, plane sections normal to its axis before deformation, remain plane subsequent to deformation. Moreover, since the center of twist of a prismatic body with circular cross section coincides with its centroid, radial straight lines remain straight and radial.

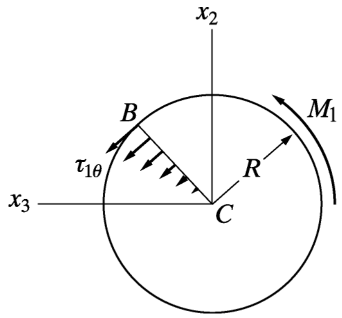


Figure d Stress distribution on the cross section of a prismatic body of circular cross section subjected to torsional moments at its ends.

Finally, the twist per unit length of a prismatic body of circular cross sections subjected to torsional moments at its ends may be obtained by substituting relation (u) in relation (i). That is, referring to relation (6.53), we have

$$\alpha = \frac{d\theta_1}{dx_1} = \frac{M_1}{GI_p} \tag{w}$$

6.6 Multiply Connected Prismatic Bodies Subjected to Equal and Opposite Torsional Moments at Their Ends

Consider a multiply connected (with N holes) prismatic body (see Fig. 6.5) made from a homogeneous, isotropic, linearly elastic material subjected to a distribution of external traction on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) which is statically equivalent to a torsional moment of magnitude M_1. In Section 6.3 we show that the stress distribution in a prismatic body (simply or multiply connected) subjected to torsional moments at its ends can be established from a stress function $\psi(x_2, x_3)$ on the basis of relations (6.34) provided that the function $\psi(x_2, x_3)$ satisfies the following requirements:

1. The differential equation (6.31) at every point of the body.
2. It is constant on the lateral surfaces of the body.
3. Relation (6.32) with (6.33).

Requirement 3 ensures that the components of stress satisfy the boundary conditions on the end surfaces ($x_1 = 0$ and $x_1 = L$) of the body. Requirement 2 ensures that the components of stress satisfy the boundary conditions on the lateral boundary of the body. Requirement 1 ensures that the components of strain obtained from the components of stress on the basis of the stress–strain relations (3.48) satisfy the equations of compatibility. For a multiply connected body the satisfaction of the equations of compatibility is necessary and sufficient to ensure that the strain–displacement relations (2.16) can be integrated. However, the resulting components of displacement may or may not be single-valued, continuous functions of the space coordinates. In order to ensure that the strain–displacement relations can be integrated to give single-valued, continuous components of displacement, additional restrictions must be imposed on the function $\psi(x_2, x_3)$. In this section we establish these additional restrictions. As we have indicated in Section 2.11 the necessary and sufficient condition for accomplishing this is that the components of strain satisfy relations which result from the following requirement:

$$\oint_P d\hat{\mathbf{u}} = 0 \tag{6.54}$$

where the integral is taken around any closed curve P lying entirely on a cross section of the prismatic body (see Fig. 6.5). However, on the basis of relations (6.44) the components of displacement u_2 and u_3 are single-valued continuous functions of the space coordinates. Therefore, noting that e_{11} vanishes, it is sufficient to set

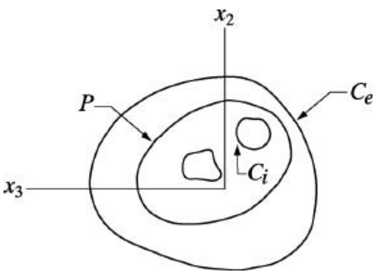


Figure 6.5 Cross section as a prismatic body with holes.

$$\oint_P d\hat{u}_1 = \oint_P \left(\frac{\partial \hat{u}_1}{\partial x_1} dx_1 + \frac{\partial \hat{u}_1}{\partial x_2} dx_2 + \frac{\partial \hat{u}_1}{\partial x_3} dx_3 \right) = \oint_P \left(\frac{\partial \hat{u}_1}{\partial x_2} dx_2 + \frac{\partial \hat{u}_1}{\partial x_3} dx_3 \right) = 0 \tag{6.55}$$

Substituting relations (6.45) into (6.55), taking into account relations (6.20a) and (6.20b) and using Green's theorem of the plane (6.19a), we get

$$\begin{aligned} \oint_P d\hat{u}_1 &= \alpha \oint_P \left[\left(\frac{\partial \psi}{\partial x_3} + x_3 \right) dx_2 + \left(-\frac{\partial \psi}{dx_2} - x_2 \right) dx_3 \right] \\ &= -\alpha \iint_{A_P} \left[\frac{\partial}{\partial x_2} \left(\frac{\partial \psi}{\partial x_2} + x_2 \right) + \frac{\partial}{\partial x_3} \left(\frac{\partial \psi}{\partial x_3} + x_3 \right) \right] dx_2 dx_3 \\ &\quad - \alpha \sum_{i=1}^{q} \oint_{C_i} \left[\left(\frac{\partial \psi}{\partial x_2} + x_2 \right) dx_3 - \left(\frac{\partial \psi}{\partial x_3} + x_3 \right) dx_2 \right] = 0 \end{aligned} \tag{6.56}$$

where A_P is the net area enclosed by the closed curve P; C_i is the perimeter of the i^{th} hole enclosed by the closed curve P; q is the number of the internal holes enclosed by curve P. In relation (6.56) the integral over the closed curve P is taken counterclockwise while the integrals over the curves C_i ($i = 1, 2, ..., N$) are taken clockwise. Using relations (6.8), (6.9), (6.20c) and (6.20d), relation (6.56) may be rewritten as

$$\begin{aligned} \oint_P d\hat{u}_1 &= -\alpha \iint_{A_P} \left[\left(\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} \right) + 2 \right] dx_2 dx_3 \\ &\quad - \alpha \sum_{i=1}^{q} \left[\oint_{C_i} \left(\frac{\partial \psi}{\partial x_3} \frac{dx_3}{dx_n} + \frac{\partial \psi}{\partial x_2} \frac{dx_2}{dx_n} \right) dx_s - \oint_{C_i} (x_3 dx_2 - x_2 dx_3) \right] \\ &= \alpha \iint_{A_p} \left[-\left(\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} \right) + 2 \right] dx_2 dx_3 - \alpha \sum_{i=1}^{q} \left[\oint_{C_i} \frac{\partial \psi}{\partial x_n} dx_s - 2A_i \right] = 0 \end{aligned} \tag{6.57}$$

Relation (6.57) is valid if the function $\psi(x_2, x_3)$ has the following attributes:

1. At every point of the cross section of the body it satisfies the following relation:

$$\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} = -2 \tag{6.58}$$

2. On the perimeter of each internal hole it satisfies the following relation:

$$\oint_{C_i} \frac{\partial \psi}{\partial x_n} dx_s = 2A_i \qquad i = 1, 2, ..., N \tag{6.59}$$

In the above relation the line integral is taken clockwise around the curves C_i. A_i is the area of the cross section of the i^{th} hole.

On the basis of the foregoing presentation we have converted the boundary value problem for computing the components of shearing stress $\tau_{12}(x_2, x_3)$ and $\tau_{13}(x_2, x_3)$ acting on the particles of the cross sections of a multiply connected (with holes) prismatic body subjected to equal and opposite torsional moments at its ends to the simpler boundary value problem of finding the stress function $\psi(x_2, x_3)$ which has the following attributes:

1. It satisfies the differential equation (6.58) at every point of the cross sections of the body.
2. It vanishes on the external lateral boundary of the cross sections of the body.
3. It satisfies relation (6.59) on the perimeter C_i ($i = 1, 2, ..., N$) of each hole of the cross section of the body.
Once the stress function $\psi(x_2, x_3)$ is established the angle of twist per unit length can be obtained on the basis of relation (6.32). Moreover, the components of stress can be computed on the basis of relations (6.35).

In what follows we compute the stress and displacement fields in a hollow prismatic body subjected to equal and opposite torsional moments at its ends. The cross sections of the body are bounded by two homothetic ellipses.

Example 2 Consider a prismatic body of hollow cross sections bounded by two homothetic ellipses, that is, similarly shaped, similarly placed and oriented, ellipses. The body is subjected to equal and opposite torsional moments at its ends of magnitude M_1. Referring to Fig. a, the equations of the curves bounding the cross sections of the body are

$$\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1 \tag{a}$$

$$\frac{x_2^2}{a_1^2} + \frac{x_3^2}{b_1^2} = 1 \tag{b}$$

Moreover,

$$\frac{a_1}{a} = \frac{b_1}{b} = m \tag{c}$$

where m is a given constant. The body is subjected to equal and opposite torsional moments at its ends. Compute the displacement and stress fields of the body.

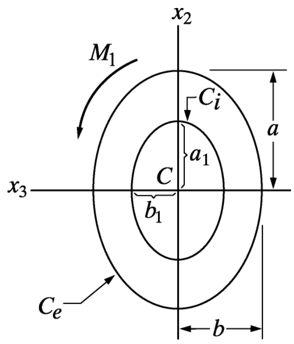


Figure a Geometry of the cross section of the prismatic body.

Solution We are looking for the function $\psi\,(x_2, x_3)$ which vanishes on the external boundary C_e, is constant ψ_{C_1} on the internal boundary C_1 and satisfies relation (6.58) at every point of the cross sections of the body. A function which appears to satisfy these requirements has the following form:

$$\psi = B\left[\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right] \tag{d}$$

Substituting relation (d) into (6.58), we get

$$B = -\frac{a^2b^2}{a^2 + b^2} \tag{e}$$

Thus, the stress function may be rewritten as

$$\psi = -\frac{a^2b^2}{a^2 + b^2}\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) = -\frac{a^2b^2}{a^2 + b^2}\left[m^2\left(\frac{x_2^2}{a_1^2} + \frac{x_3^2}{b_1^2}\right) - 1\right] \tag{f}$$

where m is defined by relation (c). On the internal boundary of the cross section of the body relation (f) gives

$$\psi_{C_1} = -\frac{a^2b^2}{a^2 + b^2}\,(m^2 - 1) \tag{g}$$

Inasmuch as the cross sections of the body under consideration are multiconnected (have holes) the satisfaction of equation (6.58) is not sufficient to ensure that the component of displacement $u_1(x_2, x_3)$ is a single-valued continuous function of the space coordinates. In order to ensure this, the stress function ψ must satisfy relation (6.59). That is,

$$\oint_{C_1} \frac{\partial \psi}{\partial x_n}\, dx_s = 2A_1 \tag{h}$$

where the line integral is taken clockwise around the internal boundary C_1; A_1 is the area of the hole. Referring to relations (6.8), (6.9), and (d) the derivative of the function ψ with respect to x_n may be expressed in terms of its derivatives with respect to x_2 and x_3 as follows:

$$\frac{\partial \psi}{\partial x_n} = \frac{\partial \psi}{\partial x_2}\frac{dx_2}{dx_n} + \frac{\partial \psi}{\partial x_3}\frac{dx_3}{dx_n} = \frac{\partial \psi}{\partial x_2}\frac{dx_3}{dx_s} - \frac{\partial \psi}{\partial x_3}\frac{dx_2}{dx_s} = 2B\left(\frac{x_2}{a^2}\frac{dx_3}{dx_s} - \frac{x_3}{b^2}\frac{dx_2}{dx_s}\right) \quad \text{(i)}$$

Substituting the above relation into (h), we obtain

$$\oint_{C_1} \frac{\partial \psi}{\partial x_n}\, dx_s = \oint_{C_1} 2B\left(\frac{x_2}{a^2}\, dx_3 - \frac{x_3}{b^2}\, dx_2\right) \quad \text{(j)}$$

Since the function ψ is not defined inside the hole, Green's theorem of the plane cannot be used to transform the line integral on the right side of relation (j) into a surface integral. Instead the line integral must be evaluated. Referring to relation (b) on the curve C_1, we have

$$x_2 = \pm \frac{a_1}{b_1}\left(b_1^2 - x_3^2\right)^{1/2}$$

and

$$x_3 = \pm \frac{b_1}{a_1}\left(a_1^2 - x_2^2\right)^{1/2}$$

Noting the line integral in relation (j) is taken clockwise we see that when we integrate from $x_3 = -b_1$ to $x_3 = b_1$ clockwise, the coordinate x_2 is negative. That is,

$$x_2 = -\frac{a_1}{b_1}\left(b_1^2 - x_3^2\right)^{1/2} \quad \text{(ka)}$$

When we integrate clockwise from $x_2 = b_1$ to $x_2 = -b_1$ we have

$$x_2 = \frac{a_1}{b_1}\left(b_1^2 - x_3^2\right)^{1/2} \quad \text{(kb)}$$

Moreover, when we integrate from $x_2 = -\,a_1$ to $x_2 = a_1$ clockwise, the coordinate x_3 is positive. That is,

$$x_3 = \frac{b_1}{a_1}\left(a_1^2 - x_2^2\right)^{1/2} \quad \text{(la)}$$

When we integrate clockwise from $x_3 = a_1$ to $x_2 = -a_2$, we have

$$x_3 = -\frac{b_1}{a_1}\left(a_1^2 - x_2^2\right)^{1/2} \quad \text{(lb)}$$

Using relations (k) and (l), we obtain

$$\begin{aligned} \frac{2B}{a^2}\oint_{C_1} x_2 dx_3 &= \frac{2Ba_1}{a^2 b_1}\left[\int_{b_1}^{-b_1} (b_1^2 - x_3^2)^{\frac{1}{2}}\, dx_3 - \int_{-b_1}^{b_1} (b_1^2 - x_3^2)^{\frac{1}{2}}\, dx_3\right] \\ &= -\frac{4Ba_1}{a^2 b_1}\left[\frac{x_3}{2}(b_1^2 - x_3^2)^{\frac{1}{2}} + \frac{b_1^2}{2}\arcsin\frac{x_3}{b_1}\right]_{-b_1}^{b_1} = \frac{2\pi a_1 b_1^3}{a_1^2 + b_1^2} \end{aligned} \quad \text{(m)}$$

and

$$\frac{2B}{b^2}\oint_{C_1} x_3 dx_2 = \frac{4Bb_1}{a^2 a_1}\left[\int_{-a_1}^{a_1}(a_1^2 - x_2^2)^{\frac{1}{2}} dx_3 - \int_{a_1}^{-a_1}(a_1^2 - x_2^2)^{\frac{1}{2}} dx_2\right]$$
$$= \frac{4Bb_1}{a^2 a_1}\left[\frac{x_2}{2}(a_1^2 - x_2^2)^{\frac{1}{2}} + \frac{a_1^2}{2}\arcsin\frac{x_2}{a_1}\right]_{-a_1}^{a_1} = \frac{2\pi a_1^3 b_1}{a_1^2 + b_1^2} \tag{n}$$

Substituting relations (m) and (n) into relation (j) and noting that $A_1 = \pi a_1 b_1$, we obtain

$$\oint_{C_1}\frac{\partial\psi}{\partial x_n}dx_s = 2\pi a_1 b_1 = 2A_1$$

Therefore, the assumed function $\psi(x_2, x_3)$ [see relation (d)] satisfies the requirement (h). Consequently, it is the stress function for the boundary value problem under consideration.

In what follows we compute the twist per unit length α in terms of the given torsional moment M_1. Substituting relation (f) and (g) into (6.33), we obtain

$$R_C = 2\iint_A \psi dA + A_1\psi_{C_1} = -2\left[\frac{a^2b^2}{a^2+b^2}\iint_A\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right)dx_2 dx_3 + \psi_{C_1}A_1\right]$$
$$= -2\frac{a^2b^2}{a^2+b^2}\left[\frac{I_{33}}{a^2} + \frac{I_{22}}{b^2} - A + A_1(m^2 - 1)\right] \tag{o}$$

Referring to the table in the inside of the back cover of the book we find that the moments of inertia about the x_2 and x_3 axes and the area of the cross sections of the body are given by

$$I_{22} = \frac{\pi ab^3}{4}(1 - m^4) \qquad I_{33} = \frac{\pi a^3 b}{4}(1 - m^4) \qquad A = \pi ab(1 - m^2) \qquad A_1 = \pi a_1 b_1 \tag{p}$$

Substituting relations (p) into (o), we get

$$R_C = \frac{\pi a^3 b^3}{a^2 + b^2}(1 - m^4) \tag{q}$$

Therefore,

$$\alpha = \frac{M_1}{GR_C} = \frac{M_1(a^2 + b^2)}{(1 - m^4)\pi G a^3 b^3} \tag{r}$$

Substituting relation (f) into (6.34) and using (r), we get

$$\tau_{12} = \alpha G \frac{\partial \psi}{\partial x_3} = -\frac{2x_3 M_1}{\pi a b^3 (1 - m^4)}$$
$$\tau_{13} = -\alpha G \frac{\partial \psi}{\partial x_2} = \frac{2x_2 M_1}{\pi a^3 b (1 - m^4)} \tag{s}$$

Consider a point D (x_2, x_3) located at a distance $CD = r$ from the centroid C of the cross section. Referring to Fig. b and relations (s) the resultant shearing stress at point D is

$$\tau_{1s} = \sqrt{\tau_{12}^2 + \tau_{13}^2} = \frac{2M_1}{\pi a b (1 - m^4)} \sqrt{\frac{x_3^2}{b^4} + \frac{x_2^2}{a^4}} = \frac{2M_1 r}{\pi a b (1 - m^4)} \sqrt{\frac{\sin^2\beta}{b^4} + \frac{\cos^2\beta}{a^4}} \tag{t}$$

Thus, we see that as in the case of a prismatic body of solid elliptical cross section the shearing stress along any radius CB has a constant direction which coincides with the direction of the tangent to the elliptical boundaries at points B and A (see Fig. b). Substituting relations (f) and (r) into (6.45), we get

$$\frac{\partial \hat{u}_1}{\partial x_2} = \alpha \frac{\partial \psi}{\partial x_3} + \alpha x_3 = -\frac{M_1 2x_3}{G\pi a b^3 (1 - m^4)} + \frac{x_3 M_1 (a^2 + b^2)}{(1 - m^4) G\pi a^3 b^3} = \frac{M_1 (b^2 - a^2) x_3}{G\pi a^3 b^3 (1 - m^4)}$$
$$\frac{\partial \hat{u}_1}{\partial x_3} = -\alpha \frac{\partial \psi}{\partial x_2} - \alpha x_2 = \frac{M_1 (b^2 - a^2) x_2}{(1 - m^4) G\pi a^3 b^3} \tag{u}$$

Integrating relations (u) and assuming that at $x_1 = x_2 = x_3 = 0$ the component of displacement $\hat{u}_1$ vanishes, we obtain

$$\hat{u}_1 = -\frac{M_1 (a^2 - b^2) x_1 x_2}{G\pi a^3 b^3 (1 - m^4)} \tag{v}$$

The contours of u_1 are hyperbolas having the principal axes of the ellipse as asymptotes (see Fig. c).

By setting $m = 0$ the results obtained for a prismatic body of hollow elliptical cross section reduce to those obtained in the example of Section 6.5 for a prismatic body of solid elliptical cross section. Moreover, when $a = b = R_e$ and $a_1 = b_1 = R_i$ the results of this example reduce to those of a prismatic body of hollow circular cross section of external radius R_e and internal radius R_i. Referring to relation (v), we see that for such a body $\hat{u}_1$ vanishes. *Thus, plane sections normal to the axis of a prismatic body of hollow circular cross section before deformation do remain plane after the body is subjected to equal and opposite torsional moments at its ends.* Moreover, for such a body relations (r) and (s) reduce to

$$\alpha = \frac{M_1}{GI_p} \qquad \tau_{12} = -\frac{M_1 x_3}{I_p} \qquad \tau_{13} = \frac{M_1 x_2}{I_p} \tag{w}$$

where I_p is the polar moment of inertia of the hollow circular cross section given as

$$I_p = R_C = \frac{\pi (R_e^4 - R_i^4)}{2} \tag{x}$$

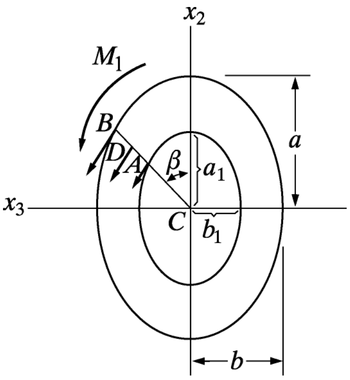


Figure b Distribution of shearing stress on the hollow elliptical cross section of a prismatic body subjected to equal and opposite torsional moments at its ends.

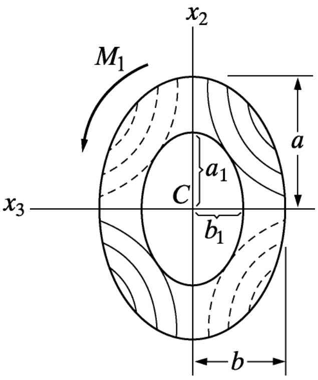


Figure c Contour lines of $\hat{u}_1$ for the hollow elliptical cross section of a prismatic body subjected to equal and opposite torsional moments at its ends.

6.7 Available Results

In Table 6.1 we tabulate formulas for the angle of twist and the maximum shearing stress of prismatic bodies having the indicated cross section subjected to equal and opposite torsional moments at their ends. These formulas have been established on the basis of the linear theory of elasticity with the assumption that the end cross sections of the bodies are not restrained from warping.

6.8 Direction and Magnitude of the Shearing Stress Acting on the Cross Sections of a Prismatic Body of Arbitrary Cross Section Subjected to Torsional Moments at Its Ends

In Fig. 6.6 we plot the stress function $\phi\,(x_2, x_3)$ on the cross section of a prismatic body; referring to this figure we see the following:

1. The integral $\iint_A \phi \; dA$ is equal to the volume of the cross-hatched regions.

2. The quantity $\phi_{Ce} A_e$ is the volume of the prism *BCDE*.

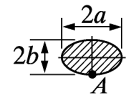
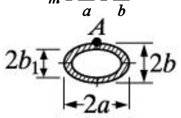
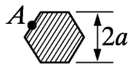
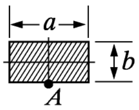
3. The quantity $\phi_{Ci} A_i$ is the volume of the prism of height ϕ_{C_i} and of cross section equal to that of the i^{th} hole.

Thus, from relation (6.23) we find that the applied torsional moment is equal to twice the volume under the surface BGHKLC.

In what follows we show that the shearing component of stress at any point of the cross section of a prismatic body subjected to torsional moments at its ends is tangent to the contour† of the stress function $\phi\,(x_2, x_3)$ passing through that point.

† A contour of the stress function is a closed curve on the cross section of a body at every point of which the value of the stress function ϕ is the same.

Table† **6.1** Angle of twist per unit length and maximum shearing stress for bodies subjected to equal and opposite torsional moments at their ends, obtained on the basis of the linear theory of elasticity.

<table>
<tr><th>Cross section</th><th colspan="2">Angle of Twist</th><th>Maximum Shearing Stress</th></tr>
<tr><td>
For circular bar : $a = b$</td><td colspan="2">$\alpha = \frac{(a^2 + b^2)M_1}{\pi a^3 b^3 G}$</td><td>$\tau_{1S}^{(A)} = \frac{2M_1}{\pi a b^2}$</td></tr>
<tr><td>Equilateral triangle
</td><td colspan="2">$\alpha = \frac{46.2 M_1}{a^4 G}$</td><td>$\tau_{1S}^{(A)} = \frac{20 M_1}{a^3}$</td></tr>
<tr><td>$m = \frac{a_1}{a} = \frac{b_1}{b}$

For circular tube : $a = b$
$a_1 = b_1$</td><td colspan="2">$\alpha = \frac{(a^2 + b^2)M_1}{4\pi a^3 b^3 G(1 - m^4)}$</td><td>$\tau_{1S}^{(A)} = \frac{2M_1}{2\pi a b^2(1 - m^4)}$</td></tr>
<tr><td>Hexagon
</td><td colspan="2">$\alpha = \frac{8.8 M_1}{a^4 G}$</td><td>$\tau_{1S}^{(A)} = \frac{5.7 M_1}{a^3}$</td></tr>
<tr><td rowspan="12">Rectangular
</td><td colspan="2">$\alpha = \frac{M_1}{GR_C} \quad R_C = \frac{K_1 h t^3}{3}$</td><td>$\tau_{1S}^{(A)} = \frac{3M_1}{K_2 h t^2}$</td></tr>
<tr><td>h/t</td><td>K_1</td><td>K_2</td></tr>
<tr><td>1.0</td><td>0.422</td><td>0.624</td></tr>
<tr><td>1.2</td><td>0.498</td><td>0.657</td></tr>
<tr><td>1.5</td><td>0.588</td><td>0.693</td></tr>
<tr><td>2.0</td><td>0.687</td><td>0.738</td></tr>
<tr><td>2.5</td><td>0.747</td><td>0.774</td></tr>
<tr><td>3.0</td><td>0.789</td><td>0.801</td></tr>
<tr><td>4.0</td><td>0.843</td><td>0.846</td></tr>
<tr><td>5.0</td><td>0.873</td><td>0.873</td></tr>
<tr><td>10.0</td><td>0.936</td><td>0.936</td></tr>
<tr><td>Infinity</td><td>1.000</td><td>1.000</td></tr>
</table>

† Taken from Timoshenko, I.S., Goodier, J.N., *Theory of Elasticity,* McGraw Hill, New York, 1951.

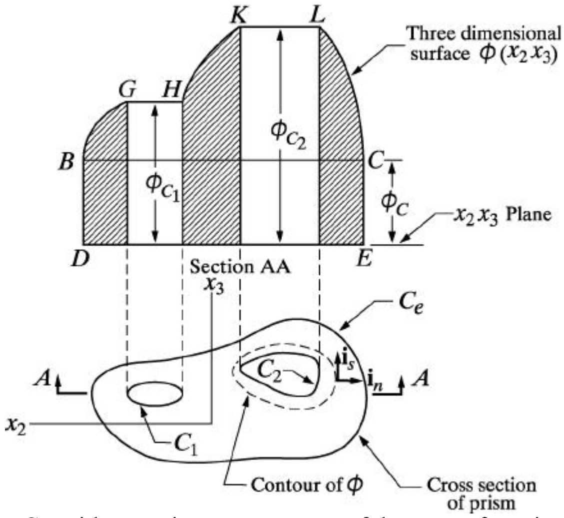


Figure 6.6 Graphical representation of the stress function $\phi(x_2, x_3)$.

Consider a point on a contour of the stress function and denote the unit vectors normal and tangential to this contour at this point by $\mathbf{i}_n$, and $\mathbf{i}_s$, respectively. We are interested to compute the shearing components of stress in the direction normal (τ_{1n}) and tangential (τ_{1s}) to the contour line. Using the transformation relations (2.48a) of the components of stress. together with relations (6.4), (6.5) and (6.16), we obtain

$$\tau_{1n} = \tau_{12}\lambda_{n2} + \tau_{13}\lambda_{n3} = \tau_{12}\frac{dx_3}{dx_s} - \tau_{13}\frac{dx_2}{dx_s} = \frac{\partial\phi}{\partial x_3}\frac{dx_3}{dx_s} + \frac{\partial\phi}{\partial x_2}\frac{dx_2}{dx_s} = \frac{d\phi}{dx_s} \tag{6.60}$$

Since ϕ must be constant on the contour line, we have

$$\frac{d\phi}{dx_s} = 0 \tag{6.61}$$

Therefore, the component of stress τ_{1n} vanishes. That is, the shear stress acting on any particle of a cross section of a prismatic body subjected to torsional moments at its ends is tangent to the contour of the stress function ϕ passing through this particle. Similarly, referring to relations (6.6), (6.7) and (6.16), we may obtain

$$\tau_{1s} = \tau_{12}\lambda_{s2} + \tau_{13}\lambda_{s3} = -\tau_{12}\frac{dx_3}{dx_n} + \tau_{13}\frac{dx_2}{dx_n} = -\frac{\partial\phi}{\partial x_3}\frac{dx_3}{dx_n} - \frac{\partial\phi}{\partial x_2}\frac{dx_2}{dx_n} = -\frac{d\phi}{dx_n} \tag{6.62}$$

where x_n is positive in the direction of the outward normal to the contour line. The shearing component of stress is taken positive if it acts along the direction of the unit vector i_s (see Fig. 6.6), that is, if it tends to turn the contour in a counterclockwise direction.

On the basis of the foregoing presentation we may conclude that the shearing component of stress acting on a particle located at any point P of a cross section of a prismatic body subjected to torsional moments at its ends is tangent to the contour of the stress function ϕ passing through point P. Moreover, it is numerically equal to $-d\phi/dx_n$. That is, if the contour lines of the ϕ surface are plotted on a cross section of a body subjected to equal and opposite torsional moments at its ends, it is possible to obtain the direction and the magnitude of the shearing component of stress acting on any particle of a cross section of this body. This finding constitutes the basis of the membrane analogy which is discussed in the next section.

6.9 The Membrane Analogy to the Torsion Problem

The solution of the torsion problem presented in Section 6.3 becomes complex for prismatic bodies whose cross sections are geometrically complicated. Exact solutions are available only for a limited number of geometries of cross sections, most of which are given in Table 6.1. However, accurate values of the components of stress and displacement may be obtained with the aid of a computer using one of the numerical methods available as for example, the finite element method. In this section, we present the "membrane analogy" to the torsion problem introduced by Prandt. This analogy is useful in visualizing the stress distribution over the cross section of prismatic bodies subjected to equal and opposite torsional moments at their ends, and in establishing experimentally the magnitude and direction of this distribution.

The membrane analogy is based on the mathematical equivalence of the boundary value problem for computing the stress function ϕ, for prismatic bodies subjected to torsional moments at their ends and the boundary value problem for computing the lateral displacement u_1 of a stretched elastic membrane, when subjected to a uniform lateral pressure. This is demonstrated below.

Consider a thin, weightless, homogeneous membrane fixed at its boundary C_e, after being subjected to an initial tension σ having the same value in all directions. That is, σ is equal to the initial tensile stress acting on any cross section of the membrane multiplied by its thickness, or to the initial tensile force per unit length acting on any cross section of the body. We consider C_e as a given curve on the x_2x_3 plane (see Fig. 6.7). When the membrane is subjected to a small uniform, lateral pressure p, it undergoes a displacement $u_1(x_2, x_3)$ in the x_1 direction. We assume that the change of the initial tension as a result of the deformation of the membrane is negligible. Referring to Fig. 6.7 consider an infinitesimal element $ABCD$ of the deformed membrane. Since, by definition, a membrane cannot resist shearing stresses, this element is subjected only to a normal force per unit length, σ, identical on all its sides. The slope of the plane tangent to the membrane which contains side AB of element $ABCD$ is equal to

$$\beta_3 = -\frac{\partial u_1}{\partial x_3} \tag{6.63}$$

The slope of the plane tangent to the membrane which contains side BC of element $ABCD$ may by approximated by

$$\beta_3 + \frac{\partial \beta_3}{\partial x_3}dx_3 = -\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial^2 u_1}{\partial x_3^2}dx_3\right) \tag{6.64}$$

Similarly, the slope of the planes tangent to the membrane which contain sides AD and BC may be approximated by

$$\beta_2 = \frac{\partial u_1}{\partial x_2} \tag{6.65a}$$

and

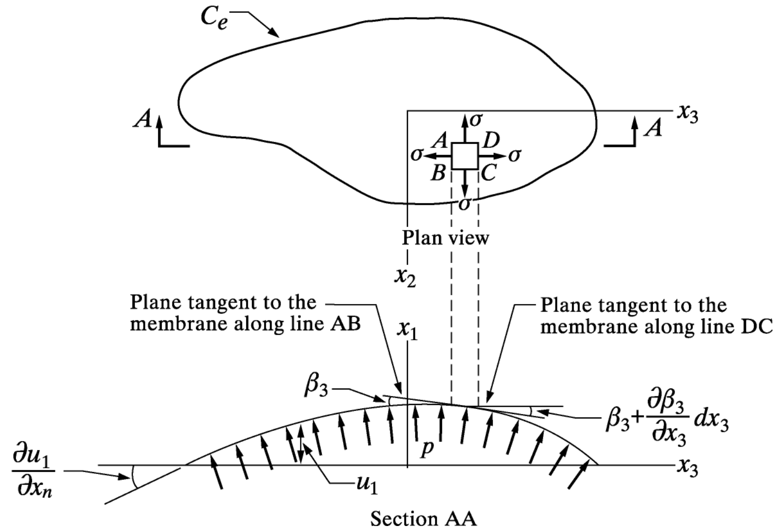


Figure 6.7 Stretched thin homogeneous membrane.

$$\beta_2 + \frac{\partial \beta_2}{\partial x_2} = -\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2}dx_2\right) \tag{6.65b}$$

Since the element under consideration is in equilibrium, setting the sum of the forces in the x_1 direction equal to zero, we obtain

$$\sigma dx_2 \frac{\partial u_1}{\partial x_3} - \sigma dx_2\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial^2 u_1}{\partial x_3^2}dx_3\right) + \sigma dx_3 \frac{\partial u_1}{\partial x_2} - \sigma dx_3\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2}dx_2\right) = p\,dx_3\,dx_2$$

Simplifying, we get

$$\frac{\partial^2 u_1}{\partial x_3^2} + \frac{\partial^2 u_1}{\partial x_2^2} = -\frac{p}{\sigma} \tag{6.66}$$

Thus, the displacement u_1 of the membrane under consideration must satisfy at every one of its points, relation (6.66) while on its boundary C_e it must vanish. Comparing relation (6.66) with (6.29) we see that the stress function ϕ, for a prismatic body having a given, simply connected cross section subjected to equal and opposite torsional moments at its ends, is identical to the displacement u_1 of a membrane stretched over a hole of the shape of the given cross section of the prismatic body and subjected to a lateral pressure p, which is equal to

$$p = 2\sigma G\alpha \tag{6.67}$$

In the case of multiply connected cross sections, the membrane is stretched between the outside boundary and weightless plates of the shape of the internal holes. These plates are kept horizontal by outside couples applied subsequent to the application of the internal pressure p. The displacement of the membrane u_1 will again satisfy relation (6.66). Moreover, since the plate is in equilibrium under the influence of the uniform pressure p and the constant tensile force per unit length σ, exerted by the membrane on its perimeter, referring to Fig. 6.8, we have

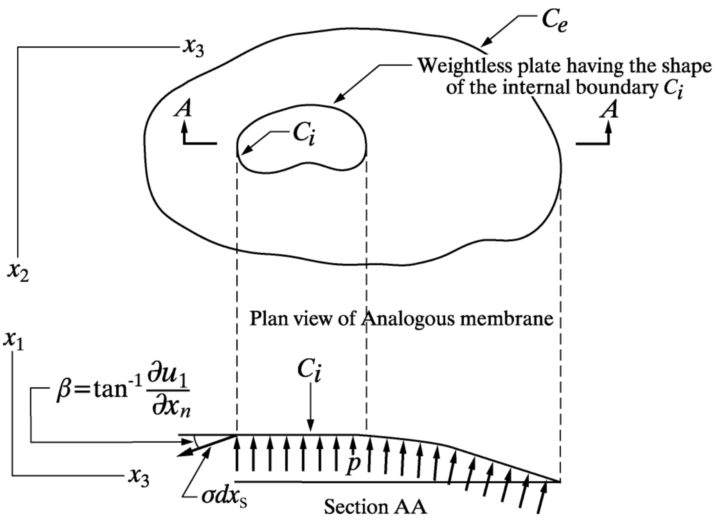


Figure 6.8 Stretched thin homogeneous membrane with weightless plate.

$$\sum F_1 = 0 \qquad \oint_{C_i} \sigma \tan \beta dx = -\oint_{C_i} \sigma \frac{du_1}{dx_n} dx_s = pA_i$$

It is apparent, that the above equation is the same as relation (6.59) if $\alpha = p/2\sigma G$. Therefore, the displacement u_1 of the membrane will be equal to the stress function ϕ if the pressure to the stress ratio of the membrane is that given by (6.67). Using relation (6.67) relation (6.32) may be written as

$$M_1 = \alpha R_T = \frac{p}{2\sigma G} R_T = \frac{p}{2\sigma} R_C \tag{6.68}$$

where R_C is the torsional constant of the prismatic body under consideration and is a property of its cross section. R_T is the torsional rigidity. Thus, if the ratio $p/2\sigma$ is known the torsional constant R_C could be established on the basis of relation (6.68). However, the ratio $p/2\sigma$ is not known.

In the laboratory the setup shown in Fig. 6.9 is employed. It consists of a rectangular container partially filled with water. The amount of water can be increased by opening the valve of the connected small water jar. The top of the container consists of a horizontal plate with two holes. The one hole is circular while the other has the geometry of the cross section of the prismatic body whose response to end torsional moments we want to establish. A soap film subjected to an unknown initial tension σ is stretched over the two holes. The pressure inside the container is subsequently increased by releasing a few drop of water from the small water jar. We denote the unknown increase of the pressure inside the container by p'. As a result of this pressure the particles of the soap film move upward forming a surface over each hole. This surface is identical to the ϕ surface for a prismatic body, whose cross section has the geometry of the hole, when subjected to equal and opposite torsional moments at its ends of unknown magnitude M_1'. The contours of the analogous membrane for the applied ratio $p'/2\sigma'$ are established experimentally using a depth gage moving freely on a horizontal plane. The volume V' under the membrane is computed from these contours. As discussed in Section 6.8 the

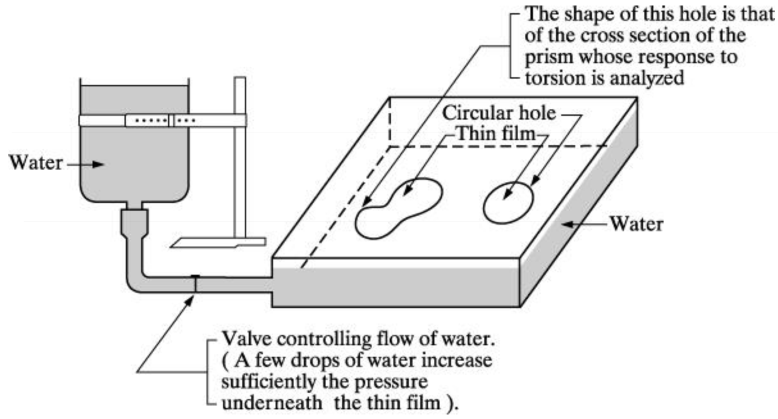


Figure 6.9 Experimental setup for membrane analogy.

torsional moment M_1' is equal to twice the volume V' under the membrane. Thus, referring to relations (6.68), we have

$$M_1' = \alpha' R_T = \frac{p'}{2\sigma'} R_C = 2(\text{volume under the membrane}) \tag{6.69}$$

This relation may be used to establish the torsional constant R_C of the body under consideration in terms of the ratio p'/σ'. That is,

$$R_C = \frac{2\ (\text{volume under the membrane})}{p'/2\sigma'} \tag{6.70}$$

However, it is difficult to measure the ratio p'/σ' in the laboratory. For this reason, the contours of the soap film extending over the circular hole are established. This soap film is subjected to the same initial stress σ' and internal pressure p' as the film extended over the hole, whose geometry is identical to that of the cross section of the body, whose response, when subjected to equal and opposite torsional moments at its ends, we are interested to establish. From the contours of the soap film, which extends over the circular hole, the volume under it is computed. In the example of Section 6.5 [see relation (u)] we have shown that the torsional constant of a prismatic body of circular cross section is equal to the polar moment of inertia of its cross sections. Taking this into account and using relation (6.70), we have

$$\frac{p'}{\sigma'} = \frac{4\ (\text{volume under the membrane over the circular hole})}{(\text{polar moment of inertia of the circular hole})} \tag{6.71}$$

Substituting relation (6.71) into (6.70) we obtain the following formula for the tortional constant R_C of the cross sections of the body under consideration.

$$R_C = \frac{(\text{volume under the membrane})\,(\text{polar moment of inertia of the circular hole})}{(\text{volume under the membrane covering the circular hole})} \tag{6.72}$$

The shearing stress at any point on a cross section of a prismatic body is tangent to the contour line of the analogous membrane at that point. In order to compute the value of the shearing stress at a point of a cross section of a prismatic body we establish the slope $\partial u_1'/\partial x_n$ in the direction normal to the contour, through that point. The measured deflection u_1' of the membrane subjected to initial tension σ' and pressure p' must satisfy a differential equation analogous to (6.66). That is,

$$\frac{\partial^2 u_1'}{\partial x_3^2} + \frac{\partial^2 u_1'}{\partial x_2^2} = -\frac{p'}{\sigma'} \tag{6.73}$$

We denote by N the following ratio:

$$N = \frac{p/\sigma}{p'/\sigma'} \tag{6.74}$$

Substituting relation (6.74) into (6.68) and using relation (6.69), we get

$$M_1 = \frac{p}{2\sigma} R_C = \frac{Np'R_C}{2\sigma'} = 2\ (\text{volume under the membrane})N \tag{6.75}$$

Thus,

$$N = \frac{M_1}{2\ (\text{volume under the membrane})} \tag{6.76}$$

Multiplying both sides of relation (6.73) by N, and using relations (6.74) and (6.67), we get

$$\frac{\partial^2 (u_1' N)}{\partial x_3^2} + \frac{\partial^2 (u_1' N)}{\partial x_2^2} = -\frac{p'N}{\sigma'} = -\frac{p}{\sigma} = -2\alpha G \tag{6.77}$$

Comparing relation (6.77) with (6.29), we see that

$$\phi = u_1' N \tag{6.78}$$

Consequently, substituting relation (6.78) into (6.62), we get

$$\tau_{1s} = -\frac{d\phi}{dx_n} = -\left(\frac{du_1'}{dx_n}\right) N \tag{6.79}$$

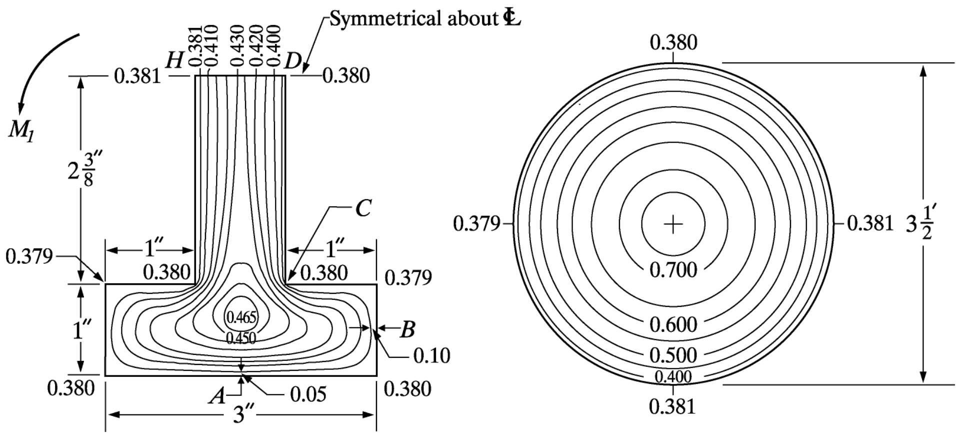
In the laboratory the contours of the membrane are established experimentally and plotted. From those the volume under the membrane is computed and used in relation (6.76) to obtain N. Moreover, from the contours of the membrane the slope $\partial u_1'/\partial x_n$ is computed. Thus, the magnitude of the shearing component of stress τ_{1s} at any point of the cross section of a prismatic body can be established using relation (6.79). Its direction is tangent to the contour passing through that point.

Example 3 A setup similar to that shown in Fig. 6.9 was employed to establish the torsional constant of a prismatic body having an I-shape cross section and made of steel ($G = 11 \times 10^6$ psi). A hole having the dimensions of the cross section of the body and a reference circular hole were made on the horizontal top plate of the rectangular container. A soap film was spread over the two holes and a few drops of water were added in the container to increase the pressure underneath the film. The contours of the soap film were established by using a depth gage moving freely on the top horizontal plane. The results are shown in Fig. a. A micrometer was employed to measure the area inside each contour of the I-section. The area inside each contour of the circular cross section was computed from the diameter of the contour. The results are given in Table a.

The volume under the membrane is equal to the sum of the volumes of the cones with bases the areas A_i and A_{i+1} enclosed by two adjacent contours [the i^{th} and the $(i+1)^{th}$] and height h equal to the difference in elevation between these two contours. Thus,

$$V = \sum_{i=1}^{n-1} (A_i + A_{i+1})\frac{h_i}{2} = (A_1 + A_2)'\frac{h_1}{2} + (A_3 + A_4)'\frac{h_2}{2} + \cdots + (A_{n-2} + A_{n-1})\frac{h_{n-2}}{2} + (A_{n-1} + A_n)\frac{h_{n-1}}{2}$$

where n is the total number of contours inside the cross section. Notice that in Fig. a the difference in elevation between most contours is constant ($h_i = h$). Using these data compute the torsional rigidity of the I-section. Compute the twist per unit length if the prismatic body is subjected to a torque of 69 in-kips (thousand lbs). Compute the shearing stress at points *A,B,C,D*.



(a) Contours of membrane on *I*-hole. (b) Contours of membrane on circular hole.

Figure a Experimentally established contours of film membranes.

Solution First the volume under the membrane of the I-section and of the circular section are computed, as the sum of the volumes of the cones having as bases the areas enclosed by two adjacent contours and height the difference in height between the two adjacent contours. Referring to Table a, we have

$$V_I = 0.01\sum_{i=1}^{n-2} \frac{1}{2}\left(A_i + A_{i+1}\right) + \frac{0.18}{2}(0.005) = (37.38)0.01 + \frac{0.18}{2}(0.005) = 0.374 \text{ in.}^3 \tag{a}$$

$$V_C = \left(\frac{9.64 + 9.11}{2}\right) 0.02 + (62.99)0.025 + (0.18)0.015 = 1.765 \text{ in.}^3 \tag{b}$$

† Notice that in this column we add only the areas, inside those of the contours which have a constant difference of elevation.

Table a Computation of the volume under the membrane.

I-Section			Circular Section		
Elev. (in)	Contour Area (in²)	Comorputations[†] for $\sum_{i=1}^{n-2} \frac{1}{2}(A_i + A_{i+1})$	Elev. (in)	Contour Area (in²)	Comorputations[†] for $\sum_{i=2}^{n-2} \frac{1}{2}(A_i + A_{i+1})$
0.380	10.75	5.38	0.380	9.64	4.56
0.390	9.13	9.13	0.400	9.11	8.46
0.400	7.60	7.60	0.425	8.46	7.83
0.410	6.15	6.15	0.450	7.83	7.15
0.420	4.46	4.46	0.475	7.15	6.65
0.430	2.76	2.76	0.500	6.65	5.94
0.440	1.31	1.31	0.525	5.94	5.16
0.450	0.50	0.50	0.550	5.16	4.50
0.460	0.18	0.09	0.575	4.50	3.98
0.465	0.00	=====	0.600	3.98	2.59
		Σ 37.38	0.625	2.59	1.77
			0.650	1.77	0.98
			0.675	0.98	0.18
			0.700	0.37	========
			0.715	0.00	Σ 62.99

Substituting result (b) into relation (6.71), we have

$$\frac{p'}{\sigma'} = \frac{4V_C}{I_P} = \frac{8\ (1.765)}{\pi(1.75)^4} = 0.479 \tag{c}$$

Substituting results (a) and (c) in relation (6.70), we get

$$R_T = R_C G = \frac{2V_I}{p'/2\sigma'} = \frac{4(0.374)G}{0.479} = 3.13\ G = 34.43\ (10^6)\ \text{psi} \tag{d}$$

The twist per unit length α may be computed from relation (6.53) as

$$\alpha = \frac{M_1}{R_T} = \frac{69{,}000}{34.43(10^6)} = 0.002\ \frac{\text{rad}}{\text{in.}} \tag{e}$$

The shearing stress at any point of the cross section of the prismatic body is tangent to the contour of the soap film membrane passing through that point. In order to find the magnitude of shearing stress, we must first establish from Fig. a the slope of the soap film membrane

$$\frac{du_1'}{dx_n} = \frac{\text{difference of height of adjacent contours}}{\text{distance between adjacent contours}} \tag{f}$$

Thus, referring to Fig. a, we have

$$\left.\frac{du_1'}{dx_n}\right|_A = \frac{-0.01}{0.05} = -0.20 \qquad \left.\frac{du_1'}{dx_n}\right|_C = \frac{-0.03}{0.06} = -0.50$$
$$\left.\frac{du_1'}{dx_n}\right|_B = \frac{-0.01}{0.10} = -0.10 \qquad \left.\frac{du_1'}{dx_n}\right|_D = \frac{-0.01}{0.06} = -0.17 \tag{g}$$

From equation (6.76) we have

$$N = \frac{M_1}{2V_I} = \frac{69.000}{0.374} = 92{,}250 \tag{h}$$

The magnitude of the shearing stress is obtained by substituting relations (g) and (h) into (6.79). That is,

$$\tau_{1s}\Big|_A = -N\left.\frac{du_1'}{dx_n}\right|_A = 92{,}250\ (0.20) = 18{,}450\ \text{psi}$$
$$\tau_{1s}\Big|_B = 9{,}200\ \text{psi} \qquad \tau_{1s}\Big|_C = 46{,}000\ \text{psi} \qquad \tau_{1s}\Big|_D = 15{,}600\ \text{psi} \tag{i}$$

6.10 Stress Distribution in Prismatic Bodies of Thin Rectangular Cross Section Subjected to Equal and Opposite Torsional Moments at Their Ends.

An important application of the membrane analogy is its use as a mental aid in visualizing the stress distribution in prismatic bodies subjected to equal and opposite torsional moments at their ends. For instance, the ϕ surface for a body having a narrow rectangular cross section has the form shown in Fig. 6.10. This analogous surface is symmetric with respect to x_3 and x_2 axes, and has a cross section that is nearly constant for a considerable portion of the width h of the cross section of the body. Actually, the ϕ function varies considerably with x_3 only in the proximity of the edges $x_3 = \mp h/2$. It is reasonable, therefore, to assume that ϕ is only a function of x_2. Clearly, the accuracy of the results obtained on the basis of this assumption increases as the h/t ratio increases. On the basis of this assumption, equation (6.29) reduces to

$$\frac{d^2\phi}{dx_2^2} = -2\alpha G \tag{6.80}$$

Integrating the above relation twice, we obtain

$$\phi = -\alpha x_2^2 G + C_1 x_2 + C_2 \tag{6.81}$$

The constants of integration are obtained by requiring that ϕ vanishes on the boundary. Thus, at $x_2 = \mp t/2$ we have

$$\phi\left(-\frac{t}{2}\right) = -G\alpha\left(-\frac{t}{2}\right)^2 - C_1\frac{t}{2} + C_2 = 0$$
$$\phi\left(+\frac{t}{2}\right) = -G\alpha\left(\frac{t}{2}\right)^2 + C_1\frac{t}{2} + C_2 = 0 \tag{6.82}$$

Consequently,

$$C_1 = 0 \qquad C_2 = \frac{G\alpha t^2}{4} \tag{6.83}$$

Substituting the values of the constants into relation (6.81), we get

$$\phi = -\alpha G\left(x_2^2 - \frac{t^2}{4}\right) \tag{6.84}$$

The angle of twist per unit length α may be obtained by equating twice the volume under the ϕ surface to the given applied torsional moment. That is,

$$M_1 = 2\iint_A \phi\, dA = -2G\alpha\int_{-\frac{t}{2}}^{\frac{t}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(x_2^2 - \frac{t^2}{4}\right) dx_3 dx_2 = G\alpha\frac{ht^3}{3} \tag{6.85}$$

Therefore,

$$\alpha = \frac{M_1}{R_T} = \frac{M_1}{GR_C} \tag{6.86}$$

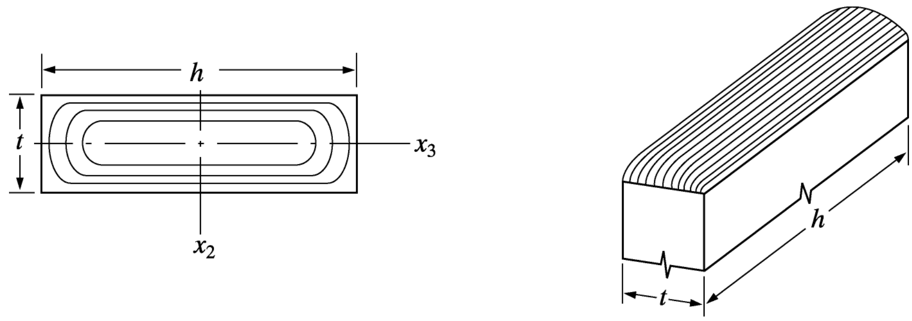
where

$$R_T = \frac{Ght^3}{3} \qquad R_C = \frac{ht^3}{3} \tag{6.87}$$

Substituting relation (6.86) with (6.87) into (6.84), we get

$$\phi = -\frac{3M_1}{ht^3}\left(x_2^2 - \frac{t^2}{4}\right) = -\frac{M_1}{R_C}\left(x_2^2 - \frac{t^2}{4}\right) \tag{6.88}$$

Substituting relation (6.88) into (6.16), we obtain



(a) Contours of the analogous membrane (b) View of the analogous membrane

Figure 6.10 Analogous membranes for a body of narrow rectangular cross section.

$$\tau_{12} = \frac{\partial \phi}{\partial x_3} = 0$$
$$\tau_{31} = -\frac{\partial \phi}{\partial x_2} = \frac{6M_1 x_2}{ht^3} \tag{6.89}$$

The distribution of the shear stress (6.89) is shown in Fig. 6.11. Substituting the second of relation (6.89) into (6.2d), we get

$$M_1 = \iint_A \tau_{13} x_2 \, dA = \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{6M_1 x_2^2}{ht^3} \, dx_2 dx_1 = \int_{-\frac{t}{2}}^{\frac{t}{2}} \frac{6M_1 x_2^2}{t^3} \, dx_2 = \frac{M_1}{2} \tag{6.90}$$

The reason for this discrepancy is that the stress component τ_{13} shown in Fig. 6.11 supplies only half of the internal moment acting on a cross section. The other half is supplied by the stress component τ_{12} which although very small occurs close to $x_3 = \pm h/2$ and therefore its moment is considerable.

The components of displacement in the x_2 and x_3 directions are given by relation (6.44) while the component of displacement u_1 may be obtained by substituting relation (6.30) into(6.88) and the resulting relation into (6.45) and using relation (6.32). Thus, taking into account that the center of twist of a rectangular cross section coincides with its centroid ($e_2 = e_3 = 0$), we have

$$\frac{\partial u_1}{\partial x_3} = \alpha x_2 \qquad \frac{\partial u_1}{\partial x_2} = \alpha x_3 \tag{6.91}$$

Integrating relations (6.91), we obtain

$$u_1 = \alpha x_2 x_3 + C_1(x_2) \qquad u_1 = \alpha x_3 x_2 + C_2(x_3) \tag{6.92}$$

These relations must be identical. Thus,

$$C_1(x_2) = C_2(x_3) = C = \text{constant} \tag{6.93}$$

If we hold the prismatic body at $x_1 = x_2 = x_3 = 0$ from moving in the x_1 direction, we have

$$u_1 = 0 \qquad \text{at} \quad x_1 = x_3 = x_2 = 0$$

Thus,

$$C = 0 \tag{6.94}$$

Substituting relations (6.94) and (6.86) into (6.92) and using (6.87), we get

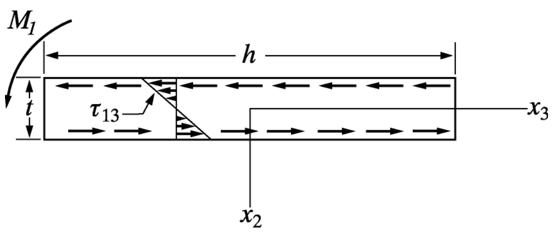


Figure 6.11 Approximate distribution of the shearing stress on the cross section of a prismatic body of thin rectangular cross section.

$$u_1 = \left(\frac{3M_1}{Ght^3} \right) x_3 x_2 \tag{6.95}$$

We define the following ratios:

$$K_1 = \frac{\alpha_A}{\alpha_E} \qquad\qquad K_2 = \frac{(\tau_{31}^A)_{\max}}{(\tau_{31}^E)_{\max}} \tag{6.96}$$

where

α_A = the angle of twist per unit length obtained on the basis of relation (6.86).

α_E = the angle of twist per unit length obtained on the basis of the theory of elasticity.

$(\tau_{13}^{A})_{\max}$ = the maximum shearing component of the stress obtained on the basis of relation (6.89).

$(\tau_{13}^{E})_{\max}$ = the maximum shearing components of stress obtained on the basis of the theory of elasticity.

The values of the correction coefficients K_1 and K_2 are tabulated in the Table 6.1 pg. 248 for various values of h/t. It can be seen, that if $h/t \geq 10$, the error of the results of the approximate analysis presented in this section is less than 8%.

6.11 Torsion of Prismatic Bodies of Composite Simply Connected Cross Sections

The results obtained for a prismatic body of thin rectangular cross section may be employed in establishing the torsional constant of prismatic bodies of composite simply connected cross sections consisting of n thin rectangles as, for example, the prismatic bodies whose cross sections are shown in Fig. 6.12. When such prismatic bodies are subjected to equal and opposite torsional moments at their ends, we note the following:

1. Each thin rectangle of a cross section rotates by the same amount. For example, the cross section of a prismatic body of I cross section consists of three thin rectangles: its two flanges and its web. As shown in Fig. 6.13 when this prismatic body is subjected to equal and opposite torsional moments at its ends, its two flanges translate and rotate while its web only rotates. The twist per unit length of the flanges is equal to that of the web.

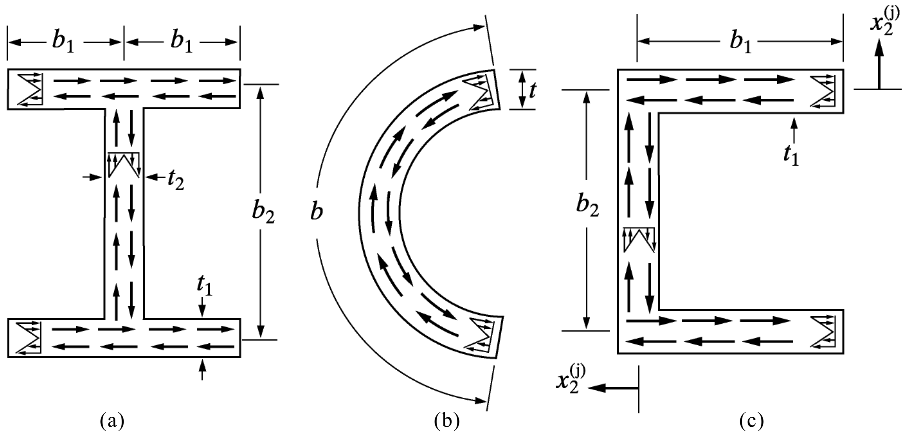


Figure 6.12 Stress distribution on bodies having thin-walled, open cross section.

2. The internal torsional moment M_1 acting on a cross section is equal to the sum of the torsional moment $M_1^{(i)}(i = 1, 2, ..., n)$ of the components of stress acting on each of the thin rectangles which constitute this cross section. That is,

$$M_1 = \sum_{i=1}^{n} M_1^{(i)} \tag{6.97}$$

Consider an auxiliary thin prismatic body having a rectangular cross section identical to that of the i^{th} thin rectangle of the prismatic body of composite cross section under consideration. Both prismatic bodies have the same length. *We assume that when the auxiliary prismatic body is subjected to equal and opposite torsional moments at its ends of magnitude* $M_1^{(i)}$ *its twist per unit length is equal to that of the prismatic body of composite cross section when subjected to equal and opposite torsional moments at its ends of magnitude* M_1. Thus referring to relation (6.85), we have

$$M_1^{(i)} = \frac{G\alpha b_i t_i^3}{3} \tag{6.98}$$

Substituting relation (6.98) into (6.97), we get

$$\alpha = \frac{M_1}{G \sum_{i=1}^{n} \frac{bt_i^3}{3}} \tag{6.99}$$

That is, referring to relation (6.53) the torsional constant R_C of a prismatic body of composite cross-section consisting of n thin rectangles is equal to

$$R_C = \sum_{i=1}^{n} \frac{b_i t_i^3}{3} \tag{6.100}$$

Substituting relation (6.99) into (6.98), we obtain

$$M_1^{(j)} = \frac{M_1 b_j t_j^3}{\sum_{i=1}^{n} b_i t_i^3} \tag{6.101}$$

Substituting relation (6.101) into (6.88), we get the following expression for the stress function of the j^{th} rectangular component of the composite cross section of the prismatic body:

$$\phi_j = -\frac{M_1^{(j)}\left((x_2^{(j)})^2 - \frac{t_j^2}{4}\right)}{\sum_{i=1}^{n} \frac{b_i t_i^3}{3}} = -\frac{3M_1^{(j)}\left((x_2^{(j)})^2 - \frac{t_j^2}{4}\right)}{\sum_{i=1}^{n} b_j t_j^3} \tag{6.102}$$

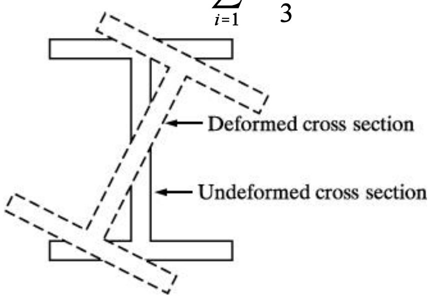


Figure 6.13 Cross section of a prismatic member subjected to torsional moments at its ends.

Where $x_2^{(j)}$ is the thickness coordinate measured from the center line of the j^{th} narrow rectangular component of the cross section; t_j is the thickness of this component of the cross section. The shearing stress ($\tau_{13}^{(j)}$) acting on the j^{th} narrow rectangular component of the cross section may be obtained by substituting relation (6.102) into (6.16). That is,

$$\tau_{13}^{(j)} = -\frac{\partial \phi_j}{\partial x_2^{(j)}} = \frac{6M_1 x_2^{(j)}}{\sum_{i=1}^{n} b_i t_i^3} = \frac{2M_1 x_2^{(j)}}{R_C} \tag{6.103}$$

6.12 Numerical Solution of Torsion Problems Using Finite Differences

In Section 6.3 we convert the boundary value problem for computing the components of stress $\tau_{12}(x_2, x_3)$ and $\tau_{13}(x_2, x_3)$ and the angle of twist per unit length of a prismatic body subjected to equal and opposite torsional moments at its ends to that of finding the stress function $\psi(x_2, x_3)$ which has the following attributes:

1. It satisfies equation (6.31) at every point of the cross sections of the body.
2. It vanishes on the curve C_e bounding the cross sections of the body.
3. It is constant and satisfies relation (6.59) on the curves C_i ($i = 1, 2, ..., N$) bounding the holes of the cross sections of the body.

Once the stress function $\psi(x_2, x_3)$ is established it can be substituted into relation (6.33) to give the torsional constant R_C, which can be substituted into relation (6.32) to give the angle of twist per unit length.

The components of stress are obtained by substituting the function $\psi(x_2, x_3)$ and the angle of twist per unit length α into relations (6.35).

In Sections 6.5 and 6.6 we obtain closed form solutions of the boundary value problem described above for prismatic bodies of solid and hollow elliptical and circular cross sections. In the literature very few more closed form or series solutions of the boundary value problem under consideration are available. They involve prismatic bodies whose cross sections have a simple geometry (i.e., equilateral triangle or square). For prismatic bodies of other geometry, it is necessary to employ a numerical procedure such as the method of finite differences or the method of finite elements. In this section we use the method of finite differences.

When the method of finite differences is applied to the boundary value problem for computing the function $\psi(x_2, x_3)$ we adhere to the following steps:

STEP 1 The cross section of the body is represented by a rectangular or other type of network called finite difference mesh (see Fig. 6.14) and the mesh points are labeled. In doing so it is important to take into account the conditions of symmetry which may exist. Cross sections with a curved boundary may require a variable mesh size close to the boundary. Fundamentally, this does not represent any difficulty but practically it causes a great deal of work. For this reason the cross section is usually approximated by a constant size mesh which as shown in Fig. 6.14 does not match its boundary. The values of the stress function at intermediate mesh points can be established by interpolation.

STEP 2 The derivatives appearing in the governing differential equation (6.31) are replaced by their finite difference approximations at the mesh points. Using a rectangular mesh of dimensions $h_2 \times h_3$ and referring to relation (D.20) of Appendix D the approximation of the governing equation (6.32) at the pivotal point mn by central differences is

$$\left(\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} + 2 \right)_{m,n} = \frac{1}{h_2^2}(\psi_{m+1,n} - 2\psi_{m,n} + \psi_{m-1,n}) + \frac{1}{h_3^2}(\psi_{m,n+1} - 2\psi_{m,n} + \psi_{m,n-1})$$
$$+ 2 + \epsilon(h^2) = 0 \tag{6.104}$$

where $\varepsilon(h^2)$ is the error term indicating that the error approaches zero as fast as h^2. For a mesh of equal spacing ($h_1=h_2=h$) relation (6.104) reduces to

$$\psi_{m+1,n} - 4\psi_{m,n} + \psi_{m-1,n} + \psi_{m,n+1} + \psi_{m,n-1} + 2h^2 + \epsilon(h^2) = 0 \tag{6.105}$$

Relation (6.105) is applied to each mesh point of the cross section. This results in a set of simultaneous algebraic equations involving the approximate values of the function $\psi(x_2, x_3)$ at the mesh points of the cross section. These equations are solved and a set of approximate values to the function $\psi(x_2, x_3)$ at the mesh points is established.

STEP 3 The values of the function $\psi(x_2, x_3)$ at the mesh points established in step 2 are used to obtain an approximation to the value of the integral in relation (6.33) using a numerical procedure such as Simpson's rule (see Fig. D.7 of Appendix D). From this an approximation to the angle of twist per unit length can be established on the basis of relation (6.32).

STEP 4 The derivatives in relations (6.35) may be approximated using finite differences.

For example, using backward differences with mesh dimensions h_2 and h_3 and referring to Fig. D.4 of Appendix D, the values of the components of stress at the mesh point m,n are approximated as follows:

$$(\tau_{12})_{m,n} = \alpha G \frac{\partial \psi}{\partial x_3} = \frac{G\alpha(3\psi_{m,n} - 4\psi_{m,n-1} + \psi_{m,n-2})}{2h_3}$$
$$(\tau_{13})_{m,n} = -\alpha G \frac{\partial \psi}{\partial x_2} = -\frac{G\alpha(3\psi_{m,n} - 4\psi_{m-1,n} + \psi_{m-2,n})}{2h_2} \tag{6.106}$$

Generally, a relatively fine mesh is required in order to obtain accurate results, especially for the components of stress. In general the accuracy of the values of the derivatives of a function is worse than the accuracy of the values of the function.

In what follows we present an example.

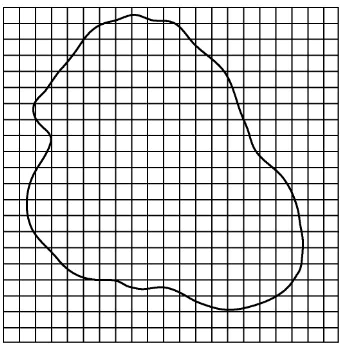


Figure 6.14 Constant size rectangular finite difference mesh which does not match the curved boundary of the cross section.

Example 4
Consider a prismatic body of square cross-section (see Fig. a) subjected to equal and opposite torsional moments M_1 at its ends. Use a mesh of equal squares of dimension $L/10$ to compute approximate maximum values for the components of stress acting at the mesh points.

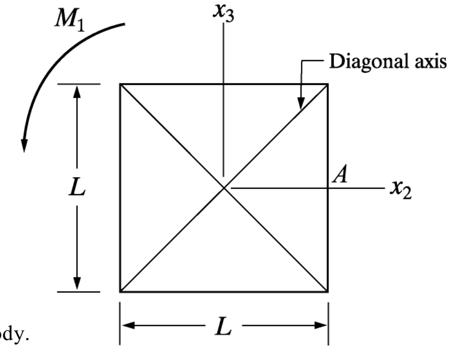


Figure a Cross section of the body.

Solution
STEP 1 The cross section of the body is represented by the square mesh and the mesh points are labeled as shown in Fig. b. In numbering these points we took into account that the stress function $\psi(x_2, x_3)$ is symmetric with respect to the x_2 and x_3 axes and with respect to the diagonal axes (see Fig. a).

STEP 2 We apply relation (6.105) to each interior mesh point taking into account that the function $\psi(x_2, x_3)$ vanishes on the boundary of the cross section. That is,

$$
\begin{aligned}
&\text{Mesh point 1} && 2\psi_2 - 4\psi_1 + \psi_6 + 0.02L^2 = 0\\
&\text{Mesh point 2} && \psi_1 - 4\psi_2 + \psi_3 + \psi_7 + 0.02L^2 = 0\\
&\text{Mesh point 3} && \psi_2 - 4\psi_3 + \psi_4 + \psi_8 + 0.02L^2 = 0\\
&\text{Mesh point 4} && \psi_3 - 4\psi_4 + \psi_5 + \psi_9 + 0.02L^2 = 0\\
&\text{Mesh point 5} && 2\psi_4 - 4\psi_5 + 0.02L^2 = 0\\
&\text{Mesh point 6} && 2\psi_7 - 4\psi_6 + \psi_1 + \psi_{10} + 0.02L^2 = 0\\
&\text{Mesh point 7} && \psi_6 - 4\psi_7 + \psi_8 + \psi_2 + \psi_{11} + 0.02L^2 = 0\\
&\text{Mesh point 8} && \psi_7 - 4\psi_8 + \psi_9 + \psi_3 + \psi_{12} + 0.02L^2 = 0\\
&\text{Mesh point 9} && 2\psi_8 - 4\psi_9 + 2\psi_4 + 0.02L^2 = 0\\
&\text{Mesh point 10} && 2\psi_{11} - 4\psi_{10} + \psi_6 + \psi_{13} + 0.02L^2 = 0\\
&\text{Mesh point 11} && \psi_{10} - 4\psi_{11} + \psi_{12} + \psi_7 + \psi_{14} + 0.02L^2 = 0\\
&\text{Mesh point 12} && 2\psi_{11} - 4\psi_{12} + 2\psi_8 + 0.02L^2 = 0\\
&\text{Mesh point 13} && 2\psi_{14} - 4\psi_{13} + \psi_{10} + \psi_{15} + 0.02L^2 = 0\\
&\text{Mesh point 14} && 2\psi_{13} - 4\psi_{14} + 2\psi_{11} + 0.02L^2 = 0\\
&\text{Mesh point 15} && 4\psi_{13} - 4\psi_{15} + 0.02L^2 = 0
\end{aligned}
\tag{a}
$$

Relations (a) may be rewritten as

$$\begin{bmatrix} -4 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -4 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & -4 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \\ \psi_{10} \\ \psi_{11} \\ \psi_{12} \\ \psi_{13} \\ \psi_{14} \\ \psi_{15} \end{Bmatrix} = -0.02L^2 \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{b}$$

From relation (b) we get

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \\ \psi_7 \\ \psi_8 \\ \psi_9 \\ \psi_{10} \\ \psi_{11} \\ \psi_{12} \\ \psi_{13} \\ \psi_{14} \\ \psi_{15} \end{Bmatrix} = \begin{Bmatrix} 0.058L^2 \\ 0.056L^2 \\ 0.051L^2 \\ 0.041L^2 \\ 0.026L^2 \\ 0.099L^2 \\ 0.096L^2 \\ 0.086L^2 \\ 0.069L^2 \\ 0.126L^2 \\ 0.122L^2 \\ 0.109L^2 \\ 0.141L^2 \\ 0.136L^2 \\ 0.146L^2 \end{Bmatrix} \tag{c}$$

x_3

	5	4	3	2	1	2	3	4	5
	4	9	8	7	6	7	8	9	4
	3	8	12	11	10	11	12	8	3
	2	7	11	14	13	14	11	7	2
	1	6	10	13	15	13	10	6	1
	2	7	11	14	13	14	11	7	2
	3	8	12	11	10	11	12	8	3
	4	9	8	7	6	7	8	9	4
	5	4	3	2	1	2	3	4	5

— x_2

Figure b Cross section of the body represented by a square mesh.

Substituting relations (c) into Simpson's rule (see Fig. D.9 of Appendix D), we obtain

$$
\begin{aligned}
\iint_A \psi dA &= \frac{L^2}{900}[4(16)\psi_5 + 8(8)\psi_4 + 8(16)\psi_3 + 8(8)\psi_2 + 4(16)\psi_1 \\
&\quad + 4(8)\psi_9 + 8(4)\psi_8 + 8(8)\psi_7 + 4(4)\psi_6 + 4(16)\psi_{12} \\
&\quad + 8(8)\psi_{11} + 4(16)\psi_{10} + 4(4)\psi_{14} + 4(8)\psi_{13} + 16\psi_{15}] \\
&= \frac{16L^2}{900}[4(\psi_1 + \psi_2 + 2\psi_3 + \psi_4 + \psi_5) + \psi_6 + 4\psi_7 + 2\psi_8 \\
&\quad + 2\psi_9 + 4\psi_{10} + 2\psi_{11} + 4\psi_{12} + 2\psi_{13} + \psi_{14} + \psi_{15}] \\
&= \frac{16(3.673)L^4}{900} = \frac{L^4}{15.314}
\end{aligned}
\tag{d}
$$

Substituting relation (d) into (6.32), we get the following approximate value for the angle of twist per unit length:

$$\alpha = \frac{M_1}{2G\iint_A \psi dA} = (7.6557)\frac{M_1}{GL^4} \tag{e}$$

Referring to Table 6.1 (see pg. 248) the exact value of the angle of twist per unit length is

$$(\alpha)_{\text{exact}} = \frac{3M_1}{0.422GL^4} = (7.109)\frac{M_1}{GL^4} \tag{f}$$

Substituting relation (c) and (e) into (6.106), we get

$$(\tau_{12})_{\text{point}A} = 0$$

$$(\tau_{13})_{\text{point}A} = (\tau_{13})_{\max} = -\frac{10G\alpha}{2L}(-4\psi_1 + \psi_6) = -\frac{10G\alpha L}{2}[-4(0.058) + 0.099] = 5.091\frac{M_1}{L^3} \tag{g}$$

Referring to Table 6.1, the exact value of the shearing stress at point A of Fig. a is

$$(\tau_{13})_{\substack{\text{point } A \\ \text{exact}}} = \frac{3M_1}{0.624L^3} = (4.8077)\frac{M_1}{L^3} \qquad \text{(h}$$

Comparing relation (e) with (f) and (g) with (h), we find that the error in the approximate value of the angle of twist per unit length is 7.69 %, while the error in the approximate value of the maximum stress is 5.88%.

6.13 Problems

1. Consider a prismatic body whose cross section is an equilateral triangle subjected to equal and opposite torsional moments at its ends. Denote by h the height of its cross section. The body is made from an isotropic linearly elastic material. Compute

(a) The twist per unit length of the body

(b) The distribution the shearing components of stress acting on the particles of the principal centroidal axes of the cross section

Hint: Referring to Fig. 6P1 the equations of the boundary of the triangle are

$$\text{For } AB \qquad x_2 - \frac{h}{3} = 0$$

$$\text{For } BD \qquad x_2 + \sqrt{3}\,x_3 + \frac{2h}{3} = 0$$

$$\text{For } AD \qquad x_2 - \sqrt{3}\,x_3 + \frac{2h}{3} = 0$$

Consequently, a function which vanishes on the boundary is

$$\psi = A\left(x_2 - \frac{h}{3}\right)\left(x_2 + \sqrt{3}\,x_3 + \frac{2h}{3}\right)\left(x_2 - \sqrt{3}\,x_3 + \frac{2h}{3}\right)$$

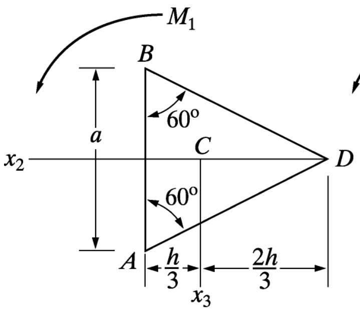


Figure 6P1

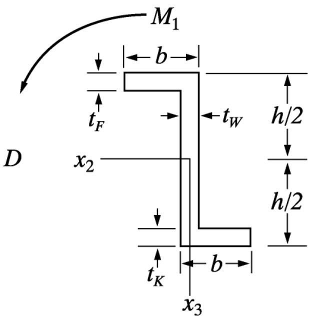


Figure 6P2

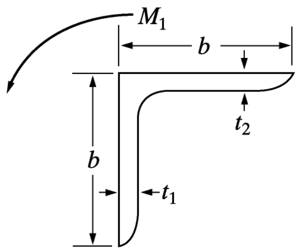


Figure 6P3

Ans. $\alpha = \dfrac{15\sqrt{3}M_1}{Gh^4}$ $\quad \tau_{12} = \dfrac{15\sqrt{3}M_1x_3}{h^5}(3x_2 - h)$ $\quad \tau_{13} = \dfrac{15\sqrt{3}M_1}{2h^5}\left[3x_2^2 + 2hx_2 - 3x_3^2\right]$

2. and **3**. A cantilever beam having the Z cross section shown in Fig. 6P2 is subjected to a torsional moment M_1. Compute and draw on a sketch the distribution of the shearing components of stress. Compute the angle of twist per unit length. Repeat with the beam having the cross section shown in Fig. 6P3.

Ans. 2 $\alpha = \dfrac{3M_1}{G[2bt_F^3 + (h - 2t_F)(t_W)^3]}$ $\quad \tau_{12}^F = \dfrac{6M_1x_2^F}{[2bt_F^3 + (h - 2t_F)(t_W)^3]}$ $\quad \tau_{13}^W = \dfrac{6M_1x_2^W}{[2bt_F^3 + (h - 2t_F)(t_W)^3]}$

Ans. 3 $\alpha = 3M_1/Gb(t_1^3 + t_2^3)$ $\quad \tau_{13}^{(i)} = 6M_1x_2^{(i)}/[b(t_1^3 + t_2^3)] \quad (i = 1, 2)$

4. A 120-mm long steel (G = 76 GPa) angle L 89 × 64 × 12.7 is subjected to equal and opposite torsional moments at its ends. Knowing that the allowable shearing stress for steel is $\tau_{allowable}$ = 50 MPa and disregarding the effect of stress concentrations determine the maximum permissible value of the applied torsional moment and the corresponding relative rotation of the two end cross sections of the angle.

Ans. $(M_1)_{max} = 377.15$ N·m $\quad \alpha = 0.052 \dfrac{\text{rad}}{\text{m}}$

5. A membrane analogy test was performed in order to establish the torsional constant of an I-beam whose cross section is shown in Fig. 6P5. Two holes were opened on a plate; the one had the geometry of the I-beam and the other was a circle of radius 100 mm. The plate was placed at the top of the rectangular container of the setup for the membrane analogy test shown in Fig. 6.9 and was sealed all around. A thin soap film was placed over the two holes. A few drops of water were added in the container and the pressure on the soap film increased. It was established that the volume of the deformed soap film over the I-shaped hole was 10^5 mm^3 while that of the deformed film over the circular hole was $5(10^5)$ mm^3. Compute the torsional constant of the I-beam and the twist per unit length when the I-beam is subjected to two equal and opposite moments at its ends of magnitude 60 kN·m. The beam is made from an isotropic, linearly elastic material (G = 100 GPa).

Ans. $R_C = 31{,}415{,}926$ mm^4 $\quad \alpha = 1.9\,(10^{-8})$ Rad/m

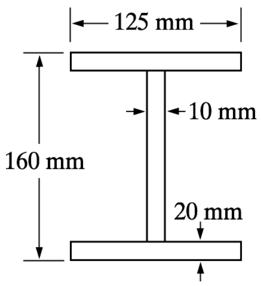


Figure 6P5

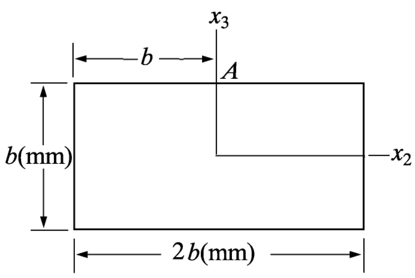


Figure 6P6

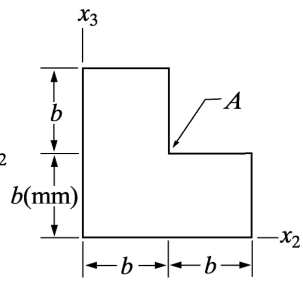


Figure 6P7

6. Consider a prismatic bar of rectangular cross section of dimension 2 b × b (mm) (see Fig. 6P6) made from an isotropic, linearly elastic material. The bar is subjected to equal and opposite torsional moments at its ends and is supported at its ends in such a way that the warping of its cross sections is not restrained. Using the method of finite differences, compute the torsional constant, the angle of twist per unit length and the components of stress acting at point A of the cross sections of this bar. Subdivide the cross section of the bar into equal rectangles of dimensions $(2b/10) \times (b/10)$.

Ans. (τ_{13}) point $A = 0$ (τ_{12}) point $A = 2.058 M_1/b^3$

7. Consider the prismatic bar whose cross section is shown (see Fig. 6P7) made from an isotropic, linearly elastic material. The bar is subjected to equal and opposite torsional moments at its ends and is supported at its ends in such a way that the warping of its cross sections is not restrained. Using the method of finite differences, compute the torsional constant, the angle of twist per unit length and the components of stress acting at point A of the cross section of this bar. Subdivide the cross section of the bar into equal squares of dimensions $b/5 \times b/5$.

Ans. $\alpha = M_1/G(0.802b^4)$ (τ_{12})point$A = -(\tau_{13})$point$A = -1.55032 M_1/b^3$

Chapter 7

Plane Strain and Plane Stress Problems in Elasticity

7.1 Plane Strain

In this section we consider prismatic (multiply or simply connected) bodies made from isotropic, linearly elastic materials and we choose the x_1 axis to coincide with the locus of the centroid of their cross sections. The bodies are originally in a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, the bodies are subjected to specified specific body forces and to specified boundary conditions and reach a second state of mechanical and thermal equilibrium at the uniform temperature T_o in which the component of displacement u_1 vanishes while their other two components of displacement are functions of only x_2 and x_3. We say that these bodies are in a *state of plane strain*. Referring to the strain–displacement relations (2.16) the components of strain of these bodies are

$$e_{11} = e_{12} = e_{13} = 0 \quad e_{22} = e_{22}(x_2, x_3) \quad e_{33} = e_{33}(x_2, x_3) \quad e_{23} = e_{23}(x_2, x_3) \tag{7.1}$$

Referring to the stress-strain relations for a state of plane strain (3.50), we see that

$$\begin{aligned} &\tau_{11} = \tau_{11}(x_2, x_3) = \nu(\tau_{22} + \tau_{33}) \quad \tau_{22} = \tau_{22}(x_2, x_3) \quad \tau_{33} = \tau_{33}(x_2, x_3) \\ &\tau_{23} = \tau_{23}(x_2, x_3) \qquad\qquad \tau_{12} = \tau_{13} = 0 \end{aligned} \tag{7.2}$$

Substituting relations (7.2) into the equilibrium equations (2.69), we have

$$\begin{aligned} B_1 &= 0 \\ \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + B_2 &= 0 \\ \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} + B_3 &= 0 \end{aligned} \tag{7.3}$$

Inasmuch as the components of stress are only functions of x_2 and x_3, referring to relations (7.3), we see that in order to have a state of plane strain in a body the distribution of the specific body force must have the following form:

$$\begin{aligned} B_1 &= 0 \\ B_2 &= B_2(x_2, x_3) \\ B_3 &= B_3(x_2, x_3) \end{aligned} \tag{7.4}$$

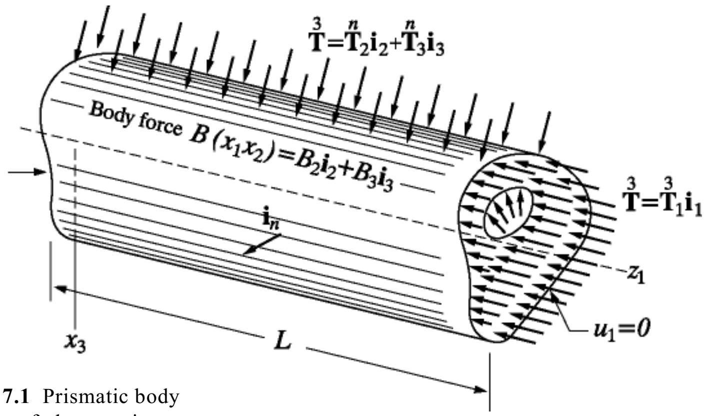


Figure 7.1 Prismatic body in a state of plane strain.

The stress distribution (7.2) when substituted into the traction–stress relations (2.73) must give components of traction which when evaluated at the points of the boundary of the body where components of traction are specified give the specified components of traction. Referring to Fig. 7.1 the unit vector outward normal to the lateral surfaces of a prismatic body is $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ $(\lambda_{n1} = 0)$. Consequently, using relations (7.2), from relations (2.73) we find that the components of traction acting on the lateral surfaces of a prismatic body, in a state of plane strain, must have the following form:

$$\overset{n}{T}_1 = 0 \tag{7.5a}$$

$$\overset{n}{T}_2(x_2, x_3) = \tau_{22}\lambda_{n2} + \tau_{23}\lambda_{n3} \tag{7.5b}$$

$$\overset{n}{T}_3(x_2, x_3) = \tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3} \tag{7.5c}$$

The unit vector normal to the end surfaces (x_1 0 and $x_1 = L$) of a prismatic body is $\mathbf{i}_n = \pm\mathbf{i}_1$ $(\lambda_{n2} = \lambda_{n3} = 0,\ \lambda_{n1} = 1)$. Consequently, using relations (2.73) we find that the components of traction acting on the end surfaces of a prismatic body in a state of plane strain, must have the following form:

$$\begin{aligned} \overset{\pm 1}{T}_1 &= \pm\tau_{11} \\ \overset{\pm 1}{T}_2 &= 0 \\ \overset{\pm 1}{T}_3 &= 0 \end{aligned} \tag{7.6}$$

We limit our discussion to bodies on the lateral surfaces of which the components of traction $\overset{n}{T}_2(x_2, x_3)$ and $\overset{n}{T}_3(x_2, x_3)$ ` $[\overset{n}{T}_1(x_2, x_3) = 0]$ are specified. That is, we do not consider boundary value problems involving bodies having one or more components of displacement specified on one or more of their lateral surfaces.

Referring to relations (7.4) to (7.6) and taking into account that the component of displacement u_1 of every particle of the body vanishes, we may conclude that a state of plane strain can be maintained in a prismatic (simply or multiply connected) body, when it is subjected to the following boundary conditions and distribution of specific body forces (see Fig. 7.1):

1. Vanishing components of traction $\overset{\pm 1}{T}_2$ and $\overset{\pm 1}{T}_3$ and component of displacement u_1 on its end surfaces ($x_1 = 0$ and $x_1 = L$). In order to satisfy this requirement the particles of the end surfaces of the body must be restrained from moving in the direction of the x_1 axis by a rigid wall which, however, allows them to move freely in the plane normal to the x_1 axis. $\overset{\pm 1}{T}_1 = \tau_{11}(x_2, x_3)$ represents the unknown distribution of the reacting traction applied on each end surface ($x_1 = 0$ and $x_1 = L$) of the prismatic body by the wall.

2. A distribution of specific body forces whose component in the direction of the x_1 axis vanishes while its other components are functions of x_2 and x_3 only.

3. A distribution of surface traction on its lateral surfaces whose component in the direction of the x_1 axis vanishes while its other components are functions of x_2 and x_3 only.

When a prismatic body is subjected to the above described boundary conditions and specific body forces, referring to relations (7.6) and taking into account that u_1 is equal to zero, we may conclude that the components of stress (7.2) satisfy the specified boundary conditions at the end surfaces of the body as well as the boundary condition (7.5a) on its lateral surfaces. Consequently, the components of stress (7.2) must be made to satisfy the following remaining requirements:

1. The equations of equilibrium at every point inside the volume of the body.

2. The boundary conditions (7.5b) and (7.5c) at every point of the lateral surfaces of the body.

3. When substituted into the stress–strain relations (3.51), they must give components of strain which when substituted into the strain–displacement relations (2.16) the resulting expressions can be integrated to give single-valued continuous components of displacement.

7.2 Formulation of the Boundary Value Problem for Computing the Stress and the Displacement Fields in a Prismatic Body in a State of Plane Strain Using the Airy Stress Function

We assume that the prismatic bodies under consideration are in a conservative specific body force field; that is, one whose components can be derived from a potential $V(x_2, x_3)$, as follows:

$$B_2 = -\frac{\partial V}{\partial x_2} \qquad B_3 = -\frac{\partial V}{\partial x_3} \tag{7.7}$$

Substituting relations (7.7) into the equations of equilibrium (7.3), we obtain

$$\begin{aligned} \frac{\partial(\tau_{22} - V)}{\partial x_2} &= \frac{\partial(-\tau_{23})}{\partial x_3} \\ \frac{\partial(\tau_{33} - V)}{\partial x_3} &= \frac{\partial(-\tau_{23})}{\partial x_2} \end{aligned} \tag{7.8}$$

The first of relations (7.8) is satisfied if the components of stress are obtained from a function $\phi_2(x_2, x_3)$ such that

$$\tau_{23} = -\frac{\partial \phi_2}{\partial x_2} \qquad \tau_{22} = \frac{\partial \phi_2}{\partial x_3} + V(x_2, x_3) \tag{7.9}$$

The second of relations (7.8) is satified if the components of stress are obtained from a function $\phi_3(x_2, x_3)$, such that

$$\tau_{23} = -\frac{\partial \phi_3}{\partial x_3} \qquad \tau_{33} = \frac{\partial \phi_3}{\partial x_2} + V(x_2, x_3) \tag{7.10}$$

From the first of relations (7.9) and (7.10) we obtain

$$\frac{\partial \phi_2}{\partial x_2} = \frac{\partial \phi_3}{\partial x_3} \tag{7.11}$$

This relation is satisfied if

$$\phi_2 = \frac{\partial \phi}{\partial x_3} \qquad \phi_3 = \frac{\partial \phi}{\partial x_2} \tag{7.12}$$

Substituting relations (7.12) into (7.9) and (7.10), we get

$$\begin{aligned} \tau_{22} &= \frac{\partial^2 \phi}{\partial x_3^2} + V(x_2, x_3) \\ \tau_{33} &= \frac{\partial^2 \phi}{\partial x_2^2} + V(x_2, x_3) \\ \tau_{23} &= -\frac{\partial^2 \phi}{\partial x_2 \partial x_3} \end{aligned} \tag{7.13}$$

Thus, the components of stress of a prismatic body in a state of plane strain obtained from a function $\phi(x_2, x_3)$ *on the basis of relations (7.13)* satisfy the equations of equilibrium. The function $\phi(x_2, x_3)$ is known as the *Airy stress function*.

In what follows we establish the restrictions which must be imposed on the Airy stress function $\phi(x_2,x_3)$ in order to ensure that when the components of stress are substituted into the stress–strain relations for plane strain (3.51) give components of strain which when substituted into the strain–displacement relations (2.16) the resulting relations can be integrated to give a set of components of displacement. In order to accomplish this it is necessary that the components of strain satisfy the equations of compatibility (2.63) at every point of the body. For simply connected bodies the satisfaction of the equations of compatibility by the components of strain is necessary and sufficient to ensure that the components of displacement are single-valued continuous functions of the space coordinates. For multiply connected bodies the satisfaction of the equations of compatibility by the components of strain is necessary and sufficient to ensure the integrability of the strain–displacement relations (2.16). However, the resulting components of displacement may or may not be single-valued continuous functions of the space coordinates. In order to ensure that the components of displacement are single-valued continuous functions of the space coordinates the components of strain must satisfy in addition to the equations of compatibility certain other relations (see Section 7.3).

Referring to the compatibility equations (2.63) we see that they are all satisfied automatically by components of strain of the form (7.1) except the third one, which is

$$\frac{\partial^2 e_{22}}{\partial x_3^2} + \frac{\partial^2 e_{33}}{\partial x_2^2} = 2\frac{\partial^2 e_{23}}{\partial x_2 \partial x_3} \tag{7.14}$$

Substituting the stress–strain relations (3.51) into the above relation and taking into account relations (7.2), we get

$$-\nu^2 \nabla_1^2(\tau_{22} + \tau_{33}) + \frac{\partial^2}{\partial x_3^2}(\tau_{22} - \nu\tau_{33}) + \frac{\partial^2}{\partial x_2^2}(\tau_{33} - \nu\tau_{22}) - 2(1 + \nu)\frac{\partial^2 \tau_{23}}{\partial x_2 \partial x_3} = 0 \tag{7.15}$$

where ∇_1^2 is the plane Laplace operator defined by

$$\nabla_1^2 = \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{7.16}$$

Substituting relation (7.13) into (7.15) we obtain†

$$\nabla_1^4 \phi = -\frac{(1 - 2\nu)}{(1 - \nu)}\nabla_1^2 V \tag{7.17}$$

where ∇_1^4 is the plane biharmonic operator defined as

$$\nabla_1^4 = \nabla_1^2(\nabla_1^2) = \frac{\partial^4}{\partial x_2^4} + 2\frac{\partial^4}{\partial x_2^2 \partial x_3^2} + \frac{\partial^4}{\partial x_3^4} \tag{7.18}$$

In case the specific body force vanishes, ($V = 0$), or V is a harmonic function ($\nabla_1^2 V = 0$) equation (7.17) reduces to the biharmonic equation

$$\nabla_1^4 \phi = 0 \tag{7.19}$$

Thus, we have reduced the problem of finding the component of stress $\tau_{22}(x_2, x_3)$, $\tau_{33}(x_2, x_3)$ and $\tau_{23}(x_2, x_3)$ in a prismatic body in a state of plane strain to that of finding the function $\phi(x_2, x_3)$ which satisfies equation (7.17) or (7.19). However, there are an infinite number of functions $\phi(x_2, x_3)$ that satisfy equation (7.17) or the biharmonic equation†† (7.19). For a unique solution, the function ϕ and its gradient $\partial \phi/\partial x_n$ must assume known values when evaluated on each curve C_k [$k = im$ or $e(m = 1, 2, ..., M)$] bounding the cross sections of a prismatic body (see Fig. 7.2). The symbol x_n represents the coordinate normal to the curve C_k [$k = im$ or $e(m = 1, 2, ..., M)$]. The values of ϕ and $\partial \phi/\partial x_n$ on each curve C_k [$k = im$ or $e(m = 1, 2, ..., M)$] bounding the cross sections of a prismatic body with M internal holes are obtained from the specified values of the components of traction $\overset{n}{T}_2(x_2, x_3)$ and $\overset{n}{T}_3(x_2, x_3)$

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† Relation (7.17) could be expanded in order to apply to prismatic bodies which reach a plane strain state of mechanical but not necessarily thermal equilibrium at a temperature $T(x_2, x_3)$. For such bodies substituting the plane strain form of the stress strain relations (3.94) with (7.1) into relation (7.14) and using relations (7.13), we get

$$\nabla_1^4 \phi = -\frac{(1 - 2\nu)}{(1 - \nu)}\nabla_1^2 V - \frac{E}{(1 - \nu)}\nabla^2 \alpha(T - T_o) \tag{7.20}$$

†† Some well-known solutions of the biharmonic equation are

$$\phi(x_2, x_3) = x_2,\ x_2^2,\ x_2^3,\ x_2 x_3,\ x_2^2 x_3,\ x_2^3 x_3,\ x_2^2 - x_3^2,\ x_2^4 - x_3^4,\ x_2^2 x_3^2,\ \cos\lambda x_3 \cos h\lambda x_2,\ x_3 \cos\lambda x_3 \cos h\lambda x_2 \tag{7.21}$$

Additional solutions may be obtained by interchanging x_2 with x_3. Any solution of the biharmonic equation is actually the Airy stress function for some prismatic body in a state of plane strain due to some appropriate distribution of surface traction on its lateral surfaces and zero body forces.

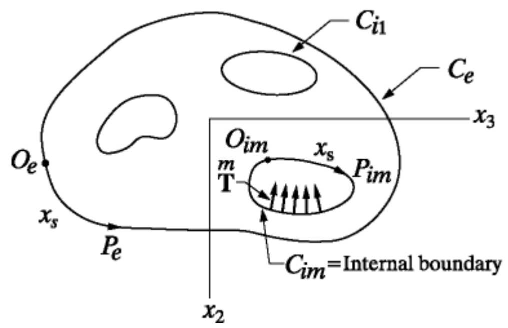


Figure 7.2 Multiply connected cross section of a prismatic body.

on the lateral surfaces of the body. Substituting relation (7.13) into relation (7.5b) and (7.5c) and using relations (6.4) and (6.5), we find that on the lateral surfaces of the body the function $\phi(x_2, x_3)$ must satisfy the following relations:

$$\overset{n}{T}{}_3^s(x_2, x_3) = -\lambda_{n2}\frac{\partial^2\phi}{\partial x_2 \partial x_3} + \lambda_{n3}\frac{\partial^2\phi}{\partial x_2^2} + \lambda_{n3}V = -\frac{dx_3}{dx_s}\left(\frac{\partial^2\phi}{\partial x_2 \partial x_3}\right) - \frac{dx_2}{dx_s}\frac{\partial^2\phi}{\partial x_2^2} + \lambda_{n3}V$$
$$= -\frac{d}{dx_s}\left(\frac{\partial\phi}{\partial x_2}\right) + \lambda_{n3}V \tag{7.22a}$$

$$\overset{n}{T}{}_2^s(x_2, x_3) = \lambda_{n2}\frac{\partial^2\phi}{\partial x_3^2} - \lambda_{n3}\frac{\partial^2\phi}{\partial x_2 \partial x_3} + \lambda_{n2}V = \frac{dx_3}{dx_s}\frac{\partial^2\phi}{\partial x_3^2} + \frac{dx_2}{dx_s}\frac{\partial^2\phi}{\partial x_2 \partial x_3} + \lambda_{n2}V$$
$$= \frac{d}{dx_s}\left(\frac{\partial\phi}{\partial x_3}\right) + \lambda_{n3}V \tag{7.22b}$$

Integrating relations (7.22) along the curves C_k [$k = im$ or $e(m = 1, 2, ..., M)$] bounding the cross sections of the prismatic body (see Fig. 7.2), from a reference point O_k to any point P_k [counterclockwise on the curve C_e, clockwise on the curves C_{im} $(m = 1, 2, ..., M)$, we obtain

$$\left.\frac{\partial\phi}{\partial x_3}\right|_{\text{at pt. } P_k} = \int_{O_k}^{P_k}(\overset{n}{T}{}_2^s - \lambda_{n2}V)\,dx_s + \left.\frac{\partial\phi}{\partial x_3}\right|_{\text{at pt. } O_k} = \beta_k + \int_{O_k}^{P_k}(\overset{n}{T}{}_2^s - \lambda_{n2}V)\,dx_s$$
$$k = im \text{ or } e \qquad m = 1, 2, ..., M \tag{7.23a}$$

and

$$\left.\frac{\partial\phi}{\partial x_2}\right|_{\text{at pt. } P_k} = -\int_{O_k}^{P_k}(\overset{n}{T}{}_3^s - \lambda_{n3}V)\,dx_s + \left.\frac{\partial\phi}{\partial x_2}\right|_{\text{at pt. } O_k} = \alpha_k - \int_{O_k}^{P_k}(\overset{n}{T}{}_3^s - \lambda_{n3}V)\,dx_s$$
$$k = im \text{ or } e \qquad m = 1, 2, ..., M \tag{7.23b}$$

where M is the total number of holes of the prismatic body; λ_{n2} and λ_{n3} are the direction cosines of the unit vector $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ which is normal to k^{th} curve C_k [$k = im$ or $e(m = 1, 2, ..., M)$] of the cross sections of the prismatic body. Multiplying relation (7.23a) by

dx_3/dx_s† and relation (7.23b) by dx_2/dx_s, adding the resulting products and taking into account relations (6.4) and (6.5), we obtain

$$\begin{aligned} \left.\frac{d\phi}{dx_s}\right|_{\text{at pt. } P_k} &= \frac{\partial\phi}{\partial x_2}\frac{dx_2}{dx_s} + \frac{\partial\phi}{\partial x_3}\frac{dx_3}{dx_s} = \alpha_k\frac{dx_2}{dx_s} + \beta_k\frac{dx_3}{dx_s} \\ &+ \frac{dx_3}{dx_s}\int_{O_k}^{P_k}(\overset{n}{T}{}_2^s - \lambda_{n2}V)dx_s - \frac{dx_2}{dx_s}\int_{O_k}^{P_k}(\overset{n}{T}{}_3^s - \lambda_{n3}V)dx_s \\ &= \alpha_k\lambda_{n2} + \beta_k\lambda_{n3} + \lambda_{n3}\int_{O_k}^{P_k}(\overset{n}{T}{}_2^s - \lambda_{n2}V)\,dx_s - \lambda_{n2}\int_{O_k}^{P_k}(\overset{n}{T}{}_3^s - \lambda_{n3}V)dx_s \end{aligned} \tag{7.24}$$

Multiplying relation (7.24) by dx_s and integrating the resulting expression, we get

$$\left.\phi(x_s)\right|_{\text{at pt. } P_k} = a_k x_2 + \beta_k x_3 + \gamma_k + \int_{O_k}^{P_k}\left[\lambda_{n2}\int_{O_k}^{P_k}(\overset{n}{T}{}_2^s - \lambda_{n2}V)\,dx_s + \lambda_{n3}\int_{O_k}^{P_k}(\overset{n}{T}{}_3^s - \lambda_{n3}V)\,dx_s\right]dx_s \tag{7.25}$$

where γ_k is a constant of integration; $\overset{n}{T}{}_2^s(x_2, x_3)$ and $\overset{n}{T}{}_3^s(x_2, x_3)$ are the given components of traction on the lateral surfaces of the body; V is the known potential from which the specific body force is obtained on the basis of relations (7.7). Relations (7.24) and (7.25) give the functions $d\phi/dx_n$ and ϕ, respectively, on each of the curves C_k [$k = im$ or $e(m = 1, 2, ..., M)$] bounding the cross sections of the body. The solution of equation (7.17) or of the plane biharmonic equation (7.19) must satisfy relations (7.24) and (7.25) on each of the curves C_k [$k = im$ or $e(m = 1, 2, ..., M)$] bounding the cross sections of the body. Thus, in the general case of a body with M holes, we will have $3(M + 1)$ unknown quantities. Namely, the constants α_k, β_k and γ_k [$k = im$ or $e(m = 1, 2, ..., M)$] for each of the M curves C_{im} and for the curve C_e. However, the constants α_k, β_k and γ_k may be set equal to zero on the curve C_e inasmuch as this choice does not affect the components of stress [see relations (7.13)].

On the basis of the foregoing presentation the solution of the boundary value problem for establishing the components of stress $\tau_{22}(x_2, x_3)$, $\tau_{33}(x_2, x_3)$ and $\tau_{23}(x_2, x_3)$ of a simply connected prismatic body in a state of plane strain subjected to the external loads described in Section 7.1 has been reduced to that of finding the function $\phi(x_2, x_3)$ which satisfies relation (7.17) at every point of the body as well as relations (7.24) and (7.25) on the curves C_k [$k = im$ or $e(m = 1, 2, ..., M)$] bounding the cross sections of the body. For multiply connected bodies the components of strain (7.1) must satisfy additional relations which we present in the next section. These relations could impose additional restrictions on the function $\phi(x_2, x_3)$.

Notice that for simply connected prismatic bodies, subjected only to specified tractions on their lateral surface [no body forces ($V = 0$)] the Airy stress function $\phi(x_2, x_3)$ is independent of the material constants E and ν [see relations (7.19), (7.24) and (7.25)]. Consequently, the components of stress τ_{22}, τ_{33}, τ_{23} of such bodies are independent of the properties of the material from which they are made; that is, the same distribution of the components of stress τ_{22}, τ_{33}, τ_{23} exists in two simply connected bodies of the same

† x_s is the coordinate measured counterclockwise along the external boundary and clockwise along any internal boundary of the cross section (see Fig. 7.2).

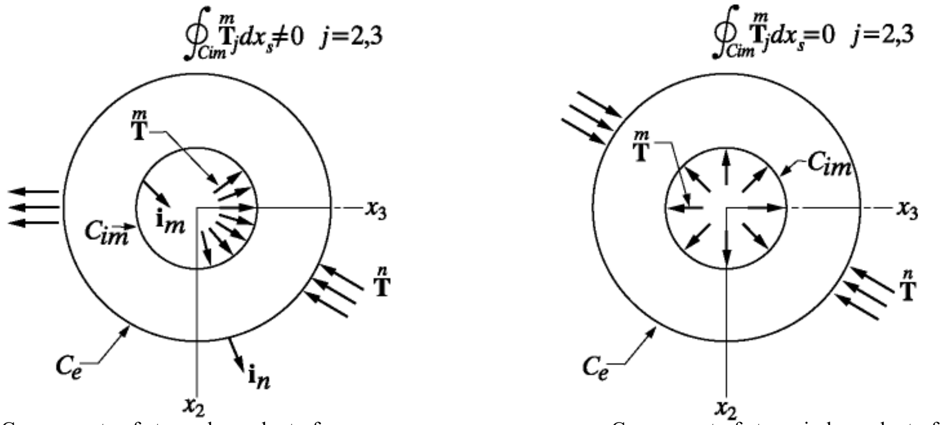
geometry but made from different materials when subjected to the same surface tractions on their lateral surface. The magnitude of the component of stress τ_{11} in the two prismatic bodies, however, is different, because as can be seen from the first of relations (7.2), the magnitude of τ_{11} depends on the value of Poisson's ratio of the material from which the prismatic body is made.

7.3 Prismatic Bodies of Multiply Connected Cross Sections in a State of Plane Strain

In Section 2.11 we conclude that a necessary and sufficient condition for ensuring that an assumed strain field for a body gives single-valued, continuous components of displacement is that the components of displacement satisfy the following relation for every closed curve of the cross sections of the body:

$$\begin{aligned}\oint d\hat{\mathbf{u}} &= \oint (d\hat{u}_1\mathbf{i}_1 + d\hat{u}_2\mathbf{i}_2 + d\hat{u}_3\mathbf{i}_3] \\ &= \oint \left[\left(\frac{\partial \hat{u}_1}{\partial x_1}dx_1 + \frac{\partial \hat{u}_1}{\partial x_2}dx_2 + \frac{\partial \hat{u}_1}{\partial x_3}dx_3 \right) \mathbf{i}_1 + \left(\frac{\partial \hat{u}_2}{\partial x_1}dx_1 + \frac{\partial \hat{u}_2}{\partial x_2}dx_2 + \frac{\partial \hat{u}_2}{\partial x_3}dx_3 \right) \mathbf{i}_2 \right. \\ &\qquad \left. + \left(\frac{\partial \hat{u}_3}{\partial x_1}dx_1 + \frac{\partial \hat{u}_3}{\partial x_2}dx_2 + \frac{\partial \hat{u}_3}{\partial x_3}dx_3 \right) \mathbf{i}_3 \right] = 0\end{aligned} \tag{7.26}$$

Using the strain–displacement relations (2.16) the derivatives of the components of displacement in relation (7.26) may be replaced by the components of strain. It can be shown that a necessary and sufficient condition for ensuring that relation (7.26) is satisfied on every closed curve of a simply connected body is that the components of strain satisfy the equations of compatibility (2.63) at every point of the body. Moreover, it can be shown that in order to ensure that relation (7.26) is satisfied for every closed curve of a multiply connected body the components of strain must satisfy certain other relations in addition to the equations of compatibility (2.63). Taking into account the stress–strain relations (3.50) and relations (7.13), these additional requirements can be written as



Components of stress dependent of v — Component of stress independent of v

Figure 7.3 Cross sections of multiply connected prismatic bodies in a state of plane strain (the body force is zero).

$$\oint_{C_{im}} \frac{d}{dx_n}\left(\nabla_1^2 \phi\right) dx_s = -\left(\frac{1-2\nu}{1-\nu}\right) \oint_{C_{im}} \frac{\partial V}{\partial x_n} dx_s$$

$$\oint_{C_{im}} \left[x_2 \frac{d}{dx_s}\left(\nabla_1^2 \phi\right) - x_3 \frac{d}{dx_n}\left(\nabla_1^2 \phi\right)\right] dx_s$$

$$= \frac{1}{1-\nu}\oint_{C_{im}} (\overset{n}{T}{}_3^s - \lambda_{n3} V) dx_s - \frac{1-2\nu}{1-\nu}\oint_{C_{im}} x_2 \frac{dV}{dx_s} dx_s + \frac{1-2\nu}{1-\nu}\oint_{C_{im}} x_3 \frac{dV}{dx_n} dx_s \tag{7.27}$$

$$\oint_{C_{im}} \left[x_3 \frac{d}{dx_s}\left(\nabla_1^2 \phi\right) + x_2 \frac{d}{dx_n}\left(\nabla_1^2 \phi\right)\right] dx_s$$

$$= -\frac{1}{1-\nu}\oint_{C_{im}} (\overset{n}{T}{}_2^s - \lambda_{n2} V) dx_s - \frac{1-2\nu}{1-\nu}\oint_{C_{im}} x_3 \frac{dV}{dx_s} dx_s - \frac{1-2\nu}{1-\nu}\oint_{C_{im}} x_2 \frac{dV}{dx_n} dx_s$$

$$m = 1, 2, ..., M$$

where C_{im} is the closed curve bounding the m^{th} hole of the multiply connected cross sections of the body, while M is the total number of holes of the cross sections of the body. In deriving relations (7.27) it has been taken into account that if the line integral of a vector field [i.e., $\hat{\mathbf{u}}(x_1, x_2, x_3)$] vanishes along a closed curve surrounding a hole of a multiply connected cross section of a body, it will vanish along any other closed curve surrounding this hole. Thus, a necessary and sufficient condition for ensuring that the components of displacement obtained from an assumed set of components of stress of a multiply connected prismatic body [with M holes] in a state of plane strain are single-valued and continuous functions of the space coordinates, is that relations (7.27) are satisfied on M closed curves each of which bounds one of the M holes of the cross sections of the body. Relations (7.27) are known as the *Mitchell conditions*†. Their derivation is rather cumbersome and for this reason it is not presented in this text. In relations (7.27) x_s and x_n are curvilinear coordinates which are measured along the curves C_{im} (m = 1, 2, ..., M) and normal to the curves C_{im}, respectively.

On the basis of the foregoing presentation the solution of the boundary value problem for establishing the distribution of the components of stress in a multiply connected prismatic body with M holes in a state of plane strain has been reduced to that of finding a function $\phi(x_2, x_3)$ which has the following attributes:

1. It satisfies relation (7.17) or (7.19) at every point of the cross sections of the body.
2. It satisfies conditions (7.24) and (7.25) at the points of the external and internal boundaries of the cross sections of the body.
3. It satisfies the Mitchell conditions (7.27) on the close curves C_{im} (m = 1, 2, ..., M) each of which bounds one of the M holes.

Notice, that the right side of relations (7.27) vanishes if the body force is zero and if the resultant of the traction acting on the surface of each hole is zero. Thus, in this case $\phi(x_2, x_3)$ and consequently, the components of stress τ_{22}, τ_{33} and τ_{23} are independent of the elastic constants E and ν (See Fig. 7.3). However, as can be seen from the first of relation (7.2) the component of stress τ_{11} depends on the value of Poisson's ratio ν.

† Mitchell, J.H., On the direct determination of stress in an elastic solid with application to the theory of plates, *Proceedings London Mathematical Society,* 31, 1899, p. 100.

7.4 The Plane Strain Equations in Cylindrical Coordinates

There are problems involving prismatic bodies which may be solved more conveniently by using cylindrical coordinates x_1, r, θ. Accordingly in this section we set in polar coordinates the equations established in cartesian coordinates in the previous sections.

In a plane strain state, the non-vanishing cylindrical components of stress are τ_{11}, τ_{rr}, $\tau_{\theta\theta}$, and $\tau_{r\theta}$. The relations between these components of stress and the Airy stress function $\phi(r, \theta)$ may be established by first converting the partial derivatives of $\phi(x_2, x_3)$ with respect to x_2 and x_3 in equation (7.13) to partial derivatives with respect to the cylindrical coordinates r and θ. Thus, using relation (2.81), we obtain

$$\begin{aligned}
\tau_{22} = \frac{\partial^2 \phi}{\partial x_3^2} + V &= \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \right)\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \right)\phi \\
&= \frac{\partial^2 \phi}{\partial r^2}\sin^2\theta + \left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \right)\cos^2\theta \\
&\quad - \left(\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \right) 2\sin\theta\cos\theta + V(r, \theta)
\end{aligned} \tag{7.28}$$

$$\begin{aligned}
\tau_{33} = \frac{\partial^2 \phi}{\partial x_2^2} + V &= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \right)\phi \\
&= \frac{\partial^2 \phi}{\partial r^2}\cos^2\theta + \left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \right)\sin^2\theta \\
&\quad + \left(\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \right) 2\sin\theta\cos\theta + V(r, \theta)
\end{aligned} \tag{7.29}$$

$$\begin{aligned}
\tau_{23} = -\frac{\partial^2 \phi}{\partial x_2 \partial x_3} &= -\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \right)\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \right)\phi \\
&= -\cos\theta\sin\theta\frac{\partial^2 \phi}{\partial r^2} - \frac{\cos^2\theta - \sin^2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \\
&\quad + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cos^2\theta - \sin^2\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial \phi}{\partial r}
\end{aligned} \tag{7.30}$$

The transformation relations between the cylindrical components of stress and the cartesian components of stress may be obtained by substituting in relation (2.48a) the direction cosines (2.77). That is,

$$\begin{aligned}
\tau_{rr} &= \tau_{22}\cos^2\theta + 2\tau_{23}\cos\theta\sin\theta \\
\tau_{\theta\theta} &= \tau_{22}\sin^2\theta + \tau_{33}\cos^2\theta - 2\tau_{12}\cos\theta\sin\theta \\
\tau_{r\theta} &= -(\tau_{22} - \tau_{33})\cos\theta\sin\theta + \tau_{23}(\cos^2\theta - \sin^2\theta)
\end{aligned} \tag{7.31}$$

$$\tau_{1r} = \tau_{12}\cos\theta + \tau_{13}\sin\theta = 0$$

$$\tau_{1\theta} = -\tau_{12}\sin\theta + \tau_{13}\cos\theta = 0 \tag{7.31}$$

Substituting relations (7.28), (7.29) and (7.30) into relations (7.31) and referring to relations (7.2), we obtain the following expressions for the cylindrical components of stress:

$$\begin{aligned}
\tau_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + V(r,\theta) \\
\tau_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} + V(r,\theta) \\
\tau_{r\theta} &= \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \\
\tau_{1r} &= \tau_{1\theta} = 0
\end{aligned} \tag{7.32}$$

Moreover, referring to first of relations (7.2) and to the first invariant of the stress tensor (1.78), we have

$$\tau_{11} = \nu(\tau_{22} + \tau_{33}) = \nu(\tau_{rr} + \tau_{\theta\theta}) \tag{7.33}$$

The harmonic operator ∇_1^2 may be expressed in terms of cylindrical coordinates, using relations (7.13), (7.16) and (7.32). Thus,

$$\nabla_1^2\phi = \left(\frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}\right) = \tau_{22} + \tau_{33} = \tau_{rr} + \tau_{\theta\theta} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi \tag{7.34}$$

Consequently, referring to relations (7.18) the biharmonic operator may be written in cylindrical coordinates as

$$\nabla_1^4\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi \tag{7.35}$$

Using relations (7.34) and (7.35), relation (7.17) becomes

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi \\
&\qquad = -\frac{1-2\nu}{1-\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)V(r,\theta)
\end{aligned} \tag{7.36a}$$

when the body forces vanish relation (7.36a) reduces to

$$\nabla_1^4\phi = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0 \tag{7.36b}$$

A solution of the biharmonic equation (7.36b) was derived by J.H. Mitchell[†(see next page)]. The following is a modified form[††(see next page)] of this solution:

$$
\begin{aligned}
\phi(r,\ \theta) &= A_o \ln r + B_o r^2 + C_o r^2 \ln r + D_o \theta + E_o r^2 \theta \\
&+ (A_1 r\theta + B_1 r^3 + C_1 r^{-1} + D_1 r \ln r) \cos\theta \\
&- (\tilde{A}_1 r\theta + \tilde{B}_1 r^3 + \tilde{C}_1 r^{-1} + \tilde{D}_1 r \ln r) \sin\theta \\
&+ \sum_{n=2}^{\infty} (A_n r^n + B_n r^{n+2} + C_n r^{-n} + D_n r^{-n+2}) \cos n\theta \\
&+ \sum_{n=2}^{\infty} (\tilde{A}_n r^n + \tilde{B}_n r^{n+2} + \tilde{C}_n r^{-n} + \tilde{D}_n r^{-n+2}) \sin n\theta
\end{aligned} \tag{7.37}
$$

The components of stress may be obtained by substituting the stress function (7.37) into relations (7.32). That is,

$$
\begin{aligned}
\tau_{rr} &= \frac{A_o}{r^2} + 2B_o + 2E_o\theta + \left(2B_1 r - \frac{2C_1}{r^3} + \frac{D_1}{r} + \frac{2\tilde{A}_1}{r} \right) \cos\theta + \left(2\tilde{B}_1 r - \frac{2\tilde{C}_1}{r^3} + \frac{\tilde{D}_1}{r} - \frac{2A_1}{r} \right) \sin\theta \\
&- \sum_{n=2}^{\infty} [(n^2 - n - 2)A_n r^n + (n^2 - n)B_n r^{n-2} + (n^2 + n)C_n r^{-n-2} + (n^2 + n - 2)D_n r^{-n}] \cos n\theta \\
&- \sum_{n=2}^{\infty} [(n^2 - 2 - n)\tilde{A}_n r^n + (n^2 - n)\tilde{B}_n r^{n-2} + (n^2 + n)\tilde{C}_n r^{-n-2} + (n^2 + n - 2)\tilde{D}_n r^{-n}] \sin n\theta
\end{aligned} \tag{7.38a}
$$

$$
\begin{aligned}
\tau_{r\theta} &= \frac{D_o}{r^2} - E_o + \left(2B_1 r - \frac{2C_1}{r^3} + \frac{D_1}{r} \right) \sin\theta - \left(2\tilde{B}_1 r - \frac{2\tilde{C}_1}{r^3} + \frac{\tilde{D}_1}{r} \right) \cos\theta \\
&+ \sum_{n=2}^{\infty} n[(n + 1)A_n r^n + (n - 1) B_n r^{n-2} - (n + 1)C_n r^{-n-2} - (n - 1)D_n r^{-n}] \sin n\theta \\
&- \sum_{n=2}^{\infty} n[(n + 1)\tilde{A}_n r^n + (-n - 1)\tilde{B}_n r^{n-2} - (-n + 1)\tilde{C}_n r^{-n-2} - (-n - 1)\tilde{D}_n r^{-n}] \cos n\theta
\end{aligned} \tag{7.38b}
$$

$$
\begin{aligned}
\tau_{\theta\theta} &= -\frac{A_o}{2r} + 2B_o + 2C_o \ln r + 4C_o + 2E_o\theta + \left(6B_1 r + \frac{6C_1}{r^3} + \frac{D_1}{r} \right) \cos\theta - \left(6\tilde{B}_1 r + \frac{6\tilde{C}_1}{r^3} + \frac{\tilde{D}_1}{r} \right) \sin\theta \\
&+ \sum_{n=2}^{\infty} \left[[(n+1)(n+2)A_n r^n + n(n-1)B_n r^{n-2} + n(n+1)C_n r^{-n-2} + (n-2)(n-1)D_n r^{-n}] \right] \cos n\theta \\
&+ \sum_{n=2}^{\infty} \left[[(n+1)(n+2)\tilde{A}_n r^n + n(n-1)\tilde{B}_n r^{n-2} + n(n+1)\tilde{C}_n r^{-n-2} + (n-2)(n-1)\tilde{D}_n r^{-n}] \right] \sin n\theta
\end{aligned} \tag{7.38c}
$$

For problems involving prismatic bodies having simply connected cross sections with boundaries adaptable to cylindrical coordinates, we retain in the expression for $\phi(r,\ \theta)$ only

---

† Mitchell, J.H., On the direct determination of stress in an elastic solid with application to the theory of plates, *Proceedings London Mathematical Society,* 31, 1899, p. 100.

†† Timoshenko, S.P. and Goodier J.N., *Theory of Elasticity,* 3rd edition, McGraw-Hill, New York, 1970, Chapter 4.

the terms of the general expression (7.37) which are required in order to satisfy the given boundary conditions. For problems involving prismatic bodies having multiply connected cross section with boundaries adaptable to cylindrical coordinates, we must retain in the general expression (7.37) for $\phi(r, \theta)$ more terms than those retained for simply connected bodies in order to satisfy the Mitchell conditions (7.27).

In what follows we present an example involving a simply connected prismatic body.

Example 1 Establish the stress field in a semi-infinite ($0 < r < \infty$) wedge subjected on its apex, to a line force of constant magnitude p given in units of force per unit length in the x_1 direction as shown in Fig. a. The particles of the end surfaces of the wedge at $x_1 = 0$ and $x_1 = L$ are restrained from moving in the x_1 direction by rigid walls which, however, do not inhibit their movement in planes parallel to the plane x_2x_3. Specialize the results to a semi-infinite body subjected to a line load of constant magnitude p (kN/m) or to a distributed load $w(x_3)$ (kN/m^2) acting on a portion of its surface.

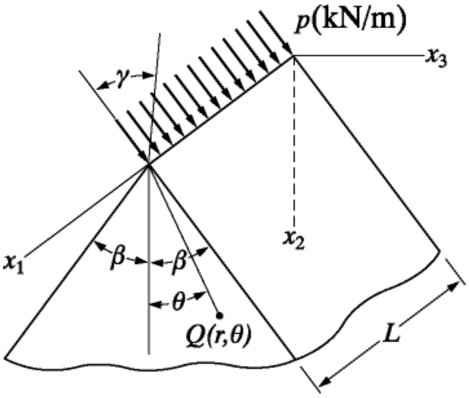


Figure a Semi-infinite wedge subjected on its apex to a line load.

Solution The wedge is in a state of plane strain ($u_1 = 0$). Theoretically at the apex of the wedge the components of stress become infinite. In reality however, the material of a small wedge of radius R near the apex yields and the force p is distributed over a small cylindrical surface as shown in Fig. b. If this small wedge of radius R is isolated, a solution on the basis of the theory of elasticity may be established for the remaining part of the wedge. Thus, referring to relations (2.73), the boundary conditions for the wedge are

$$\text{at } \theta = \pm\beta \quad \begin{cases} \overset{\pm\theta}{T}_\theta = \tau_{\theta\theta} = 0 & \text{(a)} \\ \overset{\pm\theta}{T}_r = \tau_{r\theta} = 0 & \text{(b)} \end{cases}$$

$$\text{at } r = R \quad \begin{cases} p\cos\gamma = -\int_{-\beta}^{\beta}\tau_{rr}(\cos\theta)Rd\theta + \int_{-\beta}^{\beta}\tau_{r\theta}(\sin\theta)Rd\theta & \text{(c)} \\ p\sin\gamma = -\int_{-\beta}^{\beta}\tau_{rr}(\sin\theta)Rd\theta - \int_{-\beta}^{\beta}\tau_{r\theta}(\cos\theta)Rd\theta & \text{(d)} \end{cases}$$

$$\text{at } r = \infty \qquad \tau_{rr} = \tau_{\theta\theta} = \tau_{r\theta} = 0 \qquad \text{(e)}$$

One way to satisfy the boundary conditions (a), (b) and (e) is to set the constants in the

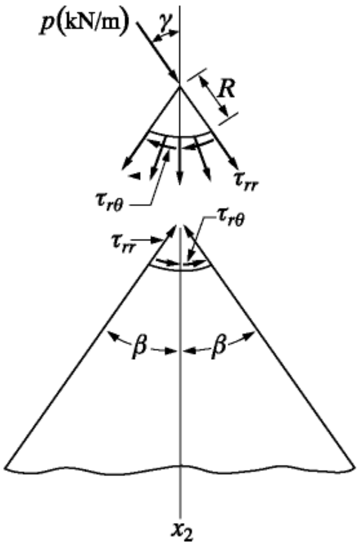


Figure b Assumed distribution of stress near the apex of a semi-infinite wedge.

expression for $\tau_{\theta\theta}$ and $\tau_{r\theta}$ in relations (7.38) equal to zero. In this case relations (7.37) and (7.38) reduce to

$$\phi(r, \theta) = A_1 r\theta \cos\theta + \tilde{A}_1 r\theta \sin\theta \tag{f}$$

and

$$\tau_{rr} = \frac{2\tilde{A}_1}{r}\cos\theta - \frac{2A_1}{r}\sin\theta \qquad \tau_{r\theta} = \tau_{\theta\theta} = \tau_{1\theta} = \tau_{1r} = 0 \tag{g}$$

Substituting relations (g) into (c) and (d), we have

$$p\cos\gamma = -\int_{-\beta}^{\beta}\tau_{rr}(\cos\theta)Rd\theta = -\tilde{A}_1(2\beta + \sin 2\beta) \tag{h}$$

$$p\sin\gamma = -\int_{-\beta}^{\beta}\tau_{rr}(\sin\theta)Rd\theta = A_1(2\beta - \sin 2\beta) \tag{i}$$

Thus,

$$\tilde{A}_1 = -\frac{p\cos\gamma}{2\beta + \sin 2\beta} \qquad A_1 = \frac{p\sin\gamma}{2\beta + \sin 2\beta} \tag{j}$$

Substituting the values of the constants (j) into relations (f) and (g) and using relation (7.33), we get

$$\phi(r, \theta) = pr\theta\left[\frac{\cos\theta\sin\gamma}{2\beta - \sin 2\beta} - \frac{\sin\theta\cos\gamma}{2\beta + \sin 2\beta}\right] \tag{k}$$

and

$$\begin{aligned}\tau_{rr} &= -\frac{2p}{r}\left[\frac{\cos\gamma\cos\theta}{2\beta + \sin 2\beta} + \frac{\sin\gamma\sin\theta}{2\beta - \sin 2\beta}\right]\\ \tau_{r\theta} &= \tau_{\theta\theta} = \tau_{1\theta} = \tau_{1r} = 0 \qquad \tau_{11} = \nu\tau_{rr}\end{aligned} \tag{l}$$

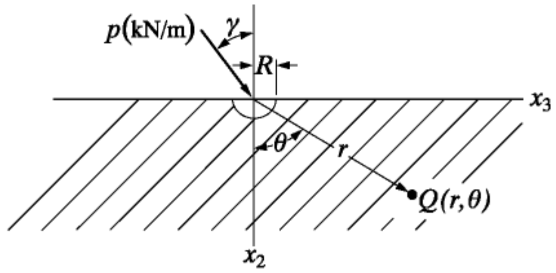


Figure c Semi-infinite body subjected to a line load.

Semi-infine body subjected to a line load

In case $\beta = \pi/2$ the infinitely long wedge becomes a semi-infinite body (see Fig. c). For this case the solutions (k) and (l) reduce to

$$\phi(r, \theta) = \frac{pr\theta}{\pi} \sin(\gamma - \theta) \tag{m}$$

and

$$\tau_{rr} = -\frac{2p}{\pi r}\cos(\gamma - \theta) \qquad \tau_{r\theta} = \tau_{\theta\theta} = \tau_{1\theta} = \tau_{1r} = 0 \qquad \tau_{11} = \nu\tau_{rr} \tag{n}$$

As expected from physical intuition the maximum stress occurs, along the line of action of the applied force $(\theta - \gamma)$.

For a vertical line load $(\gamma = 0)$ on a semi-infinite body relations (m) and (n) reduce to

$$\phi(r, \theta) = -\frac{pr\theta}{\pi}\sin\theta = -\frac{p}{\pi}x_3 \arctan\left(\frac{x_3}{x_2}\right) \tag{o}$$

and

$$\tau_{rr} = -\frac{2p}{\pi r}\cos\theta = -\frac{2px_2}{\pi(x_2^2 + x_3^2)} \qquad \tau_{r\theta} = \tau_{\theta\theta} = \tau_{1\theta} = \tau_{1r} = 0 \qquad \tau_{11} = \nu\tau_{rr} \tag{p}$$

Semi-infinite body subjected to distributed forces $w(x_3)$

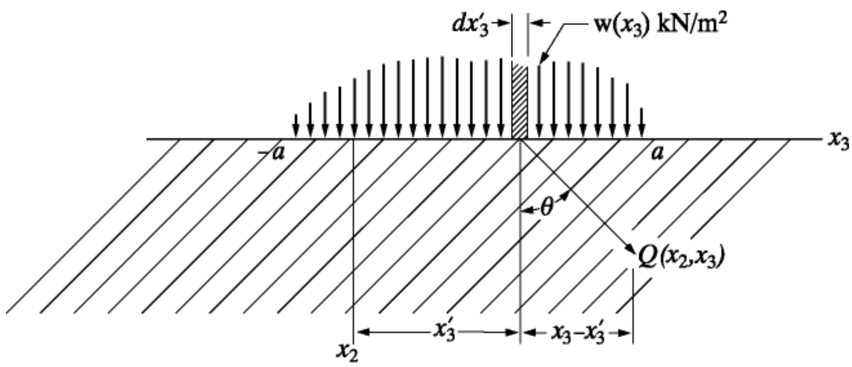


Figure d Semi-infinite body subjected to a distributed load.

Consider a semi-infinite body subjected to distributed forces $w(x_3)$ [$w(x_3)$ is given in units of forces per unit area] on a portion of its surface which, as shown in Fig d, extends from $-a \leq x_3 \leq a$, and $-\infty < x_1 < \infty$. The Airy stress function at any point $Q(x_2, x_3)$ due to a strip of force of width dx_3' may be obtained by referring to Fig d and using relation (o). Thus,

$$d\phi = -\frac{w dx_3'}{\pi}(x_3 - x_3') \arctan\left(\frac{x_3 - x_3'}{x_2}\right) \tag{q}$$

Therefore,

$$\phi(x_2, x_3) = \frac{1}{\pi}\int_{-a}^{a} w(x_3')(x_3 - x_3') \arctan\left(\frac{x_3 - x_3'}{x_2}\right) dx_3' \tag{r}$$

For a uniformly distributed load $w(x_3) = w_o$ for $-a < x_3 < a$, the above expression for $\phi(x_2, x_3)$ may be integrated to give

$$\phi(x_2, x_3) = \frac{w_o}{2\pi}\left[\left[(x_3 - a)^2 + x_2^2\right]\arctan\left(\frac{x_3 - a}{x_2}\right) - \left[(x_3 + a)^2 + x_2^2\right]\arctan\left(\frac{x_3 + a}{x_2}\right)\right] \tag{s}$$

This relation may be rewritten as

$$\phi = \frac{w_o}{2\pi}\left(r_2^2\theta_2 - r_3^2\theta_3\right) \tag{t}$$

where the quantities r_2, r_3, θ_2, θ_3 are defined in Fig. e. The components of stress for a semi-infinite body subjected to a uniformly distributed load extending from $-a \leq x_3 \leq a$, and $-\infty < x_1 < \infty$, may be obtained by substituting the expression for the Airy stress function (s) into relations (7.13). Thus,

$$\begin{aligned} \tau_{22} &= \frac{w_o}{\pi}\left[\arctan\left(\frac{x_3 - a}{x_2}\right) - \arctan\left(\frac{x_3 + a}{x_2}\right) + \frac{x_2(x_3 - a)}{x_2^2 + (x_3 - a)^2} - \frac{x_2(x_3 + a)}{x_2^2 + (x_3 + a)^2}\right] \\ \tau_{33} &= \frac{w_o}{\pi}\left[\arctan\left(\frac{x_3 - a}{x_2}\right) - \arctan\left(\frac{x_3 + a}{x_2}\right) - \frac{x_2(x_3 - a)}{x_2^2 + (x_3 - a)} + \frac{x_2(x_3 + a)}{x_2^2 + (x_3 + a)^2}\right] \end{aligned} \tag{u}$$

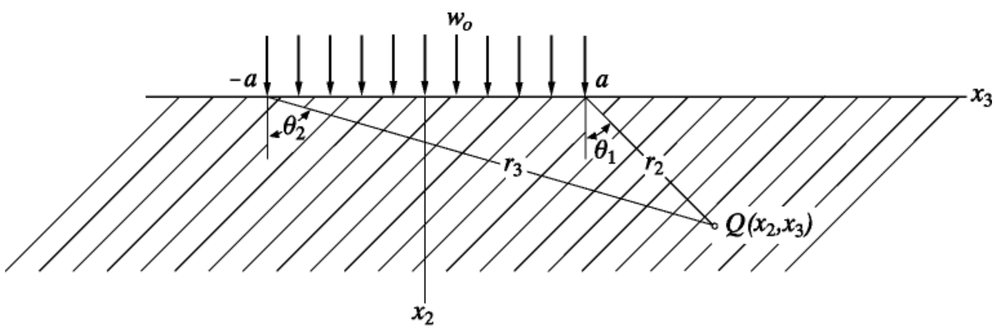


Figure e Semi-infinite body loaded with a uniformly distributed load.

$$\tau_{23} = \frac{w_o}{\pi}\left[\frac{(x_3 - a)^2}{x_2^2 + (x_3 - a)^2} - \frac{(x_2 + a)^2}{x_2^2 + (x_3 + c)^2}\right] \tag{u}$$

7.5 Plane Stress

In this section we consider prismatic bodies made from isotropic, linearly elastic materials whose length we denote by $2b$. For each body we choose a system of axes with origin at the centroid of its middle cross section while the x_1 axis coincides with its axis. Thus, its end surfaces are at $x_1 = \mp b$. We first consider bodies with no restriction on the magnitude of b. However, subsequently, we focus our attention on thin plates (b small). The bodies are originally in a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, the bodies are subjected to specific body forces and boundary conditions and reach a second state of mechanical and thermal equilibrium at the uniform temperature T_o wherein the three components of stress on the plane normal to the x_1 axis vanish, while the other components of stress are functions of x_1, x_2, x_3. That is,

$$\tau_{11} = \tau_{12} = \tau_{13} = 0 \quad \tau_{22} = \tau_{22}(x_1, x_2, x_3) \quad \tau_{33} = \tau_{33}(x_1, x_2, x_3) \quad \tau_{23} = \tau_{23}(x_1, x_2, x_3) \tag{7.39}$$

We say that these bodies *are in a state of plain stress.* Notice that in this case we cannot assume, as we have done in the case of plane strain, that the non-zero components of stress are functions only of the x_2 and x_3 coordinates. If we do so, we find that the components of strain obtained from the components of stress (7.39) on the basis of the stress–strain relations (3.52) for states of plane stress cannot be made to satisfy all the equations of compatibility.

Substituting relations (7.39) into (2.73) we find that the components of traction acting on the surfaces of the prismatic bodies under consideration must have the following form:

$$\begin{aligned} \overset{n}{T}_1 &= 0 \\ \overset{n}{T}_2 &= \tau_{22}\lambda_{n2} + \tau_{23}\lambda_{n3} \\ \overset{n}{T}_3 &= \tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3} \end{aligned} \tag{7.40}$$

Thus, in order to maintain a state of plane stress in a prismatic body the x_1 component of traction acting on its lateral surfaces must vanish. Moreover, taking into account that the unit vector normal to the end surfaces of a prismatic body is $\mathbf{i}_n = \pm\mathbf{i}_1$ ($\lambda_{n2} = \lambda_{n3} = 0$ and $\lambda_{n1} = 1$) referring to relations (2.73) and using relations (7.39) we see that the end surfaces ($x_1 = \pm b$) (see Fig. 7.4) of a prismatic body in a state of plane stress must be traction free ($\overset{\pm 1}{T}_1 = \overset{\pm 1}{T}_2 = \overset{\pm 1}{T}_3 = 0$). Furthermore, in Appendix F we show that in order to maintain a state of plane stress (7.39) in a prismatic body the specific body forces acting on its particles must be normal to its axis ($B_1 = 0$), must not be functions of its axial coordinate x_1 and must be obtained from a potential $V(x_2, x_3)$ as

$$B_2 = -\frac{\partial V}{\partial x_2} \qquad B_3 = -\frac{\partial V}{\partial x_3} \tag{7.41a}$$

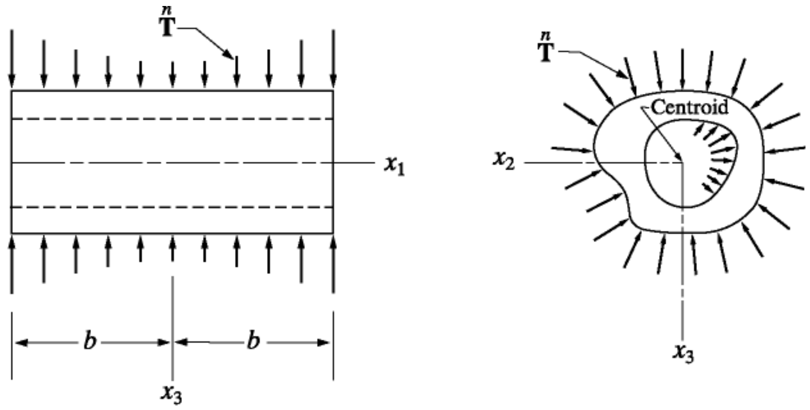


Figure 7.4 Prismatic body in a state of plane stress subjected to a distribution of traction on its lateral surface which is symmetric with respect to the plane $x_1 = 0$ and has an x_1^2 variation.

where the potential V must satisfy the following relation:

$$\nabla_1^2 V = \text{constant} \tag{7.41b}$$

The plane Laplace operator ∇_1^2 is defined by relation (7.16).

In Appendix F we show [see relations (F.6)] that the components of stress in a prismatic body subjected to the external forces described above can be obtained from a plane stress function $X(x_1, x_2, x_3)$ on the basis of the following relations:

$$\begin{aligned}
\tau_{22}(x_1, x_2, x_3) &= \frac{\partial^2 X}{\partial x_3^2} + V(x_2, x_3) \\
\tau_{33}(x_1, x_2, x_3) &= \frac{\partial^2 X}{\partial x_2^2} + V(x_2, x_3) \\
\tau_{23}(x_1, x_2, x_3) &= -\frac{\partial^2 X}{\partial x_2 \partial x_3}
\end{aligned} \tag{7.42a}$$

For some problems it is convenient to use cylindrical coordinates. For such problems referring to relations (7.13) and (7.32), relations (7.42a) can be written in cylindrical coordinates as

$$\begin{aligned}
\tau_{rr}(x_1, r, \theta) &= \frac{1}{r}\frac{\partial X}{\partial r} + \frac{1}{r^2}\frac{\partial^2 X}{\partial \theta^2} + V(r, \theta) \\
\tau_{\theta\theta}(x_1, r, \theta) &= \frac{\partial^2 X}{\partial r^2} + V(r, \theta) \\
\tau_{r\theta}(x_1, r, \theta) &= \frac{1}{r^2}\frac{\partial X}{\partial \theta} - \frac{1}{r}\frac{\partial^2 X}{\partial r \partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial X}{\partial \theta}\right)
\end{aligned} \tag{7.42b}$$

In Appendix F we show [see relation (F.38)] that the function $X(x_1, x_2, x_3)$ is obtained from the following relation:

$$X(x_1, x_2, x_3) = Y_2(x_2, x_3) + x_1 Y_1(x_2, x_3) - \frac{\nu x_1^2 \nabla_1^2 Y_2}{2(1+\nu)} - \frac{\nu x_1^2 V(x_2, x_3)}{(1+\nu)} + \frac{\nu (x_2^2 + x_3^2) x_1^2 \nabla_1^2 V}{4(1-\nu)} \tag{7.43}$$

where referring to relations (F.36) and (F.37) the functions $Y_2(x_1, x_2)$ and $Y_1(x_2, x_3)$ satisfy at every point of the body the following relations:

$$\nabla_1^4 \, Y_2 = -(1 - \nu)\nabla_1^2 \, V \tag{7.44}$$

and

$$\nabla_1^4 \, Y_1 = 0 \tag{7.45}$$

The plane biharmonic operator ∇_1^4 is defined by relation (7.18).

Relation (7.43) has been established by requiring that the components of stress, obtained from the plane stress function $X(x_1, x_2, x_3)$ on the basis of relation (7.42), when substituted into the stress–strain relations for plane stress (3.52) give components of strain which satisfy the equations of compatibility (2.63). For simply connected bodies the satisfaction of the equations of compatibility is necessary and sufficient to ensure that the strain–displacement relations (2.16) can be integrated to yield single-valued continuous components of displacement. For multiply connected bodies the satisfaction of the equations of compatibility is necessary and sufficient to ensure only that the strain–displacement relations (2.16) can be integrated. However, the resulting components of displacement may or may not be single-valued continuous functions of the space coordinates. In order to ensure that the components of displacement obtained from a set of components of strain by integrating the strain–displacement relations are single-valued continuous functions of the space coordinates, additional restrictions must be imposed on the stress function $X(x_1, x_2, x_3)$. These restrictions as well as the equations of compatibility result from the requirement that the components of displacement must satisfy relation (7.26) around every closed curve on the cross sections of the body.

From relations (7.43), (7.42) and (7.40) we see that in order to maintain a state of plane stress in a simply connected prismatic body the components of stress and, consequently, the specified components of traction ($\overset{n}{T}{}_2^s, \overset{n}{T}{}_3^s$) acting on its lateral surfaces must be the sum of two parts — an antisymmetric in x_1 and a symmetric in x_1. The first is a linear function of x_1 while the second has an x_1^2 variation (see Fig. 7.4). *If they do, the stress function $X(x_1, x_2, x_3)$ obtained from relations (7.43) when substituted into relations (7.42) gives the exact components of stress in the body under consideration.* However, there are not many cases of practical interest which involve simply connected prismatic bodies subjected to a distribution of the components of traction on their lateral surface which is required in order to maintain a state of plane stress in them. Therefore, the plane stress solution for a general thick simply connected prismatic body is of little practical value. *Nevertheless, if the thickness of a prismatic body is small compared to its other dimensions, on the basis of the principle of Saint Venant, we may conclude that for any given distribution of traction on its lateral surfaces which is statically equivalent to that required for a state of plane stress, the stress distribution at the parts of the body away from its lateral surfaces will approach a state of plane stress.* Hence, the function $X(x_1, x_2, x_3)$ given by relation (7.43) can be used to obtain the components of stress acting on particles sufficiently removed from the lateral surface of simply connected thin prismatic bodies made from isotropic, linearly elastic materials and subjected to the following forces:

1. A distribution of specific body forces exists, whose component in the direction of the x_1 axis vanishes, while its other components are only functions of x_2 and x_3 obtained from a potential $V(x_2, x_3)$ on the basis of relation (7.41a). The potential satisfies relation (7.41b).
2. A distribution of traction occurs on its lateral surface which is statically equivalent to the distribution required in order to maintain the body in a state of plane stress.
3. Vanishing components of traction occur on their end surfaces $x_1 = \pm b$.

7.6 Simply Connected Thin Prismatic Bodies (Plates) in a State of Plane Stress Subjected on Their Lateral Surface to Symmetric in x_1 Components of Traction $\overset{n}{T}_2$ and $\overset{n}{T}_3$

Consider a simply connected (without holes) thin prismatic body (plate), subjected to the following forces:

1. A distribution of components of specific body forces $\mathbf{B} = B_2(x_2, x_3)\mathbf{i}_2 + B_3(x_2, x_3)\mathbf{i}_3$ is obtained from a potential $V(x_2, x_3)$ on the basis of relations (7.41a). The potential satisfies relation (7.41b).
2. A distribution of transverse components of traction $\overset{n}{T}_2(x_1, x_2, x_3)$ and $\overset{n}{T}_3(x_1, x_2, x_3)$ on its lateral surface, which are symmetric with respect to the middle plane ($x_1 = 0$) of the plate.
3. Vanishing components of traction on the end surfaces ($x_1 = \pm b$) of the plate.

For this loading the function $Y_1(x_2, x_3)$ must vanish and relation (7.43) reduces to

$$X(x_1,x_2,x_3) = Y_2(x_2,x_3) - \frac{\nu x_1^2 \nabla_1^2 Y_2}{2\,(1+\nu)} - \frac{\nu x_1^2 V(x_2,x_3)}{(1+\nu)} + \frac{\nu(x_2^2+x_3^2)x_1^2\nabla_1^2 V}{4\,(1-\nu)} \tag{7.46}$$

In what follows we replace the function $Y_2(x_2, x_3)$ in relation (7.46) with the function $\psi(x_2, x_3)$ which represents the average value in x_1 of the function $X(x_1, x_2, x_3)$. We do this because as we will show in the next section the function $\psi(x_2, x_3)$ can be established from the plane strain function $\phi(x_2, x_3)$ for an auxiliary plate. Using relation (7.46), we get

$$\psi(x_2, x_3) = \frac{1}{2b}\int_{-b}^{b} X dx_1 = Y_2(x_2, x_3) - \frac{\nu b^2}{6(1+\nu)}\nabla_1^2 Y_2 - \frac{\nu b^2}{3(1+\nu)}V + \frac{\nu b^2}{12(1-\nu)}(x_2^2 + x_3^2)\nabla_1^2 V \tag{7.47}$$

Taking the Laplacian of both sides of relation (7.47) and using relation (7.44), we obtain

$$\nabla_1^2\psi = \nabla_1^2 Y_2 + \frac{\nu b^2(1+\nu)}{6\,(1-\nu)}\nabla_1^2 V \tag{7.48}$$

Taking the Laplacian of both sides of relation (7.48) and using relation (7.44), we have†

$$\nabla_1^4\psi(x_2, x_3) = -(1-\nu)\nabla_1^2 V \tag{7.49}$$

Thus, if V vanishes or $\nabla_1^2 V = 0$ the function $\psi(x_2, x_3)$ is biharmonic. Using relation (7.48) to eliminate $\nabla_1^2 Y_2$ from relation (7.47), we obtain

$$Y_2 = \psi + \frac{\nu b^2}{6(1+\nu)}\nabla_1^2\psi + \frac{\nu b^2 V}{3(1+\nu)} - \left(\frac{\nu b^2}{3} + x_2^2 + x_3^2\right)\frac{\nu b^2}{12(1-\nu)}\nabla_1^2 V$$

Substituting the above relation into (7.46) and using relation (7.48), we get

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† Relation (7.49) could be written to include the effect of temperature. Using the plane stress form of the stress–strain relations (2.95) we obtain

$$\nabla_1^4\psi(x_2, x_3) = -(1-\nu)\nabla_1^2 V - E\nabla_1^2\alpha(T - T_o) \tag{7.50}$$

$$X(x_1, x_2, x_3) = \psi(x_2, x_3) + \frac{\nu(b^2 - 3x_1^2)}{6(1+\nu)}\nabla_1^2\psi + \frac{\nu V}{3(1+\nu)}(b^2 - 3x_1^2) + \frac{\nu\nabla_1^2 V}{12(1-\nu)}(3x_1^2 - b^2)\left(x_2^2 + x_3^2 + \frac{\nu b^2}{3}\right) \tag{7.51}$$

On the basis of the foregoing presentation, the stress distribution of a thin prismatic simply connected body (plate), subjected to the external forces described at the beginning of this section, may be established by adhering to the following steps:

STEP 1 We establish the function $\psi(x_2, x_3)$ which satisfies the differential equations (7.49) at every point of the body. The resulting expression for the function $\psi(x_2, x_3)$ involves unknown constants.

STEP 2 We substitute the function $\psi(x_2, x_3)$ established in step 1 into relation (7.51) to obtain the stress function $X_2(x_1, x_2, x_3)$.

STEP 3 We substitute the stress function $X_2(x_1, x_2, x_3)$ established in step 2 into relations (7.42) to obtain the components of stress acting on the particles of the body.

STEP 4 We establish the components of traction acting on the lateral surface of the body by substituting the components of stress established in step 3 in the traction–stress relations (7.40). The resulting expressions have an x_1^2 variation and involve the unknown constants introduced in step 1. We evaluate these constants by requiring that at the lateral surface of the body the computed components of traction are statically equivalent to the given components of traction.

This approach yields reasonably accurate components of stress for particles sufficiently removed from the lateral boundary of thin plates. The thinner the plate the better the accuracy of the results.

7.6.1 Computation of the Plane Stress Function $\psi(x_2, x_3)$ for a Thin Plate Subjected to Symmetric in x_1 Components of Traction on Its Lateral Boundary from the Plane Strain Function $\phi(x_2, x_3)$ for an Auxiliary Plate

Consider a thin prismatic body (plate) made from an isotropic, linearly elastic material of modulus of elasticity E and Poisson's ratio ν subjected to the loading described in Section 7.6. Moreover, consider an auxiliary thin plate having the same geometry as the actual plate and made from an isotropic linearly elastic material with modulus of elasticity† E and Poison's ratio given by the following relation:

$$\nu_{\text{aux}} = \frac{\nu}{1 + \nu} \tag{7.52}$$

Assume that the auxiliary plate is in a state of plain strain due to the following external

† If the affect of temperature was included, the modulus of elasticity E_{aux} of the auxiliary plate should be

$$E_{\text{aux}} = \frac{E}{1 + \nu} \tag{7.52a}$$

forces, acting on it:

1. A distribution of components of specific body force $B_2(x_2, x_3)$ and $B_3(x_2, x_3)$ equal to that to which the actual thin plate is subjected.

2. Components of traction $\overset{n}{T}_2(x_2, x_3)$ and $\overset{n}{T}_3(x_2, x_3)$ on its lateral surface equal to the average in x_1 of the specified components of traction $\overset{n}{T}_2(x_1, x_2, x_3)$ and $\overset{n}{T}_3(x_1, x_2, x_3)$ acting on the lateral surface of the actual plate.

3. The end surfaces of the plate ($x_1 = \pm b$) are restrained from moving in the x_1 direction. However, they can move freely in the x_2 and x_3 directions. That is, the components of traction $\overset{\pm 1}{T}_2$ and $\overset{\pm 1}{T}_3$ vanish on the end surfaces ($x_1 = \pm b$) of the auxiliary plate.

For the auxiliary plate under consideration, using relation (7.52), relation (7.17) becomes

$$\nabla_1^4 \phi_{\text{aux}} = -(1 - \nu)\nabla_1^2 V \tag{7.53}$$

where $V(x_2, x_3)$ is the potential from which the components of specific body force acting on the real plate are obtained on the basis of relation (7.41a). Thus, the Airy stress function $\phi_{\text{aux}}(x_2, x_3)$ for the auxiliary plate must satisfy relation (7.53) at every point of the plate and when substituted into relation (7.21) must give components of traction $\overset{n}{T}_2(x_2, x_3)$ and $\overset{n}{T}_3(x_2, x_3)$ which are the average in x_1 of the components of traction acting on the actual plate. The average components of traction $\overset{n}{T}_2(x_2, x_3)$ and $\overset{n}{T}_3(x_2, x_3)$ may be expressed as

$$\overset{n}{T}_2(x_2, x_3) = \frac{1}{2b}\int_{-b}^{b} \overset{n}{T}{}_2^{s} dx_1 = \frac{1}{2b}\int_{-b}^{b} (\tau_{22}\lambda_{n2} + \tau_{23}\lambda_{n3})dx_1 = \frac{1}{2b}\int_{-b}^{b}\left(\frac{\partial^2 X}{\partial x_3^2}\lambda_{n2} - \frac{\partial^2 X}{\partial x_2 \partial x_3}\lambda_{n3} + V\lambda_{n2}\right)dx_1$$

$$\overset{n}{T}_3(x_2, x_3) = \frac{1}{2b}\int_{-b}^{b} \overset{n}{T}{}_3^{s} dx_1 = \frac{1}{2b}\int_{-b}^{b} (\tau_{23}\lambda_{n2} + \tau_{33}\lambda_{n3})dx_1 = \frac{1}{2b}\int_{-b}^{b}\left(\frac{\partial^2 X}{\partial x_2^2}\lambda_{n3} - \frac{\partial^2 X}{\partial x_2 \partial x_3}\lambda_{n2} + V\lambda_{n3}\right)dx_1$$

(7.54)

Notice that the same average transverse components of traction given by relations (7.54) are obtained from any distribution of transverse components of traction of the form

$[\overset{n}{T}{}_2^{s}(x_1, x_2, x_3) + \overset{n}{F}_2(x_1, x_2, x_3)]$, and $[\overset{n}{T}{}_3^{s}(x_1, x_2, x_3) + \overset{n}{F}_3(x_1, x_2, x_3)]$ provided that $\overset{n}{F}_2$ and $\overset{n}{F}_3$ satisfy the following relations:

$$\int_{-b}^{b} \overset{n}{F}_2 \, dx_1 = \int_{-b}^{b} \overset{n}{F}_3 \, dx_1 = 0 \tag{7.55}$$

However, for thin prismatic bodies (plates) on the basis of the principle of Saint Venant the effect of the components of traction $\overset{n}{F}_2$, and $\overset{n}{F}_3$ on the components of stress acting on

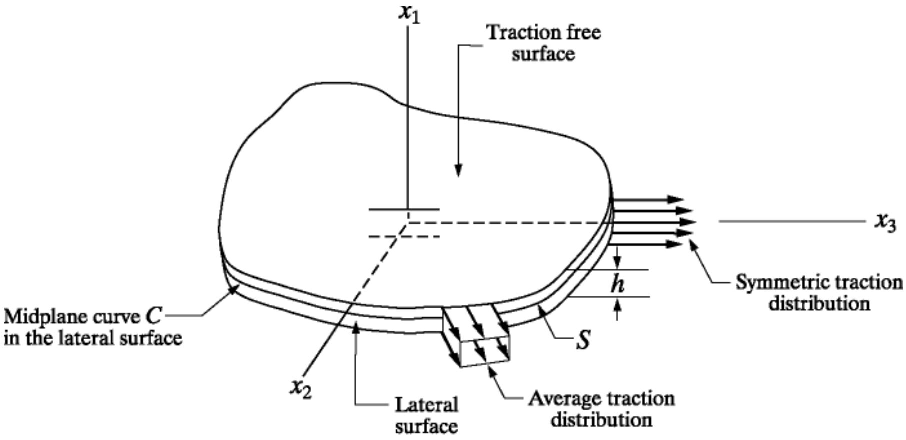


Figure 7.5 Thin plate subjected to transverse components of traction, which are symmetric with respect to the plane $x_1 = 0$.

particles located at points sufficiently away from the lateral surface of the plates is negligible.

Using relation (7.47), relations (7.54) can be rewritten as

$$\overset{n}{T}_2(x_2, x_3) = \frac{\partial^2 \psi(x_2, x_3)}{\partial x_3^2}\lambda_{n2} - \frac{\partial^2 \psi(x_2, x_3)}{\partial x_2 \partial x_3}\lambda_{n3} + V\lambda_{n2} \tag{7.56a}$$

$$\overset{n}{T}_3(x_2, x_3) = -\frac{\partial^2 \psi(x_2, x_3)}{\partial x_2 \partial x_3}\lambda_{n2} + \frac{\partial^2 \psi(x_2, x_3)}{\partial x_2^2}\lambda_{n3} + V\lambda_{n3} \tag{7.56b}$$

Comparing relation (7.53) with (7.49) and relation (7.21) with (7.56) we see that

$$\psi(x_2, x_3) = \phi_{\text{aux}}(x_2, x_3) \tag{7.57}$$

Thus, the function $\psi(x_1, x_2)$ defined by relation (7.47) for a simply connected thin prismatic plate subjected to the external forces described at the beginning of Section 7.6 is equal to the Airy stress function $\phi_{\text{aux}}(x_2, x_3)$ for the auxiliary prismatic plate under consideration. Notice that if the specific body force distribution acting on a simply connected body is negligible, the auxiliary body could be made from the same material as the actual body.

In the case of multiply connected thin prismatic plates in a state of plane stress, additional restrictions must be imposed on the functions $X(x_1, x_2, x_3)$ and $\psi(x_2, x_3)$ by the requirement (7.26). In Section 7.3 we have noted that the additional restrictions which this requirement imposes on the Airy stress function for plane strain is that this function must satisfy the Mitchell conditions (7.27). Thus, if the elastic constants of the auxiliary body under consideration could be chosen so that the stress–strain relations for plane stress of the actual prismatic body are identical to the stress–strain relations for plane strain of the auxiliary prismatic body, the function $\psi(x_2,x_3)$ will satisfy relations (7.27) and consequently, it will be equal to the plane strain function $\phi_{\text{aux}}(x_2, x_3)$ for the auxiliary body. However, this is not possible.

It can be shown that for a multiply connected body the function $\psi(x_2, x_3)$ obtained from the Airy stress function $\phi_{\text{aux}}(x_2, x_3)$ of the auxiliary body on the basis of relation (7.57) when substituted into relation (7.51) gives a function $X(x_1, x_2, x_3)$ which when substituted into relations (7.42) gives components of stress which on the basis of the stress–strain relation for plane stress (3.52) give components of strain which do not satisfy relation (7.26). *Thus, for a multiply connected thin plate the function* $\psi(x_2, x_3)$ *cannot be equal to the Airy stress function* $\phi_{\text{aux}}(x_2, x_3)$ *for the corresponding auxiliary plate described above. Nevertheless, for very thin multiply connected plates one could disregard this discrepancy and obtain an approximate solution which does not satisfy all the compatibility equations by taking the function* $\psi(x_2, x_3)$ *to be equal to the Airy stress function* $\phi_{\text{aux}}(x_2, x_3)$ *for the auxiliary plate.*

On the basis of the afore going presentation the components of stress of thin prismatic bodies (plates) subjected to the external forces described at the beginning of Section 7.6 can be established by adhering to the following steps:

STEP 1 *We first establish the Airy stress function* $\phi_{\text{aux}}(x_2, x_3) = \psi(x_2, x_3)$ for an auxiliary prismatic body in a state of plane strain. The auxiliary body has the same geometry as the real body and is subjected on its lateral surface to traction which is constant along the x_1 direction and is equal in magnitude to the average value of the specified traction $\overset{n}{T}{}_2^s(x_1, x_2, x_3)$, $\overset{n}{T}{}_3^s(x_1, x_2, x_3)$ acting on the lateral surface of the actual body. Moreover, the auxiliary body is subjected to the distribution of traction on its end surfaces $x_1 = \pm b$ which is required in order to maintain it in a state of plane strain. The elastic constant ν_{aux} of the auxiliary prismatic body is obtained from that of the real body on the basis of relation (7.52). If the specific body force distribution acting on a simply connected body is negligible, the auxiliary body could be made of the same material as the actual body.

STEP 2 Once the Airy strain function $\phi_{\text{aux}}(x_2, x_3) = \psi(x_2, x_3)$ for plane strain of the auxiliary body is obtained the stress function $X(x_1, x_2, x_3)$ for the actual thin prismatic body may be established from relation (7.51) and substituted into relations (7.42) to give the stress field of the body.

This method yields the exact solution, within the limitations of the linear theory of elasticity, for simply connected thin prismatic bodies (plates) subjected to a symmetric with respect to the plane $x_1 = 0$ distribution of traction on their lateral surface which is equal to

$$\overset{n}{T}_2(x_1, x_2, x_3) + \overset{n}{F}_2(x_1, x_2, x_3),\ \overset{n}{T}_3(x_1, x_2, x_3) + \overset{n}{F}_3(x_1, x_2, x_3)$$

Here $\overset{n}{F}_2(x_1, x_2, x_3)$, $\overset{n}{F}_3(x_1, x_2, x_3)$ is a symmetric with respect to the plane $x_1 = 0$ distribution of traction which when added to the average, in x_1 of the actual distribution of the components of traction $\overset{n}{T}_2(x_1, x_2, x_3)$, $\overset{n}{T}_3(x_1, x_2, x_3)$ makes it have the x_1^2 variation along the thickness of the plate needed for the plane stress solution. However, the distribution of the components of traction $\overset{n}{F}_2(x_1, x_2, x_3)$, $\overset{n}{F}_3(x_1, x_2, x_3)$ satisfies relation (7.55), and thus, on the basis of the principle of Saint Vernant for thin prismatic plates its effect is negligible at points away from their lateral surface. This method can be used only for very thin multiply connected prismatic plates, keeping in mind that for such plates the results do not satisfy relation (7.26).

7.7 Two-Dimensional or Generalized Plane Stress

Consider a very thin prismatic plate of thickness 2b, specific body forces and surface tractions described at the beginning of Section 7.6. An approximation to the components of stress of such a plate is their average value in x_1. Thus, for such a plate we use the average value in x_1 of the stress function $X(x_1, x_2, x_3)$ which we denote by $\tilde{X}(x_2, x_3)$. That is, referring to relation (7.51) and (7.57), we have

$$\tilde{X}(x_2, x_3) = \frac{1}{2b}\int_{-b}^{b} X(x_1, x_2, x_3)\,dx_1 = \psi(x_2, x_3) = \phi_{\text{aux}}(x_2, x_3) \tag{7.58}$$

This average in x_1 stress function when substituted into relation (7.42a) gives an approximation to the components of stress which when substituted into the stress–strain relations (3.52) for an isotropic, linearly elastic body in a state of plane stress give components of strain which satisfy the fourth and fifth of the compatibility relations but in general they do not satisfy the second, third and sixth of the compatibility relations (2.65).

In Appendix F we show that in order that the components of strain obtained from the stress function $X(x_1, x_2, x_3)$ satisfy the compatibility relations the body force must be obtainable from a potential $V(x_2, x_3)$ which satisfies relation (7.41b). Inasmuch as the two-dimensional or generalized plane stress solution is an approximation which does not satisfy the second, third and sixth compatibility relations, it is anticipated that if we remove the restriction that the components of the specific body force are derivable from a potential, the accuracy of the approximation will not deteriorate.

Thus, in order to establish on the basis of the two-dimensional or generalized plane stress approximation, the stress field in very thin simply or multiply connected prismatic bodies (plates) subjected to the external forces described at the beginning of Section 7.6, without the restriction that the components of the specific body force are derivable from a potential, we adhere to the following steps:

STEP 1 We first establish the Airy stress function $\phi_{\text{aux}}(x_2, x_3) = \psi(x_2, x_3)$ for an auxiliary prismatic body in a state of plane strain. The auxiliary body (plate) has the same geometry as the real body (plate) and is subjected to the same body forces as the actual body (plate) while on its lateral surfaces is subjected to traction which is constant along the x_1 direction and equal in magnitude to the average value of the specified traction $\overset{n}{T}{}^{s}_{2}(x_1, x_2, x_3)$ and $\overset{n}{T}{}^{s}_{3}(x_1, x_2, x_3)$ acting on the lateral surfaces of the actual body. The elastic constant ν_{aux} of the auxiliary body (plate) is obtained from the Poisson ratio of the real body (plate) on the basis of relation (7.52).

STEP 2 Once the plane strain function $\phi_{\text{aux}}(x_2, x_3) = \psi(x_2, x_3)$ is established the stress function $\tilde{X}(x_2, x_3)$ is obtained from relation (7.58) and substituted into relations (7.42a) or (7.42b) to give the average values in x_1 of the components of stress acting on the thin plate under consideration.

The components of stress of thin plates obtained on the basis of the two-dimensional or generalized plane stress approximation represent a good approximation of the actual components of stress acting on particles sufficiently removed from their lateral surfaces. Moreover, their accuracy increases as the thickness of the plate decreases.

In what follows we present two examples.

Example 2 Establish an approximation to the stress distribution in a cantilever prismatic beam of a very narrow rectangular cross section (small thickness to depth ratio) subjected at its free end ($x_1 = L$) to a distribution of traction which is statically equivalent to a force whose line of action lies in the x_1x_3 plane and it is equal to $P_3\mathbf{i}_3$ [see Fig (a)]. The dimensions of the cross section of the beam are small compared to its length. The free-body diagram of the beam is shown in Fig. a.

The boundary conditions of the beam under consideration do not correspond to those required to maintain a state of plane strain in it. In order that a state of plane strain ($e_{21} = e_{23} = e_{22} = 0$) exists in the beam it must be subjected to a distribution of normal component of traction on its sides $x_2 = \pm b$ of the magnitude required to render the component of displacement $\hat{u}_2(x_1, x_2, x_3)$ equal to zero. However, since the cross sectional dimensions of the beam are small compared to its length and its cross section is narrow $b << h$, we can obtain an approximate solution by considering the beam as being in a state of plane stress.

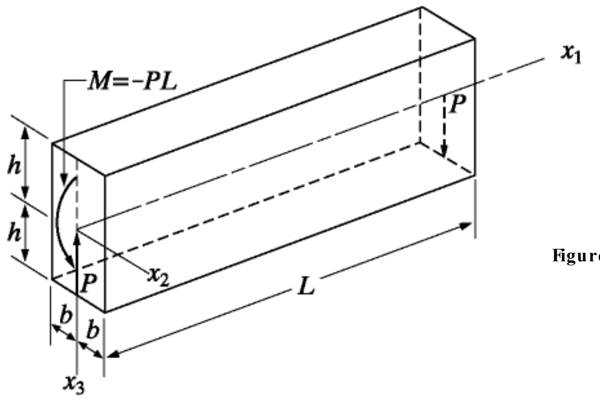


Figure a Cantilever beam of rectangular cross section.

Solution

Part a State of plane strain

In this part we assume that the beam is in a state of plane strain. That is, we assume that it is subjected in addition to the given actions on its end surfaces ($x_1 = 0$ and $x_1 = L$) to the distribution of traction on its lateral surfaces $x_2 = \pm b$ which is required in order to render the component of displacement $u_2(x_1, x_2, x_3)$ equal to zero. It can be shown† that the Airy stress function for plane strain for the beam under consideration is

$$\phi = \frac{P_3}{6I_2}\left[\,3h^2(L - x_1)x_3 - (L - x_1)x_3^3\,\right] \tag{a}$$

Where I_2 is the moment of inertia of the cross sections of the beam about the x_2 axis. Substituting relation (a) into (7.13) and taking into account relation (7.2) with subscript 1 replaced by 2 and 2 by 1, we get

† A systematic procedure for determining Airy stress functions has been presented by Neou, C.Y., Direct method for determining airy stress functions, *Journal of Applied Mechanics,* 24 Sept. 1957. Relation (a) has been established using that procedure.

$$\tau_{11} = \frac{\partial^2 \phi}{\partial x_3^2} = -\frac{P_3(L - x_1)x_3}{I_2} \qquad \tau_{13} = -\frac{\partial \phi}{\partial x_1 \partial x_3} = \frac{P_3}{2I_2}(h^2 - x_3^2) \qquad \tau_{33} = -\frac{\partial \phi^2}{\partial x_1^2} = 0$$

$$\tau_{22} = \nu(\tau_{11} + \tau_{33}) = -\frac{\nu P_3(L - x_1)x_3}{I_2} \qquad \tau_{12} = \tau_{23} = 0 \tag{b}$$

It can be shown that this distribution of the components of stress has the following attributes

1. It satisfies the equations of equilibrium (2.69).

2. It when substituted into the stress–strain relations for an isotropic, linearly elastic material, (3.48) gives components of strain which satisfy the compatibility relations.

3. It satisfies the boundary conditions at $x_3 = \pm h$.

4. It gives a distribution of the components of traction on the end surface $x_1 = L$ of the beam which is statically equivalent to the force P_3.

5. It gives a distribution of the components of traction on the end surface $x_1 = 0$ of the beam which is statically equivalent to the force $(-P_3)$ and the moment $(M_2 = -P_3L)$ which are required for the equilibrium of the beam.

6. It gives the following components of traction on any plane normal to the x_2 axis $(\lambda_{n1} = \lambda_{n3} = 0 \;\; \lambda_{n2} = \pm 1)$:

$$\overset{\pm 2}{T}_1 = \overset{\pm 2}{T}_3 = 0 \qquad\qquad \overset{\pm 2}{T}_2 = \mp \frac{\nu P_3(L - x_1)x_3}{I_2} \tag{c}$$

This distribution of traction is shown in the Fig. b. It is required in order to render the displacement component $\hat{u}_2$ equal to zero. It is statically equivalent to the following distributed moment m_1 about the x_1 axis acting on each surface $x_2 = \pm b$ of the beam

$$m_1 \bigg|_{x_2 = \pm b} = \int_{-h}^{h} \overset{\pm 2}{T}_{\pm 2} x_3 dx_3 = \mp \frac{2\nu P_3(L - x_1)h^3}{3I_2} = \frac{\mp \nu P_3(L - x_1)}{b} \tag{d}$$

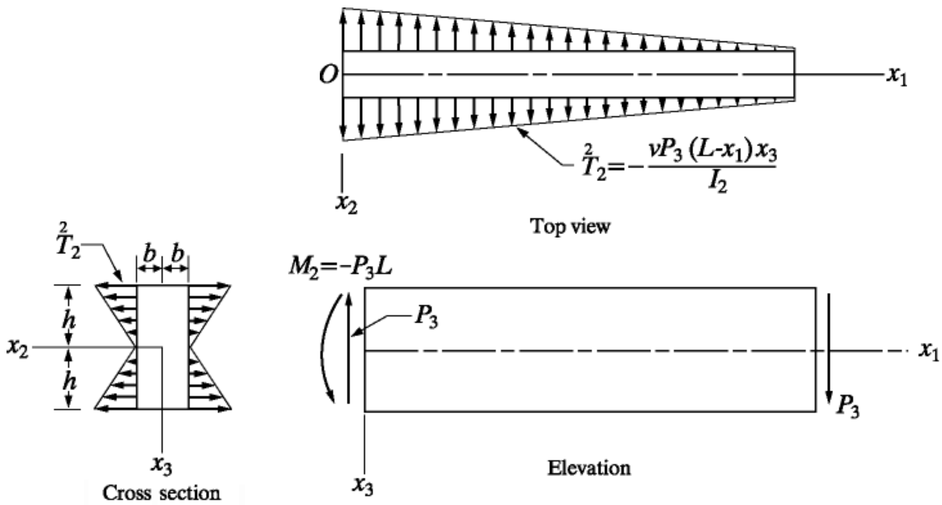


Figure b Distribution of the components of traction required for the beam to be in a state of plane.

Part b **State of plane stress**

In this part we establish the components of stress corresponding to a state of plane stress in the beam. In order to accomplish this we first substitute the plane strain function $\phi(x_1, x_3) = \psi(x_1, x_3)$ given by relation (a) into relation (7.51) to obtain the plane stress function $X(x_1, x_2, x_3)$. From relation (a) we have

$$\nabla_1^2 \phi = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_3^2} = -\frac{P_3(L - x_1)x_3}{I_2} \tag{e}$$

Substituting relations (a) and (e) in (7.51), we obtain

$$X(x_1, x_2, x_3) = \frac{P_3}{2I_2}\left[h^2(L - x_1)x_3 - \frac{(L - x_1)x_3^3}{3} - \frac{\nu(L - x_1)x_3}{3(1 + \nu)}(b^2 - 3x_2^2)\right] \tag{f}$$

The components of stress corresponding to a state of plane stress of the beam are obtained by substituting relation (f) into (7.42a). That is,

$$\begin{aligned}
\tau_{11} &= \frac{\partial^2 X}{\partial x_3^2} = -\frac{P_3(L - x_1)x_3}{I_2} \\
\tau_{33} &= \frac{\partial^2 X}{\partial x_1^2} = 0 \\
\tau_{13} &= -\frac{\partial^2 X}{\partial x_1 \partial x_3} = \frac{P_3 h^2}{2I_2}\left[1 - \left(\frac{x_3}{h}\right)^2 - \frac{\nu}{3(1 + \nu)}\left(\frac{b}{h}\right)^2\left[1 - 3\left(\frac{x_2}{b}\right)^2\right]\right] \\
\tau_{12} &= \tau_{23} = \tau_{22} = 0
\end{aligned} \tag{g}$$

On the surface $x_1 = L$ this stress distribution gives a distribution of the component of traction $\overset{1}{T}_3$ which is statically equivalent to the force P_3, while on the surface $x_1 = 0$ gives a distribution of the components of traction $\overset{-1}{T}_1$ and $\overset{-1}{T}_3$ which is statically equivalent to the

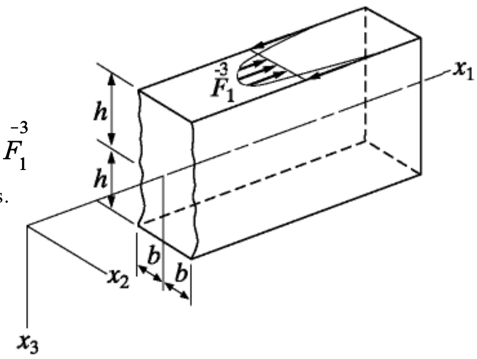


Figure c Distribution of the component of traction $\overset{-3}{F}_1$ required for the beam to be in a state of plane stress.

force $-P_3$ and the moment $M_2 = -P_3L$. Thus, the stress distribution (g) satisfies the given boundary conditions on the end surfaces of the beam $x_1 = 0$ and $x_1 = L$. However, as expected on the surfaces $x_3 = \pm h$ ($\lambda_{n1} = \lambda_{n2} = 0$ $\lambda_{n3} = \pm 1$)he stress distribution (g) does not yield the specified zero traction ($\overset{\pm 3}{\mathbf{T}} = 0$) but rather the traction $\overset{\pm 3}{\mathbf{T}} + \overset{\pm 3}{\mathbf{F}}$where

$$\overset{\pm 3}{F}_1(x_2, \pm h) = \tau_{13}\lambda_{33} = \pm\frac{\nu P_3 b^2}{6I_2(1 + \nu)}\left[1 - 3\left(\frac{x_2}{b}\right)^2\right] \qquad \text{(h)}$$

$$\overset{\pm 3}{F}_2 = 0$$

The distribution of the component of traction $\overset{-3}{F}_1$ is shown in Fig. c. The resultant force and moment per unit length of the beam of the component of traction $\overset{\pm 3}{F}_1$ are

$$\overset{\pm 3}{\mathbf{R}} = \left[\int_{-b}^{b} \overset{\pm 3}{F}_1 dx_2\right]\mathbf{i}_1 = 0 \qquad \overset{\pm 3}{\mathbf{M}} = \overset{\pm 3}{M}_2\underline{i}_2 = \left[\int_{-b}^{b} \overset{\pm 3}{F}_1 x_3 dx_2\right]\underline{\mathbf{i}}_2 = 0 \qquad \text{(i)}$$

On the basis of the Saint Venant principle, if b is small compared to h, the affect of the component of traction $\overset{3}{F}_1$ at particles away from the boundary $x_3 = \pm h$ will be negligible, and the plane stress solution will give accurate results at these particles. That is, the plane stress solution is valid for beams whose cross sectional dimensions are small as compared to their length but also whose cross section is narrow ($b << h$).

For very narrow beams the two-dimensional plane stress theory, also known as generalized plane stress theory, may be employed. In this theory the components of stress of the beam are approximated by their average in x_1 value and the stress function reduces to

$$\tilde{X}(x_1, x_3) = \psi(x_1, x_3) = \phi(x_1, x_3) \qquad \text{(j)}$$

Substituting the stress function (j) into relations (7.42a), we get

$$\tilde{\tau}_{11} = -\frac{P_3(L - x_1)x_3}{I_2} = \frac{M_2 x_3}{I_2} \qquad \tilde{\tau}_{33} = 0$$

$$\tilde{\tau}_{13} = \frac{P_3}{2I_2}(h^2 - x_3^2) \qquad \tilde{\tau}_{22} = \tilde{\tau}_{32} = \tilde{\tau}_{21} = 0 \qquad \text{(k)}$$

The value $\tilde{\tau}_{13}$ is the average in x_2 of the value of τ_{13}. Notice that the difference between solutions (k) and (g) is only in the expression for the shearing component of stress τ_{13}. Solution (k) does not satisfy all the compatibility equations but it does satisfy all the boundary conditions of the beam. It is a satisfactory approximation for beams with large h/b ratios. The stress distribution (k) is the same as that obtained on the basis of the classical theory of mechanics of materials for beams (see Example 7 Section 9.5). In general when the beam is subjected to a different loading, the results obtained on the basis of the two-dimensional or generalized plane stress theory may not be identical to those obtained on the basis of the classical theory of mechanics of materials for beams.

The maximum value of τ_{13} is obtained for $x_3 = 0$. From relations (g), we get

$$\tau_{13}\Big|_{\substack{x_2=0\\x_3=0}} = \frac{3P_3}{2A}\left[1 - \frac{\nu}{3(1+\nu)}\left(\frac{b}{h}\right)^2\right] = K\left(\frac{3P_3}{2A}\right) \tag{l}$$

and

$$\tau_{13}\Big|_{\substack{x_2=b\\x_3=0}} = \frac{3P_3}{2A}\left[1 + \frac{2\nu}{3(1+\nu)}\left(\frac{b}{h}\right)^2\right] = \overline{K}\left(\frac{3P_3}{2A}\right) \tag{m}$$

where A is the area of the cross section of the beam. From solution (k) we obtain

$$\tilde{\tau}_{13}\Big|_{\substack{x_2=0,\,b\\x_3=0}} = \frac{3P}{2A} \tag{n}$$

Table a Comparison of the exact value of the shearing component of stress with results obtained from the plane stress theory.

b/h	0.5	1.0	2.0
K ($x_2 = 0$, $x_3 = 0$)	0.983	0.933	0.733
$\overline{K}$ ($x_2 = b$ $x_3 = 0$)	1.034	1.133	1.533
% error $= 100\left(\dfrac{\tau_{13} - \tau_{13}^E}{\tau_{13}^E}\right)$ at $x_2 = b$ $x_3 = 0$	0	−0.75%	−14.2%
% error $= 100\left(\dfrac{\tau_{13} - \tau_{13}^E}{\tau_{13}^E}\right)$ at $x_2 = 0$ $x_3 = 0$	0	0.60%	10%
% error $= 100\left(\dfrac{\tilde{\tau}_{13} - \tau_{13}^E}{\tau_{13}^E}\right)$ at $x_2 = b$ $x_3 = 0$	−0.7%	-6.2%	−17.7%
% error $= 100\left(\dfrac{\tilde{\tau}_{13} - \tau_{13}^E}{\tau_{13}^E}\right)$ at $x_2 = 0$ $x_3 = 0$	3.2%	11.2%	28%

In Table a we tabulate the values of K and $\overline{K}$ for various values of the ratio b/h for $\nu = 1/4$. Moreover in this table we tabulate the percent, differences of the values of the component of stress τ_{13}given by the third of relations (g) and the third of relations (k) from its exact value† $\tau_{13}{}^{E}$ obtained on the basis of the three-dimensional theory of elasticity. It is apparent that the plane stress solution based on the assumption that the non-vanishing components of stress are functions of x_1, x_2, x_3 gives very accurate results for long beams having a thickness to depth ratio of up to $b/h = 1$. For this ratio the two-dimensional plane stress theory, based on the assumption that the non-vanishing components of stress are only functions of x_1 and x_3, gives a shearing component of stress $\tilde{\tau}_{13}$ which at $x_2 = 0, x_3 = 0$ is 11.2% larger than that computed on the basis of the exact theory of elasticity. Thus, it is on the safe side.

Example 3 Establish the stress distribution in a large very thin plate of constant thickness $2b$ with a small traction-free circular hole of radius R. The plate is subjected to a uniform axial tension p(kN/m^2) as shown in Fig. a. The effect of body forces is negligible.

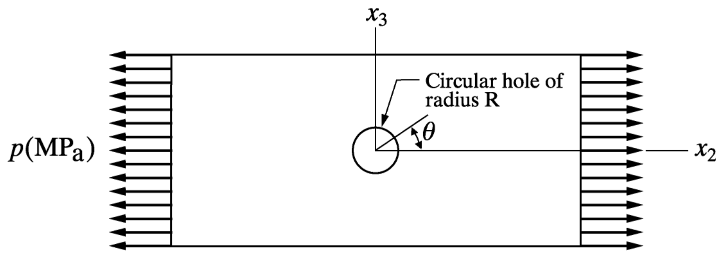


Figure a Thin large plate with a small circular hole.

Solution Inasmuch as the plate is very thin we assume that the distribution of stress in it can be approximated to that obtained on the basis of the approximate two-dimensional plane stress theory. Thus, in order to determine the approximate stress distribution in the plate, we first establish the Airy stress function for an auxiliary plate in a state of plane strain having the same geometry as the actual plate and made from the same material from which the actual plate is made. The plate is subjected on its lateral surface to the same traction as the real plate and on its end surface $x_1 = \pm b$ to the distribution of traction required in order to maintain in it a state of plane strain. Since the hole is circular, it is convenient to use cylindrical coordinates. If there was no hole in the auxiliary plate, the stress distribution would have been equal to

$$\tau_{22} = p \qquad \tau_{11} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0 \tag{a}$$

By inspection we can see that this stress distribution can be obtained from the following Airy stress function:

† See Timoshenko, I.S. and Goodier J.N., *Theory of Elasticity,* McGraw-Hill, New York, 3rd edition, 1970, p. 364. The exact solution is valid for beams whose cross sectional dimensions are small compared to their length. Their cross sections could have any b/h ratio.

$$\phi(x_2, x_3) = \frac{p}{2}x_3^2 \tag{b}$$

In cylindrical coordinates the stress function (b) may be written as

$$\phi(r, \theta) = \frac{p}{2}r^2\sin^2\theta = \frac{pr^2}{4}(1 - \cos 2\theta) \tag{c}$$

Substituting the above relation into relations (7.32), we obtain

$$\begin{aligned} \tau_{rr} &= \frac{p}{2}(1 + \cos 2\theta) \\ \tau_{\theta\theta} &= \frac{p}{2}(1 - \cos 2\theta) \\ \tau_{r\theta} &= -\frac{p}{2}(\sin 2\theta) \end{aligned} \tag{d}$$

When a small hole of radius R exists in a very large plate subjected to tension, the stress distribution far away from the hole will be the same as that in a plate without the hole. Thus,

$$\text{at } r \to \infty \quad \begin{cases} \tau_{rr} = \dfrac{p}{2}(1 + \cos 2\theta) \\ \tau_{\theta\theta} = \dfrac{p}{2}(1 - \cos 2\theta) \\ \tau_{r\theta} = -\dfrac{p}{2}\sin 2\theta \end{cases} \tag{e}$$

The boundary conditions of the plate at the hole are

$$\text{at } r = R \qquad \tau_{rr} = \tau_{r\theta} = 0 \tag{f}$$

The Airy stress function $\phi_{\text{aux}}(r, \theta)$ for this problem could have the following form:

$$\phi_{\text{aux}}(r, \theta) = f_1(r) + f_2(r)\cos 2\theta \tag{g}$$

Substituting relation (g) into the biharmonic equation (7.36b), we obtain

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left[\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \left(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} - \frac{4f_2}{r^2}\right)\cos 2\theta\right] = 0 \tag{h}$$

or

$$\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r}\right) + \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} - \frac{4f_2}{r^2}\right)\right. \\ &\qquad\qquad \left. + \left(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} - \frac{4f_2}{r^2}\right)\left(-\frac{4}{r^2}\right)\right]\cos 2\theta = 0 \end{aligned} \tag{i}$$

Since the above relation must be satisfied for all values of θ, we have

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2f_1}{dr^2}+\frac{1}{r}\frac{df_1}{dr}\right)=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]f_1=0 \tag{j}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right)\left(\frac{d^2f_2}{dr^2}+\frac{1}{r}\frac{df_2}{dr}-\frac{4f_2}{r^2}\right)=0 \tag{k}$$

After four successive integrations of equation (j), we obtain

$$f_1 = B_1 \ln r + B_2 r^2 \ln r + B_3 r^2 + B_4 \tag{l}$$

where $B_i (i = 1, 2, 3, 4)$ are constants. Equation (k) is a linear ordinary differential equation with variable coefficients and may be reduced to a linear differential equation with constant coefficients by introducing the variable ξ defined by

$$\xi = \ln r \tag{m}$$

From this definition, we have

$$\frac{d\xi}{dr}=\frac{1}{r}$$

$$\frac{dr}{d\xi}=r$$

$$\frac{df_2}{dr}=\frac{df_2}{d\xi}\frac{d\xi}{dr}=\frac{1}{r}\frac{df_2}{d\xi}$$

$$\frac{d}{dr}\left(\frac{df_2}{dr}\right)=\frac{d}{dr}\left(\frac{1}{r}\frac{df_2}{d\xi}\right)=\frac{1}{r}\frac{d}{d\xi}\left(\frac{1}{r}\frac{df_2}{d\xi}\right) \tag{n}$$

$$\frac{d^2f_2}{dr^2}=\frac{1}{r^2}\frac{d^2f_2}{d\xi^2}-\frac{1}{r^3}\frac{dr}{d\xi}\frac{df_2}{d\xi}=\frac{1}{r^2}\left(\frac{d^2f_2}{d\xi^2}-\frac{df_2}{d\xi}\right)$$

$$\frac{d^3f_2}{dr^3}=\frac{1}{r^3}\left(\frac{d^3f_2}{d\xi^3}-3\frac{d^2f_2}{d\xi^2}+2\frac{df_2}{d\xi}\right)$$

$$\frac{d^4f_2}{dr^4}=\frac{1}{r^4}\left(\frac{d^4f_2}{d\xi^4}-6\frac{d^3f_2}{d\xi^3}+11\frac{d^2f_2}{d\xi^2}-6\frac{df_2}{d\xi}\right)$$

Substituting the above relations into equation (k) and simplifying, we get

$$\frac{d^4f_2}{d\xi^4}-4\frac{d^3f_2}{d\xi^3}-4\frac{d^2f_2}{d\xi^2}+16\frac{df_2}{d\xi}=0 \tag{o}$$

This is a linear differential equation with constant coefficients and may be written in the form

$$D\,(D^3 - 4D^2 - 4D + 16)f_2 = 0 \tag{p}$$

The roots of its auxiliary equation are

$$D = 0 \qquad D = \pm 2 \qquad D = 4 \tag{q}$$

Thus, the solution of the differential equation (o) is

$$f_2 = C_1 e^{-2\xi} + C_2 e^{2\xi} + C_3 e^{4\xi} = \frac{C_1}{r^2} + C_2 r^2 + C_3 r^4 + C_4 \tag{r}$$

The right side of the above relation has been obtained by recalling that on the basis of the definition of the natural logarithm of a number from relation (m) we have $e^{\xi} = r$. Substituting relation (l) and (r) into relation (g), we obtain

$$\phi_{\text{aux}}(r,\theta) = B_1 \ln r + B_2 r^2 \ln r + B_3 r^2 + B_4 + \left(\frac{C_1}{r^2} + C_2 r^2 + C_3 r^4 + C_4 \right) \cos 2\theta \tag{s}$$

From the function $\phi_{\text{aux}}(r, \theta)$ the function $X(x_1, r, \theta)$ may be established using relation (7.57) and (7.51). The components of stress for the plane stress problem may be established by substituting the function $X(x_1, r, \theta)$ in relations (7.42a). However, since we have assumed that the plate is very thin, the components of stress may be obtained using the function $X(r, \theta)$ defined by relation (7.58). Thus, substituting the Airy stress function $\phi_{\text{aux}}(r, \theta)$ given by relation (s) into relation (7.58) and the resulting expression into relations (7.42a), we have

$$\begin{aligned}
\tau_{rr} &= B_2(1 + 2\ln r) + 2B_3 + \frac{B_1}{r^2} - \left(2C_2 + \frac{6C_1}{r^4} + \frac{4C_4}{r^2} \right) \cos 2\theta \\
\tau_{\theta\theta} &= B_2(3 + 2\ln r) + 2B_3 - \frac{B_1}{r^2} + \left(2C_2 + 12C_3 r^2 + \frac{6C_1}{r^4} \right) \cos 2\theta \\
\tau_{r\theta} &= \left(2C_2 + 6C_3 r^2 - \frac{6C_1}{r^4} - \frac{2C_4}{r^2} \right) \sin 2\theta \\
\tau_{33} &= \tau_{3r} = \tau_{3\theta} = 0
\end{aligned} \tag{t}$$

Notice, however, that the components of stress τ_{rr}, $\tau_{\theta\theta}$, $\tau_{r\theta}$ must remain finite as $r \to \infty$. Thus,

$$B_2 = C_3 = 0 \tag{u}$$

The remaining constants may be evaluated from the given boundary conditions (e) and (f). That is, from the requirement that as $r \to \infty$, the components of stress given by relations (t) must be equal to the corresponding components of stress given by relations (e). Thus, we obtain

$$B_3 = \frac{p}{4} \qquad C_2 = -\frac{p}{4} \tag{v}$$

Moreover, from the requirement that at $r = R$ the components of stress τ_{rr} and $\tau_{r\theta}$ must vanish, we get

$$2B_3 + \frac{B_1}{R^2} = 0$$

$$2C_2 + \frac{6C_1}{R^4} + \frac{4C_4}{R^2} = 0$$

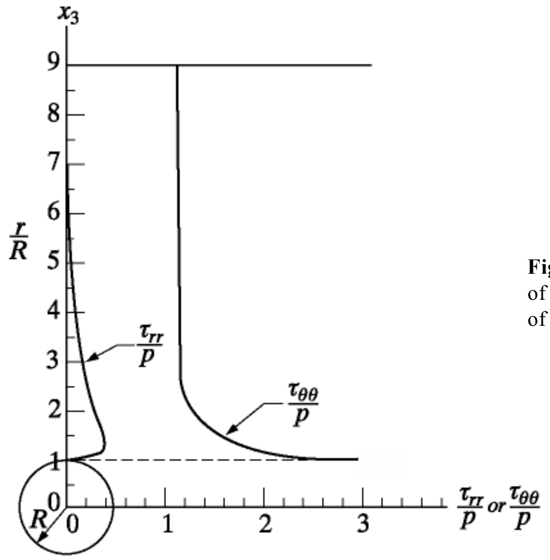


Figure b Distribution of the components of stress on the plane normal to the axis of the plate at $x_2=0$.

$$2C_2 - \frac{6C_1}{R^4} - \frac{2C_4}{R^2} = 0$$

Substituting relations (v) into the above, we obtain

$$B_1 = -\frac{R^2 p}{2} \qquad C_4 = -\frac{pR^2}{2} \qquad C_1 = -\frac{pR^4}{4} \tag{w}$$

Substituting the values of the constants (u), (v) and (w) into relations (s) and (t), we have

$$\phi(r,\theta) = -\frac{R^2 p}{2}\ln r + \frac{pr^2}{4} + B_4 + \left[-\frac{pR^4}{4r^2} - \frac{pr^2}{4} + \frac{pR^2}{2}\right]\cos 2\theta$$

$$\begin{aligned}
\tau_{rr} &= \frac{p}{2}\left(1 - \frac{R^2}{r^2}\right) + \frac{p}{2}\left(1 + \frac{3R^4}{r^4} - \frac{4R^2}{r^2}\right)\cos 2\theta \\
\tau_{\theta\theta} &= \frac{p}{2}\left(1 + \frac{R^2}{r^2}\right) - \frac{p}{2}\left(1 + \frac{3R^4}{r^4}\right)\cos 2\theta \\
\tau_{r\theta} &= -\frac{p}{2}\left(1 - \frac{3R^4}{r^4} + \frac{2R^2}{r^2}\right)\sin 2\theta
\end{aligned} \tag{x}$$

The maximum values of the components of stress τ_{rr} and $\tau_{\theta\theta}$ occur at $\theta = \pm\pi/2$ and are equal to

$$\tau_{rr} = \frac{3R^2 p}{2r^2}\left(1 - \frac{R^2}{r^2}\right) \qquad \tau_{\theta\theta} = \frac{p}{2}\left(2 + \frac{R^2}{r^2} + \frac{3R^4}{r^4}\right) \tag{y}$$

These results are plotted in Fig b. Notice that as r increases $\tau_{\theta\theta}$ decreases rapidly; it approaches asymptotically the value $\tau_{\theta\theta} = p$. Thus, stress concentration occurs only very close to the boundary of the hole. Moreover, τ_{rr} decreases to zero as r approaches R. The maximum value of $\tau_{\theta\theta}$ occurs at $\theta = \pm\pi/2$ $r = R$. From relations (y) we get

$$\tau_{rr} = 0 \qquad \tau_{r\theta} = 0 \qquad (\tau_{\theta\theta})_{\max} = 3p \tag{z}$$

At the boundary of the hole ($r=R$) relations (x) give

$$\tau_{rr} = 0 \qquad \tau_{\theta\theta} = p(1 - 2\cos 2\theta) \qquad \tau_{r\theta} = 0$$

The stress distribution (y) should not be used to determine the maximum value of the component of stress $\tau_{\theta\theta} = \tau_{22}$ acting on the cross section at $x_2 = 0$ ($\theta = \pm\pi/2$) of flat plates of finite width with a hole because it can differ considerably from the actual maximum value of $\tau_{\theta\theta} = \tau_{22}$ except when the ratio of the width D of the plate to the radius R of the hole is large ($D/R > 25$).

7.8 Prismatic Members in a State of Axisymmetric Plane Strain or Plane Stress

In this section we consider a class of problems involving prismatic bodies in a state of plane strain or plane stress which have the following attributes:

1. The axial and the tangential component of the specific body force vanishes, while the radial component is a function only of the radial coordinate and is obtained from a potential $V(r)$ as follows:

$$B_1 = 0 \qquad B_\theta = 0 \qquad B_r = -\frac{dV(r)}{dr} \tag{7.59}$$

2. The distribution of the components of traction on the lateral surfaces of these bodies is such that the components of stress and displacement are not functions of the coordinate θ. We say that these bodies are in a *state of axisymmetric plane strain or plane stress*. Notice that a body does have to be symmetric with respect to the x_1 axis in order to be in a state of axisymmetric plane strain or plane stress. For example, consider the prismatic body of square cross section subjected to the distribution of traction $\left(\overset{3}{T}_1 = \overset{-3}{T}_1 = \overset{2}{T}_2 = \overset{-2}{T}_2 = -p\right)$ shown in Fig. 7.6. It can shown that the stress distribution in this body is

$$\begin{aligned} &\tau_{22} = \tau_{33} = -p \\ &\tau_{12} = \tau_{11} = \tau_{13} = \tau_{23} = 0 \end{aligned} \tag{7.60}$$

The transformation matrix which transforms the cartesian coordinates of a point to cylindrical is given by relations (2.78). The cylindrical components of stress are obtained by substituting relation (2.78) and (7.60) into the transformation relation (2.4a). That is,

$$\tau_{rr} = \tau_{\theta\theta} = -p \qquad \tau_{r\theta} = \tau_{1\theta} = \tau_{1r} = \tau_{11} = 0 \tag{7.61}$$

Consequently, the components of stress of the prismatic body of Fig. 7.6 are not functions of the coordinate θ.

Referring to relations (7.32) we see that inasmuch as the components of stress are not functions of the coordinate θ the Airy stress function $\phi(r, \theta)$ must have the following form:

$$\phi(r, \theta) = C_o\theta + \phi_r(r) \tag{7.62}$$

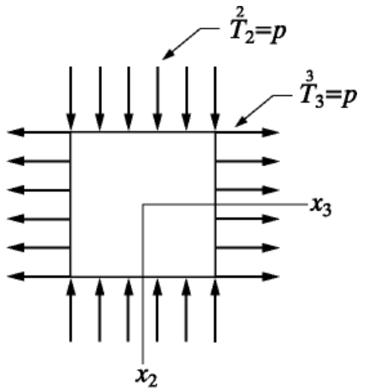


Figure 7.6 Non-symmetric with respect to the x_1 axis prismatic body whose particles are subjected to components of stress which are symmetric with respect to x_1 axis.

7.8.1 Axisymmetric Plane Strain

Referring to relations (7.2), (7.17), (7.32), (7.62) and (2.83), for a prismatic body in a state of axisymmetric plane strain, we obtain

$$\tau_{11} = \nu(\tau_{rr} + \tau_{\theta\theta}) \tag{7.63a}$$

$$\tau_{rr} = \frac{1}{r}\frac{d\phi_r}{dr} + V \tag{7.63b}$$

$$\tau_{\theta\theta} = \frac{d^2\phi_r}{dr^2} + V \tag{7.63c}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = \frac{C_o}{r^2} \tag{7.63d}$$

$$\tau_{1\theta} = \tau_{1r} = 0 \tag{7.63e}$$

$$\nabla_1^4\phi = -\frac{(1-2\nu)}{(1-\nu)}\nabla_1^2 V \tag{7.64}$$

and

$$e_{rr} = \frac{d\hat{u}_r}{dr} \tag{7.65a}$$

$$e_{\theta\theta} = \frac{\hat{u}_r}{r} \tag{7.65b}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial\hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r}\right) \tag{7.65c}$$

$$e_{11} = e_{1r} = e_{1\theta} = 0 \tag{7.65d}$$

$$\hat{u}_1 = 0 \qquad \hat{u}_r = \hat{u}_r(r) \qquad \hat{u}_\theta = \hat{u}_\theta(r) \tag{7.65e}$$

Referring to relations (7.30) and (7.34), relation (7.64) may be rewritten as

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi_r}{dr^2} + \frac{1}{r}\frac{d\phi_r}{dr}\right) = -\frac{1-2\nu}{1-\nu}\left(\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr}\right)$$

or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\left[\frac{1}{r}\frac{d}{dr}\left(\frac{rd\phi_r}{dr}\right)\right] = -\frac{1-2\nu}{1-\nu}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)\right] \tag{7.66}$$

Multiplying both sides of relation (7.66) by r and integrating, we get

$$\left(r\frac{d}{dr}\right)\left[\frac{d}{dr}\left(r\frac{d\phi_r}{dr}\right)\right] = -\frac{1-2\nu}{1-\nu}\left(r\frac{dV}{dr}\right) + C_1$$

Dividing both side of the above relation by r and integrating, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi_r}{dr}\right) = -\frac{1-2\nu}{1-\nu}V + C_1\ln r + C_2 \tag{7.67}$$

Multiplying both sides of the above relation by r and integrating and dividing by r, we have

$$\frac{d\phi_r}{dr} = -\left(\frac{1-2\nu}{1-\nu}\right)\frac{1}{r}\int_{r'=r_o}^{r'=r} r'V(r')\,dr' + \frac{C_1}{2}r\left(\ln r - \frac{1}{2}\right) + \frac{C_2 r}{2} + \frac{C_3}{r} \tag{7.68}$$

where r_o is some fixed arbitrary point on the cross section of the body. Integrating relation (7.68), we obtain

$$\phi_r = -\frac{(1-2\nu)}{1-\nu}\int_{r'=r_o}^{r'=r}\left(\frac{1}{r'}\int_{r''=r_o}^{r''=r'} r''V(r'')\,dr''\right)dr' + \frac{C_1}{4}(r^2\ln r - r^2) + \frac{C_2 r^2}{4} + C_3\ln r \tag{7.69}$$

The constant of integration is omitted inasmuch as it does not contribute to the components of stress which are derivatives of the function $\phi_r(r)$. Using relation (7.68) relation (7.67) gives

$$\frac{d^2\phi_r}{dr^2} = \left(\frac{1-2\nu}{1-\nu}\right)\frac{1}{r^2}\int_{r'=r_o}^{r'=r} r'V(r')\,dr' - \left(\frac{1-2\nu}{1-\nu}\right)V(r) + \frac{C_1}{2}\left(\ln r + \frac{1}{2}\right) + \frac{C_2}{2} - \frac{C_3}{r^2} \tag{7.70}$$

Substituting relations (7.68) and (7.70) into (7.63b) to (7.63e),we get the following formulas for the components of stress in prismatic bodies in a state of axisymmetric plain strain:

$$\tau_{rr} = \frac{1}{r}\frac{d\phi_r}{dr} + V = -\left(\frac{1-2\nu}{1-\nu}\right)\frac{1}{r^2}\int_{r'=r_o}^{r'=r} r'V(r')\,dr' + V(r) + \frac{C_1}{2}\left(\ln r - \frac{1}{2}\right) + \frac{C_2}{2} + \frac{C_3}{r^2}$$

$$\tau_{\theta\theta} = \frac{d^2\phi_r}{dr^2} + V = \left(\frac{1-2\nu}{1-\nu}\right)\frac{1}{r^2}\int_{r'=r_o}^{r'=r} r'V(r')\,dr' + \left(\frac{\nu}{1-\nu}\right)V(r) + \frac{C_1}{2}\left(\ln r + \frac{1}{2}\right) + \frac{C_2}{2} - \frac{C_3}{r^2}$$

$$\tau_{r\theta} = \frac{C_o}{r^2} \qquad \tau_{1r} = \tau_{1\theta} = 0 \tag{7.71}$$

Moreover, substituting relations (7.71) into (7.33), we obtain

$$\tau_{11} = \nu(\tau_{rr} + \tau_{\theta\theta}) = \nu\left(\frac{V}{1-\nu} + C_1 \ln r + C_2\right) \tag{7.72}$$

In what follows we compute the components of displacement of prismatic thin plates in a state of axisymmetric plane strain. The stress–strain relations for plane strain (3.51) are

$$e_{rr} = \frac{1+\nu}{E}[(1-\nu)\tau_{rr} - \nu\tau_{\theta\theta}] \tag{7.73a}$$

$$e_{\theta\theta} = \frac{1+\nu}{E}[(1-\nu)\tau_{\theta\theta} - \nu\tau_{rr}] \tag{7.73b}$$

$$e_{r\theta} = \frac{\tau_{r\theta}}{2G} \qquad e_{11} = e_{1r} = e_{1\theta} = 0 \tag{7.73c}$$

Substituting relation (7.73b) into (7.65b), we get

$$\hat{u}_r = re_{\theta\theta} = \frac{(1+\nu)r}{E}[\tau_{\theta\theta} - \nu(\tau_{rr} + \tau_{\theta\theta})] \tag{7.74}$$

Substituting the first two of relations (7.71) into (7.74), we obtain

$$\hat{u}_r = \frac{(1+\nu)r}{E}\left[\left(\frac{1-2\nu}{1-\nu}\right)\frac{1}{r^2}\int_{r'=r_o}^{r'=r} r'V(r')dr' + \frac{C_1}{2}\left[(1-2\nu)\ln r + \frac{1}{2}\right] + (1-2\nu)\frac{C_2}{2} - \frac{C_3}{r^2}\right] \tag{7.75}$$

Substituting the first of relations (7.73c) into (7.65c) and using the third of relations (7.71), we have

$$\frac{d\hat{u}_\theta}{dr} - \frac{\hat{u}_\theta}{r} = r\frac{d}{dr}\left(\frac{\hat{u}_\theta}{r}\right) = \frac{C_o}{Gr^2} \tag{7.76}$$

Integrating relation (7.76), we get

$$\hat{u}_\theta = Cr - \frac{C_o}{2Gr} \tag{7.77}$$

The first term on the right side of the above relation represents rigid-body rotation of the body about the x_1 axis which we eliminate by setting $C = 0$. Thus,

$$\hat{u}_\theta = -\frac{C_o}{2Gr} \tag{7.78}$$

For simply connected (no holes) prismatic bodies the constants C_o, C_1, C_2, C_3 are evaluated by requiring that the components of stress and displacement (7.71), (7.75) and (7.78) satisfy the following conditions:

1. They are finite at r = 0. Referring to relations (7.71), (7.75) and (7.78), we see that in order that this requirement is satisfied we must set

$$C_O = C_1 = C_3 = 0 \tag{7.79}$$

2. They satisfy the specified boundary conditions on their lateral surface.

For multiply connected bodies the constants C_o, C_1, C_2, C_3 are evaluated by requiring that the components of stress and displacement satisfy the following conditions:

1. The given boundary conditions are on their lateral surfaces.
2. The Mitchell compatibility equations (7.27) are on the boundary of the holes. For bodies with one circular hole of radius R on the boundary of the hole we have $x_n = -r$ and $x_s = r\theta$ (see Fig. 7.7). Thus,

$$\left.\frac{\partial}{\partial x_n}\right|_{r=R} = -\frac{\partial}{\partial r} \qquad \left.\frac{\partial}{\partial x_s}\right|_{r=R} = \frac{\partial}{R\partial\theta} \tag{7.80}$$

Moreover, referring to relation (7.34) and using relations (7.68) and (7.69), we have

$$\nabla_1^2\phi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}\right]\phi = C_1 \ln r + \frac{C_2}{2} \tag{7.81}$$

Thus,

$$\begin{aligned} \frac{\partial(\nabla_1^2\phi)}{\partial x_n} &= \frac{\partial(\nabla_1^2\phi)}{\partial r} = \frac{C_1}{r} \\ \frac{\partial(\nabla_1^2\phi)}{\partial x_s} &= -\frac{\partial(\nabla_1^2\phi)}{R\partial\theta} = 0 \end{aligned} \tag{7.82}$$

Substituting relations (7.82) into the Mitchell compatibility equations (7.27), we get

$$\begin{aligned} &\int_0^{2\pi}\frac{\partial(\nabla_1^2\phi)}{\partial r}R\,d\theta = C_1\int_0^{2\pi}d\theta = 2\pi C_1 = 0 \\ -&\int_0^{2\pi}R^2\sin\theta\frac{\partial(\nabla_1^2\phi)}{\partial r}d\theta = -RC_1\int_0^{2\pi}\sin\theta\,d\theta = 0 = \frac{1}{1-\nu}\int_0^{2\pi}\overset{n}{T}_3R\,d\theta \\ &\int_0^{2\pi}R^2\cos\theta\frac{\partial(\nabla_1^2\phi)}{\partial r}d\theta = -RC_1\int_0^{2\pi}\cos\theta\,d\theta = 0 = -\frac{1}{1-\nu}\int_0^{2\pi}\overset{n}{T}_2R\,d\theta \end{aligned} \tag{7.83}$$

The transformation matrix which transforms the cartesian coordinates of a point to cylindrical is given by relation (2.78). Using this relation, from relation (2.80a) we have

$$\begin{aligned} \overset{n}{T}_2 &= \overset{n}{T}_r\cos\theta - \overset{n}{T}_\theta\sin\theta \\ \overset{n}{T}_3 &= \overset{n}{T}_r s\in\theta + \overset{n}{T}_\theta\cos\theta \end{aligned} \tag{7.84}$$

Substituting relations (7.84) into (7.83) and taking into account that for axisymmetric plane strain the components of traction $\overset{n}{T}_r$ and $\overset{n}{T}_\theta$ are not functions of θ we find that the last two of relations (7.83) are automatically satisfied while the first is satisfied if

$$C_1 = 0 \tag{7.85}$$

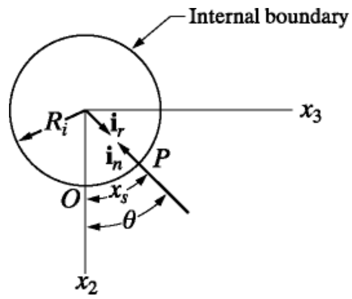


Figure 7.7 Circular internal boundary.

Example 4 Establish the stress and the displacement fields in a hollow cylinder of internal radius R_i and external radius R_e subjected to internal pressure p_i and to external pressure p_e (see Fig. a). Assume that the effect of body forces is negligible. The ends of the cylinder are restrained only against axial movement. That is, the boundary conditions at the ends of the cylinder $x_1 = 0$ and $x_1 = L$ are

$$\begin{aligned}
&\hat{u}_1(0, r, \theta) = \hat{u}_1(L, r, \theta) = 0 \\
&\overset{-1}{T}_r(0, r, \theta) = -\tau_{1r}(0, r, \theta) = 0 \\
&\overset{1}{T}_r(L, r, \theta) = \tau_{1r}(L, r, \theta) = 0 \qquad \text{(a)} \\
&\overset{-1}{T}_\theta(0, r, \theta) = -\tau_{1\theta}(0, r, \theta) = 0 \\
&\overset{1}{T}_\theta(L, r, \theta) = \tau_{1\theta}(L, r, \theta) = 0
\end{aligned}$$

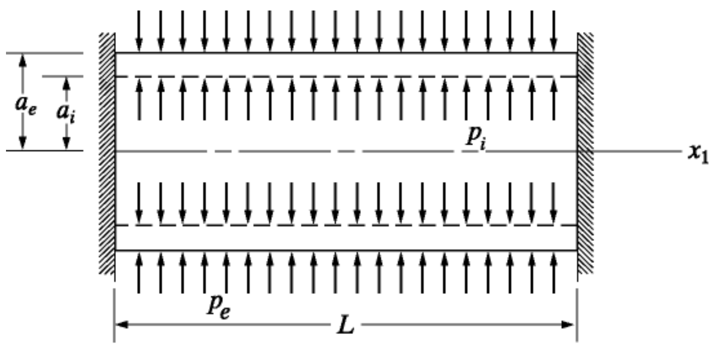


Figure a Geometry and loading of the cylinder.

Solution The cylinder under consideration is in a state of axisymmetric plane strain. That is, its components of stress and displacement are not functions of θ and are given by relations (7.71), (7.72), (7.75) and (7.78). Thus, taking into account relation (7.85), we have

$$\tau_{11} = \nu C_2 \qquad \tau_{rr} = \frac{C_2}{2} + \frac{C_3}{r^2} \qquad \tau_{\theta\theta} = \frac{C_2}{2} - \frac{C_3}{r^2} \qquad \tau_{r\theta} = \frac{C_o}{r^2} \qquad \tau_{1r} = \tau_{1\theta} = 0$$

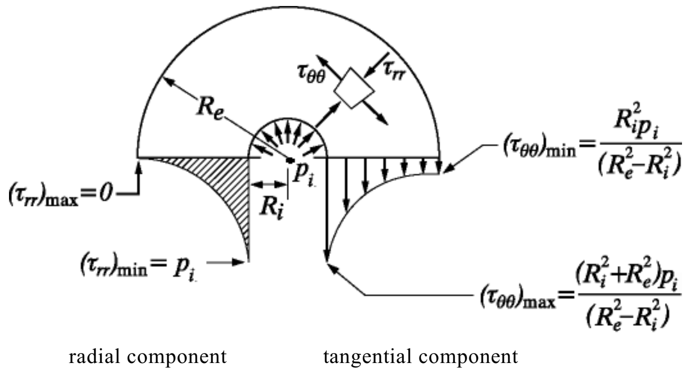


Figure b Variation of the components of stress along the radial direction when $p_e = 0$ and $p_i \neq 0$.

$$\hat{u}_r = [(1 - 2\nu)\frac{C_2}{2} - \frac{C_3}{r^2}]\frac{(1+\nu)}{E} \qquad \hat{u}_\theta = -\frac{C_o}{2Gr} \qquad \hat{u}_1 = 0 \tag{b}$$

This solution satisfies the boundary conditions (a) without imposing any restrictions on the values of the constants C_o, C_2 and C_3. These constants are evaluated by requiring that the stress distribution (b) satisfies the following boundary conditions on the lateral surfaces $\mathbf{i}_n = \pm\mathbf{r}/r$ ($\lambda_{rr} = \pm 1, \lambda_{r\theta} = \lambda_{r1} = 0$) of the cylinder

$$\begin{aligned}
\overset{-r}{T}_1(x_1, R_i, \theta) &= -\tau_{r1}(x_1, R_i, \theta) = 0 \\
\overset{r}{T}_1(x_1, R_e, \theta) &= \tau_{r1}(x_1, R_e, \theta) = 0 \\
\overset{-r}{T}_\theta(x_1, R_i, \theta) &= -\tau_{1\theta}(x_1, R_i, \theta) = 0 \\
\overset{r}{T}_\theta(x_1, R_e, \theta) &= \tau_{1\theta}(x_1, R_e, \theta) = 0 \\
\overset{-r}{T}_r(x_1, R_i, \theta) &= -\tau_{rr}(x_1, R_i, \theta) = p_i \\
\overset{r}{T}_r(x_1, R_e, \theta) &= \tau_{rr}(x_1, R_e, \theta) = -p_e
\end{aligned} \tag{c}$$

The first four of relations (c) are satisfied by the components of stress (b) without imposing any restrictions on the constants C_o, C_2 and C_3. Substituting the second of relations (b) into the last two of relation (c), we have

$$\begin{aligned}
\overset{-r}{T}_r(x_1, R_i, \theta) &= p_i = -\tau_{rr}(R_i) = -\frac{C_2}{2} - \frac{C_3}{R_i^2} \\
\overset{r}{T}_r(x_1, R_e, \theta) &= -p_e = \tau_{rr}(R_e) = \frac{C_2}{2} + \frac{C_3}{R_e^2}
\end{aligned} \tag{d}$$

Solving relations (d), we obtain

$$C_2 = \frac{2(-p_e R_e^2 + p_i R_i^2)}{R_e^2 - R_i^2} \qquad C_3 = \frac{(p_e - p_i) R_e^2 R_i^2}{R_e^2 - R_i^2} \tag{e}$$

Substituting relations (e) into (b), we get

$$\tau_{11} = \frac{2\nu(-p_e R_e^2 + p_i R_i^2)}{R_e^2 - R_i^2} \qquad \tau_{rr} = \frac{p_i R_i^2 (r^2 - R_e^2) - p_e R_e^2 (r^2 - R_i^2)}{r^2 (R_e^2 - R_i^2)}$$

$$\tau_{\theta\theta} = \frac{p_i R_i^2 (r^2 + R_e^2) - p_e R_e^2 (r^2 + R_i^2)}{r^2 (R_e^2 - R_i^2)} \qquad \tau_{1r} = \tau_{1\theta} = \tau_{r\theta} = 0 \tag{f}$$

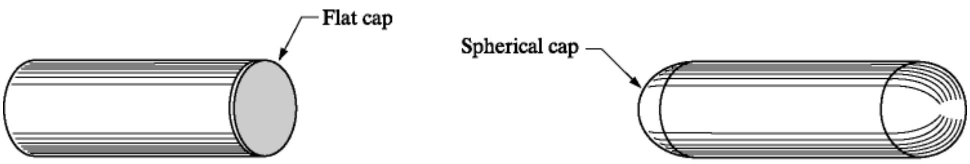
$$\hat{u}_r = \frac{(1 + \nu)}{Er(R_e^2 - R_i^2)} \left[[(1 - 2\nu) r^2 + R_e^2] R_i^2 p_i - [(1 - 2\nu) r^2 + R_i^2] R_e^2 p_e \right] \qquad \hat{u}_1 = \hat{u}_\theta = 0$$

Notice that on the basis of the Saint Venant principle, the solution is valid, at points away from the ends of a fixed at both ends cylinder ($\hat{u}_r = \hat{u}_\theta = \hat{u}_1 = 0$ at $x_1 = 0$ and $x_1 = L$) provided that the dimensions of the cross section of the cylinder are small as compared to its length. In this case, the component of traction $\overset{\pm 1}{T}_r$ acting on the end surfaces of the cylinder will not vanish. However, due to the symmetry of the geometry of the cylinder, of its loading and of its constraints, the resultant of this component of traction will vanish. That is, the distribution of the component of traction $\overset{\pm 1}{T}_r$ at $x_1 = 0$ and $x_1 = L$ required to render the displacement component u_r equal to zero is statically equivalent to zero component of traction ($\overset{\pm 1}{T}_r = 0$).

From the third of relations (f) we see that when a cylinder is subjected only to internal or only to external pressure, the maximum circumferential stress occurs at the particles of its innermost surface $r = R_i$. In Fig. b we plot the distribution of the circumferential component $\tau_{\theta\theta}$ of stress acting on a plane containing the axis of the cylinder, when $p_e = 0$ and $p_i \neq 0$. Moreover, we plot the variation of the radial component τ_{rr} of stress along the radial direction when $p_e = 0$ and $p_i \neq 0$.

Comments

Thick-walled cylinders are used extensively as pressure vessels, pipes, cannon tubes, etc. Such cylinders are usually subjected to one or more of the following loads:



(a) Cylinder with flat caps (b) Cylinder with semispherical caps

Figure c Thick-walled cylinders with end caps.

1. Uniform internal pressure
2. Uniform external pressure
3. Axial centroidal forces
4. Change of temperature

If the change of temperature of a cylinder is either uniform or only a function of the radial coordinate, its displacement and stress fields are symmetric with respect to its axis.

Usually cylindrical vessels are provided with end caps (see Fig. c) or they are fixed on rigid supports. In these cases in the vicinity of their end caps or of their supports the components of displacement and stress are functions of the axial coordinate x_1. If the external radius of a cylinder is small compared to its length, the dependence on its axial coordinate of the components of displacement and stress of particles sufficiently removed from its ends is negligible. The components of displacement and stress of such particles may be approximated by the formulas obtained in this example.

7.8.2 Axisymmetric Plane Stress

In this section, we consider a very thin prismatic plate in a state of axisymmetric plane stress. We choose a system of axes with the origin at the centroid of the middle cross section of this plate and the x_1 axis along its axis. Thus, its end surfaces are $x_1 = \pm b$. In order to establish the stress function $X(x_1, x_2, x_3)$ for plane stress we consider an auxiliary plate having the same geometry as the actual plate, being in a state of plane strain made from an isotropic, linearly elastic material whose Poisson's ratio is obtained from that of the actual plate on the basis of relation (7.52). As shown in Section 7.8, the Airy stress function $\phi_{\text{aux}}(r, \theta)$ for the auxiliary plate is equal to the function $\psi(r, \theta)$ for the real plate. Referring to relation (7.62), we have

$$\psi(r, \theta) = \phi_{\text{aux}}(r, \theta) = C_o \theta + \phi_r(r) \tag{7.86}$$

where the function ϕ_{aux} satisfies relations (7.64) with ν_{aux} replacing ν. Substituting relation (7.86) into (7.34), using relations (7.68) and (7.70) replacing ν by ν_{aux} in the resulting relations and using relation (7.52), we get

$$\nabla_1^2 \psi = \nabla_1^2 \phi_{\text{aux}} = \frac{\partial^2 \phi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_r}{\partial r} = -(1 - \nu)V(r) + C_1 \ln r + C_2 \tag{7.87}$$

Substituting relation (7.87) and (7.86) into (7.51), and noting that $x_2^2 + x_3^2 = r^2$, we have

$$X(x_1, r, \theta) = C_o \theta + \phi_r + \nu N(x_1)[C_1 \ln r + C_2] + \nu(1 + \nu)N(x_1)V(r) + \frac{\nu(1 + \nu)N(x_1)(3r^2 + \nu b^2)}{6(1 - \nu)} \nabla_1^2 V \tag{7.88}$$

where

$$N(x_1) = \frac{b^2 - 3x_1^2}{6(1 + \nu)} \tag{7.89}$$

Substituting relation (7.88) into (7.42a), replacing ν by ν_{aux} and eliminating the derivatives of ϕ_r from the resulting relations using relations (7.68) to (7.70) and eliminating ν_{aux} using relation (7.52), we obtain

$$\tau_{rr} = \frac{1}{r}\frac{\partial X}{\partial r} + \frac{1}{r^2}\frac{\partial^2 X}{\partial \theta^2} + V = \frac{C_1}{2}\left[\ln r - \frac{1}{2} + \frac{2\nu N(x_1)}{r^2}\right] + \frac{C_2}{2} + \frac{C_3}{r^2} + V$$
$$- \frac{1-\nu}{r^2}\int_{r'=0}^{r'=r} r' V(r')\,dr' + \frac{\nu(1+\nu)\ N(x_1)}{r}\left(\frac{dV}{dr}\right)$$
$$+ \frac{\nu\,(1+\nu)\ N(x_1)\nabla_1^2 V}{(1-\nu)} + \left[\frac{\nu(1+\nu)\ N(x_1)(3r^2+\nu b^2)}{6(1-\nu)r}\right]\nabla_1^2\left(\frac{dV}{dr}\right)$$

$$\tau_{\theta\theta} = \frac{\partial^2 X}{\partial r^2} + V = \frac{C_1}{2}\left(\ln r + \frac{1}{2} + \nu N(x_1)\ln r\right) + \frac{C_2}{2}\left[1 + 2\nu N(x_1)\right] - \frac{C_3}{r^2} + \nu V$$
$$+ \frac{1-\nu}{r^2}\int_{r'=0}^{r'=r} r' V(r')\,dr' + \nu(1+\nu)N(x_1)\left(\frac{d^2V}{d^2r}\right)$$
$$+ \frac{\nu(1+\nu)N(x_1)}{(1-\nu)}\nabla_1^2 V + \frac{\nu(1+\nu)N(x_1)(3r^2+\nu b^2)}{6(1-\nu)}\nabla_1^2\left(\frac{d^2V}{dr^2}\right)$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial X}{\partial \theta} - \frac{1}{r}\frac{\partial^2 X}{\partial r\partial\theta} = \frac{C_o}{r^2} \tag{7.90}$$

$$\tau_{11} = \tau_{1r} = \tau_{1\theta} = 0$$

For thin prismatic simply connected plates the constants C_1, C_2 and C_3 are evaluated by requiring that when the components of stress (7.90) are substituted into the traction–stress relations (2.73), they give components of traction which on the lateral surface of the plate are equal to the specified components of traction.

For very thin prismatic multiply connected plates with one hole in order that the function ϕ_{aux} satisfies the Mitchell compatibility relations (7.27) on the boundary of the hole, as shown in Section 7.8.1, [see relation (7.85)] we have

$$C_1 = 0 \tag{7.91}$$

The constants C_2 and C_3 are evaluated by requiring that when the components of stress (7.90) with $C_1 = 0$ are substituted into the traction–stress relations (2.73), give the specified components of traction on the lateral surfaces of the body.

In what follows we compute the components of displacement of thin prismatic bodies using the axisymmetric plane stress theory. Referring to relations (3.52) the stress strain relations for plane stress are

$$e_{11} = -\frac{\nu}{E}\left(\tau_{rr} + \tau_{\theta\theta}\right) \qquad e_{rr} = \frac{1}{E}\left(\tau_{rr} - \nu\tau_{\theta\theta}\right) \qquad e_{\theta\theta} = \frac{1}{E}\left(\tau_{\theta\theta} - \nu\tau_{rr}\right)$$
$$e_{r\theta} = \frac{\tau_{r\theta}}{2G} \qquad e_{1r} = \epsilon_{1\theta} = 0 \tag{7.92}$$

Moreover, referring to relations (2.83), the strain–displacement relations for a state of axisymmetric plane stress are

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} \qquad e_{rr} = \frac{\partial \hat{u}_r}{\partial r} \qquad e_{\theta\theta} = \frac{\hat{u}_r}{r} \tag{7.93a}$$

$$e_{1r} = \frac{1}{2}\left(\frac{\partial \hat{u}_r}{\partial x_1} + \frac{\partial \hat{u}_1}{\partial r}\right) = 0 \qquad e_{1\theta} = \frac{1}{2}\left(\frac{\partial \hat{u}_\theta}{\partial x_1}\right) = 0 \qquad e_{r\theta} = \frac{1}{2}\left(\frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r}\right) \tag{7.93b}$$

Substituting the third of relation (7.92) into the third of relation (7.93a), we obtain

$$\hat{u}_r = re_{\theta\theta} = \frac{r}{E}(\tau_{\theta\theta} - \nu\tau_{rr}) \tag{7.94}$$

The component of displacement $\hat{u}_\theta$ is obtained by substituting the first two of relations (7.92) into relation (7.94) and the last of relations (7.93b) and using the third of relations (7.90). Thus,

$$\frac{\partial \hat{u}_\theta}{\partial r} - \frac{\hat{u}_\theta}{r} = \frac{\tau_{r\theta}}{G} = \frac{C_o}{Gr^2} \tag{7.95}$$

or

$$r\frac{d}{dr}\left(\frac{\hat{u}_\theta}{r}\right) = \frac{C_o}{Gr^2} \tag{7.96}$$

The solution of the equation is

$$\hat{u}_\theta = rC - \frac{C_o}{2Gr} \tag{7.97}$$

The first term in the expression for $\hat{u}_\theta$ represents rigid-body rotation of the body about the x_1 axis which we eliminate by setting $C = 0$. Thus, relation (7.97) gives

$$\hat{u}_\theta = -\frac{C_o}{2Gr} \tag{7.98}$$

7.8.3 Two-Dimensional or Generalized Axisymmetic Plane Stress

As discussed in Section 7.6.1 for very thin plates the magnitude of the components of stress does not differ much from their average in x_1 value. In this case, substituting relation (7.86) into (7.58), we have

$$\tilde{X}(r, \theta) = C_o\theta + \phi_r(r) \tag{7.99}$$

where $\phi_r(r)$is obtained from relation (7.69) by replacing ν by ν_{aux} and using relation (7.52) in the resulting relation. Substituting relation (7.99) into (7.42a), taking into account relations (7.68), and (7.70), replacing ν by ν_{aux} and using relation (7.52) in the resulting relation, we obtain

$$\tau_{rr} = \frac{1}{r}\frac{\partial \phi_r}{\partial r} + V = \frac{C_1}{2}\left(\ln r - \frac{1}{2}\right) + \frac{C_2}{2} + \frac{C_3}{r^2} + V(r) - \frac{(1-\nu)}{r^2}\int_{r'=r_o}^{r'=r} r'V(r')dr'$$

$$\tau_{\theta\theta} = \frac{\partial^2 \phi_r}{\partial r^2} + V = \frac{C_1}{2}\left(\ln r + \frac{1}{2}\right) + \frac{C_2}{2} - \frac{C_3}{r^2} + \nu V(r) + \frac{(1-\nu)}{r^2}\int_{r'=r_o}^{r'=r} r'V(r')dr'$$

$$\tau_{r\theta} = \frac{C_o}{r^2} \qquad \tau_{11} = \tau_{1r} = \tau_{1\theta} = 0 \tag{7.100}$$

Referring to relation (7.98), the component of displacement u_θ is equal to

$$\hat{u}_\theta = -\frac{C_o}{2Gr} \tag{7.101}$$

The components of displacement $\hat{u}_r$ is obtained by substituting relation (7.100) into (7.94). That is,

$$\begin{aligned}\hat{u}_r &= \frac{r}{E}(\tau_{\theta\theta} - \nu\tau_{rr}) \\ &= \frac{r}{E}\left[\frac{(1-\nu^2)}{r^2}\int_{r'=0}^{r'=r} r'V(r')dr' + \frac{C_1}{2}(1-\nu)\ln r + \frac{C_1}{4}(1+\nu) + \frac{C_2}{2}(1-\nu) - \frac{C_3}{r^2}(1+\nu)\right]\end{aligned}$$

(7.102)

For a very thin prismatic body with one hole, referring to relation (7.91), we have

$$C_1 = 0 \tag{7.104}$$

In what follows we present two examples.

Example 5 Establish the stress distribution in a plane curved beam of thin rectangular cross section of thickness $2b$ subjected at its ends to equal and opposite bending moments, about the axis normal to its plane, as shown in Fig. a. Disregard the effect of body forces.

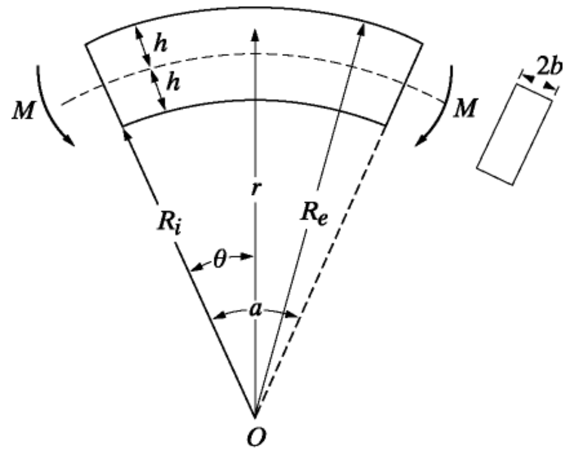


Figure a Geometry and loading of the curved beam.

Solution From physical intuition we can conclude that the components of stress and displacement of the beam are not functions of θ. Moreover, since the thickness $2b$ is small compared to the depth of the cross section of the beam, the components of stress may be approximated by relations (7.100). Substituting these relations into the traction–stress relations (2.73) (with n → r, 2 → r, 3 → θ) and noting that on the lateral surface of the beam

$\lambda_{r\theta} = \lambda_{r1} = 0,\ \lambda_{rr} = \pm 1$ the boundary conditions on this surface are

$$\begin{aligned}
\overset{r}{T}_r(R_e, \theta) &= 0 = \tau_{rr}(R_e, \theta) = \frac{C_1}{2}\left(-\frac{1}{2} + \ln R_e\right) + \frac{C_2}{2} + \frac{C_3}{R_e^2} \\
\overset{-r}{T}_r(R_i, \theta) &= 0 = -\tau_{rr}(R_i, \theta) = -\frac{C_1}{2}\left(-\frac{1}{2} + \ln R_i\right) - \frac{C_2}{2} - \frac{C_3}{R_i^2} \\
\overset{r}{T}_\theta(R_e, \theta) &= 0 = \tau_{r\theta}(R_e, \theta) = \frac{C_o}{R_e^2} \\
\overset{-r}{T}_\theta(R_i, \theta) &= 0 = -\tau_{r\theta}(R_i, \theta) = -\frac{C_o}{R_i^2}
\end{aligned} \tag{a}$$

Solving these relations, we obtain

$$\begin{aligned}
C_o &= 0 \qquad\qquad C_2 = \frac{C_1}{R_e^2 - R_i^2}\left(\frac{R_e^2 - R_i^2}{2} - R_e^2 \ln R_e + R_i^2 \ln R_i\right) \\
C_3 &= \frac{C_1 R_e^2 R_i^2}{2(R_e^2 - R_i^2)} \ln\left(\frac{R_e}{R_i}\right)
\end{aligned} \tag{b}$$

Referring to the first of relations (b) and relation (7.99), we see that

$$\tilde{X}(r, \theta) = \phi_r(r)$$

Substituting the above relation into relations (7.42a), we get

$$\tau_{rr} = \frac{1}{r}\frac{\partial \phi_r}{\partial r} \qquad \tau_{\theta\theta} = \frac{\partial^2 \phi_r}{\partial r^2} \qquad \tau_{r\theta} = 0 \tag{c}$$

On the boundaries $\theta = 0$ and $\theta = \alpha$ the resultant components of force must vanish while the resultant moment about any point must be equal to the applied moment M. Using relations (2.73), (b) and (7.100) and noting that on these boundaries $\lambda_{r\theta} = \lambda_{1\theta} = 0$, $\lambda_{\theta\theta} = \pm 1$, we have

$$\begin{aligned}
2b \int_{R_i}^{R_e} \overset{\theta}{T}_\theta dr &= 2b \int_{R_i}^{R_e} \tau_{\theta\theta} dr = 2b \int_{R_i}^{R_e} \frac{\partial^2 \phi_r}{\partial r^2} dr = 2b \left.\frac{\partial \phi_r}{\partial r}\right|_{R_i}^{R_e} = 2b\, r \tau_{rr}\Big|_{R_i}^{R_e} = 0 \\
2b \int_{R_i}^{R_e} \overset{\theta}{T}_r dr &= 2b \int_{R_i}^{R_e} \tau_{r\theta} dr = 2b\, C_o \int_{R_i}^{R_e} \frac{dr}{r^2} = 0 \\
\frac{M}{2b} &= \int_{R_i}^{R_e} \overset{\theta}{T}_\theta r dr = \int_{R_i}^{R_e} \tau_{\theta\theta} r\, dr = \int_{R_i}^{R_e} r\left[\frac{C_1}{2}\left(\ln r + \frac{1}{2}\right) + \frac{C_2}{2} - \frac{C_3}{r^2}\right] dr \\
&= -C_3 \ln\left(\frac{R_e}{R_i}\right) + \frac{C_2}{4}(R_e^2 - R_i^2) + \frac{C_1}{4}(R_e^2 \ln R_e - R_i^2 \ln R_i) = \frac{C_1}{4}\left[-\frac{2R_i^2 R_e^2}{R_e^2 - R_i^2}\left[\ln\left(\frac{R_e}{R_i}\right)\right]^2 + \frac{R_e^2 - R_i^2}{2}\right]
\end{aligned} \tag{d}$$

From relations (a) and (b) we see that $\tau_{rr}(R_e, \theta)$, $\tau_{rr}(R_i, \theta)$ and C_o are equal to zero. Thus, the first two of relations (d) are satisfied without imposing any restrictions on the value of the constant C_1, while the third reduces to

$$C_1 = \frac{4M(R_e^2 - R_i^2)}{bB} \tag{e}$$

where

$$B = (R_e^2 - R_i^2)^2 - 4R_e^2 R_i^2 \ln^2\left(\frac{R_e}{R_i}\right) \tag{f}$$

The components of stress of a prismatic thin curved beam in a state of plane stress may be obtained by substituting relations (b) and (e) into (7.100). We thus, obtain

$$\tau_{rr} = \frac{2M}{bB}\left[\frac{R_e^2 R_i^2}{r^2}\ln\left(\frac{R_e}{R_i}\right) + R_e^2\ln\left(\frac{r}{R_e}\right) + R_i^2\ln\left(\frac{R_i}{r}\right)\right]$$

$$\tau_{\theta\theta} = \frac{2M}{bB}\left[-\frac{R_e^2 R_i^2}{r^2}\ln\left(\frac{R_e}{R_i}\right) + R_e^2\ln\left(\frac{r}{R_e}\right) + R_i^2\ln\left(\frac{R_i}{r}\right) + R_e^2 - R_i^2\right] \tag{g}$$

$$\tau_{r\theta} = \tau_{11} = \tau_{1\theta} = \tau_{1r} = 0$$

The stress distribution for the case $R_e = 2R_i$ is plotted along the thickness of the beam in Fig. b.

Substituting relations (b) and (e) into (7.101) and (7.102), we have

$$\hat{u}_\theta = 0 \tag{h}$$

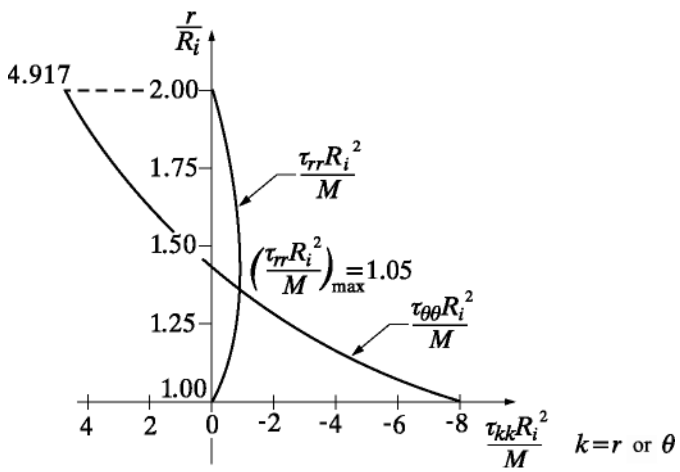


Figure b Distribution of the components of stress along the thickness of a prismatic thin curved beam with $R_e = 2R_i$.

$$\hat{u}_r = \frac{2M\left(R_e^2 - R_i^2\right)}{bB}\left[(1-\nu)\ \ln r + 1 + \frac{(1-\nu)(-R_e^2 \ln R_e \ + \ R_i^2 \ln R_i)}{(R_e^2 - R_i^2)} - \frac{(1+\nu)R_e^2 R_i^2}{r^2(R_2^2 - R_i^2)} \ln\left(\frac{R_e}{R_i}\right)\right] \tag{h}$$

Example 6 Compute the components of stress and displacement in a flat circular disk of constant thickness of radius R rotating about its axis with an angular velocity ω.

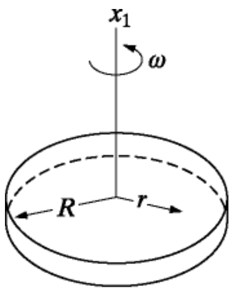


Figure a Geometry of the disk.

Solution On every particle of volume dV of the disk, a specific body force (force per unit volume) is acting in the radial direction equal to the centrifugal force. That is,

$$B_r = \rho r \omega^2 \tag{a}$$

where ρ is the mass density of the material from which the disk is made. Referring to relation (7.41a) from relation (a), we get

$$V(r) = -\frac{\rho r^2 \omega^2}{2} \tag{b}$$

Since the disk is very thin, we assume that it is in a state of two-dimensional plane stress. Moreover, since the geometry of the disk and the specific body force are not functions of the coordinate θ, we have a case of axisymmetric two-dimensional plane stress. Consequently, the components of stress and displacement are given by relations (7.100) to (7.102). Notice that in order that the components of stress τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{r\theta}$ be finite at $r = 0$, we must set in relations (7.100) to (7.102) the following:

$$C_2 = C_1 = C_3 = 0 \tag{c}$$

Substituting relation (b) into (7.100) to (7.102) and taking into account relation (c), we get

$$\begin{aligned} \tau_{rr} &= \frac{C_2}{2} - \frac{\rho r^2 \omega^2}{2} + \frac{(1-\nu)}{r^2}\int_{r'=0}^{r'=r} \frac{\rho (r')^3 \omega^2}{2} dr' = \frac{C_2}{2} - \frac{(3+\nu)\rho r^2 \omega^2}{2} \\ \tau_{\theta\theta} &= \frac{C_2}{2} - \frac{\nu \rho r^2 \omega}{2} - \frac{(1-\nu)}{r^2}\int_{r'=0}^{r'=r} \frac{\rho (r')^3 \omega^2 dr'}{2} = \frac{C_2}{2} - \frac{(1 \ + \ 3\ \nu)\rho r^2 \omega^2}{8} \\ \tau_{11} &= \tau_{1r} = \tau_{1\theta} = \tau_{r\theta} = 0 \end{aligned} \tag{d}$$

$$\hat{u}_\theta = 0$$

$$\hat{u}_r = \frac{r}{E}\left[\frac{(1-\nu^2)}{r^2}\int_{r'=0}^{r'=r} r' V(r')\,dr' + \frac{C_2(1-\nu)}{2}\right] = \frac{r}{8E}\left[-(1-\nu^2)\rho\omega^2 r^2 + 4(1-\nu)C_2\right] \tag{e}$$

The constant C_2 is evaluated by requiring that when the components of stress (d) are substituted into the traction–stress relations (2.73), they give components of traction which are equal to the specified components of traction on the lateral surface ($r = R$) of the disk. Taking into account that on the lateral surface of the disk $\lambda_{r1} = \lambda_{r\theta} = 0, \lambda_{rr} = 1$, we have

$$\begin{aligned}
\overset{r}{\mathrm{T}}_1 (x_1, R, \theta) &= \tau_{1r}(R) = 0 \\
\overset{r}{\mathrm{T}}_r (x_1, R, \theta) &= \tau_{rr}(R) = 0 \\
\overset{r}{\mathrm{T}}_\theta (x_1, R, \theta) &= \tau_{r\theta}(R) = 0
\end{aligned} \tag{f}$$

It is apparent that the components of stress (d) satisfy the first and third of relations (f) without imposing any restriction on the value of the constant C_2. Substituting the first of relations (d) into the second of relations (f), we obtain

$$C_2 = \frac{(3+\nu)\rho R^2\omega^2}{4} \tag{g}$$

Substituting the value of C_2 from relation (g) into relations (d) and (e), we have

$$\tau_{rr} = \frac{(3+\nu)\rho\omega^2}{8}(R^2 - r^2) \qquad \tau_{\theta\theta} = \frac{\rho\omega^2}{8}\left[(3+\nu)R^2 - (1+3\nu)r^2\right]$$

$$\tau_{11} = \tau_{1r} = \tau_{1\theta} = \tau_{r\theta} = 0 \tag{h}$$

$$\hat{u}_\theta = 0 \qquad \hat{u}_r = \frac{(1-\nu)r\rho\omega^2}{8E}\left[(3+\nu)R^2 - (1+\nu)r^2\right] \tag{i}$$

The maximum values of the components of stress occur at $r = 0$ and are equal to

$$\tau_{rr}(0) = \tau_{\theta\theta}(0) = \frac{(3+\nu)\rho\omega^2 R^2}{8} \tag{j}$$

The maximum radial component of displacement, $\hat{u}_r$, occurs at $r = R$ and it is equal to

$$\hat{u}_r(R) = \frac{(1-\nu)\rho\omega^2 R^3}{4E} \tag{k}$$

7.9 Problems

1. Given the polynomial

$$\phi = A_1 x_3^2 + A_2 x_3 x_2 + A_3 x_2^2$$

establish the distribution of surface traction which we must have on a prismatic body of rectangular cross section so that ϕ is the Airy stress function for plane strain. Show the required distribution of surface traction on a sketch.

Ans. $\overset{\pm 1}{\mathbf{T}} = \pm 2\,\nu(A_1 + A_2)\mathbf{i}_1 \quad \overset{\pm 2}{\mathbf{T}} = \pm 2A_1\mathbf{i}_2 \mp A_2\mathbf{i}_3 \quad \overset{\pm 3}{\mathbf{T}} = \mp A_2\mathbf{i}_2 \pm 2A_3\mathbf{i}_3$

2. Given the polynomial

$$\phi = A x_3^2 x_2^2 - 2x_2^4$$

establish the distribution of surface traction which we must have on a prismatic body of rectangular cross section so that ϕ is the Airy stress function for plane strain. Show the required distribution of surface traction on a sketch.

Ans. $\overset{\pm 1}{\mathbf{T}} = \pm 2A\nu x_3^2\mathbf{i}_1 \quad \overset{\pm 2}{\mathbf{T}} = \pm 24 x_2^2\mathbf{i}_2 \mp 48 x_3 x_2 \mathbf{i}_3 \quad \overset{\pm 3}{\mathbf{T}} = \pm 24(x_3^2 - x_2^2)\mathbf{i}_3 \mp 48 x_3 x_2 \mathbf{i}_2$

3. Derive relation (7.20) following a procedure analogous to that adhered to in Section 7.2 for the derivation of relation (7.17).

4. A long dam of triangular cross section is subjected to a linearly varying hydrostatic pressure on its surface *OA* as shown in Fig. 7P4. Surface *OB* is traction free while on the base *AB* act the normal and shearing components of stress required to balance the weight of the dam and the hydrostatic pressure. Show that for this problem $\nabla^2 V = 0$. Establish the stress distribution in the dam. Assume that the specific weights of the water and the dam are γ_w and γ_d, respectively. Use the following Airy stress function

$$\phi = \frac{C_1}{6}x_2^3 + \frac{C_2}{2}x_2^2 x_3 + \frac{C_3}{2}x_2 x_3^2 + \frac{C_4}{6}x_3^3$$

Ans. $\tau_{22} = \left[(\gamma_w/\tan^2\beta) - \gamma_d\right]x_2 + \left[\gamma_d/\tan\beta - 2\,\gamma_w/\tan^3\beta\right]x_3 \qquad \tau_{23} = -\gamma_w x_3/\tan^2\beta \qquad \tau_{33} = -\gamma_w x_2$

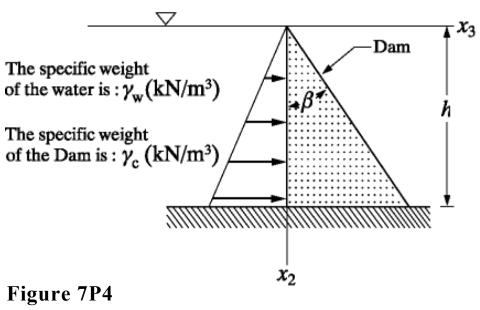


Figure 7P4

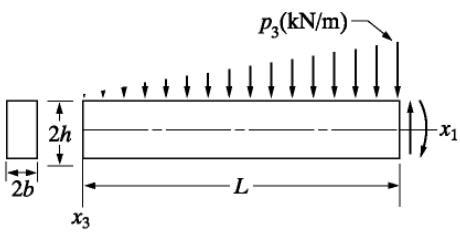


Figure 7P5

5. A thin cantilever beam is loaded as shown in Fig. 7P5. Establish the stress distribution in the beam assuming that the beam is in a state of plane stress. Disregard the effect of the weight of the beam. Assume a plane strain function for the auxiliary beam in a state of plane strain of the form

$$\phi_{\text{aux}} = A_1 x_1 x_3 + A_2 x_1^3 + A_3 x_1^3 x_3 + A_4 x_1 x_3^3 + A_5 x_1^3 x_3^3 + A_6 x_1 x_3^5$$

Ans. $\tau_{11} = (p_0/bLh)\left[(3/20)\,x_1x_3 + x_1^3x_3/8h^2 - x_1x_3^3/(2h)^2\right]$ $\quad\tau_{33} = (p_0/4bL)\left(x_1 - 3x_1x_3/2h + x_1x_3^3/2h^3\right)$

$\tau_{13} = -(p_0/bL)\left(-h/80 - 3x_1^2/16h + 3x_3^2/40h + 9x_1^2x_3^2/48h^3 - x_3^4/16h^3\right)$ $\quad\tau_{22} = 0$

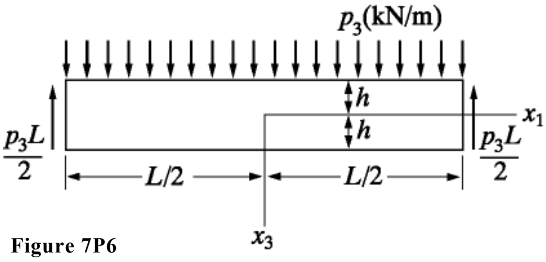


Figure 7P6

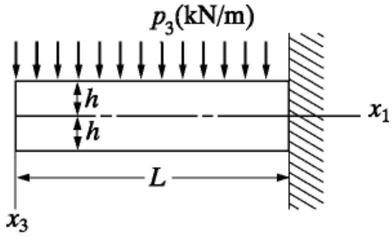


Figure 7P7

6. and **7**. A simply supported thin prismatic beam of width $2b$ and depth $2h$ is loaded with a uniform force (see Fig. 7P6). Establish the components of stress acting on the particles of the beam assuming that it is in a state of plane strain ($u_2 = 0$). Disregard the effect of body forces. Plot the distribution of the components of stress on the cross section of the beam assuming that it is in a state of plane stress. Repeat with the cantilever beam of Fig. 7P7. *Hint:* For Problem 6 assume

$$\phi_{\text{aux}} = Ax_1^2 + Bx_3x_1^2 + Cx_1^2x_3^3 + Dx_3^3 + Ex_3^5$$

For Problem 7 assume:

$$\phi_{\text{aux}} = \frac{x_1^2}{2}\left(\frac{Ax_3^3}{6} + \frac{Bx_3^2}{2} + Cx_1 + D\right)$$

Ans. 6 $\quad\tau_{11} = (p_3x_3/2bh^3)\left(-0.3h^2 + 0.18L^2 + 0.5x_3^2 - 0.75x_1^2\right)$ $\quad\tau_{13} = (0.75p_3x_1/2bh^3)\left(x_3^2 - h^2\right)$

$\tau_{33} = (p_3/2bh^3)\left(-0.5h^3 + 0.75h^2x_3 - 0.25x_3^3\right)$ $\quad\tau_{22} = 0$

Ans. 7 $\quad\tau_{11} = -M_3x_3/I_2$ $\quad\tau_{33} = (p_3/4b)\left(-x_3^3/2h^3 + 3x_3/2h - 1\right)$ $\quad\tau_{13} = -(x_1p_3/2I_2)\left(h^2 - x_3^2\right)$

8. Establish the stress distribution in a thin semi-infinite plate due to a moment M (kN·m per meter of width) as shown in Fig. 7P8. (*Hint:* Superimpose the results due to two equal and opposite line forces p kN/m [one at $(-\Delta x_3/2)$ the other at $(\Delta x_3/2)$] and take the limit as $\Delta x_3 \to 0$ and $p\Delta x_3 \to M$.

Ans. $\quad\tau_{rr} = -4x_2x_3M/\pi(x_2^2 + x_3^2)^2$

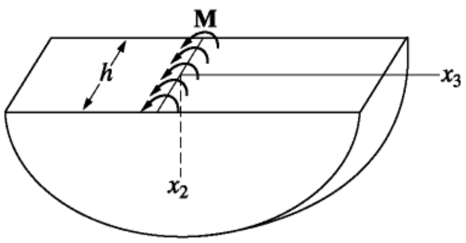


Figure 7P8

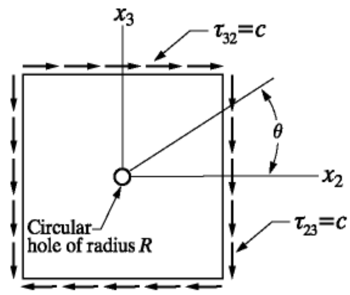


Figure 7P9

9. Consider a very large thin square plate with a small circular hole of radius R subjected to uniform distribution of the shearing, component of stresses $\tau_{23} = c$ as shown in Fig. 7P9. Establish the stress field in the plate. Plot the distribution of the components of stress τ_{22}, τ_{33}, τ_{23} at the cross section $x_2 = 0$.

Ans. $\tau_{rr} = \left(c + 3cR^4/r^4 - 4cR^2/r^2\right)\sin 2\theta \quad \tau_{\theta\theta} = \left(-c - 3cR^4/r^4\right)\sin 2\theta \quad \tau_{r\theta} = \left(c - 3cR^4/r^4 + 2cR^2/r^2\right)\cos 2\theta$

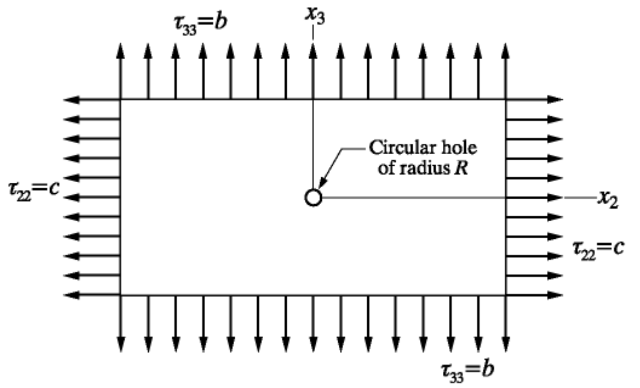


Figure 7P10

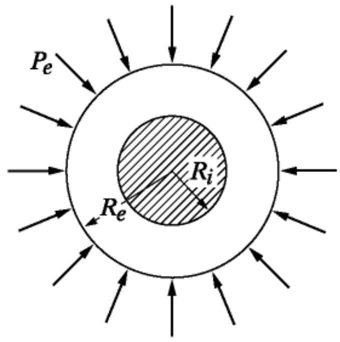


Figure 7P11

10. Consider a very large thin plate with a small circular hole of radius R subjected to a uniform distribution of the components of stress $\tau_{22} = c$, $\tau_{33} = b$ (see Fig. 7P10). Establish the stress distribution in the plate. Plot the distribution of the components of stress τ_{22}, τ_{33}, and τ_{23} at the cross section $x_2 = 0$.

Ans. $\tau_{rr} = (b+c)/2\left[1 - R^2/r^2 - (b-c)/2\right]\left[1 + 3R^4/r^4 - 4R^2/r^2\right]\cos 2\theta$

$\tau_{\theta\theta} = (b+c)/2\left[1 + R^2/r^2 + (b-c)/2\right]\left[1 + 3R^4/r^4\right]\cos 2\theta \quad \tau_{r\theta} = (b-c)/2\left[1 - 3R^4/r^4 + 2R^2/r^2\right]\sin 2\theta$

11. A long thick-walled elastic cylinder of outer radius R_e and inner radius R_i is bounded to a rigid circular cylindrical core as shown in Fig. 7P13. The end surfaces of the cylinder at $x_1 = 0$ and $x_1 = L$ are restrained from moving in the direction of its axis. However, they can move freely in direction normal to its axis. If the cylinder is subjected to a uniform external pressure p, determine the components of stress and displacement of the cylinder assuming that the body force is negligible.

Ans. $\tau_{11} = 2p_e\nu\left[\left[(1-2\nu)R_i^2\right]\Big/\left[(1-2\nu)R_i^2 + R_e^2\right] - 1\right] \qquad \tau_{1r} = \tau_{r\theta} = \tau_{1\theta} = 0$

$\tau_{rr} = p_e\left[\left[r^2(1-2\nu)R_i^2\right] - r^2\left[(1-2\nu)R_i^2 + R_e^2\right] - (1-2\nu)R_i^2R_e^2\right]\Big/\left[r^2(1-2\nu)R_i^2 + r^2R_e^2\right]$

$\tau_{\theta\theta} = p_e\left[r^2(1-2\nu)R_i^2 - r^2\left[(1-2\nu)R_i^2 + R_e^2\right] + (1-2\nu)R_i^2R_e^2\right]\Big/\left[r^2(1-2\nu)R_i^2 + r^2R_e^2\right]$

$\hat{u}_r = p_e(1-2\nu)\left\{\left[(1-2\nu)R_i^2\Big/\left[(1-2\nu)R_i^2 + R_e^2\right]\right] - 1 + \left[R_i^2R_e^2\Big/\left[(1-2\nu)r^2R_i^2 + R_e^2r^2\right]\right]\right\} \qquad \hat{u}_\theta = 0$

12. Establish the distribution of the components of stress in the circular cantilever beam of thin rectangular cross sections of width $2b$ subjected to a shearing force at its unsupported end as shown in Fig. 7P12. Use the following Airy stress function:

$$\phi(r,\theta) = \left(Ar^3 + \frac{B}{r} + Cr + Dr\ln r\right)\sin\theta$$

Ans. $\tau_{rr} = \left[Q_r/(2bK)\right]\left[r + (a^2b^2/r^3) - (a^2+b^2)/r\right]\sin\theta \qquad \tau_{\theta\theta} = \left[Q_r/(2bK)\right]\left[3r - (a^2b^2/r^3) - (a^2+b^2)/r\right]\sin\theta$

$\tau_{r\theta} = -\left[Q_r/(2bK)\right]\left[r + (a^2b^2/r^3) - (a^2+b^2)/r\right]\cos\theta$

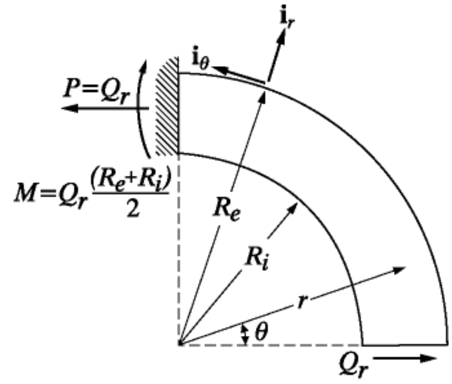


Figure 7P12

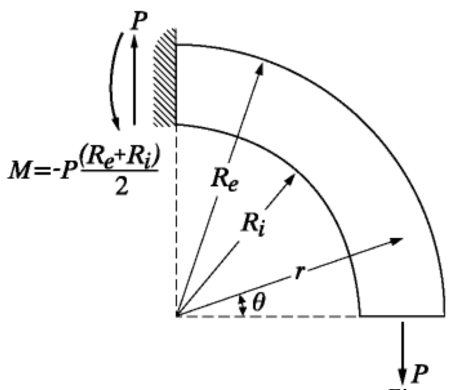


Figure 7P13

13. Establish the distribution of the components of stress in the circular beam of thin rectangular cross section of width $2b$. The beam is subjected to a tangential to its axis force at its unsupported end as shown in Fig. 7P13. Use an Airy stress function of the following form:

$$\phi\,(r,\theta) = \phi_r\,(r)\cos\theta$$

Ans. $\tau_{rr} = \left[P/(2bK)\right]\left[-r - (a^2b^2/r^3) - (a^2+b^2)/r\right]\cos\theta$

$\tau_{\theta\theta} = \left[P/(2bK)\right]\left[-3r + (a^2b^2/r^3) + (a^2+b^2)/r\right]\cos\theta \qquad \tau_{r\theta} = \left[P/(2bK)\right]\left[-r - (a^2b^2/r^3) + (a^2+b^2)/r\right]\sin\theta$

14. Consider a prismatic thick-walled circular cylinder consisting of a steel (E = 210 GPa $\nu = 0.3$) circular tube, fitted into an aluminum alloy (E = 80 GPa, $\nu = 0.3$) tube. The inner radius of the steel tube is 0.3 m while its outer radius is 0.45 m. The inner radius of the aluminum is 0.45 m while its outer radius is 0.6 m. The cylinder is subjected to internal pressure p_i and external pressure p_e. The ends of the cylinder are restrained from moving only in the axial direction. That is, the component of displacement $\hat{u}_1$ and the components of traction $\overset{\pm 1}{T}_r$ and $\overset{\pm 1}{T}_\theta$ vanish at $x_1 = 0$ and $x_1 = L$. Assuming that the body forces are negligible, compute the pressure exerted by the inner surface of the steel cylinder on the outer surface of aluminum cylinder.

Ans. $p_u = 0.126p_i + 0.947p_e$ KN/m^2

15. It is possible to increase the strength of a composite thick-walled circular cylinder by inducing beneficial initial stresses in its walls. For example consider a composite circular cylinder consisting of an inner steel (E = 210 GPa, ν = 0.3) and an outer aluminum (E = 80 GPa, ν = 0.3) circular tubes. Before assembling the cylinder the inner radius of the steel tube is 0.3 m and its outer radius is 0.454 m, while the inner radius of the aluminum tube is 0.45 m and its outer radius is 0.6 m. That is, the outer radius of the inner tube is slightly larger than the inner radius of the outer tube. The outer tube is heated in order to expand and the inner tube is pressed inside it. As the temperature of the tubes becomes uniform the inner tube resists the shrinkage of the outer tube and a pressure is created between the outer surface of the inner tube and the inner surface of the outer tube. The ends of the cylinder are restrained from moving only in the axial direction. That is, the component of displacement $\hat{u}_1$ and the components of traction $\overset{\pm 1}{T}_r$ and $\overset{\pm 1}{T}_\theta$ vanish at $x_1 = 0$ and $x_1 = L$. Assuming that the body forces are negligible, compute the components of displacement and stress of the particles of the cylinder, when it is subjected to a uniform internal pressure p_i.

Ans. $\tau_{11}^{\text{ext}} = 125{,}264 + 0.09516 p_i \text{ KN/m}^2$

$$\tau_{rr}^{\text{ext}} = (208{,}774 + 0.15859 p_i)\left[r^2 - (0.6)^2\right]/r^2$$

$$\tau_{\theta\theta}^{\text{ext}} = (208{,}774 + 0.15859 p_i)\left[r^2 + (0.6)^2\right]/r^2$$

16. A long circular cylinder is constructed from two hollow circular cylinders made from the same isotropic linearly elastic material (E = 200 GPa, v = 0.3). Before assembly the inner radius of the outside cylinder was 0.193 m while its outer radius was 0.3 m. The outer radius of the inside cylinder was 0.2 m while its inner radius was 0.1 m. The inside cylinder was inserted by heating the outside cylinder until it expanded enough. The particles of the end surfaces of the combined cylinder at $x_1 = 0$ and $x_1 = L$ are restrained from moving in the direction of the x_1 axis. However, they were free to move in the plane normal to the axis of the cylinder.

(a) Establish the contact pressure between the two cylinders.

Ans. Contact pressure = 5.16 Pa

(b) Establish the stress distribution in the two cylinders.

(c) If a pressure of 120 MPa is applied at the inside of the inner cylinder, compute the stress distribution of the combined cylinder.

Ans. Contact pressure = 2023.88 MPa

Chapter 8

Theories of Mechanics of Materials

8.1 Introduction

In this book we consider solid bodies initially in a reference stress-free, strain-free state of mechanical† and thermal†† equilibrium at a uniform temperature T_o. In this state the bodies are not subjected to external loads and heat does not flow in or out of them. Subsequently the bodies are subjected to specified external loads described in Section 2.2, as a result of which they deform and reach a second state of mechanical, but not necessarily thermal, equilibrium.

In Chapters 5 to 7 we formulate and solve boundary value problems for computing the displacement and stress fields of solid bodies subjected to external loads, on the basis of the linear theory of elasticity. This theory can be employed to formulate boundary value problems for computing the displacement, and stress fields of bodies of any geometry subjected to any loading. However, only a few such problems involving bodies of

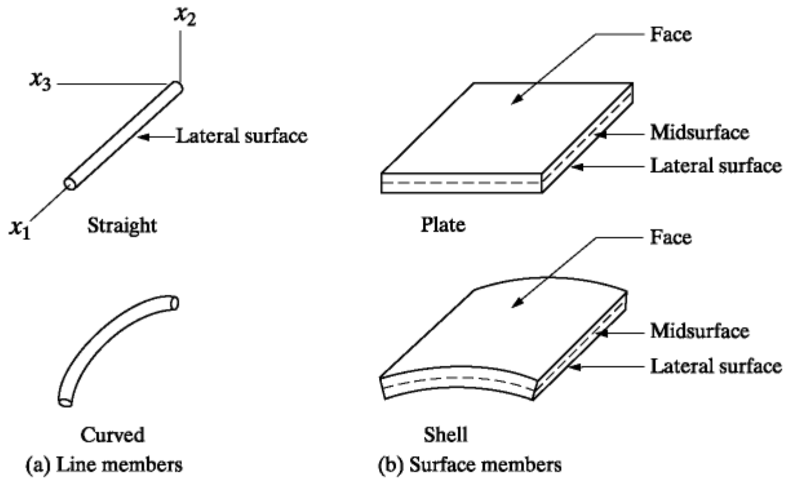


Figure 8.1 One-dimensional line and two-dimensional surface members.

† When a body is in a state of mechanical equilibrium, its particles do not accelerate. That is, the sum of the forces acting on any portion of the body and the sum of their moments about any point vanish.
†† When a body is in a state of thermal equilibrium heat, does not flow in or out of it. That is, the temperature of all its particles is the same.

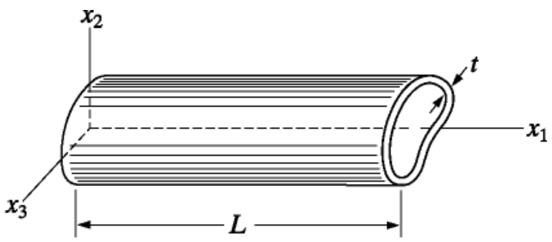


Figure 8.2 Thin-walled tubular member.

simple geometry supported in an idealized convenient way and subjected to external loads which induce states of stress having some vanishing components have been solved exactly. The rest are solved approximately using one of the modern numerical methods suitable for programming their solution on an electronic computer. The finite elements method is the most popular of these methods.

In Chapters 8 to 18 we formulate and solve boundary value problems for computing the displacement and stress fields of solid bodies subjected to external loads, using the theories of mechanics of materials. These are approximate theories which can be used only for bodies whose geometry is such that certain assumptions can be made as to the equilibrium of their particles, as to their deformed configuration and as to the distribution of the components of stress acting on their particles. Most problems can be solved exactly by hand calculation if the geometry and loading of the body is simple or with the aid of a computer if the geometry and loading of the body is complex.

Bodies for which mechanics of materials theories have been developed include:

1. *Thin surface members* — One dimension of these bodies, called their *thickness*, is considerably smaller than their other two dimensions (see Fig. 8.1b). The locus of the midpoints of their thickness is a surface known as their *midsurface.* If the midsurface of a surface member is a plane, the member is called a *plate*, while if it is a surface of higher degree, it is called a *shell* (see Fig. 8.1b).
2. *Line members* — These bodies have one dimension called their length, which is considerably larger than their other two dimensions (see Fig. 8.1a). The plane surface cut by a plane perpendicular to the larger dimension of a line member is called its *cross section*. Moreover, the locus of the centroids of the cross sections of a line member made from one material is called its *axis*. The end surfaces of a line member are perpendicular to its axis. The axis of a line member could be straight or curved. We limit our attention to straight-line members which either have constant cross sections or cross sections whose geometry changes in a way that the direction of their principal centroidal axes remains constant throughout their lengths. Moreover, we limit our attention to curved line members whose axis lies in one plane, whereas one of the principal centroidal axes of their cross sections is normal to this plane. For straight-line members we choose their axis as the x_1 axis and the centroid of one of their end cross sections as the origin of the axes of reference x_1, x_2, x_3 (see Fig. 8.1a).
3. *Thin-walled tubular member* — These are thin-walled prismatic bodies with one or more holes (see Fig. 8.2).

The theories of mechanics of materials are established for surface members of constant thickness and for line members and thin-walled, tubular members of constant cross section. Such line and tubular members are called *prismatic*. Moreover, with reduction of accuracy, the theories of mechanics of materials have been applied to surface members of variable thickness, and to line members of variable cross sections (see Chapter 10).

8.2 Fundamental Assumptions of the Theories of Mechanics of Materials for Line Members

In the theories of mechanics of materials for line members, made from any material, the following assumptions† are made:

Assumption 1. *The behavior of a line member may be approximated by that of the continuum model (see Section 2.1).*

Assumption 2. The effect of the deformation of the bodies which we are considering on their temperature is negligible. On the basis of this assumption the temperature distribution of a body can be computed independently of its deformation. In this book we assume that it has been computed and it is known.

Assumption 3. *The normal components of stress τ_{22} and τ_{33} acting on the particles of line members on the planes normal to the x_2 and x_3 axes, respectively, and the shearing component of stress $\tau_{22} = \tau_{33}$ acting on these planes are considered negligible compared to the other components of stress (see Fig. 8.3).* That is,

$$\tau_{22} \approx \tau_{33} \approx \tau_{23} \approx 0 \tag{8.1a}$$

or in cylindrical coordinates

$$\tau_{rr} \approx \tau_{\theta\theta} \approx \tau_{r\theta} \approx 0 \tag{8.1b}$$

where τ_{rr} is the normal component of stress acting on the cylindrical surface r = constant (see Fig. 2.16); $\tau_{\theta\theta}$ is the normal component of stress acting on the plane θ = constant (see Fig. 2.16). Actually, the component of stress τ_{22} or τ_{33} may not vanish at some particles of line members subjected to a distribution of transverse components of traction on their lateral surface. However, the values of these components of stress are negligible

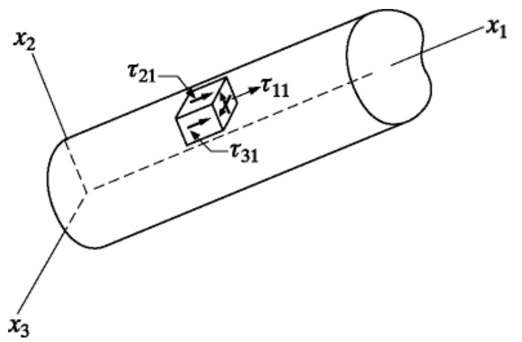


Figure 8.3 Components of stress acting on a particle of a line member.

† In this book we do the following:
(a) We limit our attention to bodies which are made from isotropic, linearly elastic materials.
(b) Except in Chapter 18 we limit our attention to bodies subjected to loads of such magnitudes that the deformation of their particles is within the range of validity of the assumption of small deformation (see Sections 2.3 and 2.4).

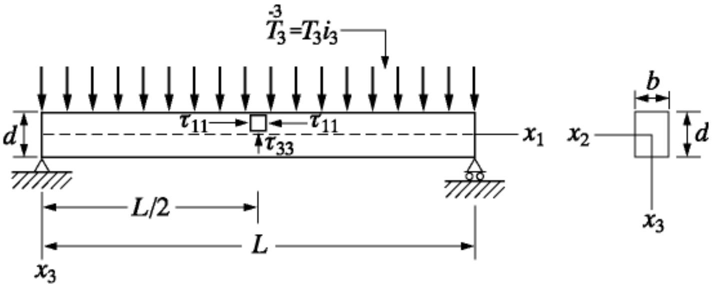


Figure 8.4 Beam subjected to a uniform distribution of the transverse component of traction.

compared to the maximum value of the normal component of stress τ_{11}. For example, consider the beam of constant rectangular cross section of width b and depth d shown in Fig. 8.4. The beam is subjected on its upper surface to a uniformly distributed traction $\bar{\mathbf{T}}^3 = \bar{T}^3_3 \mathbf{i}_3$. From physical intuition we may deduce† that the normal component of stress τ_{33} is zero at the particles of the bottom surface of the beam and maximum at the particles of its upper surface where

$$(\tau_{33})_{\max} = -\bar{T}^3_3 \tag{8.2}$$

In Section 9.2 we show that for the beam of Fig. 8.4 the value of the normal component of stress τ_{11} is equal to

$$\tau_{11} = \frac{M_2 x_3}{I_2} \tag{8.3}$$

where I_2 is the moment of inertia of the cross section of the beam about the principal centroidal axis x_2. The maximum value of the moment in the beam occurs at $x_1 = L/2$ and it is equal to

$$(M_2)_{\max} = \frac{\bar{T}^3_3 b L^2}{8}$$

Substituting the above relation into (8.3) and recalling that, for the beam of Fig. 8.4, $I_2 = bd^3/12$, the maximum value of the compressive normal component of stress τ_{11} is equal to

$$(\tau_{11})_{\max}\Big|_{x_3 = -d/2} = -\frac{3\bar{T}^3_3 L^2}{4d^2}$$

Substituting relation (8.2) into the above, we obtain

† It can be shown that for the beam of Fig. 8.4 we have $\tau_{33} = \frac{\bar{T}^3_3}{4}\left(-\frac{8x_3^3}{d^3} + \frac{6x_3}{d} - 2\right)$

$$(\tau_{33})_{\max} = \frac{4d^2}{3L^2}(\tau_{11})_{\max} \tag{8.4}$$

Inasmuch as for line members $d/L << 1$ it is apparent that for such members $(\tau_{33})_{\max}$ is very small compared to the maximum value of τ_{11}.

Assumption 3, reduces the number of unknown components of stress and simplifies the stress–strain relations for line members (see Section 8.11).

Assumption 4. The distribution of the components of traction, on the end surfaces ($x_1 = 0$ and $x_1 = L$) of line members usually is not known. However, since the dimensions of the cross sections of line members are in general small compared to their length, it is assumed that all distributions of traction on the end surfaces of a line member which are statically equivalent† have the same effect at particles sufficiently removed from the end surfaces of the member. This assumption is an application of the principle of Saint Venant which is discussed in Section 5.3. Consequently, at an end surface of a line member at which a component of displacement is not specified, it is sufficient to specify the corresponding resultant force and moment of the tractions acting on it. The resultant force and moment obtained from the calculated distribution of the components of stress on each end surface ($x_1 = 0$ and $x_1 = L$) of a member, should be equal to the specified resultant force and moments of the tractions acting on it.

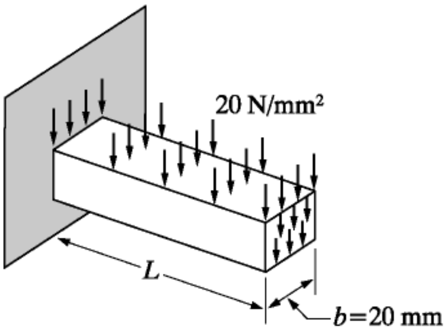
Assumption 5. When the theory of elasticity is employed to establish the distribution of the components of displacement and stress in a body in equilibrium under the influence of external forces (surface tractions and body forces) and/or change of temperature, it is required that the distribution of the components of stress satisfies the conditions for equilibrium for all the particles of the body.

In the theories of mechanics of materials for line members we do not ensure that every particle of a body is in equilibrium. Instead we ensure that each segment of infinitesimal length cut from the member by two imaginary planes normal to its axis is in equilibrium.

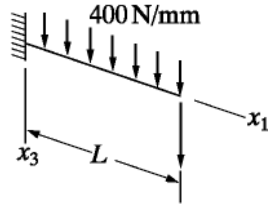
As a result of Assumption 5 in the theories of mechanics of materials it is not necessary to specify the distribution of the components of traction acting on the lateral surface of a line member or the distribution of the body force acting on the particles of a line member. Instead it is sufficient to specify the statically equivalent distribution of external forces and moments acting along the length of the member. Moreover, as discussed previously, on the basis of the principle of Saint Venant the distribution of the components of traction acting on each of the end surfaces ($x_1 = 0$ and $x_1 = L$) of a member may be replaced by statically equivalent concentrated components of force and moment. Thus, as shown in Fig. 8.5, a line member can be represented schematically by a line (its axis) subjected to specified distributions of external forces along its length and to specified concentrated forces and moments on its ends.

When external forces and moments of high intensity are distributed over a very small portion of the length of a member, they are replaced, depending upon the nature of their distribution, by an equivalent concentrated force and/or moment. *Thus, line members are represented by a line subjected to specified concentrated external forces $P_i^{(n)}$ ($i = 1, 2, 3$)($n = 1, 2, ..., n_i$) and moments $M_i^{(m)}$ ($i = 1,2,3$)($m = 1, 2, ..., m_i$) and to specified distributed*

† Two statically equivalent distributions of tractions have the same resultant force and moment about the same point.



(a) Cantilever beam subjected to a distribution of traction on its lateral surface.



(b) Schematic representation of the beam of Fig. 8.5a subjected to a specified statically equivalent distribution of external forces along its length and on its unsupported end.

Figure 8.5 Schematic representation of a line member.

forces $p_i(x_1)$ $(i = 1, 2, 3)$ *and moments* $m_i(x_1)$ $(i = 1, 2, 3)$ *along their length as well as to concentrated forces and moments on their ends.* The distributed external forces and moments are given in units of force and moment, respectively, per unit length of the axis of the member. The external forces and moments acting on a member are called the *external actions*.

Assumption 6. *Plane sections normal to the axis of a line member before deformation remain plane after deformation.* Consequently, the movement of a cross section of a member, due to its deformation, is specified by the three components of the translation vector of its centroid, referred to a set of rectangular axes, and by the three components of its rotation† about the same axes (see Fig. 8.6). The components of the displacement vector of the centroid of a cross section of a line member are referred to as the *components of translation of this cross section.* We denote the components of translation in the direction of the x_1, x_2 and x_3 axes by $u_1(x_1)$, $u_2(x_1)$ and $u_3(x_1)$, respectively. Moreover, we denote the components of rotation of a cross section about the x_1, x_2 and x_3 axes by $\theta_1(x_1)$, $\theta_2(x_1)$ and $\theta_3(x_1)$, respectively, and we consider them as positive if they are represented by a vector† acting in the direction of the positive x_1, x_2 and x_3 axes, respectively. Inasmuch as a cross section of a member is represented by a point on the line diagram of the structure, we refer to the components of translation and rotation of the cross section represented by point A on the line diagram of a member as the *components of translation and rotation of point A of this member*. The components of translation u_1, u_2, u_3 and the components of rotation θ_1, θ_2, θ_3 of a cross section of a member of a structure are called its *components of displacement.*

On the basis of Assumption 6 referring to Fig. 8.6 we have

$$\hat{u}_1(x_1, x_2, x_3) = u_1(x_1) - x_2\theta_3(x_1) + x_3\theta_2(x_1) \tag{8.5a}$$

Substituting relation (8.5) into the first of the strain–displacement relations (2.16), we

† A rotation about an axis is represented by a vector acting along this axis and pointing in the direction in which a right-hand screw moves when subjected to this rotation. Small rotations are vector quantities, while large rotations are not. Except in Chapter 18, in this text we consider bodies whose deformation involves only small rotations.

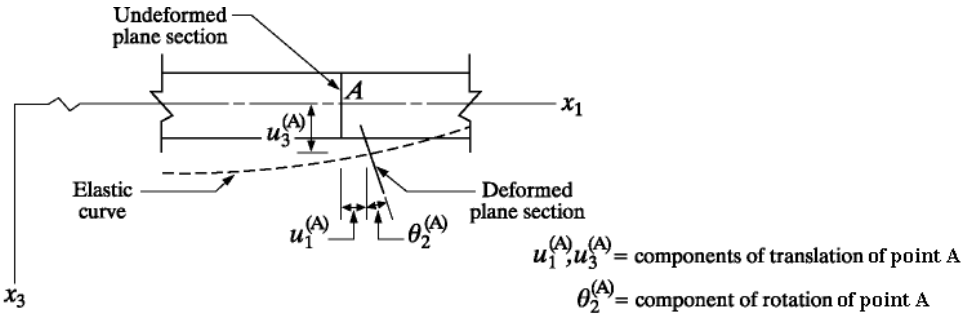


Figure 8.6 Components of displacement in the x_1x_3 plane of a cross section of a member.

obtain

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} = \frac{du_1}{dx_1} - x_2\frac{d\theta_3}{dx_1} + x_3\frac{d\theta_2}{dx_1} \tag{8.5b}$$

Assumption 7. When the theory of elasticity is employed to establish the distribution of the components of displacement and stress in a body in equilibrium under the influence of external forces and/or change of temperature, it is required that the components of strain obtained from the components of stress satisfy the compatibility equations at every particle of the body. That is, the deformation of every particle is compatible with that of its neighbors. *In the theories of mechanics of materials we do not ensure that the deformation of every particle of the body is compatible with that of its neighbors. Instead we ensure that the deformation of every segment of infinitesimal length cut from the member by two imaginary planes normal to its axis is compatible with that of its neighboring segments. This is accomplished by requiring that the components of translation $u_1(x_1)$, $u_2(x_1)$ and $u_3(x_1)$ and the components of rotation $\theta_1(x_1)$, $\theta_2(x_1)$ and $\theta_3(x_1)$ are continuous throughout the length of the member.*

In Example 1 of Section 5.5 we have shown that when a prismatic line member is subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to a distribution of traction, which is statically equivalent to an axial centroidal force, plane cross sections normal to the axis of the member prior to deformation remain plane after deformation provided that they are sufficiently removed from its end surfaces. Moreover, the cross sections of the member do not rotate and the particles of its axis do not translate in the transverse directions. Its axis only elongates or shrinks. That is,

$$u_1(x_1) \neq 0 \qquad \theta_1(x_1) = \theta_2(x_1) = \theta_3(x_1) = u_2(x_1) = u_3(x_1) = 0 \tag{8.6a}$$

while the stress distribution on the cross section of the member is

$$\tau_{11} \neq 0 \qquad \tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0 \tag{8.6b}$$

It is apparent that in this case the assumptions of the theories of mechanics of materials are satisfied and, consequently, these theories give exact results.

In the examples of Sections 6.5 and 6.6 we have shown that when a prismatic line member of solid or hollow circular cross section is subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to a distribution of traction which is statically equivalent to a torsional moment, its cross sections do not warp. That is, the axial component of translation $\hat{u}_1(x_1, x_2, x_3)$ of the particles of this member is equal to zero. Moreover, the

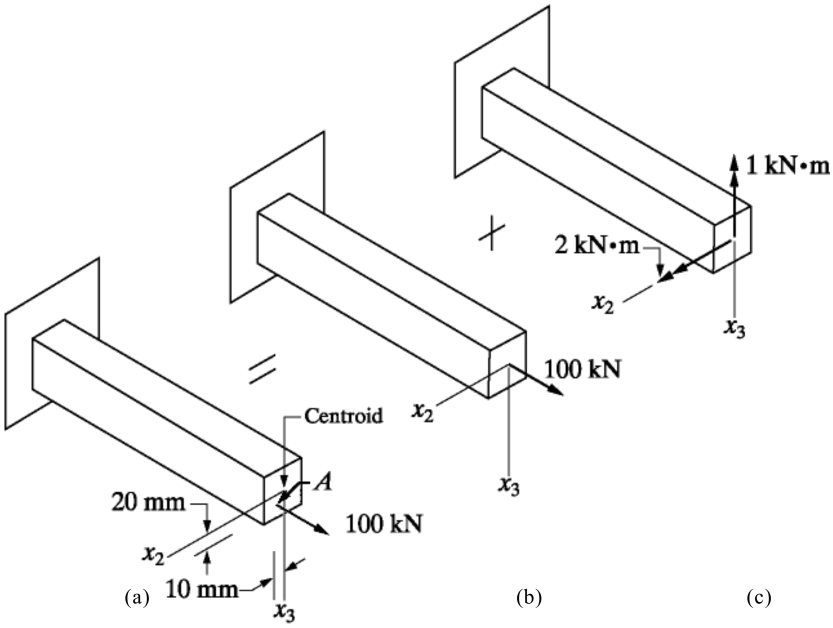


Figure 8.7 Cantilever member subjected to an axial non-centroidal force.

particles of its axis do not translate and its cross sections do not rotate about the x_2 and x_3 axes. That is, in this case, we have

$$\theta_1(x_1) \neq 0 \qquad u_1(x_1) = u_2(x_1) = u_3(x_1) = \theta_2(x_1) = \theta_3(x_1) = 0 \tag{8.7a}$$

While the stress distribution on the cross sections of the member is

$$\tau_{11} = \tau_{rr} = \tau_{\theta\theta} = \tau_{1r} = \tau_{r\theta} = 0 \qquad\qquad \tau_{1\theta} = \frac{M_1 r}{I_P} \tag{8.7b}$$

where $\tau_{1\theta}$ is the component of stress acting on the cross sections of the member in the direction normal to the radial direction. It is apparent that in this case the assumptions of the theories of mechanics of materials are satisfied and, consequently, these theories give exact results. However, when a prismatic line member of non-circular cross section is subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to a distribution of traction which is statically equivalent to a torsional moment, as discussed in Section 6.4, its cross sections warp. That is, $\hat{u}_1(x_1, x_2, x_3)$ does not vanish and plane sections normal to the axis of the member prior to deformation do not remain plane after deformation. Moreover, the effect of warping on the components of displacement $\hat{u}_2(x_1, x_2, x_3)$ and $\hat{u}_3(x_1, x_2, x_3)$ and the shearing components of strain and stress cannot be disregarded. *For this reason in the theories of mechanics of materials only line members of circular (solid or hollow) cross section are considered subjected to torsional moments.*

In Example 3 of Section 5.5 we have shown that when a prismatic member is subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to a distribution of traction which is statically equivalent to a moment whose vector is normal to the axis of the member plane, sections normal to the axis of the member prior to deformation remain plane after deformation provided that they are sufficiently removed from its end surfaces. Moreover, the translation of the particles of its axis, in the x_1 direction, is negligible†. That is, for this loading, we have

$$u_1(x_1) = \theta_1(x_1) = 0 \tag{8.8}$$

When prismatic line members are subjected only to transverse forces, whose line of action lies in a plane which contains *the shear centers of their cross sections*, their cross sections do not rotate (twist) about the x_1 axis ($\theta_1 = 0$) and relation (8.8) is valid (see Section 9.7). However, their cross sections warp. That is, plane sections normal to the axis of the member do not remain plane after deformation Nevertheless, when the length of a member is considerably larger than its other dimensions, the warping of its cross sections, due to the transverse forces, does not affect appreciably the components of displacement, strain and stress of its particles and can be disregarded.

When prismatic line members are subjected only to transverse forces whose line of action does not lie in a plane which contains the shear center of their cross sections, they twist ($\theta_1 \neq 0$).

For the reasons discussed above in the theories of mechanics of materials, the axial component of traction acting on the surface of a member and the axial component of body force acting on the particles of a member are replaced by an equivalent system consisting of

1. Distributed axial centroidal forces $p_1(x_1)$ and concentrated axial centroidal forces $P_1^{(n)}$(n = 1, 2, ..., n_1) applied on the axis of the member. These forces produce only an axial component of translation and only an axial component of stress acting on the cross sections of a member.

2. Distributed bending moments $m_2(x_1)$ and $m_3(x_1)$ and concentrated bending moments $M_2^{(m)}$ (m = 1, 2, ..., m_2) and $M_3^{(m)}$ (m = 1, 2, ..., m_3). These moments do not produce an axial component of translation or rotation.

For example, consider a cantilever beam subjected to a distribution of traction on its unsupported end ($x_1 = L$) which is statically equivalent to a force of 100 kN acting at point A as shown in Fig. 8.7a. This distribution of traction is replaced by the following statically equivalent system of an axial centroidal force P_1 and bending moments M_2 and M_3 about the x_2 and x_3 axes, respectively

$$P_1 = 100 \text{ kN}$$

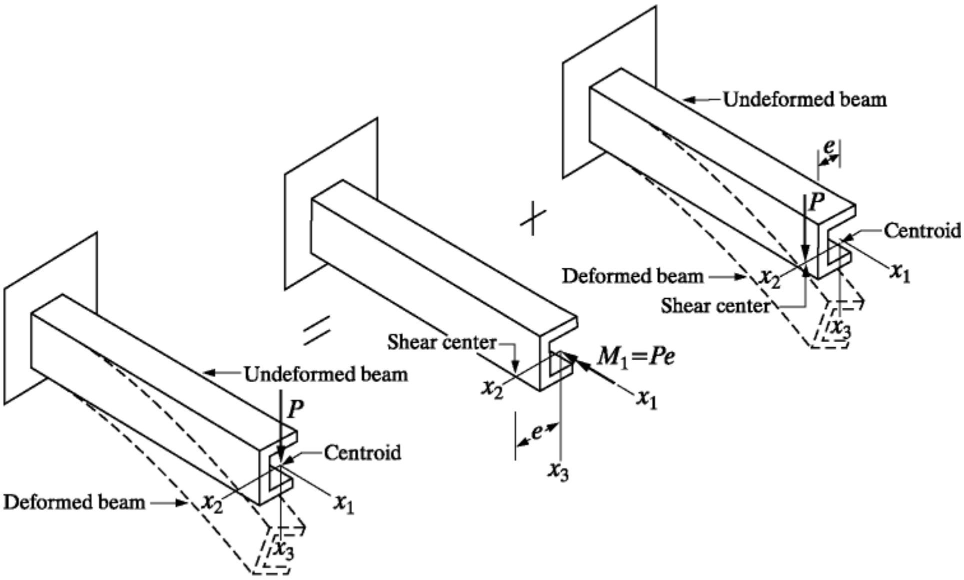
$$M_2 = 100(0.02) = 2 \text{ kN·m}$$

$$M_3 = 100(0.01) = 1 \text{ kN·m}$$

Moreover, the transverse components of traction acting on the particles of the surface of a member and the transverse components of body forces acting on its particles are replaced by an equivalent system of forces and moments consisting of

1. Distributed forces $p_2(x_1)$ and $p_3(x_1)$ and concentrated forces $P_2^{(n)}$(n = 1, 2, ..., n_2) and $P_3^{(n)}$(n = 1, 2, ..., n_3) acting in the direction of the x_2 and x_3 axis, respectively. The line of

† The axial component of translation $u_1(x_1)$ may be different than zero when a line member, made from an isotropic linearly elastic, ideally plastic material, is subjected to transverse forces and bending moments inducing components of plastic strain at some of its particles (see Section 16.2).



(a) Beam subjected to a force which does not pass through the shear center of its cross section

(b) Beam subjected to a torsional moment

(c) Beam subjected to a force which passes through the shear center of its cross section

Figure 8.8 Cantilever beam subjected to a transverse force whose line of action does not lie on a plane which contains the shear centers of its cross sections.

action of each one of these forces lies in a plane which contains the shear centers of the cross sections of the member. These forces do not produce axial component or rotation (twist) of the cross sections of the member and axial component of translation of the particles of the axis of the member.

2. Distributed moments $m_1(x_1)$ and concentrated moments $M_1^{(m)}(m = 1,2...m_1)$ about an axis parallel to the x_1 axis and passing through the shear center of the cross sections of the member. These moments produce only axial component of rotation of the cross sections of the member and only shearing component of stress on its cross section. For example, when the cantilever beam of Fig. 8.8 is subjected to a distribution of traction on its end surface $x_1 = L$ which is statically equivalent to a transverse concentrated force of magnitude P passing through its centroid, its cross sections translate vertically downward and rotate about an axis parallel to the x_1 axis, passing through the shear centers of the cross sections of the member. The force P is statically equivalent to a force $P_3 = P$ lying on a plane which contains the shear centers of the cross section of the beam and a torsional moment of magnitude $M_1 = Pe$; where e is the distance of the shear center of the cross section of the beam from its centroid. The force P_3 translates the cross sections of the beam vertically downward (see Fig. 8.8c) while the moment Pe rotates (twists) the cross sections of the beam about an axis parallel to the x_1 axis passing through the shear center of its cross sections (see Fig. 8.8c).

The afore described assumptions are considered valid for prismatic line members made from any material (as, for example, elastic, plastic, viscoelastic). Moreover, with some restrictions, they are extended to members of variable cross section (see Chapter 10).

8.3 Internal Actions Acting on a Cross Section of Line Members

When a line member is subjected to external loads, a distribution of normal and shearing components of stress could exist on any of its cross sections. This distribution of stress on a cross section is statically equivalent to one or more of the following concentrated internal forces and moments known as the *stress resultants* or the *internal actions acting on this cross section*:

1. A force applied to the centroid of the cross section acting in the direction of the x_1 axis. It is called *internal axial centroidal force* and we denote it by $N(x_1)$ (see Fig. 8.9).
2. Two forces acting in the plane of the cross section whose line of action passes through a point known as the shear center of the cross section (see Section 9.7). They are called *shearing forces*. The one acts in the direction of the x_2 axis and we denote it by $Q_2(x_1)$. The other acts in the direction of the x_3 axis and we denote by $Q_3(x_1)$ (see Fig. 8.9).
3. A moment about an axis parallel to the x_1 axis. It is called *torsional moment* and we denote it by $M_1(x_1)$ (see Fig. 8.9).
4. A moment whose vector acts in the plane of the cross section. It is called *bending moment*. It has two components: one about the x_2 axis which we denote by $M_2(x_1)$ and the other about the x_3 axis which we denote by $M_3(x_1)$.

We consider as positive the components of internal force and moment acting on a positive† cross section of a member if their sense coincides with the positive sense of its local axes x_1, x_2, x_3 (see Fig. 8.9). Furthermore, we consider as positive the components of internal force and moment acting on a negative cross section of a member if their sense coincides with the negative sense of its axes x_1, x_2, x_3 (see Fig. 8.9). Thus, a tensile axial force is considered positive, while a compressive axial force is considered negative. On the basis of the sign convention described previously, the components of the internal action acting on a cross section are related to the components of stress acting on it by the following relations:

$$N = \iint_A \tau_{11}\, dA \tag{8.9a}$$

$$Q_2 = \iint_A \tau_{12}\, dA \tag{8.9b}$$

(8.9c)

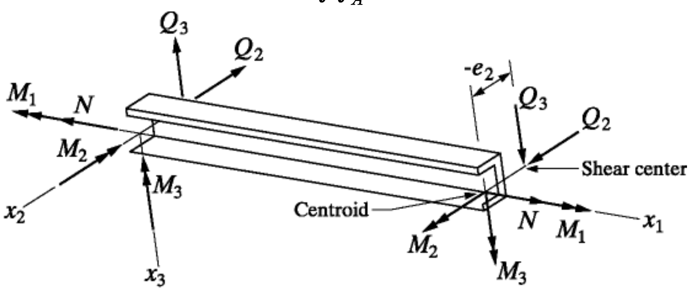


Figure 8.9 Positive internal actions acting on the cross sections of a member.

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† We call a cross section of a member space positive or negative if the unit vector normal to it is directed along the positive or negative x_1 axis, respectively.

$$M_1 = \iint_A \left[\tau_{13}(x_2 - e_2) - \tau_{12}(x_3 - e_3) \right] dA \tag{8.9d}$$

$$M_2 = \iint_A \tau_{11} x_3 \, dA \tag{8.9e}$$

$$M_3 = -\iint_A \tau_{11} x_2 \, dA \tag{8.9f}$$

where e_2 and e_3 are the x_2 and x_3 coordinates, respectively, of the shear center of a cross section of a member.

8.4 Framed Structures

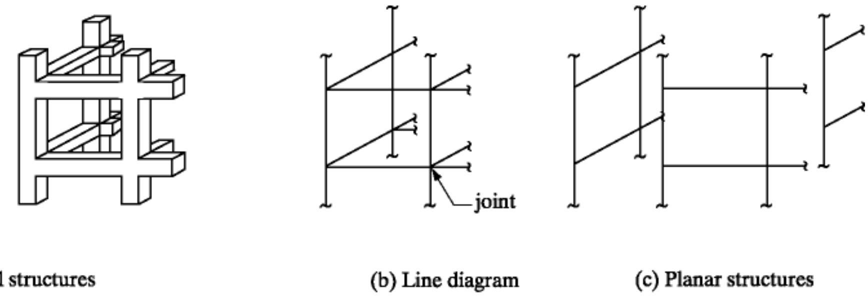
Bodies made of line members joined together at their ends are called *framed structures*. The configuration of a framed structure is conveniently described by a line diagram (see Fig. 8.10). Therein a line member is represented by a line (its axis) and a cross section by a point. Moreover, a connection of two or more members is represented by a point called a *joint*.

In general, framed structures have a three-dimensional configuration. Often, however, for purposes of analysis and design, a framed structure may be broken down into planar parts called *planar* framed structures, whose response can be considered as two-dimensional (see Fig. 8.10). The axes of the members of planar structures lie in one plane and one principal centroidal axis of the cross sections of their members is normal to this plane. Moreover, they are subjected to external loads which do not induce displacement of their axes in the direction normal to their plane. Thus, framed structures may be classified as

1. Planar
2. Space

The supports of a planar framed structure are idealized as

1. *Roller support.* This support permits the supported ends of the members of the structure to rotate about an axis normal to the plane of the structure and to move only in one direction, referred to as the direction of rolling. It can exert a reacting force on the structure acting in the direction normal to the direction of rolling and of magnitude equal



(a) Actual structures (b) Line diagram (c) Planar structures

Figure 8.10 Idealizations of a framed structure.

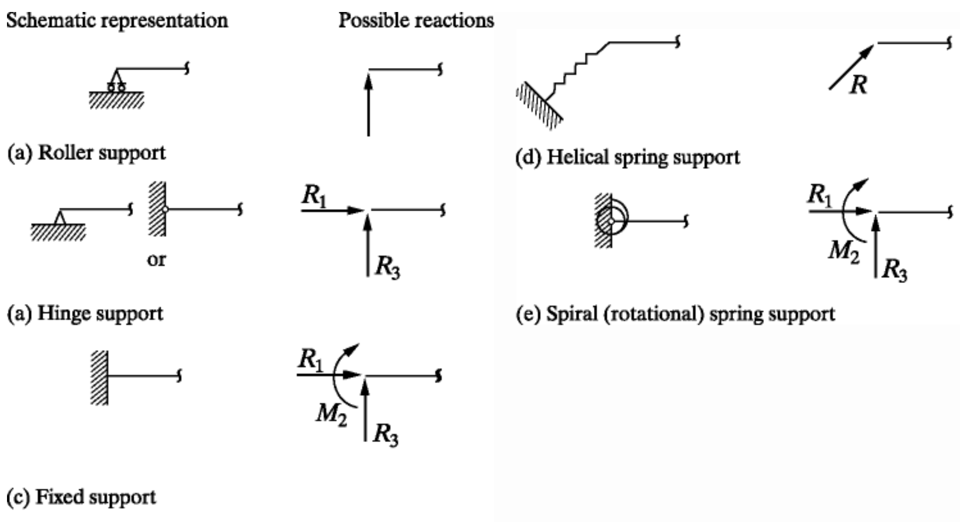


Figure 8.11 Supports of planar framed structures.

to that required to counteract the applied loads. The schematic representation of this support is shown in Fig. 8.11a.

2. *Hinge support.* This support restrains the supported end of the members of the structure from translating. However, it permits it to rotate about an axis normal to the plane of the structure. A hinged support can exert a reacting force **R** on the structure, passing through the center of the hinge and having the magnitude and direction required to counteract the applied loads. The schematic representation of this support is shown in Fig. 8.11b.

3. *Fixed support.* This support restrains the supported end of the members of the structure from translating and rotating. It can exert a reacting force on the structure acting in any required direction in the plane of the structure and a moment whose vector is normal to the plane of the structure. The schematic representation of this support is shown in Fig. 8.11c.

4. *Helical spring support.* This support partially restrains the supported end of the members of the structure from moving in the direction of the axis of the spring. However, it permits it to translate freely in the direction normal to the axis of the spring and to rotate about an axis normal to the plane of the structure. This support can exert a reacting force on the structure in the direction of the axis of the spring whose magnitude is a known function (usually a linear function) of the deformation of the spring. The schematic representation of this support is shown in Fig. 8.11d.

5. *Spiral spring support.* This support restrains the supported end of the members of the structure from translating and partially from rotating. It can exert a reacting moment on the structure whose magnitude is a known function (usually linear) of the rotation of the connected end of the members. The schematic representation of this support is shown in Fig. 8.11e.

The supports of space framed structures restrain one or more of the components of translation and-or rotation of the supported end of their members. Supports which permit rotation of the supported end of the members of a space structure about any axis are referred to as *ball-and-socket supports* (see Fig. 8.12). Supports which permit rotation of the supported end of the members of a space structure about only one axis are referred to as *cylindrical* or *pin supports.* Supports which do not permit rotation of the supported end of the members of a space structure are referred to as *fixed-against-rotation supports*. Each of the forementioned types of supports can be either non-translating or translating in one or two directions. Usually, however, supports fixed against rotation are non-translating and are referred to as *fixed supports*. These supports can exert a reacting force and a reacting moment on the structure, both acting in any direction required to counteract the applied loads.

We analyze framed structures on the basis of the theories of mechanics of materials for line members presented in this and the subsequent chapter. In the analysis of framed structures we are interested in establishing the following:

1. The internal actions (forces and moments) acting on the cross sections of their members. The components of stress at any point of a cross section of a member can be computed from the internal forces and moments acting on this cross section using the formulas established in this and the subsequent chapter.
2. Components of translation and rotation of certain cross sections.

8.5 Types of Framed Structures

We distinguish the following types of framed structures:

1. Planar trusses
2. Planar beams
3. Planar frames
4. Space trusses
5. Space beams
6. Space frames
7. Grids

Trusses are framed structures whose members are straight and assumed connected by frictionless pins; moreover, the axes of their members which are connected to the same joint are assumed to intersect at a point. Trusses are loaded by concentrated forces acting on their joints (see Fig. 8.12a). The weight of the members of trusses is usually neglected or considered as acting on their joints. Thus, it is assumed that the members of a truss are not subjected to external actions along their length or to end moments. Consequently, they are subjected only to internal axial centroidal forces, inducing a uniform state of uniaxial tension or compression.

Beams are framed structures whose line diagram is a straight line. They are subjected to external loads which induce internal forces and moments on their cross sections. Planar beams are subjected to tranverse external forces lying on a plane which passes through the shear center of their cross sections. This plane is parallel to a plane which contains their axis and a principal centroidal axis of their cross sections. Moreover, the vector of the external moments acting on planar beams is normal to the plane of the external forces. Consequently, every cross section of a planar beam rotates only about the axis normal to the plane of the external forces. It does not twist, and it does not translate in the direction normal to the plane of the external forces. When the external forces and moments acting on a beam do not meet one or more of the requirements described previously, the beam is called a *space beam.*

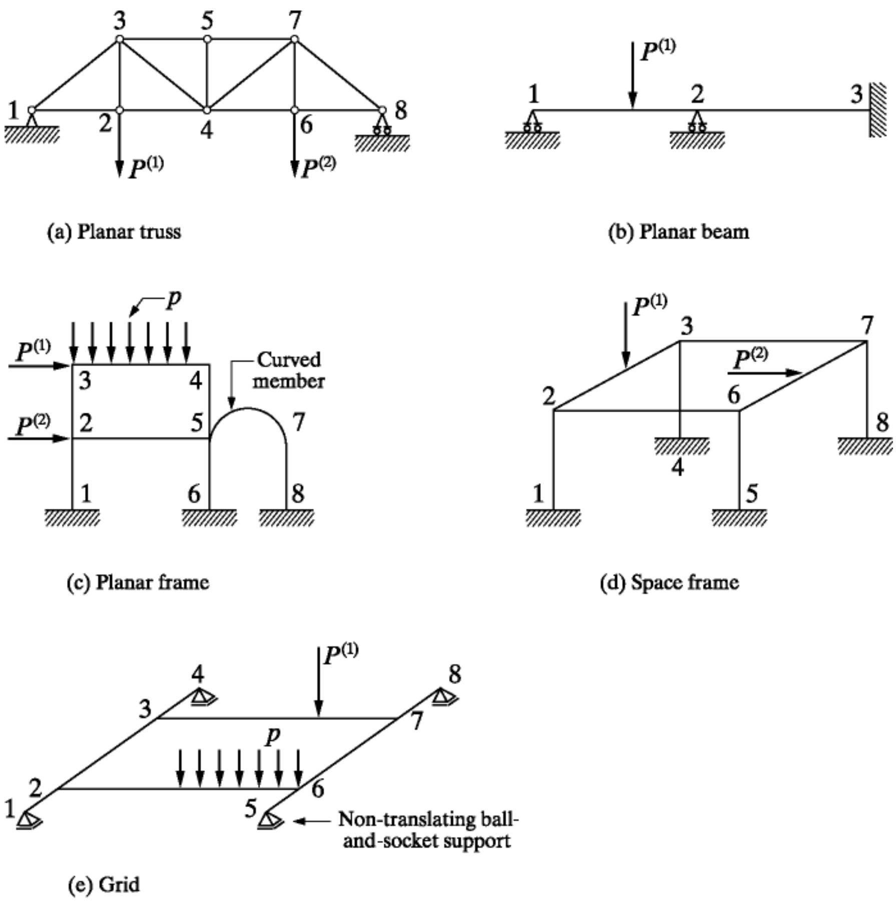


Figure 8.12 Types of framed structures.

Frames are the most general type of framed structures. Their members can be subjected to axial and shearing forces, bending moments and torsional moments. They can have both rigid and non-rigid joints and can be loaded in any way. Usually, frames are space structures. Frequently, however, they can be analyzed by being broken down into planar frames and/or grids (see Fig. 8.12c and e). The members of a planar frame lie in one plane, and one of the principal centroidal axes of their cross sections is normal to this plane. Moreover, the external forces acting on the members of a planar frame lie on a plane which contains the shear centers of their cross sections and is parallel to the plane of the frame. Furthermore, the vector of the external moments is normal to this plane. Thus the cross sections of a member of a planar frame do not twist — they translate only in the plane of the frame and rotate only about an axis normal to the plane of the frame.

The members of a grid also lie in one plane. However, the external forces are normal to the plane of the grid, and the vector of the external moments lies in this plane (see Fig. 8.12e).

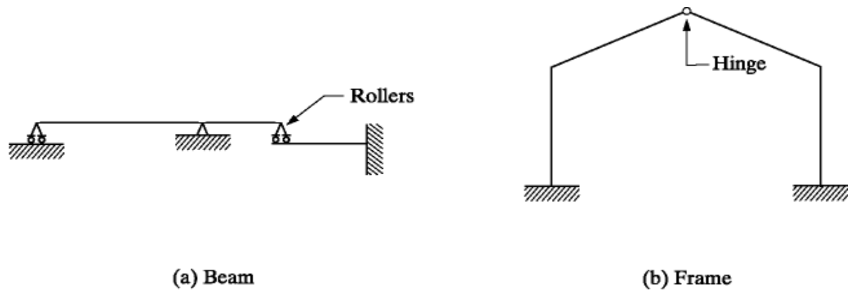


Figure 8.13 Internal action release mechanisms.

8.6 Internal Action Release Mechanisms

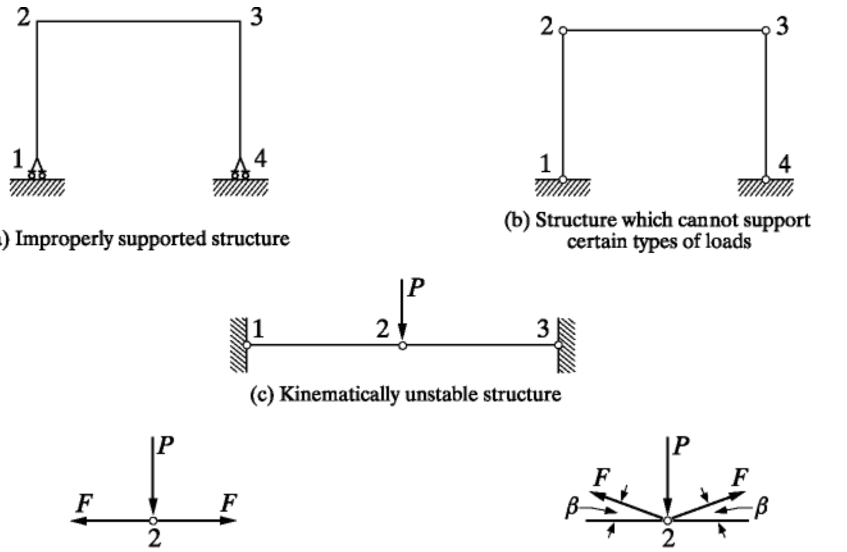
In certain cases, a mechanism is introduced at a point of a member of a beam or frame which renders one or more of the internal actions at this point equal to zero. We refer to this mechanism as an *internal action release mechanism*. For example, the internal action release mechanism (rollers) of the beam of Fig. 8.13a renders the internal axial force and the bending moment equal to zero.

Pinned joints of planar or space beams and frames and ball-and-socket joints of space beams and frames are also internal action release mechanisms. For example, the pin at the apex of the frame of Fig. 8.13b is an internal action release mechanism. It renders the bending moment at that point equal to zero.

8.6.1 Structures Which Constitute a Mechanism

When a structure or a group of its members can move without deforming when subjected to certain types of loads, we say that the structure constitutes a mechanism. We distinguish the following three types of mechanisms:

1. Structures which are not supported properly and can move as rigid bodies under certain types of loading (see Fig. 8.14a).
2. Structures which cannot support certain types of loads because when they are subjected to such loads, some of their members move without deforming until the structure collapses (see Fig. 8.14b).
3. Kinematically unstable structures move instantaneously without deforming when subjected to certain types of loads until they reach a configuration which allows them to carry the applied loads. For example, consider the structure of Fig. 8.14c subjected to a transverse force. Referring to Fig. 8.14d it can be seen that when the transverse force is applied, the members of the structure cannot be subjected to any internal force and, consequently, they do not deform. Thus, the members of the structure rotate instantaneously without deforming until they reach a configuration which allows them to carry the applied force in tension. Consequently, in order to compute the internal force in the members of this structure, their geometry in the deformated state must be taken into account (see Fig. 8.14e).



(a) Improperly supported structure

(b) Structure which cannot support certain types of loads

(c) Kinematically unstable structure

(d) Free-body diagram of joint 2 of the structure of Fig. 8.14c not taking into account the change of direction of this members due to its deformation

(e) Free-body diagram of joint 2 of the structure of Fig. 8.14c not taking intc account the change of direction of this members due to its deformation

Figure 8.14 Structures which constitute a mechanism.

8.7 Statically Determinate and Indeterminate Framed Structures

If the reactions (forces and moments) of a framed structure and the internal actions in its members can be computed using the equations of equilibrium alone, then we say that the structure is *statically determinate*. That is, the number of unknown reactions and internal actions of a statically determinate frame structure is equal to the number of independent equations of statics which can be written for this structure. For example, a beam whose one end is hinged to a rigid support and whose other end is connected to a roller support is statically determinate (see Fig. 8.15c). The first support can exert on the beam a reacting force that has two independent components, while the second support can exert on the beam a reacting force that is normal to the direction of rolling. These reacting forces can be computed by considering the free-body diagram of the beam and using the equations of equilibrium ($\Sigma F_1 = 0$, $\Sigma F_3 = 0$, $\Sigma M_2 = 0$). Moreover, the internal actions acting at any cross section of the beam can be computed by considering the equilibrium of a part of the beam. Additional examples of statically determinate structures are shown in Fig. 8.15. The reactions and internal actions of the members of these structures are obtained by using only the equations of equilibrium.

If the sum of the independent components of the reactions of a structure and of the internal actions of its members exceeds the number of independent equations of equilibrium, we say that the structure is *statically indeterminate.* The degree of static indeterminacy (IND) of a structure is equal to the number by which the sum of the independent components of its reactions and the independent components of the internal actions in its members exceeds the number of independent equations of equilibrium.

The number of independent components of internal actions in a member of a structure is equal to the number of components of internal actions which must be specified in order

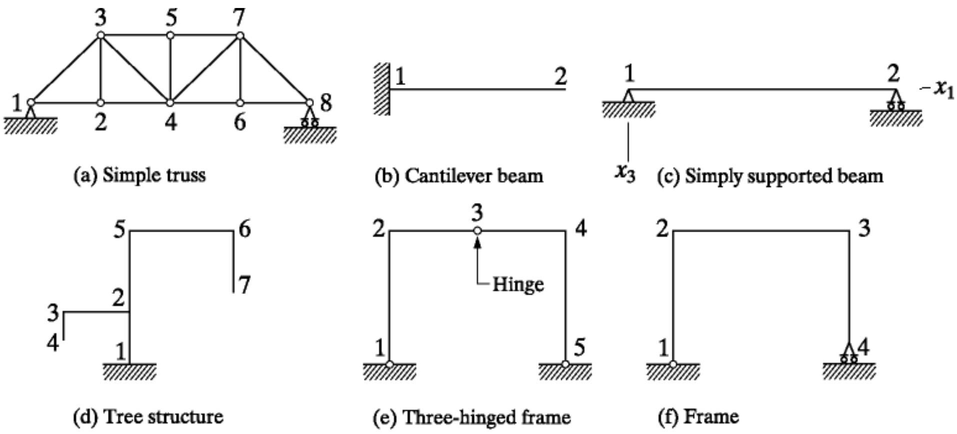


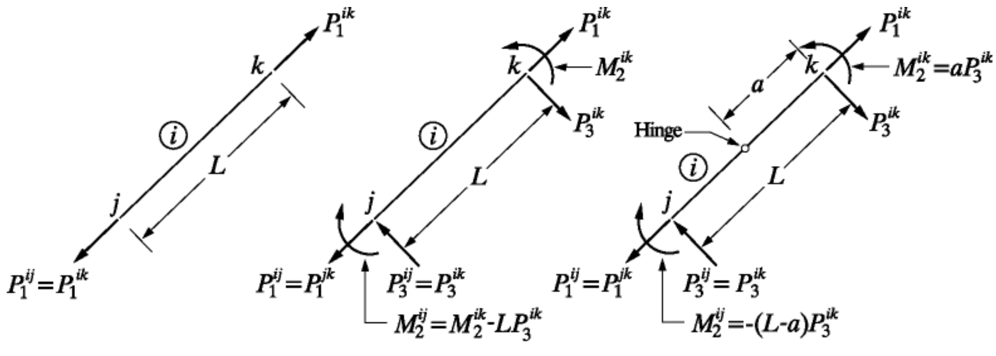
Figure 8.15 Statically determinate structures.

to be able to establish uniquely the internal actions acting at any cross section of the member by considering the equilibrium of appropriate parts of the member. Thus, referring to Fig. 8.16a, there is only one independent action in a member of a planar or space truss (the axial force acting at one end of the member). Moreover, referring to Fig. 8.16b, there are three independent components of internal actions in a member of a planar frame, if internal release mechanisms do not exist in this member. Furthermore, there are six independent components of internal actions in a member of a space beam or frame if internal release mechanisms do not exist in this member. If internal release mechanisms exist in a member, the number of independent components of its internal actions decreases by the number of actions released by these mechanisms. For example, an internal hinge at a point of a member releases (renders it equal to zero) the moment about the axis of the hinge at that point of the member. Thus, the member of a planar structure shown in Fig. 8.16c has two independent components of internal actions because it has a hinge at some point along its length which renders the internal moment at that point equal to zero.

Inasmuch as the structures that we are considering are in equilibrium, the forces acting on their joints must satisfy the equations of equilibrium. The forces acting on a joint of a planar truss must satisfy two equations of equilibrium ($\Sigma F_i = 0$, $i = 1, 3$), while the forces acting on a joint of a space truss must satisfy three equations of equilibrium ($\Sigma F_i = 0$, $i = 1, 2, 3$). Moreover, the forces acting on a joint of a planar beam or frame must satisfy three equations of equilibrium ($\Sigma F_i = 0$, $i = 1, 3$, $\Sigma M_2 = 0$), while the forces acting on a joint of a space beam or frame must satisfy six equations of equilibrium ($\Sigma F_i = 0$, $\Sigma M_i = 0$, $i = 1, 2, 3$). The number of independent equations of equilibrium that can be written for any structure is equal to the sum of independent equations of equilibrium that can be written for its joints. Thus, for a planar truss with NJ joints, we have 2NJ independent equations of equilibrium, while for a space truss with NJ joints we have 3NJ independent equations of equilibrium. Moreover, for a planar beam or frame with NJ joints, we have 3NJ independent equations of equilibrium, while for a space beam or frame with NJ joints we have 6NJ independent equations of equilibrium. Thus, the IND degree of a planar or a space truss having NR independent components of reactions and NM members is equal to

$$IND = NR + NM - 2NJ \qquad \text{for a planar truss} \tag{8.10a}$$

$$IND = NR + NM - 3NJ \qquad \text{for a space truss} \tag{8.10b}$$



(a) Member of a truss (b) Member of a planar beam or frame (c) Member of a planar beam or frame with a hinge

Figure 8.16 Independent components of internal actions in members of planar structures.

Moreover, the *IND* degree of a planar or a space beam or frame having *NR* independent components of reactions, *NM* members and *NAR* actions released by release mechanisms is equal to

$$IND = NR + 3NM - 3NJ - NAR \quad \text{for a planar beam or frame} \tag{8.11a}$$

$$IND = NR + 6NM - 6NJ - NAR \quad \text{for a space beam or frame} \tag{8.11b}$$

Notice, that

1. If *IND* < 0, the structure is a mechanism (Section 8.6.1).

2. If *IND* = 0 or if *IND* ≥ 0, then we cannot say whether the structure is or is not a mechanism. However, if the structure is not a mechanism and *IND* = 0, then it is statically determinate, while if the structure is not a mechanism and *IND* ≥ 0, it is statically indeterminate to the *IND* degree.

Relations (8.10) and (8.11) may be employed to establish the degree of static indeterminacy of a structure. For example, for the planar truss of Fig. 8.17a we have

$$NR = 4 \qquad NJ = 12 \qquad NM = 21 \tag{8.12}$$

Substituting the above values into relation (8.10a), we obtain

$$IND = 4 + 21 - 2(12) = 1 \tag{8.13}$$

Consequently, the truss of Fig. 8.17a is statically indeterminate to the first degree.

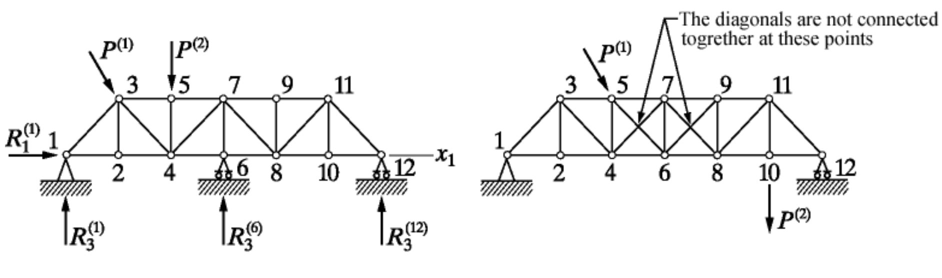


Figure 8.17 Statically indeterminate planar trusses.

8.8 Computation of the Internal Actions of the Members of Statically Determinate Framed Structures

We consider structures originally in a reference stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. Due to application of external loads (surface tractions, body forces, change of temperature or movement of their supports) the structures reach a second state of mechanical equilibrium. In this state the sum of the forces acting on any part of the structure and the sum of their moments about any convenient point vanish. That is,

$$\sum \mathbf{F} = 0 \qquad \sum \mathbf{M} = 0 \tag{8.14a}$$

These equations are called the *equations of statics*.

A vector in a three-dimensional space may be resolved into three components with respect to a rectangular system of axes x_1, x_2, x_3. Moreover, a necessary and sufficient condition for the sum of a number of vectors to be equal to zero is that the sums of their components in the x_1, x_2 and x_3 directions vanish. Thus, equations (8.14a) may be rewritten as

$$\begin{aligned} &\sum F_1 = 0 \qquad \sum F_2 = 0 \qquad \sum F_3 = 0 \\ &\sum M_1 = 0 \qquad \sum M_2 = 0 \qquad \sum M_3 = 0 \end{aligned} \tag{8.14b}$$

In the above equations, $\Sigma F_i (i = 1, 2, 3)$ represents the algebraic sum of the components, along the x_i axis, of the forces acting on the part of the structure under consideration. Moreover, $\Sigma M_i (i = 1, 2, 3)$ represents the algebraic sum of the components, along the x_i axis, of the moments, about any conveniently chosen point, of the actions acting on the part of the structure under consideration.

The internal forces acting on a part of a planar structure do not have a component in the direction normal to the plane of the structure and the moments do not have components in the plane of the structure. Thus, for a planar structure in the x_1x_3 plane the equations of statics (8.14b) reduce to

$$\sum F_1 = 0 \qquad \sum F_3 = 0 \qquad \sum M_2 = 0 \tag{8.15a}$$

Often, it is convenient to employ a different set of equilibrium equations which are equivalent to equations (8.15a). For example, for any part of a planar structure in the x_1x_3 plane, we can use one of the following sets of equilibrium equations:

$$\sum F_1 = 0 \qquad \sum M_2^{(i)} = 0 \qquad \sum M_2^{(j)} = 0 \tag{8.15b}$$

or

$$\sum M_2^{(i)} = 0 \qquad \sum M_2^{(j)} = 0 \qquad \sum M_2^{(k)} = 0 \tag{8.15c}$$

where in relation (8.15b), the moments are taken about two different points i and j which are not located on the same line normal to the x_1 axis. Moreover, in relations (8.15c), the moments are taken about three different points i, j and k which are not located on the same line.

In order to compute its internal actions of the members of a structure we consider the free body diagrams of parts of the structure which usually extend between a cross section of the structure on which unknown internal actions act and a cross section subjected to known actions. The first step when manually computing the internal actions in the members of statically determinate structures is to compute their reactions. For simple structures, this is accomplished by considering the equilibrium of the whole structure. That is, the free-body diagram of the whole structure is drawn, and the equations of equilibrium involving the unknown reactions are written and solved. The graphical representations of the internal axial force, shearing forces and moments acting on a member of a structure are known as its *axial force*, its *shear* and its *moment* diagrams, respectively.

In what follows we present three examples.

Example 1 Compute the force in each member of the simple truss shown in Fig. a.

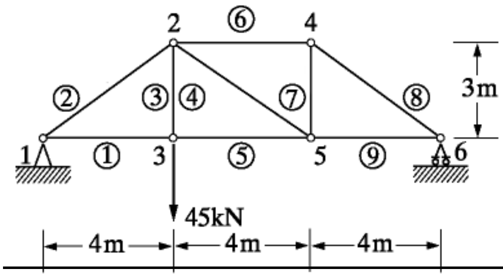


Figure a Geometry and loading of the truss.

Solution First, we compute the reactions of the truss by referring to the free-body diagram of the whole truss shown in Fig. b and write the equations of equilibrium. Thus,

$$\sum F_h = 0 = R_h^{(1)} \tag{a}$$

$$\sum F_v = 0 = R_v^{(1)} + R_v^{(6)} - 45\text{ kN} \tag{b}$$

$$\sum M^{(6)} = 0 = 12R_v^{(1)} - 45(8) \tag{c}$$

or

$$R_v^{(1)} = 30 \text{ kN}$$

Substituting the value of $R_v^{(1)}$ into relation (b), we get

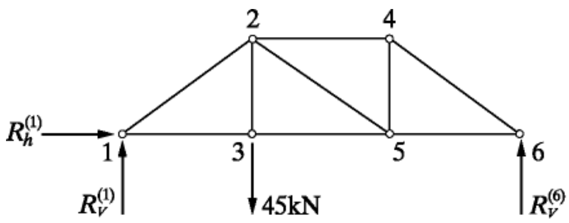


Figure b Free-body diagram of the whole truss.

$$R_v^{(6)} = 15 \text{ kN}$$

The reaction $R_v^{(6)}$ may also be computed by setting the sum of moments about joint 1 equal to zero. Thus,

$$\sum M^{(1)} = 12R_v^{(6)} - 45(4) = 0$$

or

$$R_v^{(6)} = 15 \text{ kN} \qquad \text{check}$$

The coincidence of the above result with that obtained by solving relations (b) and (c) is a check on the calculations of $R_v^{(6)}$. It is advisable to check the values of the reactions prior to proceeding to the calculation of the internal forces in the members of a structure.

The forces in the members of the truss may be obtained by drawing the free-body diagrams of its joints and considering their equilibrium. Notice that in drawing free-body diagrams we assume, for convenience, that all unknown internal forces are tensile forces.

Equilibrium of joint 1

Since the reactions of the truss have been established as shown in Fig. c, there are only two unknown internal forces ($N^{(2)}$, $N^{(1)}$) acting on joint 1; consequently, these forces may be established by considering the equilibrium of this joint. The horizontal and vertical components of the force $N^{(2)}$ may be obtained from the similarity of the force triangle shown in Fig. d and the triangle of the truss 1, 2, 3 (see Fig. a). Thus,

$$N_h^{(2)} = \frac{4}{5}N^{(2)} \qquad N_v^{(2)} = \frac{3}{5}N^{(2)}$$

Consequently, referring to Fig. c and setting equal to zero the sum of the vertical and horizontal components of the forces acting on joint 1, we obtain

$$\sum F_h = 0 = \frac{4}{5}N^{(2)} + N^{(1)}$$

$$\sum F_v = 0 = \frac{3}{5}N^{(2)} + 30$$

Thus,

$$N^{(2)} = -\left(\frac{5}{3}\right)30 = -50 \text{ kN} \quad \text{or} \quad 50 \text{ kN} \quad \text{compression}$$

and

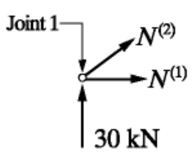


Figure c Free-body diagram of joint 1.

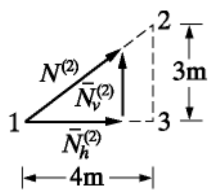


Figure d Global components of force $N(2)$.

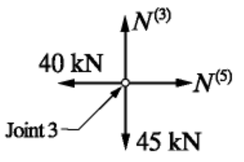


Figure e Free-body diagram of joint 3.

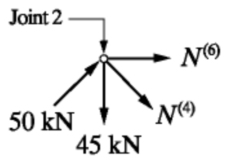


Figure f Free-body diagram of joint 2.

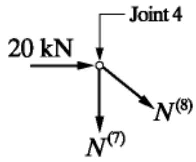


Figure g Free-body diagram of joint 4.

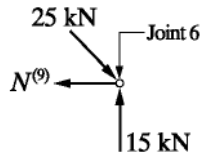


Figure h Free-body diagram of joint 6.

$$N^{(1)} = 40 \text{ kN tension}$$

Equilibrium of joint 3

Since the internal force $N^{(1)}$ has been established, as shown in Fig. e, there are only two unknown internal forces ($N^{(3)}$, $N^{(5)}$) acting on joint 3. Moreover, since the force $N^{(3)}$ and the external force of 45 kN are collinear and normal to the other forces acting at joint 3, it is apparent that

$$N^{(5)} = 40 \text{ kN} \quad \text{tension} \qquad\qquad N^{(3)} = 45 \text{ kN} \quad \text{tension}$$

Equilibrium of joint 2

As shown in Fig. f, since the internal force $N^{(2)}$ and $N^{(3)}$ are known, there are only two unknown forces ($N^{(6)}$, $N^{(4)}$) acting on joint 2. Consequently, these forces may be established by considering the equilibrium of joint 2. Thus,

$$\sum F_h = 0 = \left(\frac{4}{5}\right) 50 + N^{(6)} + \frac{4}{5} N^{(4)}$$

$$\sum F_v = 0 = \left(\frac{3}{5}\right) 50 - 45 - \frac{3}{5} N^{(4)}$$

or

$$N^{(4)} = -25 \text{ kN} \quad \text{or} \quad 25 \text{ kN compression}$$

$$N^{(6)} = -20 \text{ kN} \quad \text{or} \quad 20 \text{ kN compression}$$

Equilibrium of joint 4

Referring to Fig. g, we have

$$\sum F_h = 0 \qquad \frac{4}{5}N^{(8)} + 20 = 0 \qquad \text{or} \qquad N^{(8)} = 25 \text{ kN} \quad \text{compression}$$

$$\sum F_v = 0 \qquad \frac{3}{5}N^{(8)} + N^{(7)} = 0 \qquad \text{or} \qquad N^{(7)} = 27 \text{ kN} \quad \text{tension}$$

Equilibrium of joint 6

Referring to Fig. h, we have

$$\sum F_h = 0 = \left(\frac{4}{5}\right)25 + N^{(9)} \qquad \text{or} \qquad N^{(9)} = 20 \text{ kN} \quad \text{tension}$$

$$\sum F_v = 0 = \left(\frac{3}{5}\right)25 - 15 \qquad \text{check}$$

Example 2 Compute the distribution of the internal shearing force $Q_3(x_1)$ and bending moment $M_2(x_1)$ acting on the cross sections of the one member structure (beam) subjected to the transverse forces and bending moment shown in Fig. a. The plane of the transverse forces contains the shear centers of the cross sections of the beam.

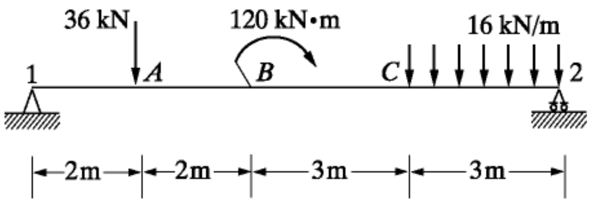


Figure a Geometry and loading of the structure.

Solution Since the structure under consideration does not have an unsupported end, it is necessary to compute its reacting forces before we establish the distribution of its internal actions. Thus, referring to Fig. b, we have

$$\sum M_2^{(1)} = 10R^{(2)} - 2(36) - 120 - 16(3)(8.5) = 0 \qquad \text{or} \qquad R^{(2)} = 60 \text{ kN} \tag{a}$$

$$\sum M_2^{(2)} = 10R^{(1)} - (36)(8) + 120 - 16(3)(1.5) = 0 \qquad \text{or} \qquad R^{(1)} = 24 \text{ kN} \tag{b}$$

$$\sum F_v = R^{(1)} + R^{(2)} = 24 + 60 = 36 + 3(16) = 84 \text{ kN check} \tag{c}$$

The shearing force $Q_3(x_1)$ and the bending moment $M_2(x_1)$ acting on the cross sections of the structure are obtained by considering the equilibrium of its parts shown in Fig. c.

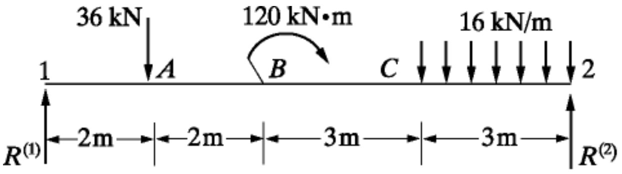


Figure b Free-body diagram of the structure.

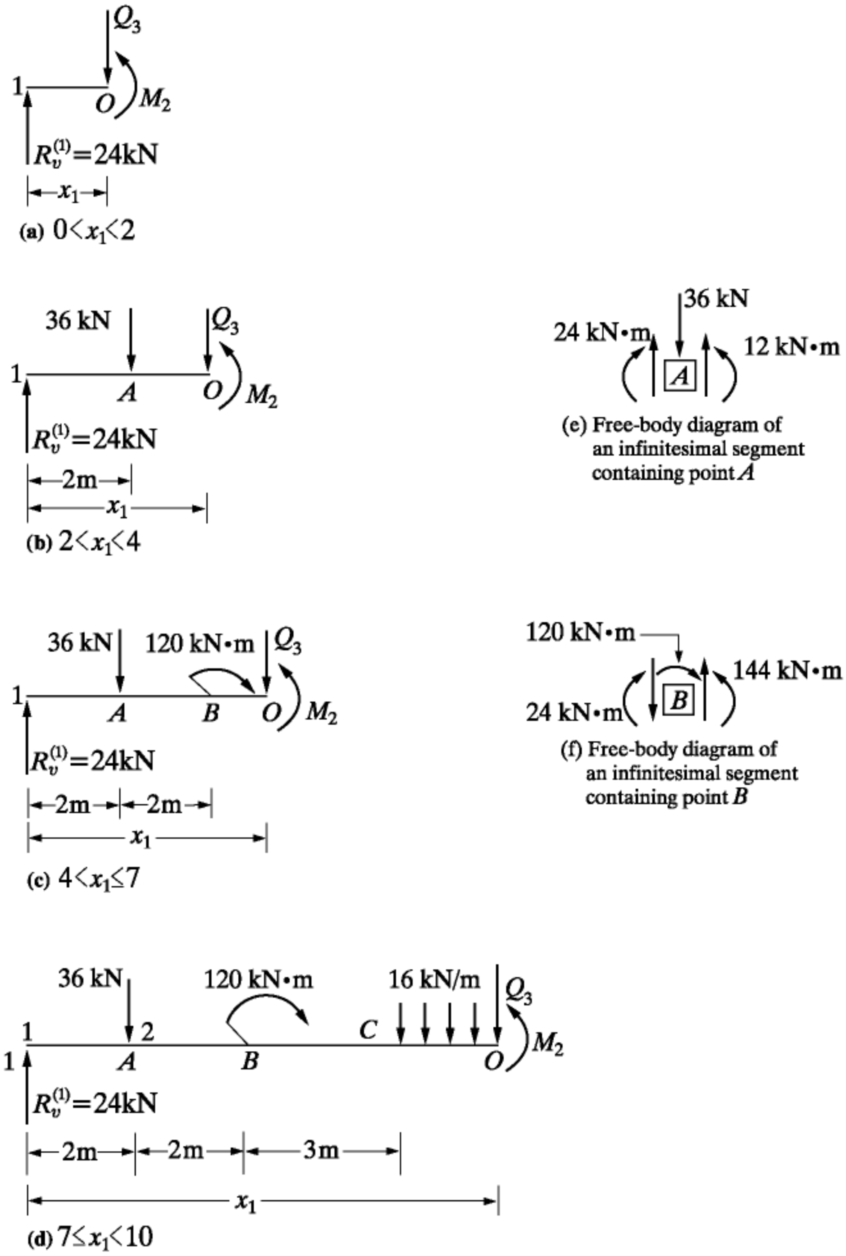


Figure c Free-body diagrams of parts of the structure.

Thus, referring to Fig. ca for $0 < x_1 < 2$, we have

$$\sum F_v = 0 \qquad Q_3(x_1) = 24 \text{ kN}$$
$$\sum M_2^{(o)} = 0 \qquad M_2(x_1) = 24x_1 \tag{d}$$

Referring to Fig. cb for $2 < x_1 < 4$, we get

$$\sum F_v = 0 \qquad Q_3(x_1) = 24 - 36 = -12 \text{ kN} \tag{e}$$

$$\sum M_2^{(o)} = 0 \qquad M_2(x_1) = 24x_1 - 36(x_1 - 2) = 72 - 12x_1 \tag{f}$$

Referring to Fig. cc for $4 < x_1 \le 7$, we obtain

$$\sum F_v = 0 \qquad Q_3 = 24 - 36 = -12 \text{ kN} \tag{g}$$

$$\sum M_2^{(o)} = 0 \qquad M_2(x_1) = 24x_1 - 36(x_1 - 2) + 120 = -12x_1 + 192 \tag{h}$$

Referring to Fig. cd for $7 \le x_1 < 10$, we have

$$\sum F_v = 0 \qquad Q_3 = 24 - 36 - 16(x_1 - 7) = 100 - 16x_1 \tag{i}$$

$$\sum M_2^{(2)} = 0 \qquad M_2(x_1) = 24x_1 - 36(x_1 - 2) + 120 - \frac{16(x_1 - 7)^2}{2} = -200 + 100x_1 - 8x_1^2 \tag{j}$$

We plot relations (c) to (j) in Figs. d and e. These are the shearing force and bending moment diagrams for the beam of Fig. a.

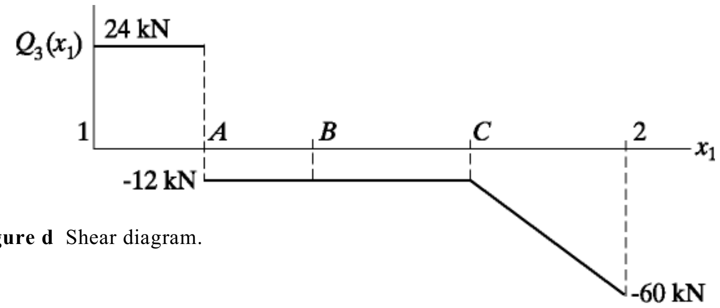


Figure d Shear diagram.

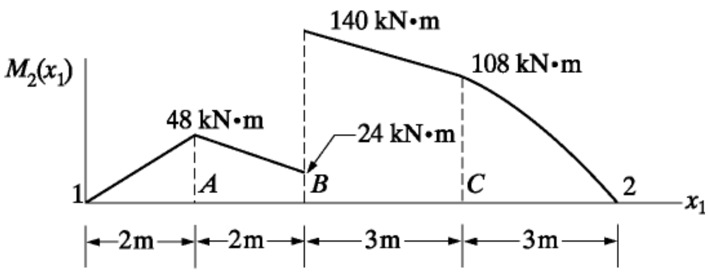


Figure e Bending moment diagram.

Notice that the distribution of the external forces acting on this structure has three points of discontinuity (points A, B and C). As a consequence of this, the distribution of the internal shearing force $Q_3(x_1)$ and bending moment $M_2(x_1)$ acting on the cross sections of the structure is specified by four different functions of x_1 — one for each of the segment $1A$, AB, BC and $C2$ [see relations (c) to (j)]. Moreover, notice that at the point of application of the concentrated force of 36 kN there is a discontinuity or jump of 36kN in the shear diagram, where as at the point of application of the concentrated moment of 120 kN·m there is a discontinuity or jump of 120 kN·m in the bending moment diagram.

Example 3 Compute the internal actions acting on the cross sections of the members of the machine component of Fig. a as functions of their axial coordinate. The machine component is subjected to the external force shown in Fig. a. Plot the moment and shear diagrams for the members of this structure.

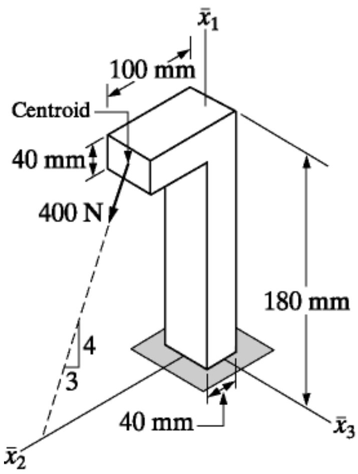


Figure a Geometry and loading of the machine component and the chosen global axes.

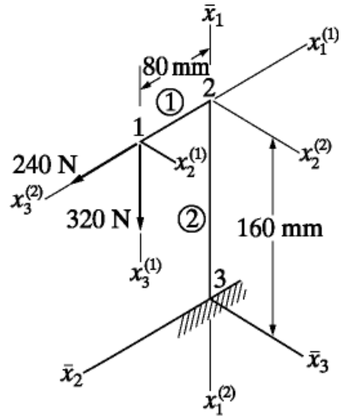


Figure b Line diagram of the machine component and the chosen local axes for its members.

Solution In Fig. b we show the line diagram of the machine component of Fig. a. Moreover, in Fig. b we give the components of the applied force in the direction of the axes $\bar{x}_1$ and $\bar{x}_2$. In Fig. c we show the free-body diagram of a part of member 1 with the unknown of internal actions at point X as assumed positive; referring to this figure, we have

$$\begin{aligned} \sum F_1 &= 0 \qquad & N^{(1)} &= 240 \text{ N} \\ \sum F_3 &= 0 \qquad & Q_3^{(1)} &= -320 \text{ N} \\ \sum M_2^{(X)} &= 0 \qquad & M_2^{(1)} &= -320x_1 \text{ N}\cdot\text{mm} \end{aligned} \tag{a}$$

In Fig. d we show the free-body diagram of a part of member 2. The internal forces acting at point 2 of member 2 have been obtained from those of point 2 of member 1 using the principle of equal and opposite action and reaction. The unknown internal actions at point X are shown as assumed positive (see Fig. 8.9). Referring to Fig. d we have

$$\sum F_1 = 0 \qquad N^{(2)}(x_1^{(2)}) = -320 \text{ N} \tag{b}$$

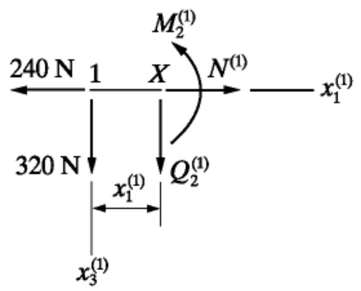


Figure c Free-body diagram of a part of member 1.

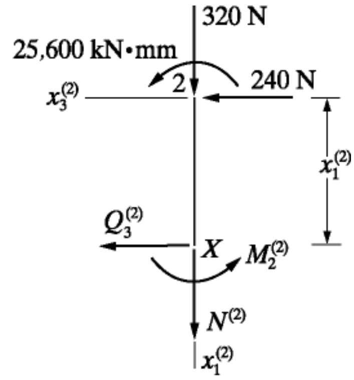


Figure d Free-body diagram of a part of member 2.

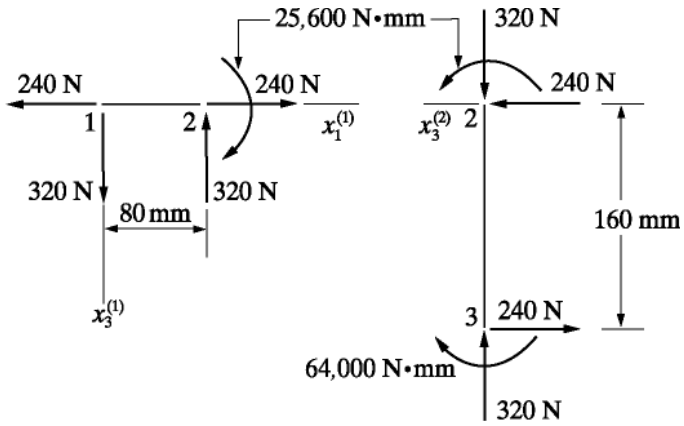


Figure e Free-body diagrams of the members of the machine component of Fig. a .

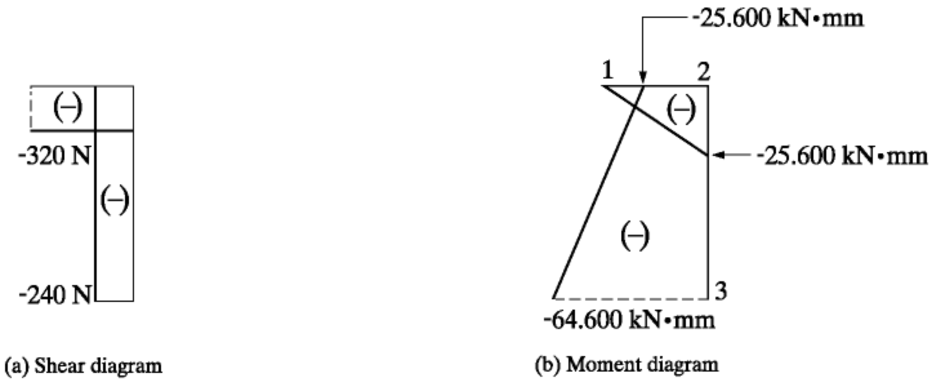


Figure f Shear and moment diagrams of the machine component.

$$\sum F_3 = 0 \qquad Q_3^{(2)}(x_1^{(2)}) = -240 \text{ N}$$
$$\sum M_2^{(0)} = 0 \qquad M_2^{(2)}(x_1^{(2)}) = -25{,}600 - 240x_1^{(2)} \tag{c}$$

In Fig. e we draw the free-body diagrams of the members of the machine component of Fig. a. In Fig. f we plot their shear and moment diagrams.

8.9 Action Equations of Equilibrium for Line Members

Consider a line member subjected to external concentrated forces $P_i^{((n)}$ ($i = 1, 2, 3$) ($n = 1, 2, ..., n_i$), and moments $M_i^{((m)}$ ($i = 1, 2, 3$)($m = 1, 2, ..., m_i$) and to external distributed forces $p_i(x_1)$ ($i = 1, 2, 3$) and moments $m_i(x_1)$ ($i = 1, 2, 3$) given in units of force or moment per unit length. The forces $P_1^{(j)}$ and $p_1(x_1)$ are axial centroidal. The forces $P_i^{(j)}$ and $p_i(x_1)$ ($i = 2, 3$) act in the direction of the x_i ($i = 2, 3$) axis. The line of action of each of these forces lies in planes which contain the shear center of the cross sections of the member. Moreover, consider a segment of this member of length Δx_1 at point B (see Fig. 8.18). This segment is subjected only to distributed forces $p_i(x_1)$ ($i = 1, 2, 3$) and moments $m_i(x_1)$ ($i = 1, 2, 3$) which are represented by one expression in the neighborhood of point B. The distributed moments are not shown in Fig. 8.18 in order to avoid cluttering it. Since the segment is in equilibrium, we have

$$\sum F_1 = 0 = -N + p_1\,\Delta x_1 + N + \Delta N$$
$$\sum F_2 = 0 = -Q_2 + p_2\,\Delta x_1 + Q_2 + \Delta Q_2$$
$$\sum F_3 = 0 = -Q_3 + p_3\,\Delta x_1 + Q_3 + \Delta Q_3$$
$$\sum M_1^{(A)} = 0 = -M_1 + m_1\Delta x_1 + M_1 + \Delta M_1$$
$$\sum M_2^{(A)} = 0 = -M_2 + m_2\Delta x_1 + M_2 + \Delta M_2 + \frac{p_3(\Delta x_1)^2}{2} - Q_3\Delta x_1$$
$$\sum M_3^{(A)} = 0 = -M_3 + m_3\Delta x_1 + M_3 + \Delta M_3 - \frac{p_2(\Delta x_1)^2}{2} + Q_2\Delta x_1$$

Dividing by Δx_1 and taking the limit as $\Delta x_1 \to 0$, the above relations reduce to

$$p_1(x_1) = -\frac{dN}{dx_1} \tag{8.16}$$

$$p_2(x_1) = -\frac{dQ_2}{dx_1} \tag{8.17}$$

$$p_3(x_1) = -\frac{dQ_3}{dx_1} \tag{8.18}$$

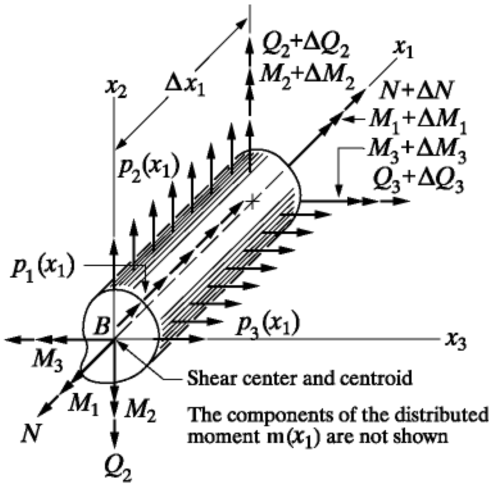


Figure 8.18 Segment of a member with positive internal and external actions.

$$m_1(x_1) = -\frac{dM_1}{dx_1} \tag{8.19}$$

$$Q_2(x_1) = -\frac{dM_3}{dx_1} - m_3 \tag{8.20}$$

$$Q_3(x_1) = \frac{dM_2}{dx_1} + m_2 \tag{8.21}$$

Differentiating relations (8.20) and (8.21) and substituting equations (8.17) and (8.18), respectively, in the resulting relations, we obtain

$$\frac{d^2M_3}{dx_1^2} + \frac{dm_3}{dx_1} = p_2 \tag{8.22}$$

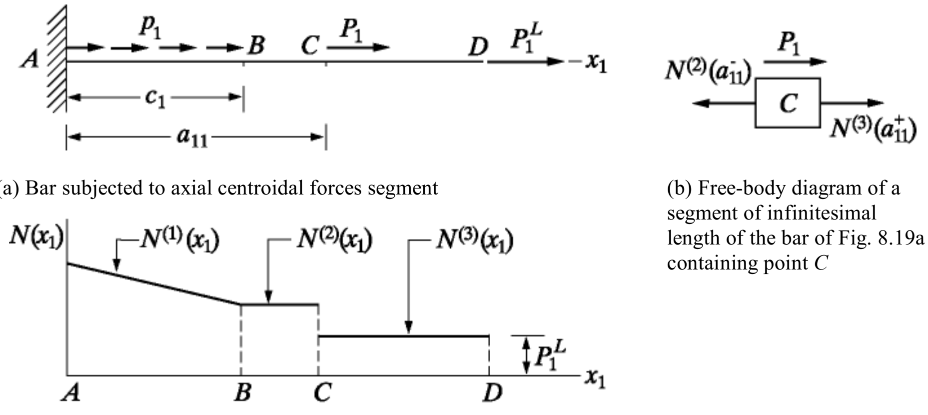
$$\frac{d^2M_2}{dx_1^2} + \frac{dm_2}{dx_1} = -p_3 \tag{8.23}$$

Relation (8.16) is the action equation of equilibrium for a line member subjected to axial centroidal forces. Relation (8.19) is the action equation of equilibrium for a line member subjected to torsional moments. Relations (8.22) and (8.23) are the action equations of equilibrium for a member subjected to transverse forces and bending moments. The line of action of each of these forces lies in a plane which contains the shear centers of the cross sections of the member. *Each one of equations (8.16) to (8.23) must be satisfied by the distribution of the corresponding internal action at every point of a line member in equilibrium which is not a point of application of the corresponding component of a concentrated external action or a point at which the expression specifying the corresponding component of distributed external actions changes.*

In practice it is difficult to apply to a member distributed moments m_2 and m_3. For this reason they are usually omitted. We retain them in this and Chapter 9 in order to explain in Section 9.3 how a function of discontinuity known as the doublet function (see Appendix G) is used to convert concentrated moments to mathematically equivalant distributed moments.

The external actions (forces and moments) acting on a member can be distributed and/or concentrated. The components of the distributed external actions are functions of x_1 specified by one or more expressions. Referring to relations (8.16) to (8.23), we see that at the points at which the expression that specifies a component of the external distributed actions changes, the expression specifying the corresponding component of internal action also changes. Moreover, at the points at which the expression specifying the internal shearing force Q_2 or Q_3 changes, the expression specifying the internal bending moment M_3 or M_2, respectively, also changes. Furthermore, it can be shown that the points at which a component of a concentrated external action is applied the corresponding component of internal action has a simple discontinuity or jump equal to the magnitude of the component of the concentrated external action. We call the points of a member at which the expressions which specify the distributed external actions change and the points at which concentrated external actions are applied, *points of load change*. Thus, a component of internal action of a member is specified by a set of expressions having continuous derivatives of any order, each of which represents the component of internal action over a segment of the member, which extends between a support and its adjacent point of load change or between two consecutive points of load change. Each one of the sets of the expressions specifying an internal action must satisfy the corresponding equation of equilibrium (8.16) to (8.23) at all points of the corresponding segment of the member. This ensures the satisfaction of the requirements for equilibrium of the actions acting on every segment of infinitesimal length of the member cut by two imaginary planes normal to its axis except the segments of infinitesimal length which contain the points of load change. Moreover, each one of the sets of expressions specifying an external action must satisfy the requirements for equilibrium of the segments of infinitesimal length which contain the points of load change.

For example, the expression describing the external axial centroidal force $p_1(x_1)$ acting on the bar of Fig. 8.19a changes at point B. Moreover, there is a concentrated axial centroidal force acting at point C. Thus, the internal axial centroidal force acting on the



(a) Bar subjected to axial centroidal forces segment

(b) Free-body diagram of a segment of infinitesimal length of the bar of Fig. 8.19a containing point C

(c) Axial force diagram for the bar of Fig. 8.19a.

Figure 8.19 Line members subjected to axial centroidal forces.

cross sections of the bar of Fig. 8.19a is represented in each of its segments $\overline{AB}$, $\overline{BC}$ or $\overline{CD}$, by the continuously differentiable function $N^{(1)}(x_1)$, $N^{(2)}(x_1)$ or $N^{(3)}(x_1)$, respectively. In order to ensure that the forces acting on each segment of infinitesimal length of the bar, except those which contain one of the points of load change B and C, are in equilibrium, referring to relation (8.16), the functions $N^{(1)}(x_1)$, $N^{(2)}(x_1)$ or $N^{(3)}(x_1)$ must satisfy the following equilibrium equations:

$$\frac{dN^{(1)}}{dx_1} = -0 \frac{\text{kN}}{\text{m}} \qquad 0 < x_1 < c_1 \tag{8.24a}$$

$$\frac{dN^{(2)}}{dx_1} = 0 \qquad c_1 \leq x_1 \leq a_1 \tag{8.24b}$$

$$\frac{dN^{(3)}}{dx_1} = 0 \qquad a_1 \leq x_1 \leq L \tag{8.24c}$$

Moreover, in order to ensure that the forces acting on each segment of infinitesimal length of the bar, which contains point B or C, are in equilibrium, referring to Fig. 8.19b, the functions $N^{(j)}(x_1)$ ($j = 1, 2, 3$) must satisfy the following equilibrium relations:

$$N^{(1)}(c_1) = N^{(2)}(c_1) \tag{8.24d}$$

$$N^{(2)}(a_{11}^-) = N^{(3)}(a_{11}^+) + P_1 \tag{8.24e}$$

It should be emphasized that the satisfaction of relations (8.16) to (8.23) by a distribution of the components of internal actions in a member does not ensure that every particle of the member is in equilibrium. As we pointed out previously in the theories of mechanics of materials it is assumed that it is not necessary to satisfy this requirement for every particle inside the volume of a member or on its boundary. In fact in some cases the components of stress obtained on the basis of the theories of mechanics of materials do not satisfy this requirement at all particles of a member.

8.10 Shear and Moment Diagrams for Beams by the Summation Method

The method of plotting shear and moment diagrams for beams presented in Section 8.8 is time-consuming. For this reason, in this section we describe a more convenient method in which the relation between the external distributed force and the shearing force (8.17) or (8.18), and the relation between the shearing force and the bending moment (8.20) or (8.21) are employed. These relations have been derived in Section 8.9 by considering the equilibrium of an infinitesimal segment of the beam. For a planar beam in the $x_1 x_3$ plane they are

$$p_3 = -\frac{dQ_3}{dx_1} \tag{8.25}$$

$$Q_3 = \frac{dM_2}{dx_1} \tag{8.26}$$

Consider the planar beam shown in Fig. 8.20, subjected to external actions. If we integrate relation (8.25) over an arbitrary interval of the length of the beam from a point $x_1 = a$ to a point $x_1 = b$, we obtain

$$\int_a^b dQ_3 = -\int_a^b p_3 dx_1 \tag{8.27a}$$

Referring to Fig. 8.20a, we see that the quantity $p_3\, dx_1$ represents the cross-hatched portion of the area under the diagram of external distributed forces. Thus, the right integral of relation (8.27a) is equal to the area under the diagram of external distributed forces from point $x_1 = a$ to point $x_1 = b$. If the shearing force Q_3 is a continuous function of x_1 between points $x_1 = a$ and $x_1 = b$ relation (8.27a) gives

$$Q_3(b) - Q_3(a) = -\text{ (Area under the diagram of external forces from } x_1 = a \text{ to } x_1 = b) \tag{8.27b}$$

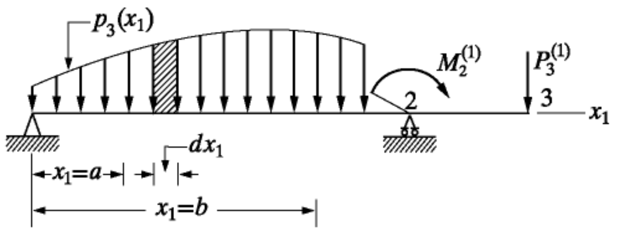
The shearing force is discontinuous only at the points of a member where a concentrated external force acts (see Fig. c of Example 2 of Section 8.8). Thus, if we know the shearing force $Q_3(a)$ at a point $x_1 = a$ of a beam and if a concentrated force does not act on the beam in the interval from $x_1 = a$ to $x_1 = b$, then from relation (8.27b), we can find directly the shearing force $Q_3(b)$ at point $x_1 = b$ of the beam.

Integrating relation (8.26) from $x_1 = a$ to $x_1 = b$, we get

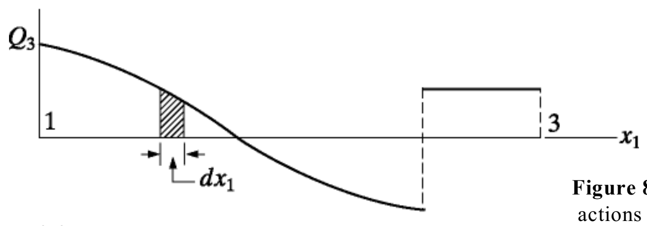
$$\int_a^b dM_2 = \int_a^b Q_3 dx_1 \tag{8.28a}$$

Referring to Fig. 8.20b, we see that the quantity $Q_3 dx_1$ represents the cross-hatched portion of the area under the shear diagram. Thus, the integral at the right side of relation (8.28a) is equal to the area under the shear diagram from point $x_1 = a$ to $x_1 = b$. If the moment M_2 is a continuous function of x_1 between points $x_1 = a$ and $x_1 = b$, relation (8.28a) gives

$$M_2(b) - M_2(a) = \text{(area under the shear diagram from } x_1 = a \text{ to } x_1 = b) \tag{8.28b}$$



(a) Geometry and loading of the beam



(B) Shear diagram

Figure 8.20 Beam subjected to external actions and its shear diagram.

The moment is discontinuous only at its points where a concentrated external moment acts (see Fig. e of Example 2 of Section 8.8). Thus, if we know the moment $M_2(a)$ at any point $x_1 = a$ of a beam, and if a concentrated moment does not act on the beam in the interval from $x_1 = a$ to $x_1 = b$, we can find directly from relation (8.28b) the moment $M_2(b)$ at point $x_1 = b$ of the beam.

On the basis of the foregoing presentation, it is clear that the shear and moment diagrams for a beam can be plotted starting from the one end (usually the left) of the beam where the shear and moment are known and proceeding to the other end using relations (8.27b) and (8.28b), respectively. This method of plotting the shear and moment diagrams is referred to as the *summation method.*

In what follows we plot the shear and moment diagrams of a beam by the summation method.

Example 4 Plot the shear and moment diagrams of the planar beam shown in Fig. a.

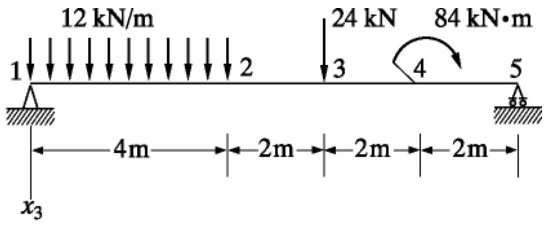


Figure a Geometry and loading of the beam.

Solution We first compute the reactions of the beam. Referring to the free-body diagram of the beam shown in Fig. b, we have

$$\sum M_2^{(1)} = (12)(4)(2) + 24(6) + 84 - 12R_v^{(5)} = 0 \quad \text{or} \quad R_v^{(5)} = 27 \text{ kN} \tag{a}$$

$$\sum M_2^{(5)} = (12)(4)(10) + 24(6) - 84 - 12R_v^{(1)} = 0 \quad \text{or} \quad R_v^{(1)} = 45 \text{ kN} \tag{b}$$

$$\sum F_v = (12)(4) + 24 = 27 + 45 \quad \text{check} \tag{c}$$

Referring to Fig. b we know that at point 1 the shearing force is equal to the reaction $R_v^{(1)}$ (45 kN). Using relation (8.27b), we find that the shearing force at point 2 is smaller than at point 1 by 48 kN which is the area under the diagram of the external forces from point 1 to point 2; that is, the shearing force at point 2 is equal to 45 − 48 = −3 kN. The shear diagram from point 1 to point 2 is a straight line inasmuch as the external force, which is equal to minus the slope of the shear diagram, is constant ($dQ_3/dx_1 = -p_3 = -12$

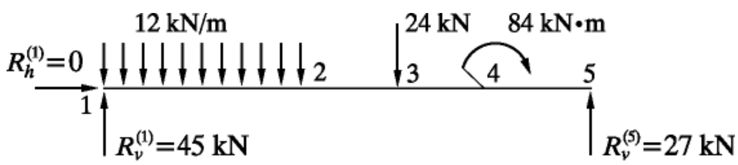


Figure b Free-body diagram of the beam.

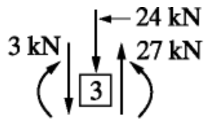


Figure c Free-body diagram of an infinitesimal segment of the beam containing point 3.

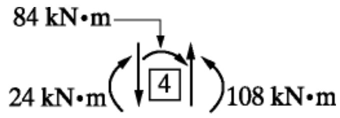


Figure d Free-body diagram of an infinitesimal segment of the beam containing point 4.

kN/m). From point 2 to point 3 the shearing force remains constant since the area under the diagram of external forces is zero. At point 3 there is a discontinuity in the shear diagram equal to the concentrated external force of 24 kN. The shearing force changes from −3 to the left of point 3 to −27 kN to the right of point 3. This becomes clear by considering the free-body diagram of the segment of infinitesimal length which includes point 3 of the beam shown in Fig.c. On the left face of this segment a shearing force of 3 kN acts downward. It is apparent that an upward shearing force of 27 kN must act on its right face in order for the segment to be in equilibrium. From point 3 to point 5 the shearing force remains constant and equal to −27 kN. Thus, the value of the shearing force at point 5 is equal to the value of the reaction at this support. This in fact is a check of the algebra required for computing the shearing forces. The shear diagram is plotted in Fig. e. Notice that the shearing force Q_3 vanishes at point A located between points 1 and 2. Referring to equation (8.26) we see that at this point $dM_2/dx_1 = 0$. Consequently, at this point $M_2(x_1)$ assumes an extremum value which is maximum in this case. The position of point A may be established from the similarity of the two triangles under the shear diagram between points 1 and A and points A and 2 [$x_1^{(A)}/45 = (4 - x_1^{(A)})/3$]. That is, the distance $x_1^{(A)}$ from point 1 to point A is equal to 3.75 m. Thus, the area of the shear diagram from point 1 to A is equal to (45) (3.75/2) = 84.375 kN·m whereas the area under the shear diagram from point A to point 2 is equal to (3)(0.25)/2 = 0.375 kN·m.

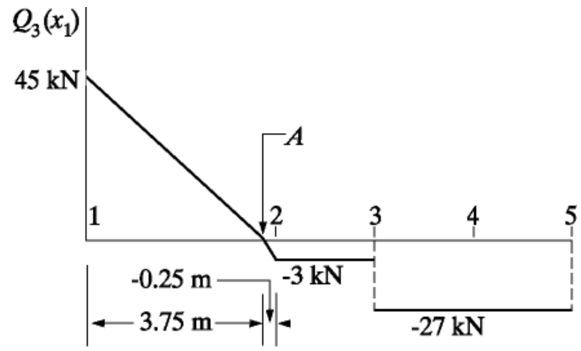


Figure e Shear diagram.

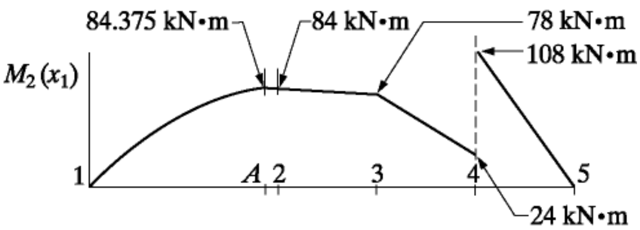


Figure f Moment diagram.

Referring to Fig. b we see that the moment at point 1 is zero. Therefore, referring to Fig. e and using equation (8.28b) at point A the moment is equal to 84.375 kN·m which is the area under the shear diagram from point 1 to point A. At point 2 the moment is equal to the moment at point A plus the area under the shear diagram from point A to point 2 which is equal to -0.375 kN·m. Therefore, the moment at point 2 is equal to 84.375 − 0.375 = 84.0 kN·m.

From point 1 to point 2 the moment diagram is a second degree curve because the shear diagram in this interval is a linear function of x_1 ($dM_2/dx_1 = Q_3$). Inasmuch as Q_3 becomes less positive as x_1 increases, the slope of the moment diagram must decrease with x_1. The moment at point 3 differs from the moment at point 2 by −6 kN·m which is the area under the shear diagram from point 2 to point 3. That is, the moment at point 3 is 78 kN·m. Since $Q_3 = dM_2/dx_1$ is constant from point 2 to point 3, the moment diagram between these two points is a straight line. Similarly, the moment just to the left of point 4 differs from the moment at point 3 by −54 kN·m which is the area under the shear diagram from point 3 to point 4. Therefore, the moment just to the left of point 4 is 24 kN·m. At point 4 there is a discontinuity in the moment diagram of 84 kN·m. The moment changes from 24 kN·m just to the left of point 4 to 108 kN·m just to the right of point 4. This becomes clear by considering the equilibrium of the infinitesimal segment of the beam, which includes point 4, whose free-body diagram is shown in Fig. d. A clockwise moment of 24 kN·m acts on the left face of this segment. For the equilibrium of the segment a counterclockwise moment of 108 kN·m must act on its right face. From point 4 to point 5 the shearing force is constant. Consequently, the moment diagram is a straight line. The moment at point 5 is equal to the moment just to the right of point 4 plus the area under the shear diagram from point 4 to point 5. This area is equal to −108 kN·m. That is, the moment at point 5 is equal to zero. This concurs with the actual moment at the pin end of the beam and it is a partial check of the algebra done in plotting the moment diagram. The moment diagram is plotted in Fig. f.

8.11 Stress–Strain Relations for a Particle of a Line Member Made from an Isotropic Linearly Elastic Material

The relations among the components of stress and strain of a particle of a line member made from a homogeneous, isotropic, linearly elastic material are obtained by substituting relations (8.1a) into the stress–strain relations (3.93). That is,

$$\begin{aligned} e_{11} &= \frac{\tau_{11}}{E} + \alpha(T - T_0) \\ e_{22} &= -\frac{\nu\tau_{11}}{E} + \alpha(T - T_0) \\ e_{33} &= -\frac{\nu\tau_{11}}{E} + \alpha(T - T_0) \end{aligned}$$

$$e_{12} = \frac{\tau_{12}}{2G} \qquad e_{13} = \frac{\tau_{13}}{2G} \qquad e_{23} = 0 \tag{8.29}$$

In general it is not possible to measure the temperature $T(x_1, x_2, x_3)$ of the particles of a member which are not located on its surface. For this reason in what follows we express the temperature of any particle of a member in terms of the temperature of certain particles located on the surface of this member. For this purpose we introduce the following notation:

1. $T_2^{(+)}(x_1)$ and $T_2^{(-)}(x_1)$ are the temperatures at the points of the two lines of intersection of the plane $x_1\ x_2$ and the lateral surface of the member (see Fig. 8.21a). We define the difference in temperature $\delta T_2(x_1)$ as

$$\delta T_2(x_1) = T_2^{(+)}(x_1) - T_2^{(-)}(x_1) \tag{8.30}$$

2. $T_3^{(+)}(x_1)$ and $T_3^{(-)}(x_1)$ are the temperatures at the points of the two lines of intersection of the plane $x_1\ x_3$ and the lateral surface of the member (see Fig. 8.21a). We define the difference in temperature $\delta T_3(x_1)$ as

$$\delta T_3(x_1) = T_3^{(+)}(x_1) - T_3^{(-)}(x_1) \tag{8.31}$$

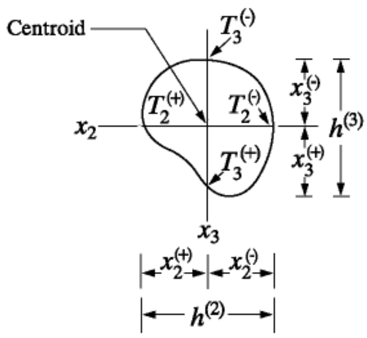
3. $T_c(x_1)$ is the temperature of the points of the centroidal axis of a member. We define the change of temperature $\Delta T_c(x_1)$ at the points of the centroidal axis of a member as

$$\Delta T_c(x_1) = T_c(x_1) - T_o \tag{8.32}$$

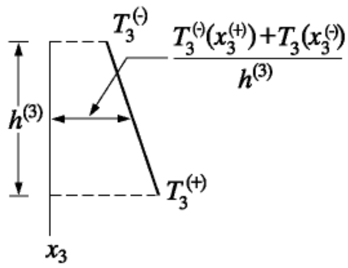
4. We denote α is the coefficient of linear thermal expansion of the material from which the member is made. Thus, the change in length of a fiber of a member of length dx_1 due to an increase of temperature ΔT is equal to

$$\Delta L = \alpha\, \Delta T \mathrm{d}x_1 \tag{8.33}$$

We assume that the variation of the temperature $T(x_1, x_2, x_3)$ is such that plane sections normal to the axis of a member, before the temperature changes, remain plane after the temperature changes. This implies that the change of temperature is a linear function of the coordinates x_2 and x_3 (see Fig. 8.21b), and consequently, it can be expressed as



(a) Temperature at the points of intersection of the x_1x_2 or x_1x_3 plane with the lateral surface of a member



(b) Distribution of temperature along the x_3 axis

Figure 8.21 Distribution of temperature on a cross section of a member.

$$T - T_o = A_1 + A_2x_2 + A_3x_3 \tag{8.34}$$

where A_1, A_2, A_3 are functions of x_1 only. Referring to relation (8.34) the change of temperature ΔT_c at the points of the centroidal axis ($x_2 = x_3 = 0$) of a member is equal to

$$\Delta T_c = T_c - T_o = A_1 \tag{8.35}$$

Referring to Fig. 8.21a, we see that

$$T = \begin{cases} T_2^{(+)} & \text{at} \quad x_3 = 0, x_2 = x_2^{(+)} \\ T_2^{(-)} & \text{at} \quad x_3 = 0, x_2 = -x_2^{(-)} \end{cases} \tag{8.36}$$

Substituting condition (8.36) into relation (8.34) and using (8.35), we obtain

$$\begin{aligned} T_2^{(+)} - T_o &= A_2x_2^{(+)} + \Delta T_c \\ T_2^{(-)} - T_o &= - A_2x_2^{(-)} + \Delta T_c \end{aligned} \tag{8.37}$$

Moreover, referring to Fig. 8.21a, we see that

$$T = \begin{cases} T_3^{(+)} & \text{at} \quad x_2 = 0, x_3 = x_3^{(+)} \\ T_3^{(-)} & \text{at} \quad x_2 = 0, x_3 = -x_3^{(-)} \end{cases} \tag{8.38}$$

Substituting condition (8.38) into relation (8.34) and using (8.35), we get

$$\begin{aligned} T_3^{(+)} - T_o &= A_2x_3^{(+)} + \Delta \mathrm{T_c} \\ T_3^{(-)} - T_o &= -A_2x_3^{(-)} + \Delta \mathrm{T_c} \end{aligned} \tag{8.39}$$

Multiplying the first of relations (8.37) by $|x_2^{(-)}|$ and the second by $|x_2^{(+)}|$ and adding, we obtain

$$T_c = \frac{T_2^{(+)}|x_2^{(-)}| + T_2^{(-)}|\, x_2^{(+)}|}{h^{(2)}} \tag{8.40}$$

Multiplying the first of relation (8.39) by $|x_2^{(-)}|$ and the second by $|x_3^{(+)}|$ and adding, we obtain

$$T_c = \frac{T_3^{(+)}|x_3^{(-)}| + T_3^{(-)}|\, x_3^{(+)}|}{h^{(3)}} \tag{8.41}$$

Notice that as expected when three of the temperatures $T_2^{(+)}$, $T_2^{(-)}$, $T_3^{(+)}$ or $T_3^{(-)}$ are given, the fourth may be established from relations (8.40) and (8.41). Moreover, notice that $\Delta T_c = T_c - T_o$ can be established if $T_2^{(+)}$ and $T_2^{(-)}$ or $T_3^{(+)}$ and $T_3^{(-)}$ are given. Subtracting the second of relations (8.37) from the first and using relation (8.30), we get

$$A_2 = \frac{\delta T_2}{h^{(2)}} \tag{8.42a}$$

Subtracting the second of relation (8.39) from the first and using relation (8.31), we obtain

$$A_3 = \frac{\delta T_3}{h^{(3)}} \tag{8.42b}$$

Substituting relations (8.35), (8.42a) and (8.42b) into relation (8.34), we have

$$T - T_o = \delta T_c + \frac{\delta T_2 x_2}{h^{(2)}} + \frac{\delta T_3 x_3}{h^{(3)}} \tag{8.43}$$

Substituting relation (8.43) into (8.29), we get

$$\begin{aligned} e_{11} &= \frac{\tau_{11}}{E} + \alpha\,\delta T_c + \frac{\alpha \delta T_2 x_2}{h^{(2)}} + \frac{\alpha \delta T_3 x_3}{h^{(3)}} \\ e_{22} &= e_{33} = -\frac{\nu \tau_{11}}{E} + \alpha \delta T_c + \frac{\alpha \delta T_2 x_2}{h^{(2)}} + \frac{\alpha \delta T_3 x_3}{h^{(3)}} \\ e_{12} &= \frac{\tau_{12}}{2G} \qquad e_{13} = \frac{\tau_{13}}{2G} \qquad e_{23} = 0 \end{aligned} \tag{8.44}$$

Relations (8.44) represent a convenient form of the stress–strain relations for a particle of a line member made of a homogeneous isotropic linearly elastic material. ΔT_c, δT_2 and δT_3 are functions only of the axial coordinate of the member. Eliminating τ_{11} from the first two of relations (8.44), we obtain

$$e_{22} = e_{33} = -\nu e_{11} + (1+\nu)\left(\alpha \delta T_c + \frac{\alpha \delta T_2 x_2}{h^{(2)}} + \frac{\alpha \delta T_3 x_3}{h^{(3)}}\right) \tag{8.45}$$

8.12 The Boundary Value Problems in the Theories of Mechanics of Materials for Line Members

We consider a line member of arbitrary constant cross sections made from an isotropic linearly elastic material. The end surfaces of the member are perpendicular to its axis. Initially the member is in a stress-free strain-free, state of mechanical and thermal equilibrium at a uniform temperature T_o. In this state the member is not subjected to external actions and heat does not flow in or out of it. Subsequently, the member is subjected to specified loads and to specified constraints as a result of which it does not move as a rigid body but it deforms and reaches a second state of mechanical but not necessarily thermal equilibrium.

We assume that the effect of the deformation of a line member on the distribution of its temperature is negligible. Consequently, the distribution of temperature in a line member can be established independently of its deformation. We assume that it has been established and it is part of the data of the problem under consideration. If the given temperature distribution is a function of the axial coordinate, the member does not reach a state of thermal equilibrium, because heat will be flowing in or out of it.

We consider only external loads whose magnitude is such that the deformation of the member is within the range of validity of the assumption of small deformation (see Section 2.4). These loads could include the following:

1. A specified change of temperature. We assume that the distribution of temperature inside a member is a linear function of x_2 and x_3. Thus, the change of temperature can be specified by $\Delta T_c(x_1)$, $\delta T_2(x_1)$ and $\delta T_3(x_1)$ (see Section 8.11).
2. Specified external axial centroidal distributed forces $p_1(x_1)$ and specified external axial centroidal concentrated forces $P_1^{(n)}$ ($n = 1, 2, ..., n_1$) acting along the length of the member. An axial force whose line of action does not pass through the centroids of the cross sections of the member on which it acts is replaced by a statically equivalent system consisting of an equal force whose line of action passes through the centroids of the cross sections of the member and moments about the x_2 and x_3 axes (see Section 8.2).
3. Specified external transverse distributed forces $p_2(x_1)$ and $p_3(x_1)$ and specified external transverse concentrated forces $P_2^{(n)}$ ($n = 1, 2, ..., n_2$) and $P_3^{(n)}$ ($n = 1, 2, ..., n_3$). The line of action of each one of these forces lies in a plane which contains the shear centers of the cross sections of the member. A transverse force whose line of action does not meet this requirement is replaced by a statically equivalent system consisting of an equal force whose line of action lies in a plane containing the shear centers of the cross sections of the member and a torsional moment which is equal to the moment of the force about an axis parallel to the x_1 axis and passing through the shear centers of the cross sections of the member (see Section 8.2).
4. Specified external distributed moments $m_i(x_1)$ ($i = 1, 2, 3$) and specified external concentrated moment $M_i^{(m)}$ ($i = 1, 2, 3$) ($m = 1, 2, ..., m_i$). However, the theories of mechanics of materials cannot handle line members of non-circular cross section, subjected to torsional moments $m_1(x_1)$ and $M_1^{(m)}$ ($m = 1, 2, ..., m_1$), because as shown in Sections 6.5 and 6.6 for such members plane sections normal to their axis before deformation do not remain plane after deformation but they warp and the effect of their warping cannot be disregarded.
5. A distribution of traction on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) which is statically equivalent to

(a) An axial centroidal concentrated force P_1^{O} or P_1^{L} (see Fig. 8.22).
(b) Transverse concentrated forces P_2^{O} or P_2^{L} and P_3^{O} or P_3^{L} (see Fig. 8.22). The line of action of each one of these forces lies in a plane containing the shear centers of the cross sections of the member.
(c) A torsional moment M_1^{O} or M_1^{L} (see Fig. 8.22).
(d) Bending moments M_2^{O} and M_3^{O} or M_2^{L} and M_3^{L} (see Fig. 8.22),

For a correctly formulated problem one quantity from each of the following pairs of quantities must be specified at each end surface ($x_1 = 0$ or $x_1 = L$) of a member:

$$\begin{array}{lll} u_1^q \text{ or } P_1^q & \theta_1^q \text{ or } M_1^q & \\ u_2^q \text{ or } P_2^q & \theta_2^q \text{ or } M_2^q & q = 0 \text{ or } L \\ u_3^q \text{ or } P_3^q & \theta_3^q \text{ or } M_3^q & \end{array} \tag{8.46}$$

The specification of u_i^q ($i = 1, 2, 3$) or θ_i^q ($i = 1, 2, 3$) is called an essential boundary condition, while the specification of P_i^q ($i = 1, 2, 3$) or M_i^q ($i = 1, 2, 3$) is called a natural

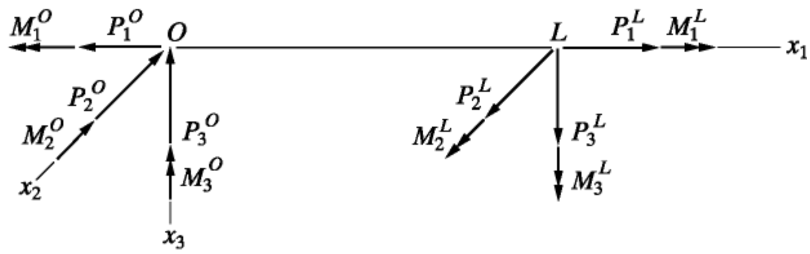


Figure 8.22 Positive actions at the ends of a member.

boundary condition. For a unique solution the rigid-body motion of the member must be specified. Consequently, the component of translation $u_1(x_1)$, $u_2(x_1)$ and $u_3(x_1)$ and the component of rotation θ_1 must be specified at least at one end of the member. Moreover, either θ_i $(i = 2, 3)$ must be specified at the same end of the member or u_j [$(j = 2, 3\ \ j \neq$ i)] must be specified at the other end of the member.

When a component of displacement is specified at an end of a member, the corresponding component of action at this end is unknown and it is called the reaction of the member at this end. When a component of action is specified at an end of a member, the corresponding component of displacement at this end is unknown.

We are interested in establishing the distribution of the components of displacement $u_i(x_1)$ $(i = 1, 2, 3)$ of the particles of the axis of the member, of the components of rotation θ_i (x_1) $(i = 1, 2, 3)$ of the cross sections of the member and of the components of stress $\tau_{11}(x_1, x_2, x_3)$, $\tau_{12}(x_1, x_2, x_3)$ and $\tau_{13}(x_1, x_2, x_3)$ acting on the particles of the member. The other components of stress are assumed negligible [see relation (8.1)].

On the basis of our discussion in Section 8.2 the boundary value problem described above is reduced into the following three uncoupled boundary value problems.

Boundary value problem 1

Find the component of translation $u_1(x_1)$ of a prismatic line member, which is restrained at one or both ends from moving as a rigid body, and is subjected to axial centroidal forces and/or change of temperature ΔT_c [see relation (8.32)].

Boundary value problem 2

Find the component of rotation θ_1 (x_1) of a prismatic line member of solid or hollow circular cross sections which is restrained at one or both ends from moving as a rigid body and is subjected to torsional moments.

Boundary value problem 3

Find the components of translation $u_i(x_1)$ $(i = 2$ or $3)$ and the components of rotation $\theta_i(x_1)$ $(i = 3$ or $2)$ of a prismatic line member which is restrained at one or both ends from moving as a rigid body and is subjected to transverse forces, bending moments and change of temperature $\delta T_2(x_1)$ and $\delta T_3(x_1)$ [see relations (8.30) and (8.31)]. The line of action of each of the forces lies in a plane which contains the shear centers of the cross sections of the member. Consequently, the cross sections of the member do not rotate (twist) about the x_1 axis. We say that the member is subject to bending without twisting. Members subjected to such loading are called beams.

In the next two sections we formulate the first two boundary value problems and we solve several examples. In Chapter 9 we formulate the third boundary value problem, and we solve several examples.

8.13 The Boundary Value Problem for Computing the Axial Component of Translation and the Internal Force in a Member Made from an Isotropic, Linearly Elastic Material Subjected to Axial Centroidal Forces and to a Uniform Change of Temperature

Consider a straight-line member, having constant cross sections of arbitrary geometry, made from a homogeneous (E = constant) isotropic, linearly elastic material. We choose the x_1 axis to be the axis of the member, that is, the locus of the centroids of its cross sections. The member is initially at a reference stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. Subsequently, the member reaches a second state of mechanical equilibrium due to the application on it of one or more of the following loads:

1. A distribution of the axial component of body force throughout its volume and a distribution of the axial component of traction on its lateral surface which are statically equivalent to a set of specified axial centroidal forces. That is, forces applied to the axis of the member. These could be concentrated forces $P_1^{(n)}$ ($n = 1, 2, ..., n_1$) and distributed forces $p_1(x_1)$ given in units of force per unit length of the member.
2. Specified change of temperature [$\Delta T_c(x_1)$ (see Section 8.11)] which does not vary over the cross sections of the member. Notice that if in the second state of mechanical equilibrium the temperature of the member is a function of its axial coordinate, heat will flow into or out of it and, consequently, it will not be in a state of thermal equilibrium.
3. Axial components of traction acting on each end surfaces ($x_1 = 0$ and $x_1 = L$) of the member which are statically equivalent to an axial force acting on the centroid of each end surface of the member.

For a correctly formulated problem either the axial component of force or the axial component of translation u_1 must be specified at each end of a member. However, for a unique solution it is essential that the rigid-body motion of a member is specified. Thus, the axial component of translation must be specified at least at one end of a member. For example, the boundary conditions for the one member structure of Fig. 8.23a are

$$u_1(0) = 0 \tag{8.47}$$

$$N(L) = P_1^L \tag{8.48}$$

where P_1^L is the specified axial component of external axial centroidal force at the end L of the member. The boundary condition (8.47) is essential, while the boundary condition (8.48) is natural. The boundary conditions for the one-member structure of Fig. 8.23b are

$$u_1(0) = 0 \tag{8.49}$$

$$u_1(L) = 0 \tag{8.50}$$

Thus, for the one-member structure of Fig. 8.23b both boundary conditions are essential.

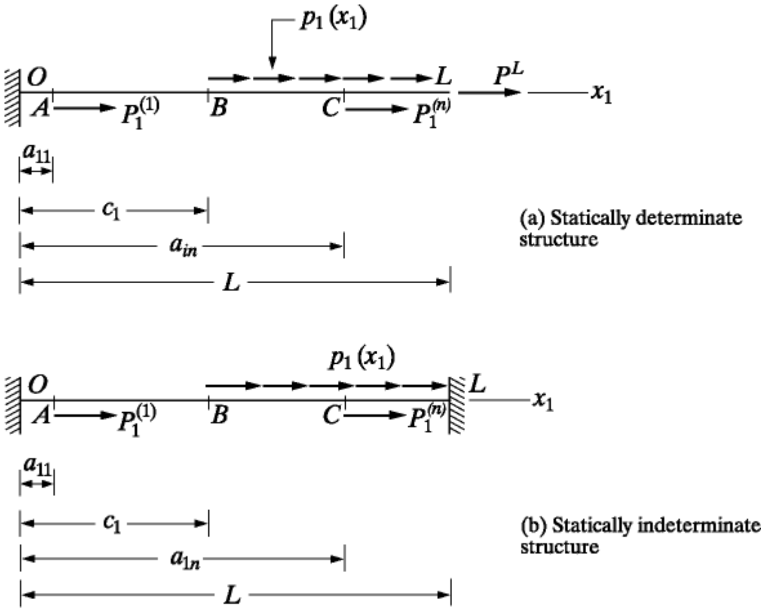


Figure 8.23 One-member structures subjected to external axial forces.

Notice that the number of essential boundary conditions specified for the one-member structure of Fig. 8.23a is the minimum required in order to prevent it from moving as a rigid body. In this case the structure is statically determinate. That is, its reactions and the distribution of the internal forces $N(x_1)$ can be established by considering the equilibrium of appropriate segments of the structure. Moreover, notice that there is one more essential boundary condition specified for the structure of Fig. 8.23b than the minimum required in order to prevent it from moving as a rigid body. In this case the structure is statically indeterminate to the first degree. That is, its reactions and the distribution of its internal force $N(x_1)$ cannot be established by considering only the equilibrium of segments of the member.

We are interested in computing the displacement and stress fields of the member under consideration when it reaches the second state of mechanical equilibrium due to the application on it of one or more of the loads described above. When a line member is subjected to one or more of these loads, as discussed in Section 8.2, its cross sections do not rotate and the particles of its axis do not translate in the transverse direction. That is,

$$u_2(x_1) = u_3(x_1) = \theta_1(x_1) = \theta_2(x_1) = \theta_3(x_1) = 0 \tag{8.51}$$

Consequently, relations (8.5) and (8.6) reduce to†

$$\hat{u}_1(x_1,x_2,x_3) = u_1(x_1) \tag{8.52}$$

and

† $\hat{u}_1(x_1,x_2,x_3)$ is the component of the displacement in the direction of the x_1 axis of a particle of the member while $u_1(x_1)$ is the component of the displacement in the direction of the x_1 axis of a particle of the centroidal axis ($x_2 = x_3 = 0$) of the member.

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} = \frac{du_1}{dx_1} \tag{8.53}$$

That is, the axial component of displacement of the particles of the member does not vary with x_2 and x_3.

In addition to the assumptions described in Section 8.2 for the member under consideration we assume that

$$e_{12} = e_{13} = 0 \tag{8.54}$$

In Example 1 of Section 5.5 we show that when a prismatic line member is subjected on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) to an uniform distribution of the axial component of traction, plane sections normal to its axis prior to deformation remain plane and normal to its axis after deformation. Thus, in this case the shearing components of strain $e_{12}(x_1, x_2, x_3)$ and $e_{13}(x_1, x_2, x_3)$ vanish. When the lateral surface of a prismatic member is subjected to a distribution of the axial component of traction, the shearing components of strain e_{12} and e_{13} do not vanish. However, they are very small compared to the normal component of strain e_{11} and can be neglected. Consequently, referring to the stress–strain relations (3.47) and taking into account assumption (8.1), the stress distribution in the member under consideration is

$$\tau_{11} \neq 0 \qquad \tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0 \tag{8.55}$$

Substituting relation (8.53) into the first of relations (8.44), we get

$$\tau_{11} = E\left(\frac{du_1}{dx_1} - H_1\right) \tag{8.56}$$

where

$$H_1 = \alpha \Delta T_c \tag{8.57}$$

Substituting relation (8.56) into relation (8.9a) and integrating, we get the following internal action–displacement relation for a line member made from a homogeneous isotropic, linearly elastic material

$$N(x_1) = \iint_A \tau_{11} dA = EA\left(\frac{du_1}{dx_1} - H_1\right) \tag{8.58}$$

where A is the area of the cross section of the member.

Notice that, by eliminating $(du_1/dx_1 - H_1)$ from relations (8.58) and (8.56), as expected, we obtain the following stress–internal action relation for the member under consideration

$$\tau(x_1) = \frac{N}{A} \tag{8.59}$$

where $N(x_1)$ is the resultant axial centroidal force acting on the cross sections of the member.

Referring to relations (8.55) and (8.59) we see that the state of stress at the particles of a member subjected to external axial centroidal forces is specified by the following matrix which gives the components of the stress tensor referred to the x_1, x_2 and x_3 axes.

$$[\tau] = \begin{bmatrix} \tau_{11} = \dfrac{N}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.60}$$

In Example 1 of Section 5.5 we find that the stress distribution (8.60) is equal to that obtained on the basis of the linear theory of elasticity for a prismatic member subjected only to a uniform distribution of the axial component of traction on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) which is equivalent to an axial centroidal force. Moreover, it can be shown that the stress distribution (8.60) is a satisfactory approximation to the stress distribution in prismatic members subjected to axial centroidal forces along their length, except in the neighborhood of the discontinuities of the external forces, where there may be stress concentrations.

Substituting relation (8.58) into (8.16), we get

$$\frac{d}{dx_1}\left(EA\frac{du_1}{dx_1}\right) = -p_1(x_1) + \frac{d}{dx_1}(EAH_1) \tag{8.61}$$

On the basis of relations (8.58) and (8.61) and referring to our discussion in Section 8.9, the axial component of translation $u_1(x_1)$ is represented by a set of continuously differentiable functions $u_1^{(s)}(x_1)$ [$s = 1, 2, ..., (S_1+1)$], where S_1 is equal to the number of points of load change of the axial component of the external forces acting on the member. Each one of these functions gives the axial component of translation in one segment of the member extending between a support and its adjacent point of load change of the axial component of the external forces or between two consecutive points of load change of the axial component of the external forces. Thus, the boundary value problem for computing the axial component $u_1(x_1)$ of translation and the axial component $N(x_1)$ of internal centroidal force in a line member (statically determinate or indeterminate) subjected to a distribution of external axial centroidal forces which have S_1 points of load change and to a uniform change of temperature ($\Delta T_c \neq 0$, $\delta T_2 = \delta T_3 = 0$), can be formulated in the so-called strong or classical form cited below:

Find the functions $u_1^{(s)}(x_1)$ [$s = 1, 2, ..., (S_1+1)$] which have the following properties:

1. They have continuous derivatives of any order.
2. They satisfy the essential boundary conditions at the ends of the member, when such conditions are specified.
3. They yield a continuous axial component of translation throughout the length of the member. This requirement is met if at each point of load change of the external axial centroidal forces, the axial component of translation u_1 obtained from the function $u_1^{(s)}(x_1)$ for the segment at the left of the point of load change is equal to that obtained from the functions $u_1^{(s+1)}(x_1)$ for the segment at the right.
4. Each satisfies the displacement equation of equilibrium (8.61) at all points of the segment of the member whose axial component of translation it represents. That is,

$$\frac{d}{dx_1}\left(EA\frac{du_1^{(s)}}{dx_1}\right) = -p_1^{(s)}(x_1) + \frac{d}{dx_1}(EAH_1^{(s)}) \qquad s = 1, 2, ..., (S+1) \tag{8.62}$$

This ensures that every segment of infinitesimal length of the member cut by two imaginary planes perpendicular to its axis, which does not contain a point of load change of the axial component of the external forces, is in equilibrium.

5. When substituted into the internal force, translation relation (8.58), they give a set of functions $N^{(s)}(x_1)[s = 1, 2, ..., (S_1 + 1)]$ each one of which represents the axial component of internal force in the s^{th} segment of the member. Each of these segments extends between a support and its adjacent point of load change of the external axial centroidal forces or between two adjacent points of load change of the external axial centroidal forces. These functions satisfy

(a) The natural boundary condition at the end of the member where a specified external axial centroidal force is applied.

(b) The equilibrium of the segments of infinitesimal length cut by two imaginary planes normal to the axis of the member, which contain a point of load change of the external axial centroidal forces.

On the basis of the foregoing presentation, the axial component of translation $u_1(x_1)$ and the axial component of the internal force $N(x_1)$ of any member can be established by adhering to the following steps:

STEP 1 The set of differential equations (8.62) is solved to obtain the axial component of translation $u_1^{(s)}(x_1)$ $[s = 1, 2, ..., (S_1 + 1)]$ of each segment of the member extending between a support and its adjacent point of load change of the external axial centroidal forces or between two points of load change of the external axial centroidal forces. Each of the functions $u_1^{(s)}(x_1)$ $[s = 1, 2, ..., (S_1 + 1)]$ involves two constants.

STEP 2 The constants are evaluated by requiring that the axial components of translation $u_1^{(s)}(x_1)$ $[s = 1, 2, ..., (S_1 + 1)]$ satisfy properties 2 and 3 described above, and when substituted into the internal-force-translation relation (8.58) give a set of functions $N(x_1)$ $[s = 1, , ..., (S_1+1)]$ which satisfies the natural boundary condition of the member if any and the requirements for equilibrium of the segments of infinitesimal length cut by two imaginary planes normal to the axis of the member, which contain a point of load change of the external axial centroidal forces.

The component of translation $u_1(x_1)$ of statically determinate members can also be established by adhering to the following simpler procedure:

STEP 1 The set of functions $N_1^{(s)}(x_1)$ $[s=1,2...(S_1+1)]$ representing the axial component of the internal forces in the statically determinate member is established by considering the equilibrium of appropriate segments of the member. It is apparent that the set of functions $N_1^{(s)}(x_1)$ satisfies the natural boundary condition of the problem and the requirements for equilibrium of any segment of infinitesimal length cut by two imaginary planes normal to the axis of the member.

STEP 2 The functions $N_1^{(s)}(x_1)$ established in step 1 are substituted into relation (8.58) and the resulting differential equations are solved to obtain the functions $u_1^{(s)}(x_1)$ $[s = 1, 2, ..., (S_1 + 1)]$ involving one constant each.

STEP 3 The constants are evaluated by requiring that the solution satisfies the essential boundary conditions of the member as well as the continuity conditions at the points of load change of external axial centroidal forces.

In what follows we present an example.

Example 5 Establish the component of translation $u_1(x_1)$ and the internal axial centroidal force $N(x_1)$ for the prismatic bar [$A = 10(10^3)$ mm^2] of Fig. a subjected to the external axial centroidal forces shown in this figure and to a uniform increase of temperature ΔT_c = 20°C. The bar is made from a homogeneous, isotropic, linearly elastic material [E = 200 GPa, $\alpha = 10^{-5}$/°C]. Evaluate the reactions of the bar for P_1 = 60kN, p_1 = 36 kN/m and L = 2 m. Show the results on a sketch.

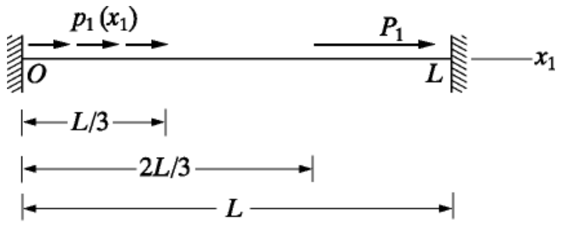


Figure a Geometry and external forces acting on the bar.

Solution This is a statically indeterminate bar; consequently, we cannot establish the distribution of the internal axial centroidal force $N(x_1)$ in it by considering the equilibrium of its segments. Thus, we must use the displacement equations of equilibrium (8.62).

In this problem we have two points of load change of the external axial centroidal forces, one at $x_1 = L/3$ and another at $x_1 = 2L/3$. Thus, the component of translation is

$$\begin{aligned} \frac{d}{dx_1}\left(EA\frac{du_1^{(1)}}{dx_1}\right) &= -p_1 \qquad 0 < x_1 < \frac{L}{3} \\ \frac{d}{dx_1}\left(EA\frac{du_1^{(2)}}{dx_1}\right) &= 0 \qquad \frac{L}{3} < x_1 < \frac{2L}{3} \\ \frac{d}{dx_1}\left(EA\frac{du_1^{(3)}}{dx_1}\right) &= 0 \qquad \frac{2L}{3} < x_1 < L \end{aligned} \tag{a}$$

expressed in terms of three different functions: $u_1^{(1)}(x_1)$ in the segment $0 \leq x_1 \leq L/3$, $u_1^{(2)}(x_1)$ in the segment $L/3 \leq x_1 \leq 2L/3$ and $u_1^{(3)}(x_1)$ in the segment $2L/3 \leq x_1 \leq L$. Each of these functions must satisfy one of the following differential equations:
Integrating the above equations twice, we obtain

$$\left.\begin{aligned} N^{(1)}(x_1) = EA\frac{du_1^{(1)}}{dx_1} &= -p_1x_1 + C_1 \\ EAu_1^{(1)} &= -\frac{p_1x_1^2}{2} + C_1x_1 + C_2 \end{aligned}\right\} \quad 0 < x_1 \leq \frac{L}{3}$$

$$\left.\begin{aligned} N^{(2)}(x_1) = EA\frac{du_1^{(2)}}{dx_1} &= C_3 \\ EAu_1^{(2)} &= C_3x_1 + C_4 \end{aligned}\right\} \quad \frac{L}{3} < x_1 < \frac{2L}{3} \tag{b}$$

$$\left.\begin{aligned} N^{(3)}(x_1) &= EA\frac{du_1^{(3)}}{dx_1} = C_5 \\ EAu_1^{(3)} &= C_5x_1 + C_6 \end{aligned}\right\} \quad \frac{2L}{3} < x_1 < L \tag{b}$$

The six constants $C_i (i = 1, 2, ..., 6)$ are evaluated by requiring that the expressions for the axial component of displacement (b) satisfy the following conditions:

1. The essential boundary conditions. That is,

$$u_1(0) = u_1^{(1)}(0) = 0 \qquad u_1(L) = u_1^{(3)}(L) = 0 \tag{c}$$

2. The continuity conditions at the points of load change of the given external axial centroidal forces. That is,

$$\begin{aligned} u_1^{(1)}\left(\frac{L}{3}\right) &= u_1^{(2)}\left(\frac{L}{3}\right) \\ u_1^{(2)}\left(\frac{2L}{3}\right) &= u_1^{(3)}\left(\frac{2L}{3}\right) \end{aligned} \tag{d}$$

3. The internal axial centroidal forces $N^{(s)}(x_1)$ $(s = 1, 2, 3)$, obtained by substituting in relation (8.58) the expressions for $EA\, du_1^{(s)}/dx_1$ $(s = 1, 2, 3)$ from relations (b) must satisfy the requirements for equilibrium of the segments of infinitesimal length of the bar at points $x_1 = L/3$ and $x_1 = 2L/3$. That is,

$$\begin{aligned} N^{(1)}\left(\frac{L}{3}\right) &= EA\left(\left.\frac{du_1^{(1)}}{dx_1}\right|_{x_1=\frac{L}{3}} - H_1\right) = N^{(2)}\left(\frac{L}{3}\right) = EA\left(\left.\frac{du_1^{(2)}}{dx_1}\right|_{x_1=\frac{L}{3}} - H_1\right) \\ N^{(2)}\left(\frac{2L^-}{3}\right) &= EA\left(\left.\frac{du_1^{(3)}}{dx_1}\right|_{x_1=\left(\frac{2L}{3}\right)^-} - H_1\right) = N^{(3)}\left(\frac{2L^+}{3}\right) + P_1 = EA\left(\left.\frac{du_1^{(3)}}{dx_1}\right|_{x_1=\left(\frac{2L}{3}\right)^+} - H_1\right) + P_1 \end{aligned} \tag{e}$$

Substituting relations (b) into relations (c), (d) and (e) we obtain a set of six linear algebraic equations whose solution gives the following values of the constants $C_i (i = 1, 2, .., 6)$:

$$\begin{aligned} C_1 &= \frac{5p_1L}{18} + \frac{P_1}{3} & C_4 &= \frac{p_1L^2}{18} \\ C_2 &= 0 & C_5 &= -\frac{p_1L}{18} - \frac{2P_1}{3} \\ C_3 &= -\frac{p_1L}{18} + \frac{P_1}{3} & C_6 &= \frac{p_1L^2}{18} + \frac{2P_1L}{3} \end{aligned} \tag{f}$$

Substituting the values of the constants in relations (b), we obtain

$$EAu_1^{(1)}(x_1) = \frac{p_1x_1}{2}\left(\frac{5L}{9} - x_1\right) + \frac{P_1x_1}{3} \qquad 0 < x_1 < \frac{L}{3} \tag{g}$$

$$EAu_1^{(2)}(x_1) = \frac{p_1 L\,(L - x_1)}{18} + \frac{P_1 x_1}{3} \qquad \frac{L}{3} < x_1 < \frac{2L}{3}$$

$$EAu_1^{(3)}(x_1) = \frac{p_1 L\,(L - x_1)}{18} + \frac{2P_1\,(L - x_1)}{3} \qquad \frac{2L}{3} < x_1 < L \tag{g}$$

and

$$N^{(1)}(x_1) = EA\left[\frac{du_1^{(1)}}{dx_1} - H_1\right] = p_1\left(\frac{5L}{18} - x_1\right) + \frac{P_1}{3} - EAH_1 \qquad 0 < x_1 < \frac{L}{3}$$

$$N^{(2)}(x_1) = EA\left[\frac{du_1^{(2)}}{dx_1} - H_1\right] = -\frac{p_1 L}{18} + \frac{P_1}{3} - EAH_1 \qquad \frac{L}{3} < x_1 < \frac{2L}{3} \tag{h}$$

$$N^{(3)}(x_1) = EA\left[\frac{du_1^{(3)}}{dx_1} - H_1\right] = -\frac{p_1 L}{18} - \frac{2P_1}{3} - EAH_1 \qquad \frac{2L}{3} < x_1 < L$$

Where, referring to relation (8.57), we have

$$EAH_1 = EA\alpha\Delta T_c = 200\ (10^4)\ (10^{-5})\ 20 = 400\ \text{kN} \tag{i}$$

Referring to Fig. b and using relations (h) and (i), the reactions of the structure are

$$R_1^0 = -N^{(1)}(0) = -\frac{5p_1 L}{18} - \frac{P_1}{3} + EAH_1 = -\frac{5(36)(2)}{18} - \frac{60}{3} + 400 = 360\ \text{kN} \tag{j}$$

$$R_1^L = N^{(3)}(L) = -\frac{p_1 L}{18} - \frac{2P_1}{3} - EAH_1 = -\frac{36(2)}{18} - \frac{2(60)}{3} - 400 = -444\ \text{kN} \tag{k}$$

Using relations (j) and (k) we get

$$\sum F_h = 0 = R_1^0 + \frac{p_1 L}{3} + P_1 + R_1^L = 360 + 24 + 60 - 444 = 0 \ \text{ check}$$

The maximum stress occurs at $2L/3 < x_1 < L$ and for A = $10(10^3)$ mm^2 it is equal to

$$(\tau_{11})_{\max} = \frac{R_1^L}{A} = \frac{444}{10(10^3)} = 0.0444\ \frac{\text{kN}}{\text{mm}^2} = 44.4\ \text{MPa} \tag{l}$$

The yield stress of ordinary structural steel is about 210 MPa.

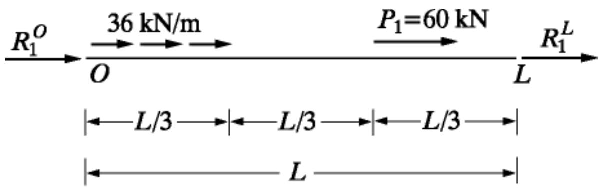


Figure b Reactions of the one-member structure of Fig. a.

Notice that for this example the internal force and the component of stress resulting from the increase of temperature are considerably larger than the corresponding quantities resulting from the external forces.

For $p_1 = P_1 = 0$ relations (g) and (h) reduce to

$$u_1^{(1)} = u_1^{(2)} = u_1^{(3)} = 0$$

$$N^{(1)} = N^{(2)} = N^{(3)} = N = -EAH_1 = -EA\alpha\Delta T_c = -400 \text{ kN}$$

Consequently,

$$\tau_{11} = \frac{N}{A} = -E\alpha\Delta T_c = E\alpha(T-T_o) = -\frac{400}{(10^4)} = -0.04 \frac{\text{kN}}{\text{mm}^2} = -40 \text{ MPa} \tag{m}$$

Referring to relations (q) of Example 2 of Section 5.5, we see that the results obtained for the same problem using the linear theory of elasticity are identical to those obtained using the theory of mechanics of materials.

8.13.1 Solution of the Boundary Value Problem for Computing the Axial Component of Translation and of the Internal Force in a Member Subjected to Axial Centroidal Forces and to a Uniform Change of Temperature Using Functions of Discontinuity

The formulation and the solution of the boundary value problem for computing the axial components of translation $u_1(x_1)$ and of the internal force $N(x_1)$ presented in the previous section are cumbersome. In this section we present a more efficient approach to the same problem using functions of discontinuity. These functions are used to write a discontinuous function as, for example, the external or internal forces acting on a member by a single expression. Using these functions we can write one displacement equation of equilibrium (8.61), as if the distribution of the external axial centroidal forces was continuous. That is, referring to Appendix G, we can convert a concentrated force $P_1^{(n)}$ applied at $x_1 = a_{1n}$ to an equivalent distributed force $p_1(x_1)$ as follows:

$$p_1(x_1) = P_1^{(n)}\delta(x_1 - a_{1n}) \tag{8.63}$$

where $\delta(x_1 - a_{1n})$ is the Dirac delta-unction defined by relation (G.10) of Appendix G. Referring to this definition, we see that the right side of relation (8.63) vanishes everywhere except at the point a_{1n}, where it is not specified. However, referring to relation (G.13), the resultant of the external forces acting on the member from $x_1 = a_{1n} - \epsilon$ to $x_1 = a_{1n} + \epsilon$ is equal to

$$\int_{a_{1n}-\epsilon}^{a_{1n}+\epsilon} p_1(x_1)\, dx_1 = \int_{a_{1n}-\epsilon}^{a_{1n}+\epsilon} P_1^{(n)}\delta(x_1 - a_{1n})\, dx_1 = P_1^{(n)} \tag{8.64}$$

where ϵ is a small positive number.

Using relation (8.63) and referring to relation (G.1) of Appendix G, the displacement equation of equilibrium for the one-member structure of Fig. 8.23a can be written as

$$\frac{d}{dx_1}\left[EA\left(\frac{du_1}{dx_1} - H_1\right)\right] = -p_1(x_1)\Delta(x_1 - c_1) - P_1^{(1)}\delta(x_1 - a_{11}) - P_1^{(2)}\delta(x_1 - a_{12})$$

(8.65)

where $\Delta(x_1\text{-}c_1)$ is the unit step function defined by relation (G.1) and $\delta(x_1\text{-}a_{1n})(n$=1 or 2) is the Dirac delta-function defined by relation (G.10) of Appendix G. Using the rules for integration of the functions of discontinuity presented in Appendix G, relation (8.65) may be integrated to give the axial component of translation $u_1(x_1)$ as a function of the axial coordinate of the member involving two constants. The constants can be evaluated from the specified boundary conditions at the ends of the member. As mentioned previously, for a properly posed problem either the internal axial centroidal force $N(x_1)$ or the axial component of translation $u_1(x_1)$ must be specified at each end of a member.

In what follows we illustrate the solution of the boundary value problem described in this section, using functions of discontinuity, by an example.

Example 6 Establish the component of translation $u_1(x_1)$ and the internal axial force $N(x_1)$ for the one-member structure of Fig. a subjected to the external axial centroidal forces shown in this figure and to a uniform increase of temperature $\Delta T_c = 20°\text{C}$. The member has a constant cross section and is made from a homogeneous, isotropic, linearly elastic material [$E = 200$ kN/mm², $A = 10\ (10^3)$ mm², $\alpha = 10^{-5}/°\text{C}$]. Evaluate the reactions of the member for $P_1 = 60$ kN, $p_1 = 9$ kN/m and $L = 2$ m. Show the results on a sketch.

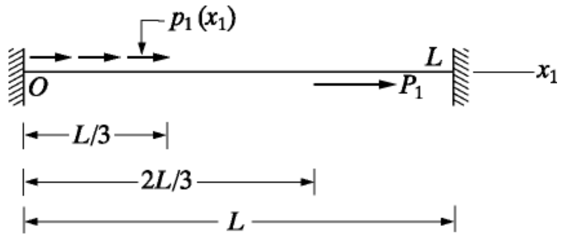


Figure a Geometry and external forces acting on the one-member structure.

Solution Referring to relation (8.65), the displacement equation of equilibrium for the one-member structure of Fig. a is

$$\frac{d}{dx_1}\left[EA\left(\frac{du_1}{dx_1} - H_1\right)\right] = -p_1\left[1 - \Delta\left(x_1 - \frac{L}{3}\right)\right] - P_1\delta\left(x_1 - \frac{2L}{3}\right) \tag{a}$$

Moreover, the boundary conditions for the one-member structure of Fig. a are

$$u_1(0) = 0 \qquad\qquad u_1(L) = 0$$

Referring to relations (G.4) and (G.12) of Appendix G and integrating equation (a) twice, we get

$$N(x_1) = EA\left(\frac{du_1}{dx_1} - H_1\right) = -p_1\left[x_1 - \left(x_1 - \frac{L}{3}\right)\Delta\left(x_1 - \frac{L}{3}\right)\right] - P_1\Delta\left(x_1 - \frac{2L}{3}\right) + C_1 \tag{b}$$

$$EAu_1(x_1) = -p_1\left[\frac{x_1^2}{2} - \frac{1}{2}\left(x_1 - \frac{L}{3}\right)^2\Delta\left(x_1 - \frac{L}{3}\right)\right] - P_1\left(x_1 - \frac{2L}{3}\right)\Delta\left(x_1 - \frac{2L}{3}\right) + C_1x_1 + C_2 + EAH_1x_1 \tag{c}$$

The constants are evaluated by requiring that the solution (c) satisfies the boundary conditions. Thus,

$$
\begin{aligned}
u_1(0) &= 0 \quad \Longrightarrow \quad C_2 = 0 \\
u_1(L) &= 0 \quad \Longrightarrow \quad EAH_1 L - p_1\left[\frac{L^2}{2} - \frac{1}{2}\left(\frac{2L}{3}\right)^2\right] - \frac{P_1 L}{3} + C_1 L = 0
\end{aligned} \tag{d}
$$

or

$$
C_1 = \frac{5Lp_1}{18} + \frac{P_1}{3} - EAH_1 \tag{e}
$$

Substituting the values of the constants (d) and (e) into relations (b) and (c), we get

$$
N(x_1) = -p_1\left[x_1 - \left(x_1 - \frac{L}{3}\right)\Delta\left(x_1 - \frac{L}{3}\right)\right] - P_1\Delta\left(x_1 - \frac{2L}{3}\right) + \frac{5Lp_1}{18} + \frac{P_1}{3} - EAH_1 \tag{f}
$$

and

$$
\begin{aligned}
EAu_1(x_1) = {} & -p_1\left[\frac{x_1^2}{2} - \frac{1}{2}\left(x_1 - \frac{L}{3}\right)^2 \Delta\left(x_1 - \frac{L}{3}\right)\right] \\
& - P_1\left(x_1 - \frac{2L}{3}\right)\Delta\left(x_1 - \frac{2L}{3}\right) + \frac{5Lp_1 x_1}{18} + \frac{P_1 x_1}{3}
\end{aligned} \tag{g}
$$

As expected, these results are identical to the corresponding results (h) and (i) of the example of the previous section.

8.14 The Boundary Value Problem for Computing the Angle of Twist and the Internal Torsional Moment in Members of Circular Cross Section Made from an Isotropic, Linearly Elastic Material Subjected to Torsional Moments

Straight-line members subjected to torsional moments are encountered in many engineering applications as, for example, power transmission shafts. These are members of solid or hollow circular cross sections and are used to transmit power from one device or machine to another.

The assumptions of the theories of mechanics of materials discussed in Section 8.2 are valid for prismatic line members of solid or hollow circular cross sections subjected to tractions on each of their end surfaces ($x_1 = 0$ and $x_1 = L$) which are statically equivalent to a torsional moment. However, as we show in Sections 6.5 and 6.6, plane sections normal to the axis of prismatic members of non-circular cross sections subjected to equal and opposite torsional moments at their ends, do not remain plane after deformation; they warp and the effect of warping on the components of stress acting on their particles and on the angle of twist of their cross sections cannot be disregarded. Consequently, members having non-circular cross sections subjected to torsional moments cannot be analyzed on the basis of the theory of mechanics of materials.

Consider a member of constant solid or hollow circular cross sections made from a homogeneous, isotropic, linearly elastic material (see Fig. 8.24). The member is initially

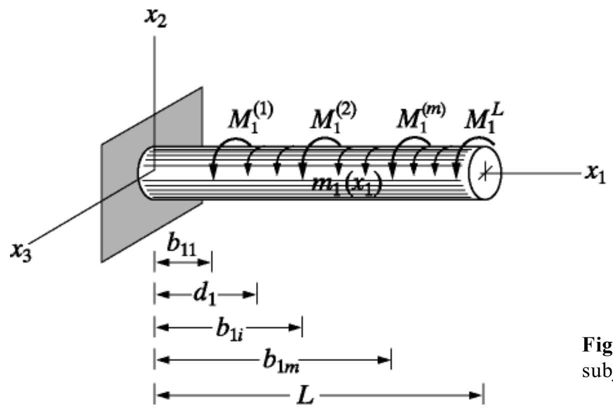


Figure 8.24 One-member structure subjected to positive torsional moments.

in a reference stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. The member reaches a second state of mechanical and thermal equilibrium due to the application on it of the following loads :

1. A distribution of external tractions acting on its lateral surface which are statically equivalent to specified torsional moments. These could be distributed torsional moments $m_1(x_1)$ and concentrated torsional moments $M_1^{(m)}(m = 1, 2, ..., m_1)$ applied at $x_1 = b_{1m}(m = 1, 2, ..., m_1)$.
2. A distribution of shearing components of traction acting on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) which is equivalent to a torsional moment M_1^0 and M_1^L, respectively.

For a correctly formulated problem, either the component of rotation about the x_1 axis or the torsional component of moment must be specified at each end of a member. However, for a unique solution it is essential that the rigid-body motion of the member is specified. That is, the component of rotation about its x_1 axis must be specified at least at one end of the member. For example, the boundary conditions for the one member structure of Fig. 8.24 are

$$\begin{aligned} \theta_1(0) &= 0 \\ M_1(L) &= M_1^L \end{aligned} \tag{8.66}$$

In addition to the fundamental assumptions of the theories of mechanics of materials discussed in Section 8.2, we assume that

1. All material lines of a cross section of a member rotate due to its deformation by the same angle known as the angle of twist which we denote by $\theta_1(x_1)$.

2. When the loads described above are applied to a member, the particles of its axis do not translate and its cross sections do not rotate about the x_2 and x_3 axes. That is,

$$u_1(x_1) = u_2(x_1) = u_3(x_1) = \theta_2(x_1) = \theta_3(x_1) = 0 \qquad \theta_1(x_1) \neq 0 \tag{8.67}$$

3. Radial lines do not elongate. That is,

$$\hat{u}_r = 0 \tag{8.68}$$

In Section 6.3 and in the examples of Sections 6.5 and 6.6 we show, using the theory of

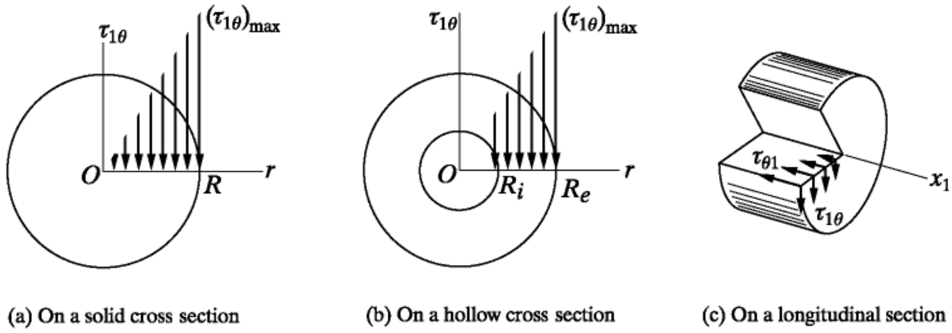


Figure 8.25 Distribution of the shearing component of stress in a member of circular cross section subjected to torsional moments.

elasticity, that the assumptions discussed in Section 8.2 and the ones described above are valid for members having constant solid or hollow circular cross sections, when they are fixed on their one end surface and are subjected to a distribution of traction on their other end surface which is equivalent to a torsional moment. However, the torsional moment must be applied in such a way that the end surface itself remains plane and undistorted. This may be accomplished by applying the torsional moment to a rigid plate which is solidly attached to the unsupported end of these members. In the theories of mechanics of materials the above stated assumptions are considered a good approximation for prismatic members subjected to torsional moments along their length. Thus, referring to relations (w) and (x) of the example of Section 6.6,† we get

$$\alpha = \frac{d\theta_1}{dx_1} = \frac{M_1}{GI_P} \tag{8.69}$$

where referring to Fig. 8.18 for members of hollow circular cross sections of external radius R_e and internal radius R_i, we have

$$I_p = \iint_A r^2 dA = \frac{\pi(R_e^4 - R_i^4)}{2} \tag{8.70}$$

while for members of solid circular cross sections of radius R, we obtain

$$I_p = \frac{\pi R^4}{2} \tag{8.71}$$

Moreover,

$$\tau_{11} = \tau_{rr} = \tau_{\theta\theta} = \tau_{1r} = \tau_{r\theta} = 0 \qquad \tau_{1\theta} = \frac{M_1 r}{I_P} \tag{8.72}$$

The last of relations (8.72) indicates that the shearing components of stress $\tau_{1\theta}$ acting on

†A proof of relations (8.69), and (8.70) based on the assumptions of Mechanics of Materials, can be found in
1. Beer, F., Johnson, R. Jr., *Mechanics of Materials,* 2nd edition, McGraw-Hill, New York, 1982
2. Gere, J. M., Timoshenko, S.P., *Mechanics of Materials,* Brooks/Cole Engineering Division, Monterey, CA, 1984.

the cross sections of a member of circular cross section subjected to torsional moments vary linearly with the radial distance r (see Fig. 8.25a and b). As shown in Fig. 8.25c, there is a distribution of the component of stress $\tau_{\theta 1}$ of the same magnitude as $\tau_{1\theta}$ acting on the longitudinal planes of the member. This is due to the fact that, as shown in Section 2.13 the stress tensor is symmetric. That is,

Table 8.1 Comparison of the boundary value problems for computing the displacement and stress fields of a member subjected to axial centroidal forces in an environment of constant temperature or to torsional moments.

Tabulated Item	Members Subjected to	
	Axial Centroidal	**Torsional Moments**
External Actions	$p_1(x_1)$ $P_1^{(n)} (n = 1, 2, ..., n_1)$	$m_1(x_1)$ $M_1^{(m)} (m = 1, 2, ..., m_1)$
Component of Displacement	Axial Component of Translation $u_1(x_1)$	Axial Component of Rotation (Twist) $\theta_1(x_1)$
Internal Action	Axial Centroidal Force $N(x_1)$	Torsional Moment $M_1(x_1)$
Differential Equation	$\frac{d}{dx_1}\left(EA\frac{du_1}{dx_1}\right) = -p_1(x_1)$	$\frac{d}{dx_1}\left(GI_p\frac{d\theta_1}{dx_1}\right) = -m_1(x_1)$
Action–Displacement Relation	$N(x_1) = EA\frac{du_1}{dx_1}$	$M_1(x_1) = GI_p\frac{d\theta_1}{dx_1}$
Boundary Conditions	either $\left.\begin{matrix} u_1 = \text{specified} \\ \text{or} \\ N = \text{specified}\end{matrix}\right\}$ at each end	either $\left.\begin{matrix} \theta_1 = \text{specified} \\ \text{or} \\ M_1 = \text{specified}\end{matrix}\right\}$ at each end
State of Stress	$\begin{bmatrix} \tau_{11} = \frac{N}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \tau_{1\theta} = \frac{M_1 r}{I_p} \\ 0 & 0 & 0 \\ \tau_{\theta 1} & 0 & 0 \end{bmatrix}$

$$\tau_{1\theta} = \tau_{\theta 1} \tag{8.73}$$

Differentiating both sides of relation (8.69) and using the equilibrium equation (8.14), we get

$$\frac{d}{dx_1}\left[GI_p\left(\frac{d\theta_1}{dx_1}\right)\right] = -m_1(x_1) \tag{8.74}$$

where $m_1(x_1)$ is the distributed torsional moment acting along the length of the member.

As illustrated in Table 8.1, the boundary value problem for computing the axial component of rotation $\theta_1(x_1)$ of a prismatic member of circular cross section subjected to torsional moments is mathematically identical to the boundary value problem for computing the axial component of translation $u_1(x_1)$ of a prismatic member subjected to axial centroidal forces. The domain of both problems is a line (the length of the member) and the boundary consists of its two end points ($x_1 = 0$ and $x_1 = L$). For this reason these problems are called two point boundary value problems. Moreover, both problems involve a linear ordinary differential equation of the second order. Thus, the procedures for establishing the axial component of rotation of members subjected to torsional moments are identical to those described in Sections 8.13 for establishing the axial component of translation of members subjected to axial centroidal forces.

8.14.1 Analysis of Power Transmitting Prismatic Circular Shafts

The most common use of prismatic members of circular cross section subjected to torsional moments is to transmit mechanical power from one machine or device to another as rotating shafts. The torsional moments to which a shaft is subjected depend on the magnitude of the transmitted power and the angular velocity of the rotational motion of the shaft. The work done by a torsional moment of constant magnitude M_1 acting on a cross section of a member when it rotates by an angle θ_1 is equal to

$$W = M_1\,\theta_1 \tag{8.75}$$

If M_1 is expressed in Newton-meters and θ_1 in radians, then the work is expressed in joules. That is, one Joule is one Newton-meter.

Consider a motor-driven shaft rotating with an angular velocity of ω radians per second which transmits a torsional moment M_1. The power P transmitted by the shaft is the time rate at which work is done. That is, recalling from dynamics that $\omega = d\theta_1/dt$, we have

$$P = \frac{dW}{dt} = M_1\frac{d\theta_1}{dt} = M_1\,\omega \tag{8.76}$$

If M_1 is expressed in newton-meters and ω in radians per second, then the power is expressed in watts (W) or joules per second. From dynamics we know that the angular speed ω may be written as

$$\omega = 2\pi f \quad \text{rad/sec} \tag{8.77}$$

where f is the frequency of revolution, that is, the number of revolutions per unit time. The unit of frequency is the hertz (Hz) which is equal to one revolution per second. Substituting relation (8.77) into (8.76) we obtain

$$P = 2\pi f M_1 \tag{8.78}$$

If M_1 is expressed in newton meters and f in hertz, then P is in watts. This relation is often written as

$$P = \frac{2\pi n M_1}{60} \tag{8.79}$$

where n is the number of revolutions per minute (rpm). That is,

$$n = 60f \tag{8.80}$$

In the English system power is often expressed in horsepower (hp). One horsepower is equal to 550 $\text{ft} \cdot \text{lb/sec}$. Thus, the horsepower transmitted by a shaft subjected to a moment of M_1 $\text{ft} \cdot \text{lbs}$ at n rpm is equal to

$$H = \frac{2\pi n M_1}{60(550)} = \frac{2\pi n M_1}{33{,}000} \tag{8.81}$$

In what follows we present an example.

Example 7 The assembly of two solid steel shafts and gears shown in Fig. a is used to transmit to machine D, a power of 16 kW from motor A, which rotates at a frequency of 25 Hz . Determine the maximum shearing stress in each shaft.

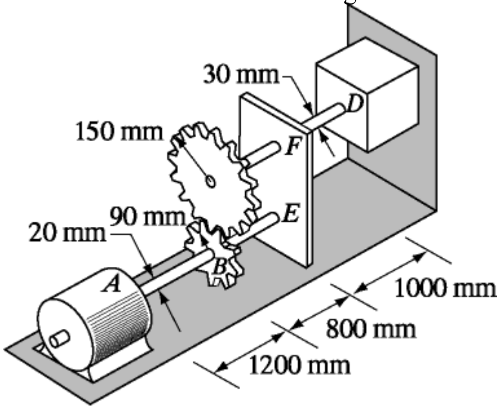


Figure a Assembly of power transmitting shafts.

Solution Shaft AB transmits the 16 kW power at a frequency of 25 Hz. Using relation (8.78), the torsional moment in this shaft is equal to

$$M_1^{(AB)} = \frac{P}{2\pi f^{(AB)}} = \frac{16}{2\pi 25} = 0.1019 \quad \text{kN} \cdot \text{m} \tag{a}$$

In Fig. ba the gears are shown at time t when particle $H^{(B)}$ of gear B is in contact with particle $H^{(C)}$ of gear C. At time $t + \Delta t$ as shown in Fig. bb, particles $H^{(B)}$ and $H^{©}$, traveled a distance $AH^{(B)}$ and $AH^{(C)}$ respectively. From physical intuition it is clear that the distance $AH^{(B)}$ must be equal to distance $AH^{(C)}$. Referring to Fig. bb, we see that

$$AH^{(B)} = 90\Delta\theta_1^{(B)} = AH^{(C)} = 150\Delta\theta_1^{(C)} \tag{b}$$

or

$$\Delta\theta_1^{(B)} = \frac{5}{3}\Delta\theta_1^{(C)} \tag{c}$$

Dividing by Δt both sides of the above relation, taking the limit as Δt goes to zero and using relation (8.77), we obtain

$$2\pi f^{(AB)} = \omega^{(AB)} = \frac{d\theta_1^{(B)}}{dt} = \frac{5}{3}\frac{d\theta_1^{(C)}}{dt} = \frac{5}{3}\omega^{(CD)} = \frac{5}{3}(2\pi)f^{(CD)} \tag{d}$$

Thus,

$$f^{(AB)} = \frac{5}{3}f^{(CD)} = 25 \text{ Hz}$$

or

$$f^{(CD)} = 15 \quad \text{Hz} \tag{e}$$

Substituting result (e) into (8.78), we get

$$M_1^{(CD)} = \frac{P}{2\pi f^{(CD)}} = \frac{16}{2\pi 15} = 0.16977 \quad \text{kN}\cdot\text{m}$$

Referring to relation (8.72), the maximum values of the shear stress in shafts AB and CD are

$$\left(\tau_{1\theta}^{(AB)}\right)_{\max} = \frac{M_1^{(AB)}r^{(AB)}}{I_p^{(AB)}} = \frac{(101.9)10(2)}{\pi(10)^4} = 0.0649 \ \frac{\text{kN}}{\text{mm}^2} = 64.9 \quad \text{MPa} \tag{f}$$

$$\left(\tau_{1\theta}^{(CD)}\right)_{\max} = \frac{(169.85)(15)(2)}{\pi(15)^4} = 0.032 \ \frac{\text{kN}}{\text{mm}^2} = 32.0 \quad \text{MPa} \tag{g}$$

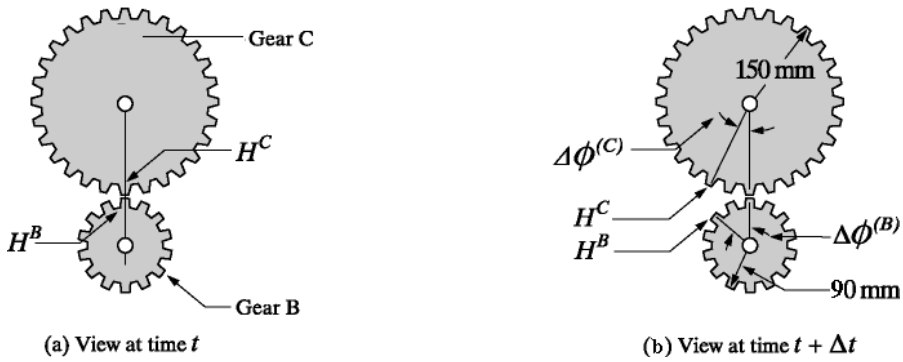


Figure b Gear kinematics.

8.15 Problems

1. to **4.** Consider the beam subjected to the transverse forces shown in Fig. 8P1. The line of action of each force passes through the shear center of the cross section on which it acts. The x_2 and x_3 axes are principal centroidal.

(a) Write the expressions for the internal shearing force $Q_3(x_1)$ and the internal bending moment $M_2(x_1)$ acting on the cross sections of the beam and plot them.

(b) Write the equations which the internal actions must satisfy in order to ensure that every segment of infinitesimal length of the beam cut by two planes normal to its axis is in equilibrium. Describe the types of discontinuities of the internal actions.

Repeat with the beams of Figs. 8P2 to 8P4. Ans. 4 (a) $0 \leq x_1 < 3$ $Q_3^{(1)}(x_1) = 30$ kN $M_2^{(1)}(x_1) = 30x_1$ $3 < x_1 < 6$ $Q_3^{(2)}(x_1) = -20(x_1-3)$ $M_2^{(2)}(x_1) = 60x_1 - 10x_1^2$ (b) $0 < x_1 < 3$ $\frac{dQ_3^{(1)}}{dx_1} = 0$ $\frac{d^2M_2^{(1)}}{dx_1^2} = 0$ $3 < x_L < 6$

$\frac{dQ_3^{(2)}}{dx_1} = -20\frac{\text{kN}}{\text{m}}$ $\frac{d^2M_2^{(2)}}{dx_1^2} = -20\frac{KN}{m}$ $Q_3^{(1)}(3^-) = Q_3^{(2)}(3^+) + 30$ $M_2^{(1)}(3) = M_2^{(2)}(3)$ $M_2^{(1)}(0) = 0$ $M_2^{(2)}(L) = 0$

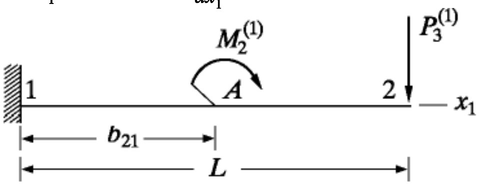


Figure 8P1

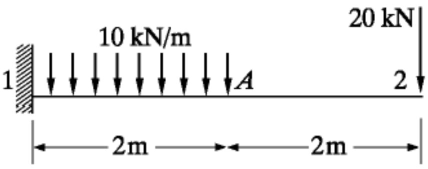


Figure 8P2

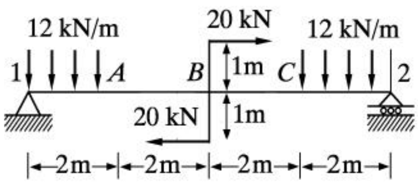


Figure 8P3

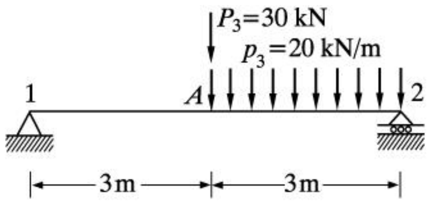


Figure 8P4

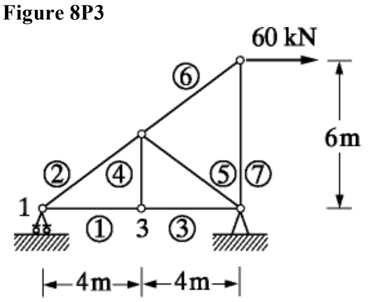


Figure 8P5

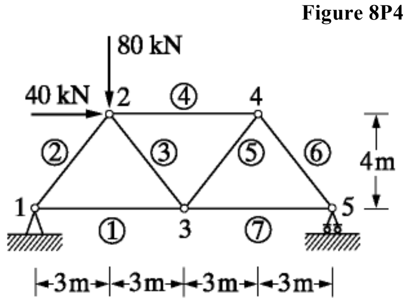


Figure 8P6

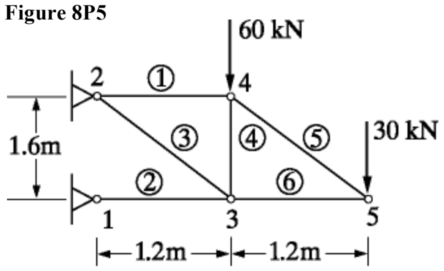


Figure 8P7

5. to **7.** Compute the internal forces in the members of the truss subjected to the forces shown in Fig. 8P5. Repeat with the truss of Figs. 8P6 and 8P7.

Ans. 5 $N^{(2)} = N^{(6)} = 75$ kN Tension, $N^{(1)} = N^{(3)} = 60$ kN Comp., $N^{(7)} = 45$ kN comp.

Ans 6 $N^{(1)} = 75$ kN tension, $N^{(2)} = 58.34$ kN tension, $N^{(3)} = 41.66$ kN tension,
$N^{(4)} = 50$ kN comp., $N^{(5)} = 41.66$ kN tension, $N^{(6)} = 41.66$ kN comp., $N^{(7)} = 25$ kN tension

Ans. 7 $N^{(1)} = 22.5$ kN tension, $N^{(2)} = 90$ kN comp., $N^{(3)} = 112.5$ kN tension, $N^{(5)} = 37.5$ kN tension

8. to **11.** Plot the moment diagram of the beam subjected to the external actions shown in Fig. 8P1 using the method of summation. Repeat with the beams of Figs. 8P2 to 8P4.

12. to **19**. Compute the internal actions in each member of the structure of Fig. 8P12 as functions of its axial coordinate. Plot the shear and moment diagrams of the members of this structure. Repeat with the structures of Figs. 8P13 to 8P19.

Ans. 12 $R_h^{(1)} = 98$ kN $\leftarrow$, $R_v^{(1)} = 64$ kN $\uparrow$, $M_2^{(2)} = -192$ kN·m, $M_2^{(1)} = -557$ kN·m

Ans. 13 $R_h^{(1)} = \dfrac{4P}{5} \leftarrow$, $R_v^{(1)} = \dfrac{5P}{10} \downarrow$, $M_2^{(2)} = \dfrac{4PL}{5}$, $R_v^{(3)} = \dfrac{11P}{10} \uparrow$

Ans. 14 $R_v^{(1)} = 35$ kN, $R_v^{(3)} = 35$ kN, $R_h^{(3)} = 0$, $M_2^{(1)} = 60$ kN·m, $Q_3^{(3)} = 21$ kN $\nearrow$

Ans. 15 $R_v^{(2)} = \dfrac{5P_3}{4}$, $R_v^{(3)} = \dfrac{3P_3}{4}$, $M_2^{(3)} = -P_3L$

Ans. 16 $R_v^{(1)} = 52.73$ kN $\uparrow$, $R_h^{(1)} = 30$ kN $\leftarrow$, $R_v^{(4)} = 47.27$ kN $\uparrow$, $M_2^{(2L)} = 129.11$ kN·m, $M_2^{(3)} = -50.9$ kN·m

Ans. 17 $R_v^{(1)} = 2.5$ kN $\uparrow$, $R_h^{(1)} = 40$ kN $\leftarrow$, $R_v^{(2)} = 32.5$ kN $\downarrow$, $M_2^{(2)} = 10$ kN·m

Ans. 18 $R_V^{(1)} = 10$ kN $\uparrow$ $R_V^{(4)} = 70$ kN $\uparrow$ $M_2^{(2)} = 360$ kN · m

Ans. 19 $M_2^{(2)} = 120$ kN·m $N^{1,2} = 29.615$ kN $Q_3^{(1)} = 17.275$ kN $Q_3^{(3)} = -17.275$ kN

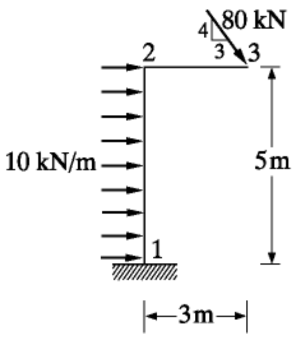
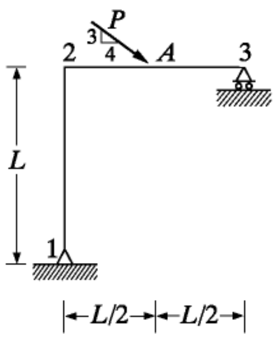


Figure 8P12



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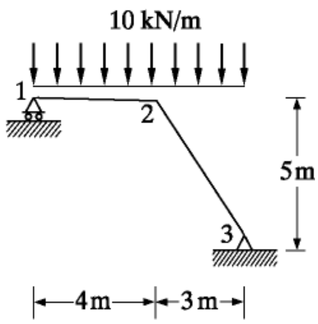

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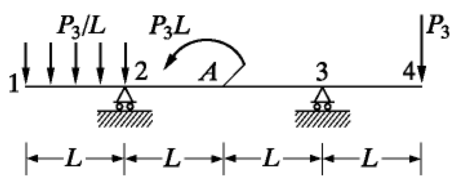

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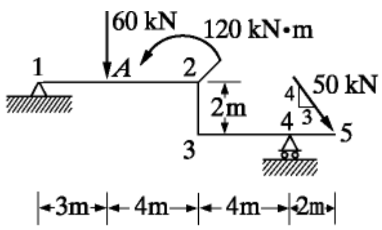


Figure 8P16

20. Determine the internal actions acting on the cross sections of each member of the pipe assembly shown in Fig. 8P20 as functions of its axial coordinate. Plot the shear and moment diagrams of the members of the pipe assembly.

Ans. $N^{(AB)} = 4$ kN $\quad Q_2^{(AB)} = 2$ kN $\quad Q_3^{(AB)} = 0 \quad M_1^{(AB)} = 480$ kN·mm

$M_2^{(B)} = -960$ kN·mm $\quad M_3^{(AB)} = 1{,}600 - 2x_1^{(AB)}$ kN·mm

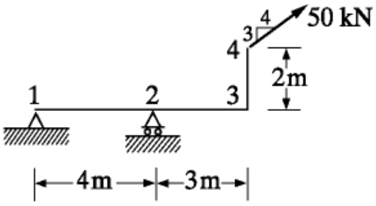


Figure 8P17

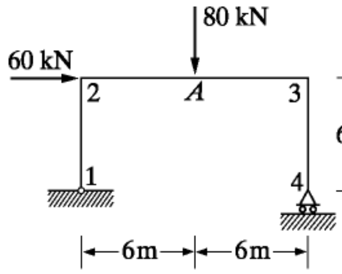


Figure 8P18

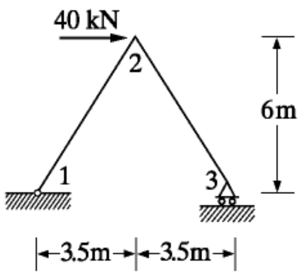


Figure 8P19

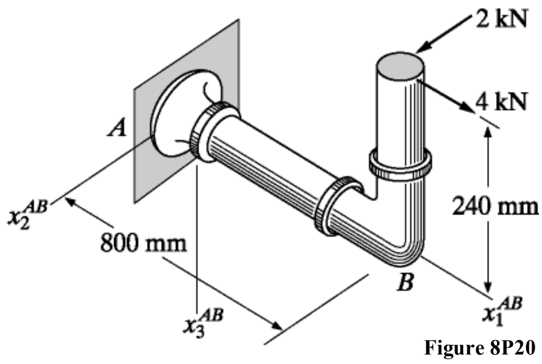


Figure 8P20

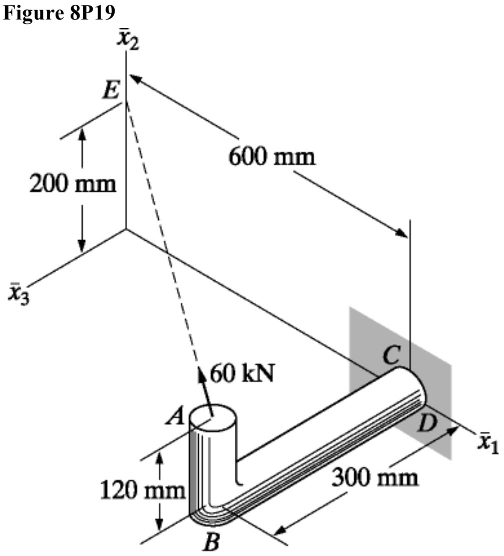


Figure 8P21

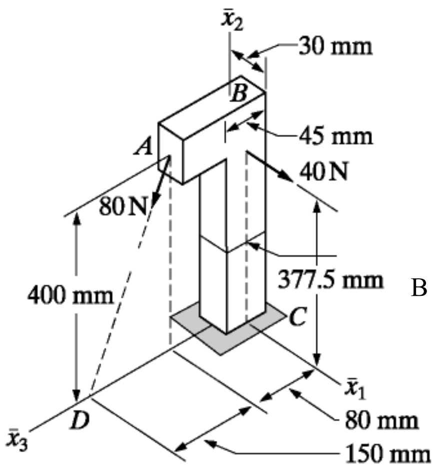


Figure 8P22

21. A steel hanger is subjected to the forces shown in Fig. 8P21. Compute the internal actions acting on the cross sections of each member of the hanger as functions of its axial coordinate. Plot the shear and moment diagrams of the members of the hanger.

Ans. $\bar{R}_3^{(D)} = 26.6$ kN, $\bar{R}_1^{(D)} = 53.3$ kN, $\bar{R}_2^{(D)} = -7.1$ kN,

$\bar{M}_1^{(D)} = 5.322$ kN·m, $\bar{M}_3^{(D)} = -6.396$ kN·m (torsional), $\bar{M}_2^{(D)} = 15.99$ kN·m

22. Determine the internal actions acting on the cross sections of each member of the machine part of Fig. 8P22 as functions of its axial coordinate. Plot the shear and moment diagrams of the members of this machine part.

Ans. $\bar{M}_1^{(C)} = -17.232$ kN·m, $\bar{Q}_3^{(C)} = 28.1$ kN, $\bar{Q}_1^{(C)} = 40$ kN, $\bar{M}_3^{(C)} = -15.1$ kN·m

23. A bar ABC of rectangular cross section (6 x 12 mm) shown in Fig. 8P23 rotates about an axis through point B with a constant angular speed ω= 180 rpm. The bar is made from an isotropic, linearly elastic material of modulus of elasticity E = 75 GPa and density ρ= 2800 kg/m^3. Determine the elongation of part AB of the bar due to the centrifugal forces acting on its particles.

Ans. Elongation = $1.590(10^{-4})\cdot L^3$ (m)

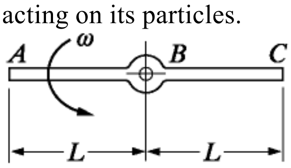


Figure 8P23

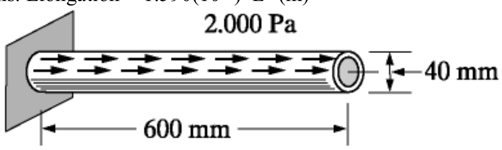


Figure 8P24

24. A 3 mm thick steel tube (E = 200 GPa) of 40 mm outside diameter and 600 mm length is fixed at its one end and is subjected to an uniformly distributed axial component of traction of 2,000 Pa on its outer lateral surface as shown in Fig. 8P24. Compute the axial component of translation $u_1(x_1)$ of the tube.

Ans. $u_1^L = 0.6486$ mm

25. A prismatic bar of length L is suspended in the vertical position from one of its end surfaces. The bar is made from an isotropic, linearly elastic material of modulus of elasticity E (kN/m^2) and specific weight w (kN/m^3). The area of cross section of the bar is A (m^2). Compute the component of translation $u_1(x_1)$ of the bar caused by its own weight.

Ans. $u_1(x_1) = \dfrac{wx_1}{2E}(2L - x_1)$

26. A 2 m long prismatic bar of rectangular cross section (80 x 40 mm) is subjected to an uniform distribution of the axial component of traction p_1 (kN/mm^2) on its lateral surface and to a concentrated force of 40 kN as shown in Fig. 8P26. The bar is made from an isotropic, linearly elastic material with modulus of elasticity E = 200 GPa and Poisson's ratio ν= 1/3. If the total decrease of the 80 -m dimension of the bar at section $\alpha - \alpha$ is 0.024 mm., compute the value of the uniform axial traction p_1 and the total elongation of the bar.

Ans. p_1 = 0.44667 kN/mm^2

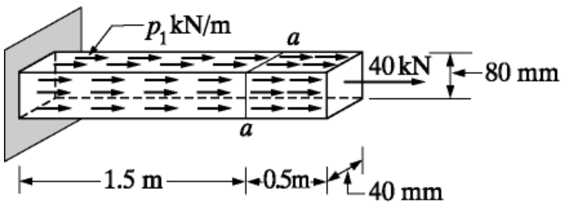


Figure 8P26

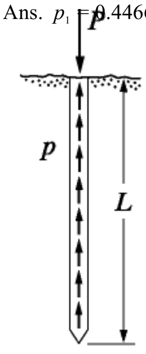


Figure 8P27

27. Friction piles support vertical forces by frictional forces exerted along their length by the surrounding soil (see Fig. 8P27). Consider a friction prismatic steel pile of modulus of elasticity E and cross sectional area A, embedded L meters into soil. Assuming that the frictional force is uniform and has a magnitude p kN/m derive a formula for the total shortening of the pile when it is subjected to a compressive force P at its one end.

Ans. Total shortening $= \dfrac{1}{EA}\left(PL - \dfrac{pL^2}{2}\right)$

28. and **29.** Consider a member of constant cross section (A = 60 mm^2) subjected to the axial centroidal forces shown in Fig. 8P28. The member is made from an isotropic, linearly elastic material with modulus of elasticity E = 200 GPa. Using functions of discontinuity determine the translation field $u_1(x_1)$ of the member. Repeat with the member of Fig. 8P29.

Ans. 28 For $0 < x_1 < 2$ $\quad u_1(x_1) = \dfrac{69.20x_1}{EA}$ $\quad$ for $2 < x_1 < 3.2$ $\quad u_1(x_1) = \dfrac{81.2x_1 - 8x_1^2 + 8}{EA}$

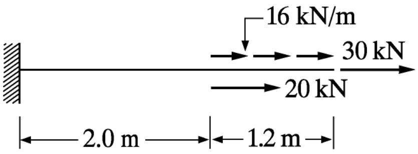


Figure 8P28

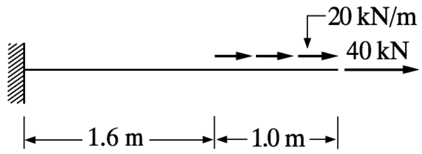


Figure 8P29

30. and **31.** A prismatic member made from an isotropic, linearly elastic material with modulus of elasticity E is subjected to the axial centroidal forces shown in Fig. 8P30. Using functions of discontinuity compute the reactions of the member and plot the axial force diagram. Repeat with the member of Fig. 8P31.

Ans. 30 $R_1^0 = 64$ kN← $R_1^L = 16$ kN← $\quad$ Ans. 31 $R_1^0 = 9.8$ kN← $R_1^L = 18.21$ kN←

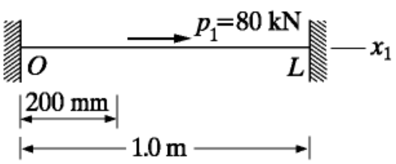


Figure 8P30

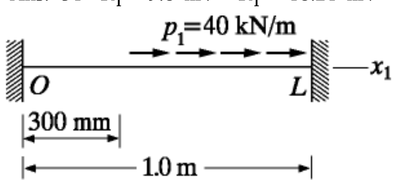


Figure 8P31

32. Compute the component of displacement and the components of stress in a fixed at one end ($x_1 = 0$) prismatic member of solid circular cross section when subjected to a torsional moment M_1^L at its other end ($x_1 = L$) and to a linearly varying distribution of torsional moment along its length $[m_1(x_1) = m_1^0(L - x_1)/L]$. The member is made from a homogeneous, isotropic, linearly elastic material and has a cross section of polar moment of inertia I_p.

Ans. $\theta_1(x_1) = \dfrac{M_1^2 x_1}{GI_p} + \dfrac{m_1^{(o)}}{6LGI_p}\left[-(L - x_1)^3 + L^3\right]$

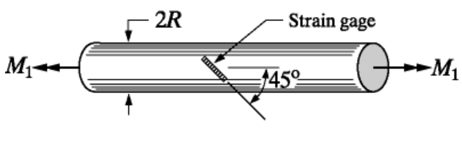


Figure 8P33

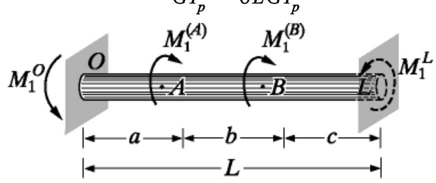


Figure 8P34

33. A prismatic shaft of solid circular cross section of radius R is subjected at its ends to equal and opposite torsional moments of magnitude M_1. The member is made from an isotropic, linearly elastic material of shear modulus G and Poison's ratio ν. The maximum normal component of strain at 45° to the axis of the member is equal to e (see Fig. 8P33). Obtain a formula for G as function of ν, M_1, e and R.

Ans. $G = \dfrac{8M(1+\nu)}{e\pi R_3}$

34. A member of solid circular cross section of radius R is fixed at both ends and is subjected to the torsional moments shown Fig. 8P34. The member is made from an isotropic, linearly elastic material of shear modulus G. Using functions of discontinuity derive formulas for the reactions of the member.

Ans. $M_1^L = -\dfrac{M_1^{(A)}a}{L} + \dfrac{M_1^{(B)}b}{L}$

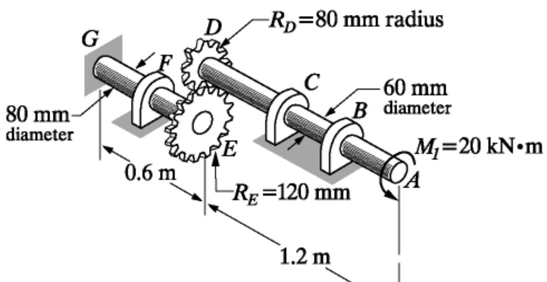


Figure 8P35

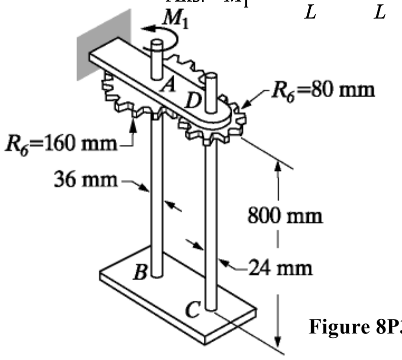


Figure 8P36

35. The assembly of two shafts connected with gears as shown in Fig. 8P35 is subjected to a torsional moment of magnitude 20 kN·m at the cross section at point A. Compute

(a) The maximum shearing stress $\tau_{1\theta}$ in shafts $ABCD$ and EFG.

Ans. $\tau_{1\theta}^{(ABCD)} = 471.6$ MPa $\quad \tau_{1\theta}^{(EFG)} = 298$ MPa

(b) The angle of twist of the cross section is at point A (G=75 GPa). Ans. $\theta_1^{(A)} = 19.54°$

(c) If the allowable shearing stress for the shafts is 60 MPa, determine the maximum torsional moment that can be applied at point A. Ans. $[M^{(A)}] = 2.54$ kN·m

(d) Redesign the assembly of shafts; that is, choose new diameters for the shafts so that you have the most economical design when the assembly is subjected to a torsional moment of 20 kN·m at point A. The most economical design of the assembly is achieved if both shafts are stressed to the maximum allowable stress.

Ans. $R^{(ABCD)} = 59.64$ mm $\quad R^{(EFG)} = 68.27$ mm

36. The steel (G = 75 GPa) shafts AB and CD are fixed at their ends B and C and connected by gears, as shown in Fig. 8P36.

(a) Compute the maximum value of the shearing stress in each shaft when a torsional moment M_1 = 3.6 kN·m is applied at point A. Ans. $\tau_{1\theta}^{(AB)} = 27.4$ MPa

(b) Compute the angle of twist of the cross section of the shaft AB at point A when a torsional moment M_1 = 3.6 kN·m is applied at point A. Ans. $\theta^{(AB)} = 0.008132$ rad

(c) Establish the maximum allowable torsional moment M_1 which can be applied at point A of shaft AB if the allowable shearing stress for steel is 60 MPa. *Hint:* Notice that the algebraic sum of the torsional moments M_1^B and M_1^C is not equal to M_1 because the board EAD exerts a force on shafts AB and CD. Ans. $M_1 = 5.902$ kN·m

Chapter 9

Theories of Mechanics of Materials for Straight Beams Made from Isotropic, Linearly Elastic Materials

9.1 Formulation of the Boundary Value Problem for Computing the Components of Displacement and the Internal Actions in Prismatic Straight Beams Made from Isotropic, Linearly Elastic Materials

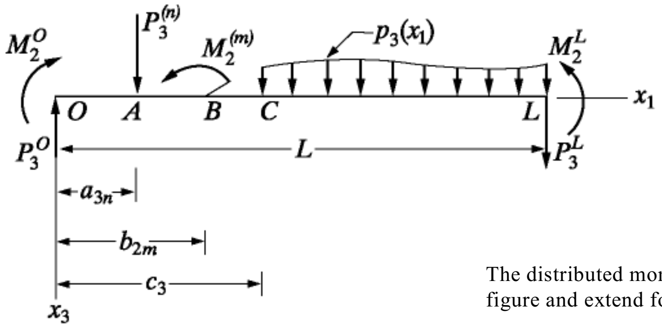
Consider a prismatic straight-line member, having cross sections of arbitrary geometry, made from a homogeneous, isotropic, linearly elastic material. We choose the x_1 axis to be the axis of the member, that is, the locus of the centroids of its cross sections. The member is initially at a reference stress-free, stain-free state (undeformed state) of mechanical and thermal equilibrium at a uniform temperature T_o. Subsequently, the member reaches a second state (deformed state) of mechanical, but not necessarily thermal, equilibrium due to the application on it of one or more of the following loads (see Fig. 9.1):

1. A distribution of body forces throughout its volume as well as a distribution of traction on its lateral surfaces which are equivalent to specified transverse forces and bending moments whose vector is normal to the axis of the member ($M_1 = 0$ and $m_1 = 0$). The forces could be distributed $p_2(x_1)$ and $p_3(x_1)$ and concentrated $P_2^{(n)}(n = 1, 2, ..., n_2)$ and $P_3^{(n)}(n = 1, 2, ..., n_3)$. The moments could be distributed $m_2(x_1)$ and $m_3(x_1)$ and concentrated $M_2^{(m)}(m = 1, 2, ..., m_2)$ and $M_3^{(m)}(m = 1, 2, ..., m_3)$. The line of action of the transverse forces lies in a plane which contains the shear centers of the cross sections of the member (see Section 9.7).
2. A specified change of temperature which is a linear function of x_2 and x_3 and moreover vanishes at the centroid ($x_2 = x_3 = 0$) of the cross section of the member [$\Delta T_c = 0$, $\delta T_2 \neq 0$, $\delta T_3 \neq 0$ (see Section 8.11)]. Notice that if in the second state of mechanical equilibrium the temperature of the surface of the member varies with the space coordinates, the temperature inside the member will be non-uniform. Thus, heat will flow into or out of the member and consequently, the member will not be in a state of thermal equilibrium.
3. A distribution of traction on each of its end surfaces ($x_1 = 0$ and $x_1 = L$) which is statically equivalent to

(a) Transverse concentrated forces P_2^0 or P_2^L and P_3^0 or P_3^L (see Fig 9.1). The line of action of each of these forces passes through the shear center of the end surface on which it acts.

(b) Bending moments M_2^0 or M_2^L and M_3^0 or M_3^L (see Fig 9.1).

Line members subjected to the loading described above are called *beams*. They are subjected to bending without twisting. Due to this loading, the originally straight axis of the beam deforms into a curve called its *elastic curve* which is specified by its translation



The distributed moments $m_2(x_1)$ not shown in the figure and extend for $d_2 \leq x_1 \leq L$.

Figure 9.1 Free-body diagram of a beam subjected to bending about the x_2 axis.

(deflection) fields $u_3(x_1)$ and $u_2(x_1)$.

In order to formulate correctly the boundary value problems for computing the components of deflection $u_2(x_1)$, and $u_3(x_1)$ of a beam one quantity from each of the following pairs of quantities must be specified at each end of a beam:

$$\begin{array}{lll} u_2{}^q & \text{or} & P_2{}^q \\ u_3{}^q & \text{or} & P_3{}^q \\ \theta_2^q & \text{or} & M_2{}^q \qquad\qquad q = 0 \text{ or } L \\ \theta_3^q & \text{or} & \mathrm{M}_3{}^q \end{array} \tag{9.1}$$

However, for a unique solution, the rigid-body motion of the beam must be specified. Consequently, at least either u_2 and θ_3 must be specified at one end of a beam or u_2 must be specified at both ends of a beam. Moreover, at least either u_3 and θ_2 must be specified at one end of a beam or u_3 must be specified at both ends of a beam. For example, the boundary conditions for the beam of Fig. 9.2a are

$$u_3(0) = 0 \tag{9.2a}$$
$$\theta_2(0) = 0 \tag{9.2b}$$
$$Q_3(L) = P_3{}^L \tag{9.2c}$$
$$M_2(L) = M_2{}^L \tag{9.2d}$$

where $P_3{}^L$ and $M_2{}^L$ are the specified transverse force and bending moment, respectively, at the end $x_1 = L$ of the beam. The boundary conditions (9.2a) and (9.2b) are essential, while the boundary conditions (9.2c) and (9.2d) are natural. The boundary conditions for the beam of Fig. 9.2b are

$$u_3(0) = 0 \tag{9.3a}$$
$$\theta_2(0) = 0 \tag{9.3b}$$
$$u_3(L) = 0 \tag{9.3c}$$
$$\theta_2(L) = 0 \tag{9.3d}$$

Thus, all boundary conditions for the beam of Fig. 9.2b are essential. Notice that the number of essential boundary conditions specified for the beam of Fig. 9.2a is the minimum required in order to prevent it from moving as a rigid body. In this, case the beam is *statically determinate*. That is, its reactions and the distribution of its internal actions $M_2(x_1)$ and $Q_3(x_1)$ can be established by considering the equilibrium of appropriate segments of the beam. Moreover, notice that there are two more essential boundary conditions specified for

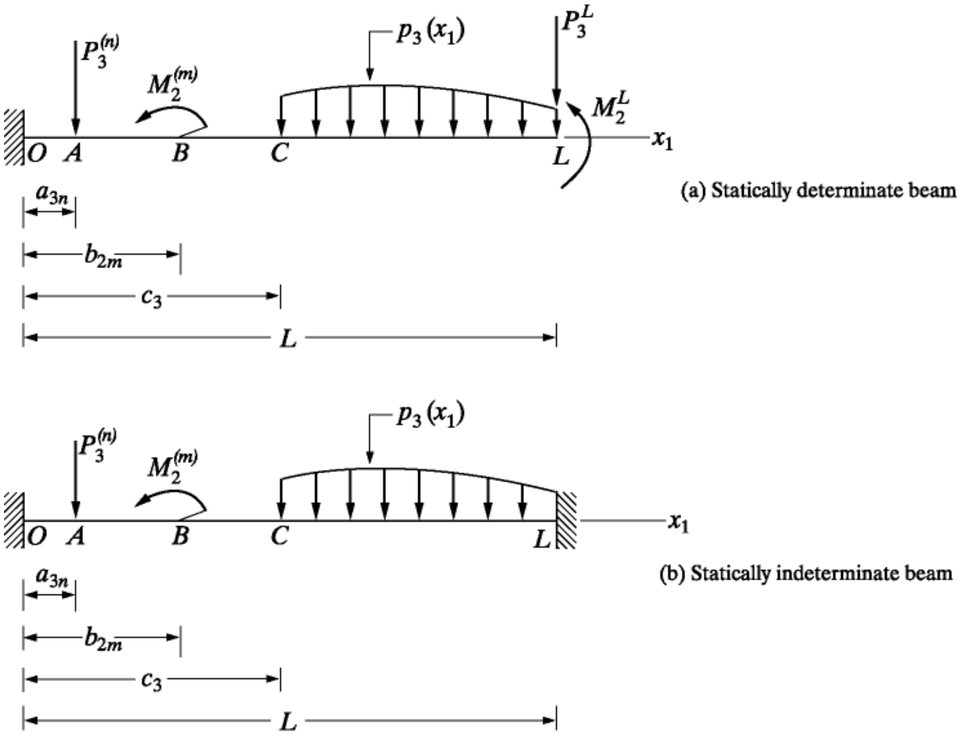


Figure 9.2 Beams subjected to external actions producing only a transverse component of translation $u_3(x_1)$.

the beam of Fig. 9.2b than the minimum required in order to prevent it from moving as a rigid body. In this case the beam is statically indeterminate to the second degree. That is, its reactions and the distribution of its internal actions $M_2(x_1)$ and $Q_3(x_1)$ cannot be established by considering only the equilibrium of segments of the beam.

We are interested in computing the components of translation $u_2(x_1)$ or $u_3(x_1)$ and rotation $\theta_3(x_1)$ or $\theta_2(x_1)$ and the internal shearing forces $Q_2(x_1)$ or $Q_3(x_1)$ and bending moments $M_3(x_1)$ or $M_2(x_1)$ acting on the cross sections of beams when they reach the second state of equilibrium due to the application on them of the loads and boundary conditions described above. When a line member is subjected to these loads and boundary conditions, as discussed in Section 8.12, its cross sections do not rotate (twist) about the x_1 axis. That is,

$$\theta_1(x_1) = 0 \tag{9.4a}$$

Moreover, referring to relations (8.5), we have

$$\hat{u}_1(x_1, x_2, x_3) = u_1(x_1) - x_2\theta_3(x_1) + x_3\theta_2(x_1) \tag{9.4b}$$

and

$$e_{11}(x_1, x_2, x_3) = \frac{\partial \hat{u}_1}{\partial x_1} = \frac{du_1}{dx_1} - x_2\frac{d\theta_3}{dx_1} + x_3\frac{d\theta_2}{dx_1} \tag{9.4c}$$

Relations (9.4b) and (9.4c) have been established on the basis of the assumption that plane sections normal to the axis of the beam remain plane after deformation. Consequently, they are valid for beams made from any material (elastic, plastic, viscoelastic, etc.).

Substituting relation (9.4c) into the first of the stress–strain relations (8.44), we obtain the following stress–displacement relations for a beam made from a homogeneous, isotropic, linearly elastic material

$$\tau_{11} = E\left[\frac{du_1}{dx_1} - x_2\left(\frac{d\theta_3}{dx_1} + H_3\right) + x_3\left(\frac{d\theta_2}{dx_1} - H_2\right)\right] \tag{9.5a}$$

where

$$H_2(x_1) = \frac{\alpha\delta T_3(x_1)}{h^{(3)}(x_1)} \tag{9.5b}$$

$$H_3(x_1) = \frac{\alpha\delta T_2(x_1)}{h^{(2)}(x_1)} \tag{9.5c}$$

$\delta T_3(x_1)$ and $\delta T_2(x_1)$ are defined in relations (8.31) and (8.30), respectively; $h^{(3)}(x_1)$ and $h^{(2)}(x_1)$ are defined in Fig. 8.21; α is the coefficient of linear thermal expansion of the material from which the beam is made.

Inasmuch as the beam is not subjected to external axial centroidal forces, the internal axial centroidal force $N(x_1)$ vanishes. Taking this into account and substituting relation (9.5) into relation (8.9a), we obtain

$$N(x_1) = \iint_A \tau_{11}\, dA = EA\frac{du_1}{dx_1} = 0$$

or

$$u_1(x_1) = \text{constant} \tag{9.6a}$$

In obtaining the above relation we have noted that the x_2 and x_3 axes are centroidal and consequently,

$$\iint_A x_2\, dA = 0 \qquad \iint_A x_3\, dA = 0 \tag{9.6b}$$

If we restrain the beam from translating in the axial direction as a rigid body at one point of its axis, taking into account relation (9.6), we have

$$u_1(x_1) = 0 \tag{9.6c}$$

From relation (9.4c) we see that

$$e_{11}(x_1, 0, 0) = 0 \tag{9.6d}$$

It should be emphasized that the above relation was obtained using relation (9.5) and, consequently, we have only shown that it is valid for beams made from an isotropic, linearly elastic material. Thus, for such beams relations (9.4) and (9.5) reduce to

$$\hat{u}_1(x_1, x_2, x_3) = -x_2\theta_3(x_1) + x_3\theta_2(x_1) \tag{9.7a}$$

$$e_{11}(x_1, x_2, x_3) = \frac{\partial \hat{u}_1}{\partial x_1} = -x_2\frac{d\theta_3}{dx_1} + x_3\frac{d\theta_2}{dx_1} \tag{9.7b}$$

$$\tau_{11} = E\left[-x_2\left(\frac{d\theta_3}{dx_1} + H_3\right) + x_3\left(\frac{d\theta_2}{dx_1} - H_2\right)\right] \tag{9.7c}$$

Substituting relation (9.7c) into (8.9e) and (8.9f), we get the following internal moment–rotation relations for prismatic, straight beams made from a homogeneous, isotropic, linearly elastic material.

$$
\begin{aligned}
M_2(x_1) &= \iint_A \tau_{11} x_3 dA = E\left[-\left(\frac{d\theta_3}{dx_1} + H_3\right)\iint_A x_2 x_3 dA + \left(\frac{d\theta_2}{dx_1} - H_2\right)\iint_A x_3^2 dA\right] \\
&= E\left[-\left(\frac{d\theta_3}{dx_1} + H_3\right) I_{23} + \left(\frac{d\theta_2}{dx_1} - H_2\right) I_{22}\right]
\end{aligned}
\tag{9.8a}
$$

$$
\begin{aligned}
M_3(x_1) &= -\iint_A \tau_{11} x_2 dA = -E\left[-\left(\frac{d\theta_3}{dx_1} + H_3\right)\iint_A x_2^2 dA + \left(\frac{d\theta_2}{dx_1} - H_2\right)\iint_A x_2 x_3 dA\right] \\
&= -E\left[-\left(\frac{d\theta_3}{dx_1} + H_3\right) I_{33} + \left(\frac{d\theta_2}{dx_1} - H_2\right) I_{23}\right]
\end{aligned}
\tag{9.8b}
$$

where $I_{22}(x_1)$ and $I_{33}(x_1)$ are the moments of inertia of the cross section of the beam about its x_2 and x_3 axes, respectively; I_{23} is the product of inertia of the cross section of the beam about the x_2 and x_3 axis. Substituting relations (9.8b) into (8.20) and (9.8a) into (8.21), we obtain the following internal shearing force–rotation relations for prismatic, straight beams made from a homogeneous, isotropic, linearly elastic material.

$$
Q_2 = -\frac{dM_3}{dx_1} - m_3 = -\frac{d}{dx_1}\left[EI_{33}\left(\frac{d\theta_3}{dx_1} + H_3\right) - EI_{23}\left(\frac{d\theta_2}{dx_1} - H_2\right)\right] - m_3 \tag{9.8c}
$$

$$
Q_3 = \frac{dM_2}{dx_1} + m_2 = \frac{d}{dx_1}\left[EI_{22}\left(\frac{d\theta_2}{dx_1} - H_2\right) - EI_{23}\left(\frac{d\theta_3}{dx_1} + H_3\right)\right] + m_2 \tag{9.8d}
$$

Solving relations (9.8a) and (9.8b) for $(d\theta_2/dx_1 - H_2)$ and $(d\theta_3/dx_1 + H_3)$, we get

$$
I_{33}M_2 + I_{23}M_3 = E\left(I_{22}I_{33} - I_{23}^2\right)\left(\frac{d\theta_2}{dx_1} - H_2\right) \tag{9.9a}
$$

$$
I_{22}M_3 + I_{23}M_2 = E\left(I_{22}I_{33} - I_{23}^2\right)\left(\frac{d\theta_3}{dx_1} + H_3\right) \tag{9.9b}
$$

Substituting relation (9.8a) and (9.8b) into the action equations of equilibrium (8.22) and (8.23), we obtain

$$
\frac{d^2}{dx_1^2}\left[EI_{33}\left(\frac{d\theta_3}{dx_1} + H_3\right) - EI_{23}\left(\frac{d\theta_2}{dx_1} - H_2\right)\right] + \frac{dm_3}{dx_1} = p_2(x_1) \tag{9.10a}
$$

$$
\frac{d^2}{dx_1^2}\left[EI_{22}\left(\frac{d\theta_2}{dx_1} - H_2\right) - EI_{23}\left(\frac{d\theta_3}{dx_1} + H_3\right)\right] + \frac{dm_2}{dx_1} = -p_3(x_1) \tag{9.10b}
$$

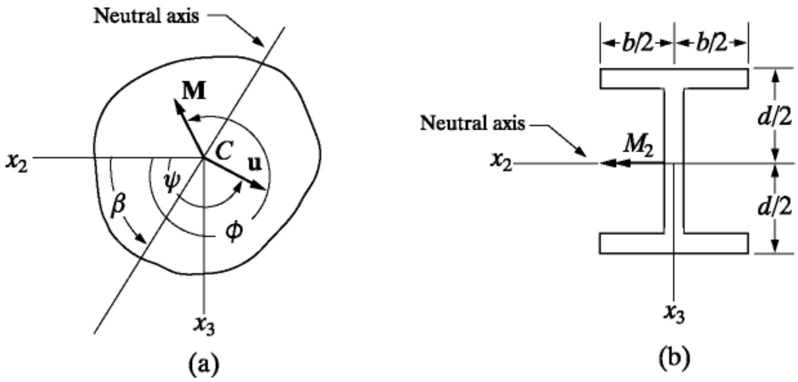


Figure 9.3 Cross sections of prismatic beams showing their neutral axis.

As discussed in Section 8.2 in the theories of mechanics of materials it is assumed that the components of stress τ_{22}, τ_{33} and τ_{23} are very small compared to the other components of stress and can be disregarded. Thus,

$$\tau_{22} = \tau_{33} = \tau_{23} = 0 \tag{9.11a}$$

Consequently, we only consider the following components of stress acting on a cross section of prismatic, straight beams

$$\begin{aligned} \tau_{11} &= \tau_{11}(x_1, x_2, x_3) \\ \tau_{12} &= \tau_{12}(x_1, x_2, x_3) \\ \tau_{13} &= \tau_{13}(x_1, x_2, x_3) \end{aligned} \tag{9.11b}$$

A formula for the normal component of stress acting on the cross sections of prismatic, straight beams made from an isotropic, linearly elastic material, subjected to bending can be established by using relations (9.9) and (9.7c). That is,

$$\tau_{11} = -\left(\frac{M_2 I_{23} + M_3 I_{22}}{I_{22} I_{33} - I_{23}^2}\right) x_2 + \left(\frac{M_2 I_{33} + M_3 I_{23}}{I_{22} I_{33} - I_{23}^{\,2}}\right) x_3 \tag{9.12a}$$

where I_{22}, I_{33}, I_{23} are the moments and product of inertia of a cross section of the beam, with respect to the x_2 and x_3 axes, defined in Appendix C. Formulas for the shearing components of stress τ_{12} and τ_{13} are derived in Section 9.5. When the axes x_2 and x_3 are principal centroidal ($I_{23} = 0$) relation (9.12a) reduces to

$$\tau_{11} = -\frac{M_3 x_2}{I_3} + \frac{M_2 x_3}{I_2} \tag{9.12b}$$

From relation (9.12a), we see that when a cross section of a beam is subjected only to M_2 ($M_3 = 0$) and its x_2, x_3 axes are not principal centroidal, the normal component of stress varies in the direction of the x_2 axis (see the example at the end of this section). This is not the case, however, when the x_2, x_3 axes are principal centroidal [see relation (9.12b)].

From relation (9.12a) we see that for every cross section of a beam, made from an isotropic, linearly elastic material there exists a straight line, which passes through the centroid of the cross section, whose particles are not subjected to axial component of stress (τ_{11}= 0). We call this line the *neutral axis* of the cross section. From relations (9.7b) and (9.7c) we deduce that the particles of the neutral axis of the cross sections of beams made from an isotropic, linearly elastic material and loaded in an environment of constant temperature ($H_2 = H_3 = 0$), do not elongate or shrink in the axial direction; that is, $e_{11} = 0$. The neutral axis of a cross section can be located by specifying the angle β shown in Fig. 9.3a. Referring to relation (9.12a), the equation of the neutral axis is

$$-\left(\frac{M_2 I_{23} + M_3 I_{22}}{I_{22} I_{33} - I_{23}^2}\right) x_2^* + \left(\frac{M_2 I_{33} + M_3 I_{23}}{I_{22} I_{33} - I_{23}^2}\right) x_3^* = 0 \tag{9.13}$$

where x_2^* and x_3^* are the coordinates of a point located on the neutral axis. Moreover, referring to Fig. 9.3a, we have

$$\tan\beta = \frac{x_3^*}{x_2^*}$$

Solving equation (9.13) for x_3^* and substituting the resulting expression in the above relation, we obtain

$$\tan\beta = \frac{M_2 I_{23} + M_3 I_{22}}{M_2 I_{33} + M_3 I_{23}} \tag{9.14}$$

In Fig. 9.3 we denote by φ the angle which the vector of the bending moment $\mathbf{M} = M_2\mathbf{i}_2 + M_3\mathbf{i}_3$ acting on a cross section makes with the x_2 axis; referring to this figure, we have

$$\begin{aligned} M_2 &= M\cos\varphi \\ M_3 &= M\sin\varphi \end{aligned} \tag{9.15a}$$

where M is the magnitude of the bending moment $\mathbf{M}$ acting on the cross section under consideration. From relation (9.15a), we get

$$\tan\varphi = \frac{M_3}{M_2} \tag{9.15b}$$

Using relation (9.15b) to eliminate the moments from relation (9.14) we obtain

$$\tan\beta = \frac{I_{23} + I_{22}\tan\varphi}{I_{33} + I_{23}\tan\varphi} \tag{9.16}$$

If the x_2 and x_3 axes are principal centroidal, relation (9.16) reduces to

$$\tan\beta = \tan\varphi\left(\frac{I_2}{I_3}\right) = \frac{M_3 I_2}{M_2 I_3} \tag{9.17}$$

Therefore, if for a cross section of a beam $I_2 = I_3$, its neutral axis is in the direction of the vector of the bending moment acting on it. Moreover, if the vector of the bending moment acting on a cross section has component, only in the direction of the principal axis x_2 or x_3 ($M_3 = 0$ or $M_2 = 0$), then the principal axis x_2 or x_3, respectively, is the neutral axis.

The moment of inertia I_2 of some manufactured rolled beams such as channels and I-beams is many times greater than I_3. As demonstrated buy the following example, the maximum value of the normal component of stress τ_{11} acting on the cross sections of such beams, when subjected to a bending moment with φ close to zero but not equal to zero, could be considerably larger than when the beam is subjected to a bending moment of the same maguitude with φ equal to zero. Referring to Appendix H, an I-beam S 380 × 74 has $I_2 = 201(10^6)\text{mm}^4$, $I_3 = 6.65(10^6)\text{mm}^4$, $b = 143$ mm and $d = 381$mm. Suppose that this beam was subjected to a moment $\mathbf{M}^* = M_2\mathbf{i}_2 + M_3\mathbf{i}_3$ whose vector was inclined 2° counterclockwise from the principal x_2 axis (φ is 2°). Referring to Fig. 9.3b and relations (9.12b and 9.15b) we see that the maximum value of the normal component of stress occurs at $x_2 = -b/2$, $x_3 = d/2$. That is,

$$(\tau_{11})_{\max} = \frac{M_2 d}{2I_2} + \frac{M_3 b}{2I_3} = \frac{M_2 d}{2I_2}\left(1 + \frac{I_2 b \tan\varphi}{I_3 d}\right) = \frac{M_2(190.5)}{201(10^6)}\left[1 + \frac{201(143)}{6.65(381)}(0.03492)\right]$$

$$= M_2(0.94776)(10^{-6})(1 + 0.39615) = 1.32322(10^{-6})M_2$$

The magnitude of the moment M^* acting on the cross sections of the beam is

$$M^* = \frac{M_2}{\cos\varphi} = 1.0006095 M_2$$

If that moment had been acting about the x_2 axis of the beam, the maximum stress would have been

$$(\tau_{11})_{\max} = \frac{M^* d}{2I_2} = \frac{1.0006095(190.5)M_2}{(201)(10^6)} = 0.948328(10^{-6})M_2$$

Thus, we see that a small inclination of the vector of the bending moment from the x_2 axis ($\varphi = 2°$) and consequently of the plane of loading from the x_1x_3 plane, increases the maximum normal component of stress by 39.54%.

In order to be able to formulate the boundary value problem under consideration, we need two relations relating the component of translation $u_2(x_1)$ or $u_3(x_1)$ of the cross sections of a beam with its component of rotation $\theta_3(x_1)$ or $\theta_2(x_1)$, respectively. There are two mechanics of materials theories available in the literature, for computing the deflection of beams. *They differ only in the relation of the component of translation $u_2(x_1)$ or $u_3(x_1)$ of a cross section of a beam with its component of rotation $\theta_3(x_1)$ or $\theta_2(x_1)$, respectively.* The one theory which we use in this text is called the *classical theory of beams.* We present it in Section 6.2. It is based on the assumption that the geometry of the beam is such that the effect of shear deformation [that is, of the shearing components of strain $e_{12}(x_1, x_2, x_3)$ and $e_{13}(x_1, x_2, x_3)$] of the particles of the beam on its components of translation $u_2(x_1)$ and $u_3(x_1)$ is negligible. The other theory for analyzing beams is called the *Timoshenko theory of beams.* We present it in Section 6.4. In this theory a portion of the effect of shear deformation of the particles of a beam on its components of translation $u_2(x_1)$ and $u_3(x_1)$ is retained.

Expressions for the shearing components of strain $e_{12}(x_1, x_2, x_3)$ and $e_{13}(x_1, x_2, x_3)$ of a beam can be obtained by substituting relation (9.7a) into the corresponding strain–displacement relations (2.16). That is,

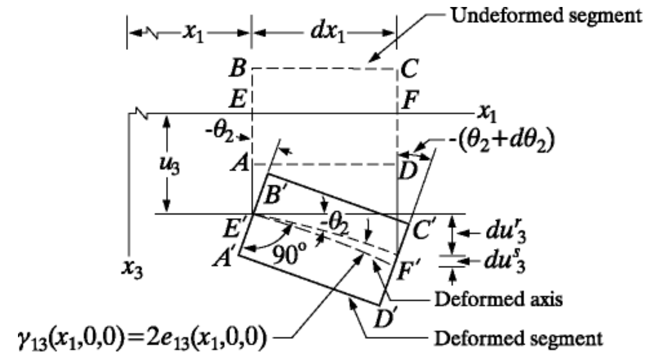


Figure 9.4 Deformed configuration of a segment of infinitesimal length of a beam subjected to bending in the x_1x_3 plane without twisting.

$$2e_{12} = \frac{\partial \hat{u}_1}{\partial x_2} + \frac{\partial \hat{u}_2}{\partial x_1} = -\theta_3 + \frac{\partial \hat{u}_2}{\partial x_1} \tag{9.18a}$$

$$2e_{13} = \frac{\partial \hat{u}_1}{\partial x_3} + \frac{\partial \hat{u}_3}{\partial x_1} = \theta_2 + \frac{\partial \hat{u}_3}{\partial x_1} \tag{9.18b}$$

However, the theories of mechanics of materials do not offer a satisfactory way for computing the components of displacement $\hat{u}_2(x_1, x_2, x_3)$ and $\hat{u}_3(x_1, x_2, x_3)$ of the particles of beams. Consequently, relations (9.18) cannot be used to establish the shearing components of strain $e_{12}(x_1, x_2, x_3)$ and $e_{13}(x_1, x_2, x_3)$. Instead these quantities are obtained from the shearing components of stress $\tau_{12}(x_1, x_2, x_3)$ and $\tau_{13}(x_1, x_2, x_3)$, respectively, using the third and fourth of the stress-strain relations (8.44). The shearing components of stress are established from the normal components of stress $\tau_{11}(x_1, x_2, x_3)$ as shown in Section 9.5.

When relations (9.18) are evaluated at $x_2 = x_3 = 0$, they become relations between the component of translation $u_2(x_1)$ or $u_3(x_1)$ and the component of rotation $\theta_3(x_1)$ or $\theta_2(x_1)$, respectively, for a beam subjected to bending without twisting. That is,

$$\theta_2(x_1) = -\frac{du_3}{dx_1} + 2e_{13}(x_1,0,0) \tag{9.19a}$$

$$\theta_3(x_1) = \frac{du_2}{dx_1} - 2e_{12}(x_1,0,0) \tag{9.19b}$$

In order to illustrate the physical significance of the terms of relations (9.19) consider the segment $ABCD$ of length dx_1 of a beam in its stress-free strain-free, state of mechanical and thermal equilibrium shown in Fig. 9.4. The segment is cut from the beam by two imaginary planes normal to its axis. When the beam is subjected to external loads which bend it only in the x_1x_3 plane without twisting it, the segment under consideration translates in the x_3 direction, rotates about an axis parallel to the x_2 axis as a rigid body and it deforms. In Fig. 9.4 we denote the deformed configuration of the segment by $A'B'C'D'$. The deformed configuration of the cross section AB is specified by its component of translation u_3 and its component of rotation $-\theta_3$, while the deformed configuration of the cross section CD is specified by its component of translation $u_3 + du_3$ and its component of rotation $-(\theta_2 + d\theta_2)$ The angle between the deformed configuration $B'E'A'$ of the cross section BEA of the beam and its deformed axis is equal to $\pi/2 - 2e_{13}(x_1, 0, 0)$. The strain $e_{13}(x_1, 0, 0)$ is positive because the before deformation right angle $\triangle AEF$ after deformation has decreased to $\triangle A'E'F'$. Referring to Fig. 9.4, we see that the increment du_3 of the translation u_3 consists of two parts. That is,

$$du_3 = du_3^r + du_3^s \tag{9.20a}$$

where, referring to Fig. 9.4 ,we have

$$du_3^r = -\theta_2 dx_1 \tag{9.20b}$$

$$du_3^s = 2e_{13}(x_1, 0, 0)dx_1 \tag{9.20c}$$

Substituting relations (9.20b) and (9.20c) into (9.20a), we obtain

$$du_3 = -\theta_2 dx_1 + 2e_{13}(x_1, 0, 0)dx_1 \tag{9.21}$$

Relation (9.21) is identical to (9.19a). Thus, relation (9.19a) indicates that the increment of the transverse component of translation du_3 of a segment of infinitesimal length of the beam may be regarded as the sum of two parts. The one du^r_3 is the result of the rotation of the segment as a rigid body while the other du^s_3 is the result of its shear deformation.

Similarly, it can be shown that relation (9.19b) indicates that the increment of the transverse component of translation du_2 of a segment of infinitesimal length of a beam may be regarded as the sum of two parts. The one du^r_2 is the result of the rotation of the segment as a rigid body while the other du^s_2 is the result of its shear deformation as follows:

$$du_2^r = \theta_3 dx_1 \tag{9.22a}$$

$$du_2^s = 2e_{12}(x_1, 0, 0)dx_1 \tag{9.22b}$$

Example 1 A 2 m long cantilever beam made from an American rolled steel angle L 51 x 51 x 9.5 is subjected on its unsupported end to a force $P_3 = 2$ kN (see Fig. a). Compute the distribution of the normal component of stress at the fixed end of the beam and locate its neutral axis.

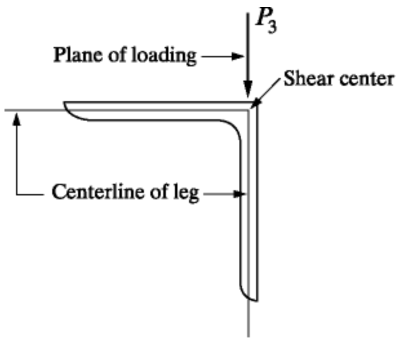


Figure a Geometry and loading of the beam.

Solution As we explain in Section 9.7, the shear center of an angle is located at the intersection of the center lines of its legs. Consequently, the beam under consideration bends without twisting.

Referring to Fig. b, notice that the pair of principal centroidal axes of the cross section of the beam are its axis of symmetry $\tilde{x}_2$ and the axis perpendicular to it. The x_2 and x_3 axes are not principal.

Referring to the table of properties of rolled steel shapes in Appendix H, we obtain

$$
\begin{aligned}
I_{22} &= I_{33} = 0.202(10^6)\ \text{mm}^4 \\
\text{thickness} &= 9.5\ \text{mm} \\
A &= 879\ \text{mm}^2 \\
r_3 &= \text{radius of gyration} = \sqrt{\frac{I_3}{A}} = 9.95\ \text{mm}
\end{aligned}
\tag{a}
$$

Using relation (C.11a) of Appendix C the moment of inertia of the cross sections of the beam about the principal centroidal axis $\tilde{x}_3$ is equal to

$$I_3 = Ar_3^2 = 879(9.95)^2 = 87{,}023.2\ \text{mm}^4 \tag{b}$$

From relations (a) and (b) we know the moments of inertia I_{22} and I_{33} about the x_2 and x_3 axes and the moment of inertia I_3 about the principal axis $\tilde{x}_3$. Consequently, we can compute the product of inertia I_{23} and the principal moment of inertia I_2 about the principal axis $\tilde{x}_2$. For this purpose we recall that as shown in Section C.4 of Appendix C the tensor of the moments and the product of inertia with respect to the x_2 and x_3 axes is

$$\begin{bmatrix} I_{22} & -I_{23} \\ -I_{23} & I_{33} \end{bmatrix} = \begin{bmatrix} 0.202 & -I_{23} \\ -I_{23} & 0.202 \end{bmatrix} 10^6\ \text{mm}^4 \tag{c}$$

Referring to relation (1.116c) the moment of inertia I_3 with respect to the principal centroidal axis $\tilde{x}_3$ (see Fig. b) can be expressed as

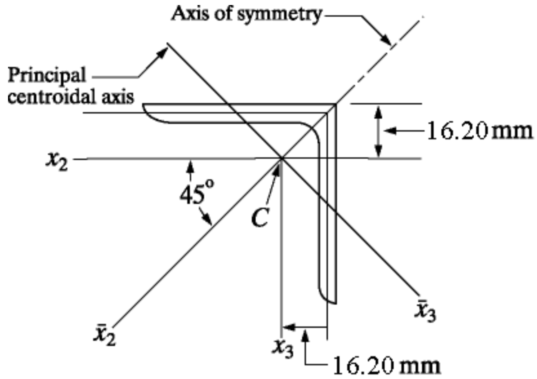


Figure b Cross section of the beam.

$$I_3 = 0.087023(10^6) = \frac{1}{2}(I_{22} + I_{33}) - \frac{1}{2}(I_{22} - I_{33})\cos 2\varphi_{11} + I_{23}\sin 2\varphi_{11}$$

$$= 0.202(10^6)\text{ mm}^4 + I_{23}$$

Thus,

$$I_{23} = -0.114977(10^6)\text{ mm}^4 \tag{d}$$

Using the first of relations (a) and relation (d), we get

$$I_{22}I_{33} - I_{23}^2 = [(0.202)^2 - (0.114977)^2]10^{12} = 27.584(10^9)\text{ mm}^4 \tag{e}$$

The distribution of the normal component of stress τ_{11} acting on the cross sections of the beam is given by formula (9.12a). Substituting the first of relations (a) and relations (d) and (e) into (9.12a) and noting that $M_2 = -P_3(L - x_1)$, we obtain

$$\begin{aligned} \tau_{11}(x_1, x_2, x_3) &= -\frac{M_2 I_{23} x_2}{I_{22}I_{33} - I_{23}^2} + \frac{M_2 I_{33} x_3}{I_{22}I_{33} - I_{23}^2} \\ &= -\frac{P_3(L - x_1)}{27.584(10^9)}[0.114977(10^6)x_2 + 0.202(10^6)x_3] \\ &= -P_3(L - x_1)(10^{-5})[0.4168x_2 + 0.7323x_3] \end{aligned} \tag{f}$$

The normal components of stress τ_{11} acting on the particles of the top surface ($x_3 = -16.15$) and of the surface of the right side ($x_2 = -16.20$) of the beam at its fixed end ($x_1 = 0$) are equal to

$$\begin{aligned} \tau_{11}(0,\ x_2, -16.20) &= P_3L(10^{-5})[0.7323(16.20) - 0.4168x_2] \\ &= P_3L(10^{-5})(11.8633 - 0.4168x_2) \end{aligned} \tag{g}$$

and

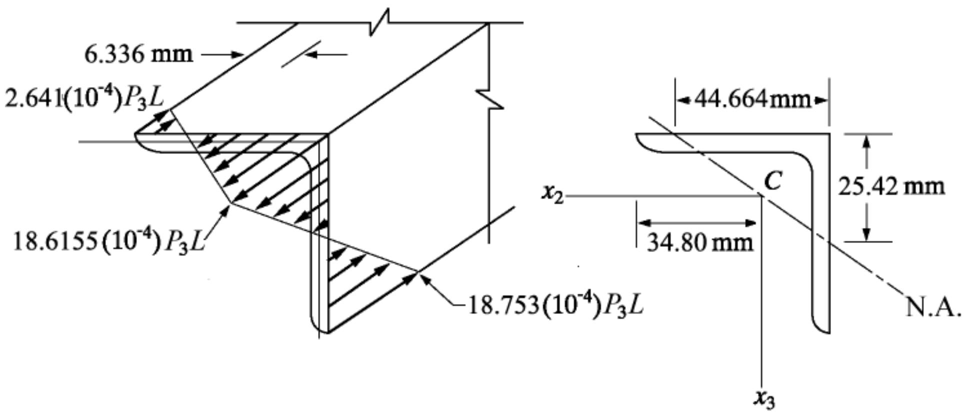


Figure c Distribution of the normal component of stress at the fixed end of the beam.

Figure d Location of the neutral axis on a cross section of the beam.

$$\tau_{11}(0, -16.20,\ x_3) = P_3L(10^{-5})(6.7313 - 0.7323x_3) \tag{h}$$

The results are plotted in Fig. c.

The equation of the neutral axis is obtained by setting $\tau_{11} = 0$ in relation (f). That is,

$$0.4168x_2 + 0.7323x_3 = 0 \tag{i}$$

The location of the neutral axis is shown in Fig. d.

9.1.1 Radii of Curvature of the Elastic Curve of a Beam

Consider two particles of the axis of a beam located before deformation at points A and B. The coordinates of these particles are x_1 and $x_1 + dx_1$, respectively. When the beam is subjected to external loads which bend it without twisting it, the particles under consideration move to points A' and B' of its elastic curve (deformed axis). In Fig. 9.5a we show the projection of the elastic curve of the beam on the x_1x_3 plane and we denote by A'' and B'' the projection on the x_1x_3 plane of points A' and B', respectively. Moreover, we denote by dx_s the distance between points A'' and B'' measured along the projection of the elastic curve on the x_1x_3 plane. Furthermore, in Fig. 9.5a we draw lines normal to the tangent lines to the projection of the elastic curve on the x_1x_3 plane at points A'' and B''. These lines intersect at point C which is called the *center of curvature* for point A'' of the projection of the elastic curve of the beam on the x_1x_3 plane. The length of line CA'' is the *radius of curvature* ρ_{13} at point A'' of the projection of the elastic curve of the beam on the x_1x_3 plane. The curvature of the projection of the elastic curve on the x_1x_3 plane is denoted by k_{13} and is defined as the reciprocal of the radius of curvature ρ_{13}. That is,

$$k_{13} = \frac{1}{\rho_{13}}$$

Finally, we denote by $\omega_2(x_1)$ the angle between the x_1 axis and the tangent to the projection of the elastic curve of the beam on the x_1x_3 plane at point A''. This angle represents the rotation about the x_2 axis of the particle which before deformation was located at point A of the axis of the beam. In accordance with our convention (see Section 8.2) ω_2 is considered positive when its vector is directed in the direction of the positive x_2 axis. Referring to Fig. 9.5a, we see that

$$\tan\,\omega_2 = -\frac{du_3}{dx_1}$$

Moreover, as shown in Fig. 9.5a the angle between lines CA'' and CB'' is equal to the increment $d\omega_2$ of the angle ω_2 from point A'' to point B''. Thus,

$$\rho_{13}d\omega_2 = dx_s$$

and

$$k_{13} = \frac{1}{\rho_{13}} = \frac{d\omega_2}{dx_s} = \frac{d[\arctan\,(-du_3/dx_1)]}{dx_1}\left(\frac{dx_1}{dx_s}\right) \tag{9.23}$$

Referring Fig. 9.5a, we have

$$A''B'' = dx_s = [(dx_1)^2 + (du_3)^2]^{\frac{1}{2}}$$

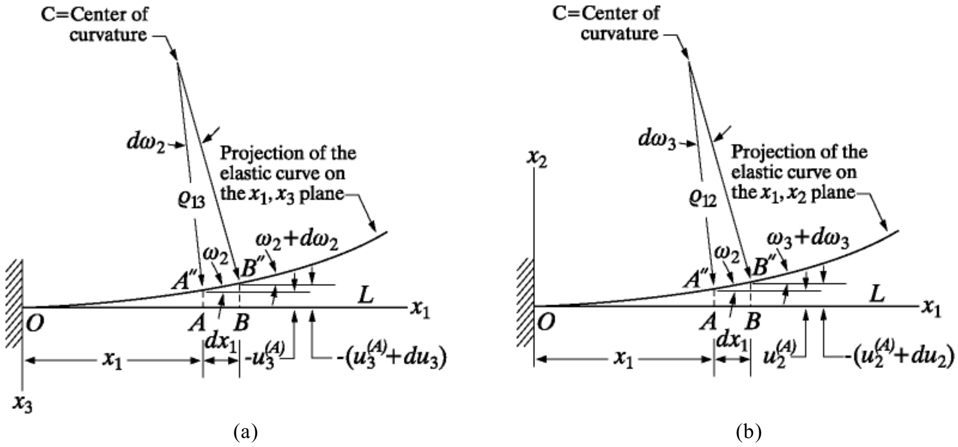


Figure 9.5 Projections of the elastic curve of a cantilever beam on the x_1x_3 and x_1x_2 planes having positive curvatures.

Thus,

$$\frac{dx_s}{dx_1} = \left[1 + \left(\frac{du_3}{dx_1}\right)^2\right]^{\frac{1}{2}} \tag{9.24}$$

Moreover, recall from calculus that the derivative of arctan (du_3/dx_1) is equal to

$$\frac{d}{dx_1}\left(\arctan \frac{du_3}{dx_1}\right) = \frac{\dfrac{d^2u_3}{dx_1^2}}{1 + \left(\dfrac{du_3}{dx_1}\right)^2} = -\frac{d}{dx_1}\left[\arctan\left(-\frac{du_3}{dx_1}\right)\right]$$

Substituting the above relation and (9.24) into (9.23), we obtain

$$k_{13} = \frac{1}{\rho_{13}} = -\frac{\dfrac{d^2u_3}{dx_1^2}}{\left[1 + \left(\dfrac{du_3}{dx_1}\right)^2\right]^{\frac{3}{2}}} \tag{9.25a}$$

Similarly, referring to Fig. 9.5b, we see that

$$\tan\omega_3 = \frac{du_2}{dx_1}$$

and

$$k_{12} = \frac{1}{\rho_{12}} = \frac{d\omega_3}{dx_s} = \frac{d[\arctan(du_2/dx_1)]}{dx_1}\left(\frac{dx_s}{dx_1}\right) = \frac{\dfrac{d^2u_2}{dx_1^2}}{\left[1 + \left(\dfrac{du_2}{dx_1}\right)^2\right]^{\frac{3}{2}}} \tag{9.25b}$$

Within the range of validity of the theory of small deformation $(du_3/dx_1)^2$ and $(du_2/dx_1)^2$ are very small compared to unity and can be disregarded. Thus, relations (9.25) reduce to

$$k_{13} = \frac{1}{\rho_{13}} = -\frac{d^2u_3}{dx_1^2} \tag{9.26a}$$

$$k_{12} = \frac{1}{\rho_{12}} = \frac{d^2u_2}{dx_1^2} \tag{9.26b}$$

9.2 The Classical Theory of Beams

In the classical theory of beams it is assumed that *the contribution of the shear deformation of the particles of a beam to its components of translation* u_3 *and* u_2 *is negligible compared to the contribution of the rotation of these segments.* That is, in relations (9.19) the components of strain $e_{12}(x_1, 0, 0)$ and $e_{13}(x_1, 0, 0)$ are considered negligible compared to the components of rotation $\theta_3(x_1)$ and $\theta_2(x_1)$, respectively. Consequently, relations (9.19) reduce to

$$\theta_2(x_1) \approx -\frac{du_3}{dx_1} \tag{9.27a}$$

$$\theta_3(x_1) \approx \frac{du_2}{dx_1} \tag{9.27b}$$

This is equivalent to assuming that plane sections normal to the axis of a beam before deformation not only remain plane after deformation but also are perpendicular to its deformed axis. Substituting relations (9.27) into (9.7a) and (9.7b) and using relations (9.26) for beams made from isotropic, linearly elastic materials, we have

$$\hat{u}_1(x_1, x_2, x_3) = -x_2\frac{du_2}{dx_1} - x_3\frac{du_3}{dx_1} \tag{9.28a}$$

and

$$e_{11}(x_1, x_2, x_3) = -x_2\frac{d^2u_2}{dx_1^2} - x_3\frac{d^2u_3}{dx_1^2} = -\frac{x_2}{\rho_{12}} + \frac{x_3}{\rho_{13}} \tag{9.28b}$$

Substituting relations (9.27) into (9.8), we obtain the following action–displacement relations for a beam made from a homogeneous, isotropic, linearly elastic material

$$M_2(x_1) = -EI_{22}\left(\frac{d^2u_3}{dx_1^2} + H_2\right) - EI_{23}\left(\frac{d^2u_2}{dx_1^2} + H_3\right) \tag{9.29a}$$

$$M_3(x_1) = EI_{33}\left(\frac{d^2u_2}{dx_1^2} + H_3\right) + EI_{23}\left(\frac{d^2u_3}{dx_1^2} + H_2\right) \tag{9.29b}$$

$$Q_2(x_1) = -\frac{d}{dx_1}\left[EI_{33}\left(\frac{d^2u_2}{dx_1^2} + H_3\right)\right] - \frac{d}{dx_1}\left[EI_{23}\left(\frac{d^2u_3}{dx_1^2} + H_2\right)\right] - m_3 \tag{9.29c}$$

$$Q_3(x_1) = -\frac{d}{dx_1}\left[EI_{22}\left(\frac{d^2u_3}{dx_1^2} + H_2\right)\right] - \frac{d}{dx_1}\left[EI_{23}\left(\frac{d^2u_2}{dx_1^2} + H_3\right)\right] + m_2 \tag{9.29d}$$

Substituting relations (9.27) into (9.9), we obtain

$$I_{33}M_2 + I_{23}M_3 = -E(I_{22}I_{33} - I_{23}^2)\left(\frac{d^2u_3}{dx_1^2} + H_2\right) \tag{9.30a}$$

$$I_{22}M_3 + I_{23}M_2 = E(I_{22}I_{33} - I_{23}^2)\left(\frac{d^2u_2}{dx_1^2} + H_3\right) \tag{9.30b}$$

Substituting relations (9.30) into (9.26), we obtain

$$k_{13} = \frac{1}{\rho_{13}} = \frac{I_{33}M_2 + I_{23}M_3}{E(I_{22}I_{33} - I_{23}^2)} + H_2 \tag{9.31a}$$

$$k_{12} = \frac{1}{\rho_{12}} = \frac{I_{22}M_3 + I_{23}M_2}{E(I_{22}I_{33} - I_{23}^2)} - H_3 \tag{9.31b}$$

If the x_2 and x_3 axes are principal centroidal ($I_{23} = 0$), relations (9.29) and (9.31) reduce to

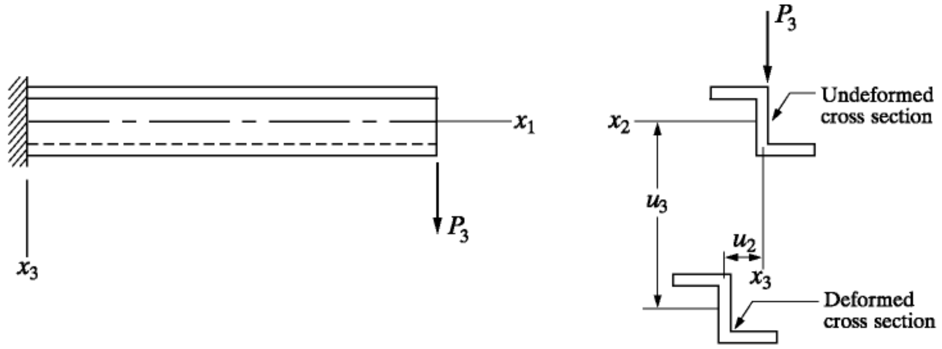


Figure 9.6 Deflection of a cantilever beam having a Z cross section.

$$M_2 = -EI_2\left(\frac{d^2u_3}{dx_1^2} + H_2\right) \tag{9.32a}$$

$$M_3 = EI_3\left(\frac{d^2u_2}{dx_1^2} + H_3\right) \tag{9.32b}$$

$$Q_2 = -\frac{d}{dx_1}\left[EI_3\left(\frac{d^2u_2}{dx_1^2} + H_3\right)\right] - m_3 \tag{9.32c}$$

$$Q_3 = -\frac{d}{dx_1}\left[EI_2\left(\frac{d^2u_3}{dx_1^2} + H_2\right)\right] + m_2 \tag{9.32d}$$

$$k_{13} = \frac{1}{\rho_{13}} = \frac{M_2}{EI_2} + H_2 \tag{9.32e}$$

$$k_{12} = \frac{1}{\rho_{12}} = \frac{M_3}{EI_3} - H_3 \tag{9.32f}$$

where I_2 or I_3 is the moment of inertia about the principal centroidal axis x_2 or x_3, respectively. Referring to relations (9.30), we see that when the x_2, x_3 axes of a beam are principal centroidal and the loads acting on the beam produce an internal moment $\mathbf{M} = M_i\mathbf{i}_i$ (i = 2 or 3) and $H_2 = H_3 = 0$, the centroids of the cross sections of the beam translate only in the direction of the x_j (j = 3 or 2, $j \neq i$) axis. That is, the component of translation u_i vanishes. However, when the x_2, x_3 axes of a beam are not principal centroidal and the external actions acting on the beam produce an internal moment $\mathbf{M} = M_i\mathbf{i}_i$ (i = 2 or 3), the centroids of the cross sections of the beam translate in the directions of the x_2 and x_3 axes. That is, the translation vector of the beam has two components $u_2(x_1)$ and $u_3(x_1)$. For example, the cantilever beam of Fig. 9.6 has a Z cross section, consequently, the x_2 and x_3 axes of the beam are not principal. When this beam is subjected to a force at its free end acting in the direction of the x_3 axis, the internal moment is $M_2 = P_3(L - x_1)$. As shown in Fig. 9.6 the movement of the cross sections of the beam has components in the directions of the x_2 and x_3 axes.

Substituting relations (9.27) into (9.10), we obtain

$$\frac{d^2}{dx_1^2}\left[EI_{33}\left(\frac{d^2u_2}{dx_1^2} + H_3\right) + EI_{23}\left(\frac{d^2u_3}{dx_1^2} + H_2\right)\right] = p_2 - \frac{dm_3}{dx_1} \tag{9.33a}$$

$$\frac{d^2}{dx_1^2}\left[EI_{22}\left(\frac{d^2u_3}{dx_1^2} + H_2\right) + EI_{23}\left(\frac{d^2u_2}{dx_1^2} + H_3\right)\right] = p_3 + \frac{dm_2}{dx_1} \tag{9.33b}$$

For prismatic beams relations (9.33) can be uncoupled. That is, they can be rewritten as a differential equation involving only $u_2(x_1)$ and a differential equation involving only $u_3(x_1)$. If the x_2 and x_3 axes are principal centroidal ($I_{23} = 0$), relations (9.33) reduce to

$$\frac{d^2}{dx_1^2}\left[EI_3\left(\frac{d^2u_2}{dx_1^2} + H_3\right)\right] = p_2 - \frac{dm_3}{dx_1} \tag{9.34a}$$

$$\frac{d^2}{dx_1^2}\left[EI_2\left(\frac{d^2u_3}{dx_1^2} + H_2\right)\right] = p_3 + \frac{dm_2}{dx_1} \tag{9.34b}$$

Relations (9.33) or (9.34) are *the displacement equations of equilibrium for the beam under consideration* .

Consider a prismatic beam subjected to a loading having S points of load change. Referring to our discussion in Section 8.9, we may conclude that the transverse component of translation $u_2(x_1)$ or $u_3(x_1)$ of this beam is represented by a set of functions $u_i^{(s)}(x_1)$ [[s = 1, 2, ..., (S + 1)] (i = 2 or 3)] having derivatives of any order. Each one of these gives the component of translation u_i(i = 2 or 3) and satisfies relations (9.33) in one segment of the beam extending either between a support and its adjacent point of load change or between two consecutive points of load change.

The boundary value problem for computing the transverse components of translation $u_3(x_1)$ and $u_2(x_1)$, the components of rotation $\theta_2(x_1)$ and $\theta_2(x_1)$, the shearing components of the internal force $Q_3(x_1)$ and $Q_2(x_1)$ and the components of the bending moment $M_2(x_1)$ and $M_3(x_1)$ for a beam (statically determinate or indeterminate) subjected to a distribution of external transverse forces and bending moments which have S points of load change and to a change of temperature ($\Delta T_c = 0$, $\delta T_2 \neq \delta T_3 \neq 0$) can be formulated as shown next.

Find the functions $u_i^{(s)}(x_1)$[s = 1, 2, ..., (S + 1)] (i = 2, 3) which have the following properties:

1. Each has derivatives of any order.
2. They satisfy the translation boundary conditions at the ends of the beam where such conditions are specified.
3. They form continuous transverse components of translation (deflection) throughout the length of the beam.
4. When substituted into relations (9.27), they give a set of components of rotation $\theta_2^{(s)}$ and $\theta_3^{(s)}$[s = 1, 2, ..., (S + 1)] which has the following properties:
(a) It satisfies the rotation boundary conditions of the beam.
(b) It yields continuous components of rotation throughout the length of the beam.
5. Each pair [$u_2^{(s)}$ and $u_3^{(s)}$] of these functions satisfies the displacement equations of equilibrium (9.33) at all points of the segment of the beam whose transverse components of translation they represent. That is,

$$\frac{d^2}{dx_1^2}\left[EI_{33}^{(s)}\left(\frac{d^2u_2^{(s)}}{dx_1^2} + H_3^{(s)}\right) + EI_{23}^{(s)}\left(\frac{d^2u_3^{(s)}}{dx_1^2} + H_2^{(s)}\right)\right] = p_2^{(s)} - \frac{dm_3^{(s)}}{dx_1} \tag{9.35a}$$

$$\frac{d^2}{dx_1^2}\left[EI_{22}^{(s)}\left(\frac{d^2u_3^{(s)}}{dx_1^2} + H_2^{(s)}\right) + EI_{23}^{(s)}\left(\frac{d^2u_2^{(s)}}{dx_1^2} + H_3^{(s)}\right)\right] = p_3^{(s)} + \frac{dm_2^{(s)}}{dx_1} \tag{9.35b}$$

$$s = 1, 2, ..., S$$

This ensures that every segment of the beam of infinitesimal length, cut by two imaginary planes perpendicular to its axis, which does not contain a point of load change, is in equilibrium.
6. When substituted into the internal action–translation relations (9.29), they give a set of functions $Q_3^{(s)}(x_1)$, $Q_2^{(s)}(x_1)$, $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ [s = 1, 2, ..., (S + 1)] each one of which represents either the shearing force or the bending moment in the s^{th} segment of the beam

extending between a support and its adjacent point of load change or between two consecutive points of load change. These functions satisfy

(a) The natural boundary conditions at the ends of the beam where specified external transverse forces and/or beading moments are applied.

(b) The requirements for equilibrium of the segments of infinitesimal length of the beam cut by two imaginary planes normal to its axis which contain a point of load change.

On the basis of the foregoing presentation, the components of translation $u_3(x_1)$ and $u_2(x_1)$, of shearing force $Q_3(x_1)$ and $Q_2(x_1)$ and of bending moment $M_2(x_1)$ and $M_3(x_1)$ of a beam can be established by adhering to the following steps:

STEP 1 The set of differential equations (9.35) are solved to obtain the transverse components of translation $u_i^{(s)}(x_1)$ $[s = 1, 2, ..., (S + 1)]$ $(i = 2, 3)$ of each segment of the beam extending between a support and its adjacent point of load change or between two consecutive points of load change. Each of the functions $u_i^{(s)}(x_1)$ $[s = 1, 2, ..., (S_i + 1)]$ $(i = 2, 3)$ involves four constants.

STEP 2 The constants are evaluated by requiring that the transverse components of translation $u_i^{(s)}(x_1)$ $[s = 1, 2, ..., (S+1)]$ $(i = 2, 3)$ satisfy the following requirements:

1. The translation equations of continuity at every point of load change.

2. The translation boundary conditions at the ends of the beam where components of translation are specified.

3. When substituted into the rotation–translation relations (9.27), they give components of rotation $\theta_2(x_1)$ and $\theta_3(x_1)$ which satisfy

(a) The rotation boundary conditions at the ends of the beam where components of rotation are specified

(b) The rotation equations of continuity at every point of load change

4. When substituted into the action–translation relations (9.29), they give internal shearing forces and bending moments which satisfy

(a) The requirement for equilibrium of each segment of infinitesimal length of the beam which contains a point of load change

(b) The natural boundary conditions at the ends of the beam at which such conditions are specified

The components of translation $u_3(x_1)$ and $u_2(x_1)$ of statically determinate beams can also be established by adhering to the following simpler procedure:

STEP 1 The sets of functions $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ $[s = 1, 2, ..., (S + 1)]$ representing the bending moments in the statically determinate beam is established by considering the equilibrium of appropriate segments of the beam.

STEP 2 The functions $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ $[s = 1, 2, ..., (S + 1)]$ established in step 1 are substituted into relations (9.30) and the resulting differential equations are solved to obtain the components of translation $u_2^{(s)}(x_1)$ and $u_3^{(s)}(x_1)$ $[s = 1, 2, ...(S + 1)]$ as functions of x_1 involving two constants each.

STEP 3 The constants are evaluated by requiring that the solution satisfies the essential boundary conditions of the beam as well as the continuity conditions at the points of load change. It is apparent that the sets of functions $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ $[s = 1, 2, ..., (S + 1)]$ obtained in step 1 satisfy the natural boundary conditions of the beam, if any, and the requirements for equilibrium of all the segments of infinitesimal length of the beam cut by two imaginary planes normal to its axis, which contain a point of load change.

In what follows we present two examples.

Example 2 Determine the components of translation $u_2(x_1)$ and $u_3(x_1)$ of the 2 m long cantilever beam subjected to a concentrated force P_3= 0.6 kN at its free end as shown in Fig. a. The beam is made of from an American rolled angle with equal legs L 51 x 51 x 9.5 (E = 200 GPa).

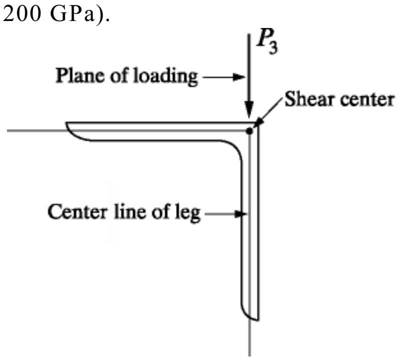


Figure a Geometry and loading of the beam.

Solution We can solve the problem under consideration either by referring our calculations to the axes x_2 and x_3 or by resolving the force P_3 into components parallel to the principal axes $\tilde{x}_2$ and $\tilde{x}_3$ and referring our calculations to these axes. In what follows we use the first approach. The beam under consideration is statically determinate; consequently, it is preferable to determine the equation of its elastic curve by using relations (9.30). Noting that $M_2 = -P_3(L - x_1)$, we have

$$-E(I_{22}I_{33} - I_{23}^2)\left(\frac{d^2u_3}{dx_1^2}\right) = I_{33}M_2 = -I_{33}P_3(L - x_1)$$

$$E(I_{22}I_{33} - I_{23}^2)\left(\frac{d^2u_2}{dx_1^2}\right) = I_{23}M_2 = -I_{23}P_3(L - x_1) \tag{a}$$

Integrating relations (a) twice, we get

$$-E(I_{22}I_{33} - I_{23}^2)\frac{du_3}{dx_1} = \frac{I_{33}P_3(L - x_1)^2}{2} + C_1$$

$$E(I_{22}I_{33} - I_{23}^2)\frac{du_2}{dx_1} = \frac{I_{23}P_3(L - x_1)^2}{2} + C_2 \tag{b}$$

and

$$-E(I_{22}I_{33} - I_{23}^2)u_3 = -\frac{I_{33}P_3(L - x_1)^3}{6} + C_1x_1 + C_3$$

$$E(I_{22}I_{33} - I_{23}^2)u_2 = -\frac{I_{23}P_3(L - x_1)^3}{6} + C_2x_1 + C_4 \tag{c}$$

The constants are evaluated by substituting relations (b) and (c) into the following essential boundary conditions of the beam:

$$u_3(0) = 0 \qquad u_2(0) = 0$$
$$\left.\frac{du_3}{dx_1}\right|_{x_1=0} = 0 \qquad \left.\frac{du_2}{dx_1}\right|_{x_1=0} = 0 \tag{d}$$

Thus, we obtain

$$C_3 = \frac{I_{33}P_3L^3}{6} \qquad C_4 = \frac{I_{23}P_3L^3}{6}$$
$$C_1 = -\frac{I_{33}P_3L^2}{2} \qquad C_2 = -\frac{I_{23}P_3L^2}{2} \tag{e}$$

Substituting the values of the constants (e) into the expressions for the components of displacement (c), we get

$$u_3(x_1) = \frac{I_{33}P_3}{6E(I_{22}I_{33} - I_{23}^2)}[(L - x_1)^3 + 3L^2x_1 - L^3]$$
$$u_2(x_1) = -\frac{I_{23}P_3}{6E(I_{22}I_{33} - I_{23}^2)}[(L - x_1)^3 + 3L^2x_1 - L^3] \tag{f}$$

The moments and products of inertia of the cross sections of the beam of Fig. b, with respect to the x_2, x_3 centroidal axes, are computed in the example of Section 9.1. Referring to this example we have

$$I_{22} = 0.202(10^6)\ \text{mm}^4 = I_{33}$$
$$I_{23} = -\ 0.114977(10^6)\ \text{mm}^4 \tag{g}$$

Thus,

$$I_{22}I_{33} - I_{23}^2 = [(0.202)^2 - (0.114977)^2]10^{12} = 27.584(10^9)\ \text{mm}^4 \tag{h}$$

Substituting relations (g) and (h) into (f) the components of translation (deflection) of the beam at its end $x_1 = L$ are

$$u_3(L) = \frac{I_{33}P_3L^3}{3E(I_{22}I_{33} - I_{23}^2)} = \frac{0.202(10^6)(0.6)(2)^3(10)^9}{3(200)27.584(10)^9} = 58.58 \quad \text{mm} \downarrow \tag{i}$$

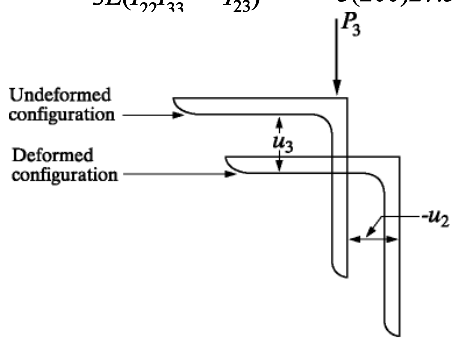


Figure b Results for the beam of Fig a.

$$u_2(L) = -\frac{I_{23}P_3L^3}{3E(I_{22}I_{33} - I_{23}^2)} = -\frac{0.114977(10^6)(0.6)(2)^3(10)^9}{3(200)27.587(10)^9} \tag{i}$$

$$= -33.34 \quad \text{mm} \leftarrow \; = 33.34 \quad \text{mm} \rightarrow$$

The results are shown in Fig b.

Example 3 Establish the equation of the elastic curve and the reactions of the beam subjected to a concentrated moment as shown in Fig. a. The beam is made from a homogeneous, isotropic, linearly elastic material with modulus of elasticity E and has a constant cross section. The x_3 axis is a principal centroidal axis of the cross sections of the beam. Thus, the beam will deflect in the x_3x_1 plane ($u_2 = 0$).

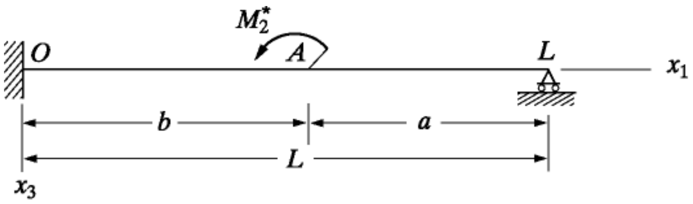


Figure a Geometry and loading of the beam.

Solution The beam of Fig. a is statically indeterminate. Thus, we cannot find its reactions and its internal actions by considering only the equilibrium of segments of the beam. Consequently, in order to establish the deflection of the beam we must use relation (9.35b). At point A of the beam there is a load change. Thus, the deflection $u_3(x_1)$ of the beam is represented in each of the segments $\overline{OA}$ and $\overline{AL}$ by a continuously differentiable function which we denote by $u_3^{(1)}(x_1)$ and $u_3^{(2)}(x_1)$, respectively. Each of these functions must satisfy relation (9.35b). That is, referring to Fig. a, we have

$$\begin{aligned} \frac{d^2}{dx_1^2}\left(EI_2\frac{d^2u_3^{(1)}}{dx_1^2}\right) &= 0 \qquad & 0 < x_1 < b \\ \frac{d^2}{dx_1^2}\left(EI_2\frac{d^2u_3^{(2)}}{dx_1^2}\right) &= 0 \qquad & b < x_1 < L \end{aligned} \tag{a}$$

Integrating each of equations (a) four times, we obtain

$$\begin{aligned} -Q_3^{(1)}(x_1) &= \frac{d}{dx_1}\left(EI_2\frac{d^2u_3^{(1)}}{dx_1^2}\right) = C_1 \qquad & 0 < x_1 < b \\ -Q_3^{(2)}(x_1) &= \frac{d}{dx_1}\left(EI_2\frac{d^2u_3^{(2)}}{dx_1^2}\right) = C_2 \qquad & b < x_1 < L \end{aligned} \tag{b}$$

$$-M_2^{(1)}(x_1) = EI_2\frac{d^2u_3^{(1)}}{dx_1^2} = C_1x_1 + C_3 \qquad 0<x_1<b$$
$$-M_2^{(2)}(x_1) = EI_2\frac{d^2u_3^{(2)}}{dx_1^2} = C_2x_1 + C_4 \qquad b<x_1<L \tag{c}$$

$$EI_2\frac{du_3^{(1)}}{dx_1} = \frac{C_1x_1^2}{2} + C_3x_1 + C_5 \qquad 0<x_1<b$$
$$EI_2\frac{du_3^{(2)}}{dx_1} = \frac{C_2x_1^2}{2} + C_4x_1 + C_6 \qquad b<x_1<L \tag{d}$$

$$EI_2u_3^{(1)} = \frac{C_1x_1^3}{6} + \frac{C_3x_1^2}{2} + C_5x_1 + C_7 \qquad 0<x_1<b$$
$$EI_3u_3^{(2)} = \frac{C_2x_1^3}{6} + \frac{C_4x_1^2}{2} + C_6x_1 + C_8 \qquad b<x_1<L \tag{e}$$

We evaluate the constants by requiring that the deflection $u_3(x_1)$ given by relations (e) satisfies.

1. The boundary conditions of the beam. That is,

$$u_3^{(1)}(0) = 0 \qquad \left.\frac{du_3^{(1)}}{dx_1}\right|_{x_1=0} = 0$$
$$u_3^{(2)}(L) = 0 \qquad M_2^{(2)}(L) = -EI_2\left.\frac{d^2u_3^{(2)}}{d^2x_1}\right|_{x_1=L} = 0 \tag{f}$$

2. The continuity relations at the point (point A) of load change. That is,

$$u_3^{(1)}(b) = u_3^{(2)}(b) \tag{g}$$

$$\left.\frac{du_3^{(1)}}{dx_1}\right|_{x_1=b} = \left.\frac{du_3^{(2)}}{dx_1}\right|_{x_1=b} \tag{h}$$

3. The equations of equilibrium for the segment of infinitesimal length of the beam which includes the point (point A) of load change. That is,

$$Q_3^{(1)}(b) = Q_3^{(2)}(b)$$
$$M_2^{(1)}(b^-) = M_2^{(2)}(b^+) + M_2^* \tag{i}$$

Substituting relations (e) into (f) and (g), (b) into the first of relations (i), (c) into the second of relations (i) and solving the resulting linear algebraic equations, we obtain

$$C_1 = C_2 = -\frac{3\, M_2^*\, b\, (2L - b)}{2L^3} \qquad C_3 = -\frac{M_2^*}{2L^2}(2L^2 + 3b^2 - 6bL)$$
$$C_4 = \frac{3M_2^* b\, (2L - b)}{2L^2} \qquad C_5 = C_7 = 0 \qquad C_6 = -M_2^* b \qquad C_8 = \frac{M_2^* b^2}{2} \tag{j}$$

Substituting relations (j) into (b), (c), and (e), we obtain

$$Q_3^{(1)} = -C_1 = \frac{3M_2^* b(2L - b)}{2L^3}$$
$$Q_3^{(2)} = -C_2 = \frac{3M_2^* b(2L - b)}{2L^3}$$
$$M_2^{(1)} = -C_1 x_1 - C_3 = \frac{3M_2^* b(2L - b)x_1}{2L^3} + \frac{M_2^*}{2L^2}(2L^2 + 3b^2 - 6bL)$$
$$M_2^{(2)} = -C_2 x_1 - C_4 = \frac{3M_2^* b(2L - b)x_1}{2L^3} - \frac{3M_2^* b(2L - b)}{2L^2}$$
$$u_3^{(1)} = -\frac{M_2^* x_1^2}{4EI_2 L^3}[b(2L - b)x_1 + L(2L^2 + 3b^2 - 6bL)]$$
$$u_3^{(2)} = -\frac{M_2^* b}{4EI_2 L^3}[(2L - b)(x_1 - 3L)x_1^2 + 2L^3(2x_1 - b)]$$

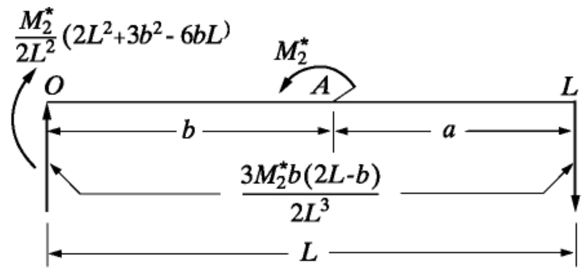


Figure b Free-body diagram of the beam.

9.3 Solution of the Boundary Value Problem for Computing the Transverse Components of Translation and the Internal Actions in Prismatic Beams Made from Isotropic, Linearly Elastic Materials Using Functions of Discontinuity

The solution of the boundary value problem for computing the transverse components of translation $u_2(x_1)$ and $u_3(x_1)$ and the internal actions of beams, presented in the previous section, is cumbersome. In this section we present a more efficient approach to the same problem using functions of discontinuity. These functions are used to write a discontinuous function, as, for example, the external forces and moments acting on a beam, by a single expression. Using these functions we write one set of displacement equations of equilibrium (9.33) for the whole beam as if the distribution of the external actions was continuous. That is, referring to Fig. 9.7 and to Appendix G, we can convert the concentrated forces $P_2^{(n)}$ and $P_3^{(n)}$ (n = 1, 2, 3, ...)to mathematically equivalent distributed forces $p_2(x_1)$ and $p_3(x_1)$, respectively, as follows:

$$p_2(x_1) = P_2^{(n)}\delta(x_1 - a_{2n}) \tag{9.36}$$

$$p_3(x_1) = P_3^{(n)}\delta(x_1 - a_{3n}) \tag{9.37}$$

Referring to the definition of the Dirac delta-function in Appendix G, we see that the right hand side of relation (9.36) [or (9.37)] vanishes everywhere except at point $x_1 = a_{2n}$ [or $x_1 = a_{3n}$] where it is not specified. However, referring to relation (G.13), the resultant of the distributed forces $p_2(x_1)$ [or $p_3(x_1)$] from $x_1 = a_{2n} - \varepsilon$ to $x_1 = a_{2n} + \varepsilon$ (or $x_1 = a_{3n} - \varepsilon$ to $x_1 = a_{3n} + \varepsilon$) is equal to

$$\int_{a_{2n}-\varepsilon}^{a_{2n}+\varepsilon} p_2(x_1)dx_1 = \int_{a_{2n}-\varepsilon}^{a_{2n}+\varepsilon} P_2^{(n)}\delta(x_1 - a_{2n})dx_1 = P_2^{(n)} \tag{9.38}$$

$$\int_{a_{3n}-\varepsilon}^{a_{3n}+\varepsilon} p_3(x_1)dx_1 = \int_{a_{3n}-\varepsilon}^{a_{3n}+\varepsilon} P_3^{(n)}\delta(x_1 - a_{3n})dx_1 = P_3^{(n)} \tag{9.39}$$

where ϵ is a small positive number.

Similarly, we can convert the concentrated moments $M_2^{(m)}$ and $M_3^{(m)}$ acting on a beam to mathematically equivalent distributed moments $m_2(x_1)$ and $m_3(x_1)$ as follows:

$$m_2(x_1) = M_2^{(m)}\delta(x_1 - b_{2m}) \tag{9.40}$$

$$m_3(x_1) = M_3^{(m)}\delta(x_1 - b_{3m}) \tag{9.41}$$

Referring to relation (G.15), we have

$$\frac{dm_2(x_1)}{dx_1} = M_2^{(m)}\delta^{\mathrm{I}}(x_1 - b_{2m}) \tag{9.42a}$$

$$\frac{dm_3(x_1)}{dx_1} = M_3^{(m)}\delta^{\mathrm{I}}(x_1 - b_{3m}) \tag{9.42b}$$

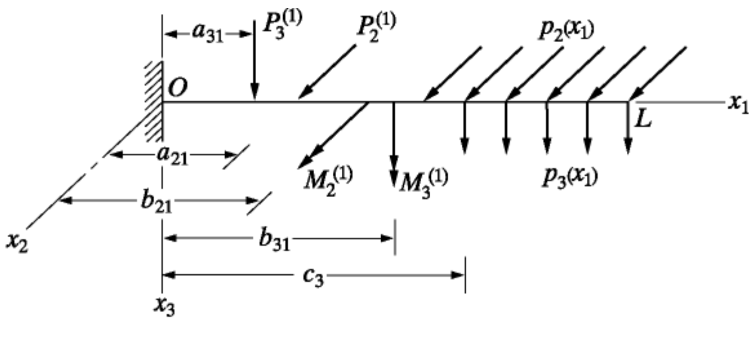


Figure 9.7 Beam subjected to transverse forces and bending moments.

where $\delta^{I}(x_1 - b_{2m})$ and $\delta^{I}(x_1 - b_{3m})$ are the doublet functions defined by relation (G.15).

Using relations (9.36), (9.37) and (9.42) and taking into account relation (G.14), the displacement equations of equilibrium (9.33) for the beam of Fig. 9.7 can be written as

$$\frac{d^2}{dx_1^2}\left[EI_{33}\left(\frac{d^2u_2}{dx_1^2} + H_3\right) + EI_{23}\left(\frac{d^2u_3}{dx_1^2} + H_2\right)\right]$$
$$= P_2^{(1)}\delta(x_1 - a_{21}) - M_3^{(1)}\delta^{I}(x_1 - b_{31}) + p_2\Delta(x_1 - c_2) \tag{9.43a}$$

$$\frac{d^2}{dx_1^2}\left[EI_{22}\left(\frac{d^2u_3}{dx_1^2} + H_2\right) + EI_{23}\left(\frac{d^2u_2}{dx_1^2} + H_3\right)\right] \tag{9.43b}$$
$$= P_3^{(1)}\delta(x_1 - a_{31}) + M_2^{(1)}\delta^{I}(x_1 - b_{21}) + p_3\Delta(x_1 - c_3)$$

where $\Delta(x_1 - c_2)$ and $\Delta(x_1 - c_3)$ are unit step functions defined by relation (G.1), $\delta(x_1 - a_{21})$ and $\delta(x_1 - a_{31})$ are Dirac Delta functions defined by relation (G.10), $\delta^{I}(x_1 - b_{21})$ and $\delta^{I}(x_1 - b_{31})$ are doublet functions defined by relation (G.15). Using the rules for integration of the functions of discontinuity presented in Appendix G, relations (9.43) may be integrated to give the transverse components of translation $u_2(x_1)$ and $u_3(x_1)$ as functions of the axial coordinate of the member involving four constants each. The constants may be evaluated from the specified boundary conditions at the ends of the beam. As mentioned previously, for a properly posed problem one quantity from each of the following pairs of quantities must be specified at each end of a beam

$$\begin{aligned} &u_2 \quad \text{or} \quad Q_2 \\ &\theta_3 \quad \text{or} \quad M_3 \\ &u_3 \quad \text{or} \quad Q_3 \\ &\theta_2 \quad \text{or} \quad M_2 \end{aligned} \tag{9.44}$$

That is, using functions of discontinuity, the requirements for continuity of the transverse components of translation $u_2(x_1)$ and $u_3(x_1)$ and of the components of rotation $\theta_2(x_1)$ and $\theta_3(x_1)$ at each point of load change are satisfied automatically. Moreover, the requirements for equilibrium of the internal shearing forces $Q_2(x_1)$ and $Q_3(x_1)$ and the internal bending moments $M_2(x_1)$ and $M_3(x_1)$ acting on each segment of infinitesimal length containing a point of load change are satisfied automatically.

In what follows we compute the transverse component of translation $u_3(x_1)$, the internal shearing force $Q_3(x_1)$ and the bending moment $M_2(x_1)$ of two prismatic beams using functions of discontinuity.

Example 4 Consider the fixed at both ends beam of constant cross section and length L subjected to the loads shown in Fig. a. The line of action of the external forces is in a plane which contains the shear centers of the cross sections of the beam and is parallel to the x_3 axis. The x_2 and x_3 axes are principal centroidal of the cross section of the beam. The beam is made of a homogeneous, isotropic, linearly elastic material. Establish the deflection $u_3(x_1)$, the internal shearing force $Q_3(x_1)$ and the internal bending moment $M_2(x_1)$ of the beam using functions of discontinuity. Compute the reactions of the beam and show the results on a sketch.

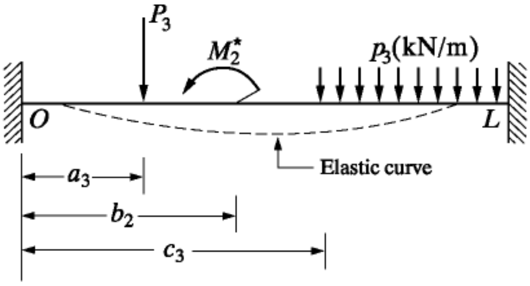


Figure a Geometry and loading of the beam.

Solution This is a statically indeterminate beam, consequently, the distribution of its internal moments cannot be established by considering only the equilibrium of parts of the beam. For this reason, the deflection $u_3(x_1)$ of the beam cannot be established by integrating relation (9.30a), but rather by solving equation (9.43b). Referring to this equation, we have:

$$\frac{d^2}{dx_1^2}\left(EI_2\frac{d^2u_3}{dx_1^2}\right) = p_3\Delta(x_1 - c_3) + P_3\delta(x_1 - a_3) + M_2^*\delta^{\mathrm{I}}(x_1 - b_2) \tag{a}$$

Moreover, the deflection u_3 must satisfy the following boundary conditions:

$$\begin{aligned} u_3(0) &= 0 \qquad \left.\frac{du_3}{dx_1}\right|_{x_1=0} = 0 \\ u_3(L) &= 0 \qquad \left.\frac{du_3}{dx_1}\right|_{x_1=L} = 0 \end{aligned} \tag{b}$$

Integrating equation (a) four times, using relations (G.5) and (G.16) and referring to relation (9.32a) and (9.32d), we obtain

$$-Q_3(x_1) = \frac{d}{dx_1}\left(EI_2\frac{d^2u_3}{dx_1^2}\right) = p_3(x_1 - c_3)\Delta(x_1 - c_3) + P_3\Delta(x_1 - a_3) + M_2^*\delta(x_1 - b_2) + C_1$$

$$\begin{aligned} -M_2(x_1) = EI_2\frac{d^2u_3}{dx_1^2} &= \frac{p_3(x_1 - c_3)^2}{2}\Delta(x_1 - c_3) + P_3(x_1 - a_3)\Delta(x_1 - a_3) \\ &\quad + M_2^*\Delta(x_1 - b_2) + C_1x_1 + C_2 \end{aligned} \tag{d}$$

$$\begin{aligned} EI_2\frac{du_3}{dx_1} &= \frac{p_3(x_1 - c_3)^3}{6}\Delta(x_1 - c_3) + \frac{P_3(x_1 - a_3)^2}{2}\Delta(x_1 - a_3) \\ &\quad + M_2^*(x_1 - b_2)\Delta(x_1 - b_2) + \frac{C_1x_1^2}{2} + C_2x_1 + C_3 \end{aligned} \tag{e}$$

$$EI_2u_3(x_1) = \frac{p_3(x_1 - c_3)^4}{24}\Delta(x_1 - c_3) + \frac{P_3(x_1 - a_3)^3}{6}\Delta(x_1 - a_3) \tag{c}$$

$$+ \frac{M_2^*(x_1 - b_2)^2}{2}\Delta(x_1 - b_2) + \frac{C_1x_1^3}{6} + \frac{C_2x_1^2}{2} + C_3x_1 + C_4 \tag{f}$$

The constants C_1, C_2, C_3 and C_4 are evaluated by requiring that the solution (e) and (f) satisfies the boundary conditions (b). Hence, substituting relations (e) and (f) into the first two of the boundary conditions (b), we obtain

$$C_3 = 0 \qquad C_4 = 0 \tag{g}$$

Moreover, substituting relations (e) and (f) into the last two of the boundary conditions (b) and using relations (g), we get

$$\frac{p_3(L - c_3)^4}{24} + \frac{P_3(L - a_3)^3}{6} + \frac{M_2^*(L - b_2)^2}{2} + \frac{C_1L^3}{6} + \frac{C_2L^2}{2} = 0$$

$$\frac{p_3(L - c_3)^3}{6} + \frac{P_3(L - a_3)^2}{2} + M_2^*(L - b_2) + \frac{C_1L^2}{2} + C_2L = 0 \tag{h}$$

Solving relations (h), we obtain

$$C_1 = -\frac{p_3(L - c_3)^3(L + c_3)}{2L^3} - \frac{P_3(L - a_3)^2(L + 2a_3)}{L^3} - \frac{6M_2^*b_2(L - b_2)}{L^3}$$

$$C_2 = \frac{p_3(L - c_3)^3(L + 3c_3)}{12L^2} + \frac{P_3a_3(L - a_3)^2}{L^2} + \frac{M_3^*(L - b_2)(3b_2 - L)}{L^2} \tag{i}$$

Substituting the values of the constants (g) and (i) into relations (c), (d) and (f), we obtain

$$Q_3(x_1) = -p_3(x_1 - c_3)\Delta(x_1 - c_3) - P_3\Delta(x_1 - a_3) - M_2^*\delta(x_1 - b_2)$$

$$+ \frac{p_3(L - c_3)^3(L + c_3)}{2L^3} + \frac{P_3(L - a_3)^2(L + 2a_3)}{L^3} + \frac{6M_2^*b_2(L - b_2)}{L^3} \tag{j}$$

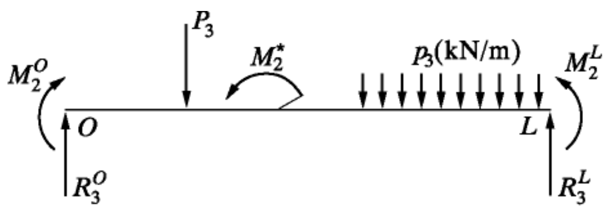


Figure b Free-body diagram of the beam.

$$
\begin{aligned}
M_2(x_1) = & -\frac{p_3(x_1 - c_3)^2}{2}\Delta(x_1 - c_3) - P_3(x_1 - a_3)\Delta(x_1 - a_3) - M_2^*\Delta(x_1 - b_2) \\
& + \frac{p_3(L - c_3)^3(L + c_3)x_1}{2L^3} + \frac{P_3(L - a_3)^2(L + 2a_3)x_1}{L^3} + \frac{6M_2^* b_2(L - b_2)x_1}{L^3} \\
& - \frac{p_3(L - c_3)^3(L + 3c_3)}{12L^2} - \frac{P_3 a_3(L - a_3)^2}{L^2} - \frac{M_2^*(L - b_2)(3b_2 - L)}{L^2}
\end{aligned} \qquad \text{(k)}
$$

$$
\begin{aligned}
EI_2 u_3(x_1) = & \frac{p_3(x_1 - c_3)^4}{24}\Delta(x_1 - c_3) + \frac{P_3(x_1 - a_3)^3}{6}\Delta(x_1 - a_3) + \frac{M_2^*(x_1 - b_2)^2}{2}\Delta(x_1 - b_2) \\
& - \frac{p_3(L - c_3)^3(L + c_3)x_1^3}{12L^3} - \frac{P_3(L - a_3)^2(L + 2a_3)x_1^3}{6L^3} - \frac{6M_2^* b_2(L - b_2)x_1^3}{6L^3} \\
& + \frac{p_3(L - c_3)^3(L + 3c_3)x_1^2}{24L^2} + \frac{P_3 a_3(L - a_3)^2 x_1^2}{2L^2} + \frac{M_2^*(L - b_2)(3b_2 - L)x_1^2}{2L^2}
\end{aligned}
$$

Referring to Fig. b, the reactions of the beam are

$$
R_3^0 = Q_3(0) = \frac{p_3(L - c_3)^3(L + c_3)}{2L^3} + \frac{P_3(L - a_3)^2(L + 2a_3)}{L^3} + \frac{6M_2^* b_2(L - b_2)}{L^3}
$$

$$
M_2^0 = M_2(0) = -\frac{p_3(L - c_3)^3(L + 3c_3)}{12L^2} - \frac{P_3 a_3(L - a_3)^2}{L^2} - \frac{M_2^*(L - b_2)(3b_2 - L)}{L^2}
$$

$$
\begin{aligned}
R_3^L = -Q_3(L) = & \frac{p_3(L - c_3)}{2L^3}[2L^3 - (L - c_3)^2(L + c_3)] + \frac{P_3}{L^3}[L^3 - (L - a_3)^2(L + 2a_3)] \\
& - \frac{6M_2^* b_2(L - b_2)}{L^3}
\end{aligned} \qquad \text{(l)}
$$

$$
M_2^L = M_2(L) = -\frac{p_3(L - c_3)^2}{12L^2}(L^2 + 2Lc_3 + 3c_3^2) - \frac{P_3 a_3^2(L - a_3)}{L^2} + \frac{M^* b_2(2L - 3b_2)}{L^2}
$$

Example 5 Using functions of discontinuity establish the deflection $u_3(x_1)$ of the simply supported beam of constant cross section subjected to a concentrated moment M_2^*, as shown in Fig. a. The x_3 axis is a principal centroidal axis of the cross sections of the beam. The beam is made from a homogeneous, isotropic, linearly elastic material with modulus of elasticity E.

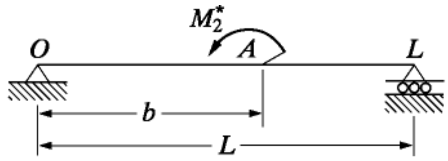


Figure a Geometry and loading of the beam.

Solution This beam is statically determinate. Thus, we can easily find its internal moment as a function of its axial coordinate by considering the equilibrium of portions of the beam. Consequently, it is preferable to establish the deflection of the beam using relation (9.32a) instead of relation (9.34b). The reaction of support 0 may be obtained by referring to Fig. b and setting the sum of moments about point L equal to zero. Thus,

$$R_3^0 = \frac{M_2^*}{L} \tag{a}$$

The moment acting on the cross sections of the beam is given by

$$\begin{aligned} M_2 &= \frac{M_2^* x_1}{L} & \qquad 0 \le x_1 \le b \\ M_2 &= M_2^*\left(\frac{x_1}{L} - 1\right) & \qquad b \le x_1 \le L \end{aligned} \tag{b}$$

These relations may be rewritten as

$$M_2(x_1) = \frac{M_2^* x_1}{L} - M_2^* \Delta(x_1 - b) \tag{c}$$

Substituting relation (c) into (9.32a), integrating twice the resulting differential equation and using relation (G.5) of Appendix G, we obtain

$$\begin{aligned} EI_2 \frac{d^2 u_3}{dx_1^2} &= -M_2 = \frac{M_2^*}{L}\left[-x_1 + L\Delta(x_1 - b)\right] \\ EI_2 \frac{du_3}{dx_1} &= -\frac{M_2^*}{L}\left[\frac{x_1^2}{2} - L(x_1 - b)\Delta(x_1 - b)\right] + C_1 \\ EI_2 u_3 &= -\frac{M_2^*}{L}\left[\frac{x_1^3}{6} - \frac{L(x - b)^2}{2}\Delta(x_1 - b)\right] + C_1 x_1 + C_2 \end{aligned} \tag{d}$$

The constants C_1 and C_2 are evaluated by requiring that the deflection $u_3(x_1)$ satisfies the essential boundary conditions at $x_1 = 0$ and $x_1 = L$. That is,

$$u_3(0) = 0 \qquad\qquad u_3(L) = 0 \tag{e}$$

Substituting the last of relations (d) into (e), we get

$$C_2 = 0$$
$$C_1 = -\frac{M_2^*}{6L}\left[-L^2 + 3(L - b)^2\right] \tag{f}$$

Substituting the values of the constants (f) in the last of relations (d), we obtain

$$EI_2 u_3 = -\frac{M_2^*}{6L}\left[x_1^3 - L^2 x_1 + 3(L - b)^2 x_1 - 3L(x_1 - b)^2 \Delta(x_1 - b)\right] \tag{g}$$

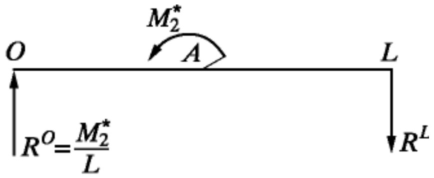


Figure b Free-body diagram of the beam.

9.4 The Timoshenko Theory of Beams

In this theory the assumption is made that relations (9.19) represent a satisfactory approximation of the magnitude of the shearing components of strain $e_{12}(x_1, 0, 0)$ and $e_{13}(x_1, 0, 0)$ of the particles which are located on the axis of a beam. Thus, substituting relations (9.19) into the fourth and fifth of relations (8.44), we obtain

$$\tau_{12}(x_1, 0, 0) = 2Ge_{12}(x_1, 0, 0) = G\left[-\theta_3(x_1) + \frac{du_2}{dx_1}\right] \tag{9.45a}$$

$$\tau_{13}(x_1, 0, 0) = 2Ge_{13}(x_1, 0, 0) = G\left[\theta_2(x_1) + \frac{du_3}{dx_1}\right] \tag{9.45b}$$

The shearing components of stress acting on the particles of the cross sections of a beam may be expressed as

$$\tau_{12}(x_1, x_2, x_3) = \tau_{12}(x_1, 0, 0)\, f_2(x_2, x_3) \tag{9.46a}$$

$$\tau_{13}(x_1, x_2, x_3) = \tau_{13}(x_1, 0, 0)\, f_3(x_2, x_3) \tag{9.46b}$$

Substituting relations (9.46) into (8.9b) and (8.9c), we get

$$Q_2 = \iint_A \tau_{12}(x_1, x_2, x_3)\, dA = \tau_{12}(x_1, 0, 0)\iint_A f_2(x_2, x_3)\, dA \tag{9.47a}$$

$$Q_3 = \iint_A \tau_{13}(x_1, x_2, x_3)\, dA = \tau_{13}(x_1, 0, 0)\iint_A f_3(x_2, x_3) dA \tag{9.47b}$$

Substituting relations (9.45) into (9.47), we obtain

$$Q_2 = G\left(-\theta_3 + \frac{du_2}{dx_1}\right)\iint_A f_2(x_2, x_3)\, dA = \lambda_2 GA\left(-\theta_3 + \frac{du_2}{dx_1}\right) \tag{9.48a}$$

$$Q_3 = G\left(\theta_2 + \frac{du_3}{dx_1}\right)\iint_A f_3(x_2, x_3)\, dA = \lambda_3 GA\left(\theta_2 + \frac{du_3}{dx_1}\right) \tag{9.48b}$$

where using relations (9.47), we have

$$\lambda_2 = \frac{\iint_A f_2(x_2, x_3)\, dA}{A} = = \frac{Q_2}{A\tau_{12}(x_1, 0, 0)} \tag{9.49a}$$

$$\lambda_3 = \frac{\iint_A f_3(x_2, x_3)\, dA}{A} = = \frac{Q_3}{A\tau_{13}(x_1, 0, 0)} \tag{9.49b}$$

Comparing relations (9.48) with (9.8c) and (9.8d), we get

$$\begin{aligned} \lambda_2 GA(-\theta_3 + \frac{du_2}{dx_1}) &= -\frac{d}{dx_1}[EI_{33}(\frac{d\theta_3}{dx_1} + H_3) - EI_{23}(\frac{d\theta_2}{dx_1} - H_2) - m_3] \\ \lambda_3 GA(\theta_2 + \frac{du_3}{dx_1}) &= \frac{d}{dx_1}[\, EI_{22}(\frac{d\theta_2}{dx_1} - H_2) - EI_{23}(\frac{d\theta_3}{dx_1} + H_3) + m_2\,] \end{aligned} \tag{9.50}$$

These are the relations between the components of rotation θ_2 and θ_3 and the components of translation u_2 and u_3 used in the Timoshenko theory of beams. For a given cross section the factors λ_2 and λ_3 can be computed by first computing the shearing components of stress $\tau_{12}(x_1, 0, 0)$ and $\tau_{13}(x_1, 0, 0)$ using formula (9.65). This formula is derived in Section 9.5 by requiring that the sum of the axial component of the forces acting on any segment of infinitesimal length and finite cross sections of a beam is equal to zero. For example referring to relation (a) of Example 7 of Section 9.5, the shearing stress $\tau_{13}(x_1, x_2, x_3)$ acting on the cross sections of a beam of rectangular cross section of width b and depth d subjected to transverse forces in the $x_1\, x_3$ plane is equal to

$$\tau_{13}(x_1, x_2, x_3) = \frac{Q_3}{2I_2}\left(\frac{d^2}{4} - x_3^2\right) \tag{9.51}$$

Consequently,

$$\tau_{13}(x_1, 0, 0) = \frac{Q_3 d^2}{8I_2} \tag{9.52}$$

Substituting relation (9.52) into (9.49b), we obtain

$$\lambda_3 = \frac{8I_2}{d^2 A} = \frac{8bd^3}{12bd^3} = \frac{2}{3} \tag{9.53}$$

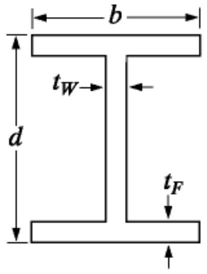


Figure 9.8 Cross section of a I-beam or a wide beam.

Moreover, referring to relation (b) of Example 2 of Section 9.5, the shearing stress $\tau_{13}(x_1, x_2, x_3)$ acting on the cross sections of a beam having an S or a W cross section (see Appendix H) shown in Fig. 9.8 is equal to

$$\tau_{13}(x_1, x_2, x_3) = \frac{Q_3}{2I_2 t_W}\left[bt_F(d - t_F) + t_W\left[\left(\frac{d}{2} - t_F\right)^2 - x_3^2\right]\right] \tag{9.54}$$

Consequently,

$$\tau_{13}(x_1, 0, 0) = \frac{Q_3}{2I_2 t_W}\left[bt_F(d - t_F) + t_W\left(\frac{d}{2} - t_F\right)^2\right] \tag{9.55}$$

Substituting relation (9.55) into (9.49b), we get

$$\lambda_3 = \frac{2I_2 t_W}{[bt_F(d - t_F) + t_W(\frac{d}{2} - t_F)^2]A} \tag{9.56}$$

Referring to Appendix H for a standard W100 x 19.3 beam, we have

$A = 2480 \text{ mm}^2$	$b = 103$ mm	$t_F = 8.8$ mm
$d = 106$ mm	$t_W = 7.1$ mm	$I_2 = 4.77\ (10^6)\ \text{mm}^4$

Substituting the properties of the cross section of the W100 x 19.3 beam given above in relation (9.56), we obtain

$$\lambda_3 = \frac{2(4.77)(10^6)7.1}{[10^3(8.8)(106 - 8.8) + 7.1(53 - 8.8)^2]2480} = 0.2677 \tag{9.57}$$

For wide flanges and I-beams λ_3 is taken as approximately equal to A_W / A where A_W is the area of the cross section of the web. Referring to Appendix H for the standard W100 × 19.3 beam, we have

$$A_W = (106 - 16.6)7.1 = 641.84$$

Thus,

$$\lambda_3 = \frac{A_W}{A} = \frac{641.84}{2470} = 0.264$$

Consider a prismatic beam subjected to external loads having S points of load change.

Recalling our discussion in Section 8.9, the transverse components of translation $u_2(x_1)$ and $u_3(x_1)$ and the components of rotation $\theta_2(x_1)$ and $\theta_3(x_1)$ of this beam are represented by a set of functions $u_i^{(s)}(x_1)$ and $\theta_i^{(s)}[[s = 1, 2, ..., (S + 1)]$ $(i = 2$ or $3)]$ having continuous derivatives, of any order. Each one of the functions $u_i^{(s)}$ and $\theta_i^{(s)}[[s = 1, 2, ..., (S + 1)]$ $(i = 2$ or $3)]$ gives the component of translation u_i or rotation θ_i $(i = 2$ or $3)$ in one segment of the beam extending either between a support and its adjacent point of load change or between two consecutive points of load change.

The boundary value problem for computing the transverse components of translation $u_3(x_1)$ and $u_2(x_1)$, the components of rotation $\theta_2(x_1)$ and $\theta_3(x_1)$, the shearing components of the internal force $Q_3(x_1)$ and $Q_2(x_1)$ and the components of the internal bending moment $M_2(x_1)$ and $M_3(x_1)$ in a beam (statically determinate or indeterminate) subjected to a change of temperature $(\Delta T_c = 0,\ \delta T_2 \neq \delta T_3 \neq 0)$ as well as to a distribution of external transverse forces and bending moments, which have S points of load change, can be formulated on the basis of the Timoshenko theory of beams as shown next:

Find the functions $\theta_i^{(s)}(x_1)[s = 1, 2, ..., (S + 1)]$ $(i = 2, 3)$ which have the following properties:

1. Each has continuous derivatives of any order.
2. They satisfy the rotation boundary conditions at the ends of the beam where rotations are specified.
3. They yield continuous components of rotation throughout the length of the beam.
4. Each pair [$\theta_2^{(s)}$and $\theta_3^{(s)}$] of these functions satisfies the rotation equations of equilibrium (9.10) at all points of the segment of the beam whose components of rotation they represent. That is,

$$\frac{d^2}{dx_1^2}\left[EI_{33}^{(s)}\left(\frac{d\theta_3^{(s)}}{dx_1} + H_3^{(s)}\right) - EI_{23}^{(s)}\left(\frac{d\theta_2^{(s)}}{dx_1} - H_2^{(s)}\right)\right] + \frac{dm_3^{(s)}}{dx_1} = p_2^{(s)}(x_1) \tag{9.58a}$$

$$\frac{d^2}{dx_1^2}\left[EI_{22}^{(s)}\left(\frac{d\theta_2^{(s)}}{dx_1} - H_2^{(s)}\right) - EI_{23}^{(s)}\left(\frac{d\theta_3^{(s)}}{dx_1} + H_3^{(s)}\right)\right] + \frac{dm_2^{(s)}}{dx_1} = -p_3^{(s)}(x_1) \tag{9.58b}$$

$$s = 1, 2, ..., (S + 1)$$

This ensures that every segment of the beam of infinitesimal length, cut by two imaginary planes perpendicular to its axis, which does not contain a point of load change, is in equilibrium.
5. When substituted into the internal action–rotation relations (9.8), they give a set of functions $Q_3^{(s)}(x_1)$, $Q_2^{(s)}(x_1)$, $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ $[s = 1, 2, ..., (S + 1)]$ each one of which represents either the shearing force or the bending moment in the s^{th} segment of the beam extending between a support and its adjacent point of load change or between two consecutive points of load change. These functions satisfy

(a) The natural boundary conditions are at the ends of the beam where specified external transverse forces and beading moments are specified.
(b) The requirements for equilibrium of the segments of infinitesimal length of the beam are cut by two imaginary planes normal to its axis which contain a point of load change.

6. When substituted into relations (9.50), the resulting equations can be solved to give a set of components of translation $u_2^{(s)}$ and $u_3^{(s)}$ $[s = 1, 2, ..., (S + 1)]$ which

(a) Satisfy the translation boundary conditions of the beam where components of translation are specified

(b) Yield continuous components of translation throughout the length of the beam

The components of rotation $\theta_2(x_1)$and $\theta_2(x_1)$, translation $u_3(x_1)$ and $u_2(x_1)$ shearing force $Q_3(x_1)$ and $Q_2(x_1)$ and bending moment $M_2(x_1)$ and $M_3(x_1)$ of a beam can be established on the basis of the Timoshenko theory of beams by adhering to the following steps:

STEP 1 The set of differential equations (9.58) are solved to obtain the transverse components of rotation $\theta_i^{(s)}(x_1)$ [s = 1, 2, ..., (S + 1)] (i = 2, 3) of each segment of the beam extending between a support and its adjacent point of load change or between two consecutive points of load change. Each of the functions $\theta_i^{(s)}(x_1)$ [s = 1, 2, ..., (S_i+ 1)] (i = 2, 3) involves three constants.

STEP 2 The constants are evaluated by requiring that the components of rotation $\theta_i^{(s)}(x_1)$ [s = 1, 2, ..., (S + 1)] have the following attributes:

1. They satisfy the equations of continuity for the components of rotation at every point of load change of the beam.
2. They satisfy the rotation boundary conditions at the ends of the beam.
3. When substituted into the action–rotation relations (9.8), they give internal shearing forces and bending moments which satisfy
   (a) The requirements for equilibrium of the segments of infinitesimal length of the beam which contain a point of load change
   (b) The natural boundary conditions at the ends of the beam at which such conditions are specified

STEP 3 The components of rotation $\theta_i^{(s)}(x_1)$and the shearing forces $Q_i^{(s)}(x_1)$ [s = 1, 2, ..., (S + 1)] (i = 2, 3) are substituted into relations (9.48) and the resulting equations can be solved to obtain a set of components of translation $u_2^{(s)}$ and $u_3^{(s)}$ [s = 1, 2, ..., (S + 1)] involving two constants which are established by requiring that the components of translation satisfy
   (a) The translation boundary conditions at the ends of the beam where such conditions are specified
   (b) The equations of continuity for the components of translation at every point of load change

The components of rotation $\theta_3(x_1)$ and $\theta_2(x_1)$ and the components of translation $u_3(x_1)$ and $u_2(x_1)$ of statically determinate beams can also be established by adhering to the following simpler procedure:

STEP 1 The set of functions $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ [s = 1, 2, ...,(S + 1)] representing the bending moments in the statically determinate beam and the set of functions $Q_2^{(s)}(x_1)$ and $Q_3^{(s)}(x_1)$ [s = 1, 2, ..., (S + 1)] representing the shear forces in the statically determinate beam are established by considering the equilibrium of appropriate segments of the beam.

STEP 2 The functions $M_2^{(s)}(x_1)$ and $M_3^{(s)}(x_1)$ [s = 1, 2, ..., (S + 1)] established in step 1 are substituted into relation (9.8a) and (9.8b) and the resulting differential equations are solved to obtain the components of rotation $\theta_3^{(s)}(x_1)$and $\theta_3^{(s)}(x_1)$[s = 1, 2 ..., (S + 1)] as functions of x_1 involving one constant each.

STEP 3 The components of rotation $\theta_2^{(s)}$and $\theta_3^{(s)}$and the shear forces $Q_2^{(s)}(x_1)$ and $Q_3^{(s)}(x_1)$

$[s = 1, 2, ..., (S + 1)]$ established in Step 1, are substituted into relations (9.48) and the resulting equations are solved to obtain the components of translation $u_2^{(s)}$ and $u_3^{(s)}$ $[s = 1, 2, ..., (S + 1)]$ involving two constants each.

STEP 4 The constants are evaluated by requiring that the solution satisfies the essential boundary conditions of the beam as well as the continuity conditions at the points of load change of the corresponding external actions. It is apparent that the set of functions $M_2^{(s)}(x_1)$, $M_3^{(s)}(x_1)$ and $Q_2^{(s)}(x_1)$, $Q_3^{(s)}(x_1)$ $[s = 1, 2, ..., (S + 1)]$ obtained in step 1 satisfy the natural boundary conditions of the problem and the requirements for equilibrium of all the segments of infinitesimal length of the beam cut by two imaginary planes normal to its axis.

In what follows we present one example.

Example 6 Using the Timoshenko theory of beams establish the equations of the elastic curve of the fixed at the one end and simply supported at the other end beam subjected to a concentrated force P_3 as shown in Fig a. Moreover, compute and show on a sketch the reactions of the beam. The x_3 axis is a principal centroidal axis of the cross sections of the beam. The line of action of the force P_3 is parallel to the x_3 axis and lies in a plane which contains the shear centers of the cross sections of the beam.

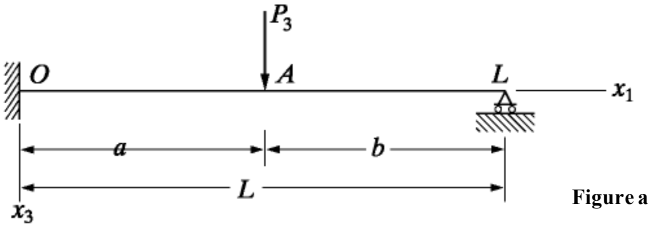


Figure a Geometry and loading of the beam.

Solution This is a statically indeterminate beam. Consequently, we must use the first procedure described in this section. The loading has one point (point A) of load change. Thus, the components of internal action $Q_3(x_1)$ and $M_2(x_1)$, the component of translation $u_3(x_1)$ and the component of rotation $\theta_2(x_1)$ are each expressed by two functions.

STEP 1 We integrate the equations of equilibrium (9.58) to obtain the components of rotation as functions of the axial coordinate involving three constants each. Referring to relations (9.58), we have

$$\frac{d^2}{dx_1^2}\left(EI_2\frac{d\theta_2^{(1)}}{dx_1}\right) = 0 \qquad 0 < x_1 < a \tag{a}$$

$$\frac{d^2}{dx_1^2}\left(EI_2\frac{d\theta_2^{(2)}}{dx_1}\right) = 0 \qquad a < x_1 < L \tag{b}$$

Integrating three times the above equations and using relations (9.8), we obtain

$$0 < x_1 < a \qquad\qquad . \qquad\qquad a < x_1 < L$$

$$Q_3^{(1)} = \frac{d}{dx_1}\left(EI_2\frac{d\theta_2^{(1)}}{dx_1}\right) = C_1 \qquad Q_3^{(2)} = \frac{d}{dx_1}\left(EI_2\frac{d\theta_2^{(2)}}{dx_1}\right) = C_2 \tag{c}$$

$$M_2^{(1)} = EI_2\frac{d\theta_2^{(1)}}{dx_1} = C_1x_1 + C_3 \qquad M_2^{(2)} = EI_2\frac{d\theta_2^{(2)}}{dx_1} = C_2x_1 + C_4 \tag{d}$$

$$EI_2\theta_2^{(1)} = C_1\frac{x_1^2}{2} + C_3x_1 + C_5 \qquad\qquad EI_2\theta_2^{(2)} = C_2\frac{x_1^2}{2} + C_4x_1 + C_6 \tag{e}$$

STEP 2 We establish relations among the constants by requiring that the functions $Q_3^{(s)}(x_1)$, $M_2^{(s)}(x_1)$ and $\theta_2^{(s)}(x_1)(s = 1, 2)$ given by relations (c), (d) and (e) satisfy the following conditions:

1. The rotation $\theta_2^{(1)}(x_1)$ must satisfy the boundary condition at $x_1 = 0$. That is,

$$\theta_2^{(1)}(0) = 0$$

Substituting the first of relations (e) in the above, we get

$$C_5 = 0 \tag{f}$$

2. The internal bending moment $M_2^{(2)}(x_1)$ must satisfy the natural boundary condition at $x_1 = L$. That is, referring to relation (d), we have

$$M_2^{(2)}(L) = 0 = C_2L + C_4 \tag{g}$$

3. The rotation $\theta_2^{(s)}(x_1)$ must satisfy the rotation continuity condition at $x_1 = a$. That is,

$$\theta_2^{(1)}(\alpha) = \theta_2^{(2)}(\alpha) \tag{h}$$

Substituting relations (e) in the above, we obtain

$$C_1\frac{a^2}{2} + C_3a = \frac{C_2a^2}{2} + C_4a + C_6 \tag{i}$$

4. The internal shearing forces and the internal bending moments must satisfy the requirements for equilibrium of an infinitesimal segment containing the concentrated force P_3. That is,

$$Q_3^{(1)}(a^-) = Q_3^{(2)}(a^+) + P_3 \tag{j}$$

$$M_2^{(1)}(a) = M_2^{(2)}(a) \tag{k}$$

Substituting relations (c) and (d) in the above, we have

$$C_1 = C_2 + P_3 \tag{l}$$

$$C_1a + C_3 = C_2a + C_4 \tag{m}$$

From relations (g), (i), (l) and (m), we obtain

$$C_1 = C_2 + P_3 \qquad C_3 = -C_2L - aP_3 \qquad C_4 = -C_2L \qquad C_6 = -\frac{a^2P_3}{2} \tag{n}$$

STEP 3 We substitute the values of the constants (f) and (n) into relations (e) and (c) and the resulting expressions into (9.48b) and we integrate the resulting relation to obtain an expression for each of the components of translation $u_3^{(1)}(x_1)$, $u_3^{(2)}(x_1)$ involving three constants. For the beam under consideration relations (9.48b) reduce to

$$\begin{aligned}
\lambda_3 GA(\theta_2^{(1)} + \frac{du_3^{(1)}}{dx_1}) &= Q_3^{(1)}(x_1) = \frac{d}{dx_1}(EI_2 \frac{d\theta_2^{(1)}}{dx_1}) \\
\lambda_3 GA(\theta_2^{(2)} + \frac{du_3^{(2)}}{dx_1}) &= Q_3^{(2)}(x_1) = \frac{d}{dx_1}(EI_2 \frac{d\theta_2^{(2)}}{dx_1})
\end{aligned} \tag{o}$$

Substituting the values of the constants (f) and (n) into relations (e) and (c) and the resulting expressions in relations (o), we get

$$\begin{aligned}
\frac{du_3^{(1)}}{dx_1} &= \frac{C_2 + P_3}{\lambda_3 GA} - \frac{(C_2 + P_3)x_1^2}{2EI_2} + \frac{(C_2 L + aP_3)x_1}{EI_2} \\
\frac{du_3^{(2)}}{dx_1} &= \frac{C_2}{\lambda_3 GA} - \frac{C_2 x_1^2}{2EI_2} + \frac{C_2 L x_1}{EI_2} + \frac{a^2 P_3}{2EI_2}
\end{aligned} \tag{p}$$

Integrating relations (p), we obtain

$$\begin{aligned}
u_3^{(1)}(x_1) &= \frac{(C_2 + P_3)x_1}{\lambda_3 GA} - \frac{(C_2 + P_3)x_1^3}{6EI_2} + \frac{(C_2 L + aP_3)x_1^2}{2EI_2} + C_7 \\
u_3^{(2)}(x_1) &= \frac{C_2 x_1}{\lambda_3 GA} - \frac{C_2 x_1^3}{6EI_2} + \frac{C_2 L x_1^2}{2EI_2} + \frac{a^2 P_3 x_1}{2EI_2} + C_8
\end{aligned} \tag{q}$$

STEP 4 We evaluate the constants C_2, C_7 and C_8 by requiring that the functions $u_3^{(1)}(x_1)$ and $u_3^{(2)}(x_1)$ satisfy the translation boundary conditions at $x_1 = 0$ and $x_1 = L$ and the translation continuity condition at $x_1 = a$. That is,

$$u_3^{(1)}(0) = 0$$

Thus,

$$C_7 = 0 \tag{r}$$

and

$$u_3^{(2)}(L) = 0$$

Hence,

$$\frac{C_2 L}{\lambda_3 GA} + \frac{C_2 L^3}{3EI_2} + \frac{a^2 L P_3}{2EI_2} + C_8 = 0 \tag{s}$$

Moreover, at $x_1 = a$, we have,

$$u_3^{(1)}(a) = u_3^{(2)}(a)$$

Substituting relations (q) into the above, we get

$$C_8 = \frac{P_3 a}{\lambda_2 GA} - \frac{P_3 a^3}{6EI_2} \tag{t}$$

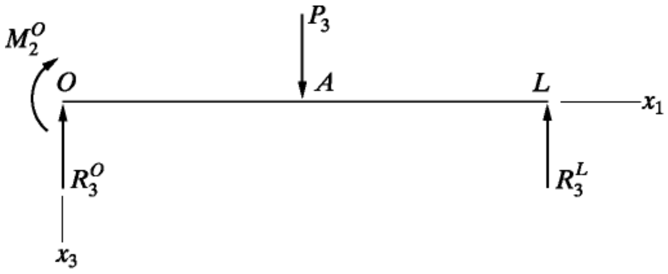


Figure b Free-body diagram of the beam.

Substituting relation (t) into (s), we obtain

$$C_2 = -\frac{P_3 a}{2L^3}\left(\frac{3aL - a^2 + 2kL^2}{1 + k}\right) \tag{u}$$

where

$$k = \frac{3EI_2}{\lambda_3 GAL^2} \tag{v}$$

For a beam of rectangular cross section of width b and depth h, we have

$$I_2 = \frac{bh^3}{12} \tag{w}$$

Substituting relations (w), (9.53) and (2.53) into relation (v), for $v = ⅓$ we get

$$k = \frac{3h^2(1 + v)}{4L^2} \tag{x}$$

The components of translation of the beam are obtained by substituting the values of the constants (r), (t) and (u) into relations (q).

The internal shearing forces and bending moments acting on the cross sections of the beam are obtained by substituting the values of the constants from relations (n) and (u) into relations (c) and (d), respectively. That is,

$$\begin{aligned}
Q_3^{(1)} &= C_1 = C_2 + P_3 = -\frac{P_3 a}{2L(1 + k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] + P_3 \\
Q_3^{(2)} &= C_2 = -\frac{P_3 a}{2L(1 + k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] \\
M_2^{(1)} &= C_1 x_1 + C_3 = -C_2(L - x_1) - P_3(a - x_1) \\
M_2^{(2)} &= C_2 x_1 + C_4 = -C_2(L - x_1)
\end{aligned} \tag{y}$$

Referring to Fig b, the reactions of the beam are

$$R^0 = Q_3^{(1)}(0) = -\frac{P_3 a}{2L(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] + P_3$$

$$R^L = -Q_3^{(2)}(L) = \frac{P_3 a}{2L(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] \tag{z}$$

$$M_2^0 = M_2^{(1)}(0) = -C_2 L - P_3 a = \frac{P_3 a}{2(1+k)}\left[\frac{3a}{L} - \left(\frac{a}{L}\right)^2 + 2k\right] - P_3 a$$

$$M_2^L = M_2^{(2)}(L) = 0$$

Usually, in practice, beams have a very small h/L ratio ($h/L < 1/10$). Consequently, k is very small compared to unity. Thus, referring to relations (z), we may conclude that the influence of the shear deformation of such beams on their internal actions is negligible.

9.5 Computation of the Shearing Components of Stress in Prismatic Beams Subjected to Bending without Twisting

When a prismatic straight beam is subjected to transverse forces whose line of action lies in a plane which contains the shear centers of its cross sections, in addition to the normal components of stress τ_{11}, shearing components of stress τ_{12} and τ_{13} act on its cross sections.

Consider a prismatic beam of a general cross section and denote by x_2 and x_3 a set of orthogonal centroidal axes not necessarily principal. The beam is subjected to external transverse forces acting in a plane parallel to the x_1x_3 plane which contains the shear centers of its cross sections. Moreover, referring to Fig. 9.9, consider particle *abcdef* located on the perimeter of a cross section of the beam. The resulting shearing stress acting on the plane *abcd* of this particle may be decomposed into a normal τ_{1n} and a tangential τ_{1t} to the perimeter components. Since the particle under consideration is in equilibrium, as we have shown in Section 2.13, a shearing stress τ_{n1} equal in magnitude to τ_{1n} must act on the plane *cdef*. This, however, is not possible because the plane *cdef* is part of the lateral surface of the body which is traction free. Thus, the component of stress τ_{1n} must vanish. That is, the shearing stress acting on a cross section of a beam on particles located on its perimeter must be tangent to the perimeter.

In Section 9.1 we establish formulas (9.12) for computing the normal component of stress τ_{11} acting on a cross section of a beam. This formula is valid for beams whose cross sections have any given geometry. In this section we establish a formula for computing the

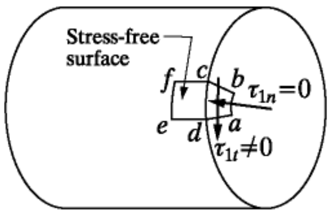


Figure 9.9 Shearing stress acting on a particle located on the perimeter of a cross section of a member.

properties.

shearing components of stress acting on the cross sections of a beam. However, as we show later, this formula can be used only for beams whose cross sections have certain geometric properties.

Consider a prismatic beam subjected to one or more of the external loads described in Section 9.1. Moreover, consider a segment of infinitesimal length dx_1 of this beam (see Fig. 9.10a). The components of shearing force and bending moment acting on the left surface of this segment of the beam are denoted by Q_2, Q_3, M_2 and M_3, while the components of shearing force and bending moment acting on its right surface are denoted by $Q_2 + dQ_2$, $Q_3 + dQ_3$, $M_2 + dM_2$ and $M_3 + dM_3$. Referring to relations (9.12a), the normal component of stress acting on the left surface of the segment under consideration is equal to

$$\tau_{11} = -\left(\frac{M_2 I_{23} + M_3 I_{22}}{I_{22} I_{33} - I_{23}^2}\right) x_2 + \left(\frac{M_2 I_{33} + M_3 I_{23}}{I_{22} I_{33} - I_{23}^2}\right) x_3 \tag{9.59}$$

Moreover, the normal component of stress acting on the right surface of the segment under consideration is equal to

$$\tau_{11} + d\tau_{11} = -\left[\frac{(M_2 + dM_2)I_{23} + (M_3 + dM_3)I_{22}}{I_{22}I_{33} - I_{23}^2}\right]x_2 + \left[\frac{(M_2 + dM_2)I_{33} + (M_3 + dM_3)I_{23}}{I_{22}I_{33} - I_{23}^2}\right]x_3 \tag{9.60}$$

Imagine that the segment under consideration is cut in two parts by a plane $ABCD$ which is normal to a unit vector $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 = \lambda_{n3}\mathbf{i}_3$ and consider the equilibrium of part $ABCDEF$ shown in Fig. 9.10d. Using relation (9.59), the resultant force F_1 of the normal component of stress τ_{11} acting on face CDE of this part is equal to

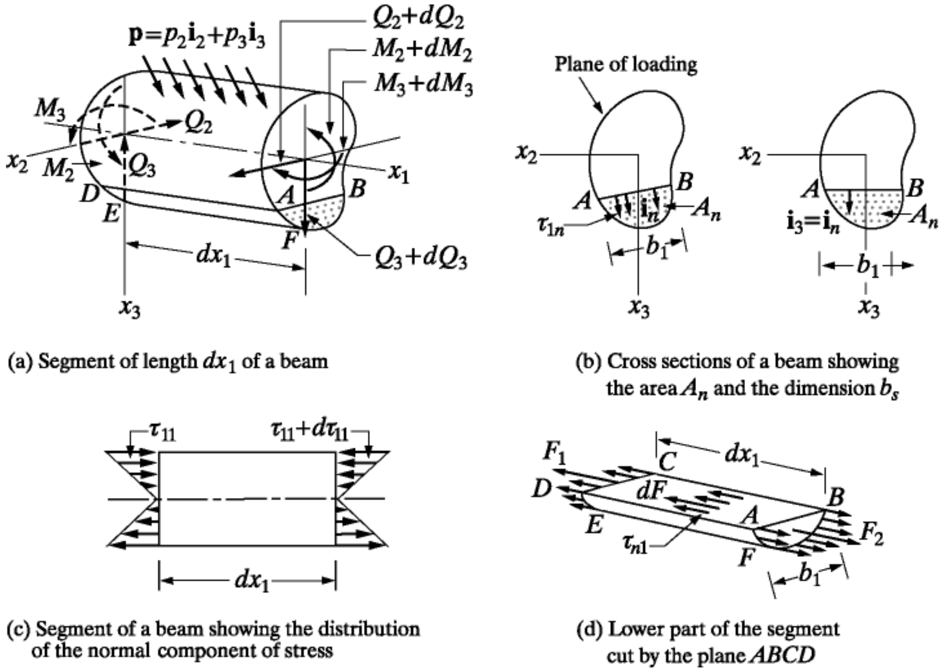


Figure 9.10 Segment of a beam subjected to bending without twisting.

$$F_1 = \iint_{A_n} \tau_{11}\, dA = -\left(\frac{M_2 I_{23} + M_3 I_{22}}{I_{22} I_{33} - I_{23}^2} \right) \iint_{A_n} x_2\, dA + \left(\frac{M_2 I_{33} + M_3 I_{23}}{I_{22} I_{33} - I_{23}{}^2} \right) \iint_{A_n} x_3\, dA \tag{9.61a}$$

where A_n is the area of face *CDE* which is equal to the area of face *ABF*. Moreover, using relation (9.60) the resultant force F_2 of the normal component of stress $\tau_{11} + d\tau_{11}$ acting on face *ABF* of part *ABCDEF* is equal to

$$F_2 = \iint_{A_n} (\tau_{11} + d\tau_{11}) dA = -\left[\frac{(M_2 + dM_2) I_{23} + (M_3 + dM_3) I_{22}}{I_{22} I_{33} - I_{23}^2} \right] \iint_{A_n} x_2 dA$$

$$+ \left[\frac{(M_2 + dM_2) I_{33} + (M_3 + dM_3) I_{23}}{I_{22} I_{33} - I_{23}{}^2} \right] \iint_{A_n} x_3 dA \tag{9.61b}$$

Referring to Fig. 9.10d, we see that the equilibrium of part *ABCDEF* requires that a force dF must exist on its surface *ABCD* given by

$$dF = F_2 - F_1 = -\left(\frac{dM_2 I_{23} + dM_3 I_{22}}{I_{22} I_{33} - I_{23}^2} \right) \iint_{A_n} x_2\, dA + \left(\frac{dM_2 I_{33} + dM_3 I_{23}}{I_{22} I_{33} - I_{23}{}^2} \right) \iint_{A_n} x_3\, dA \tag{9.62}$$

The force per unit length q_{1n} exerted on the surface *ABCD* of Fig. 9.10d is referred to as the *shear flow*. Taking into account relations (8.20) and (8.21), and disregarding m_2 and m_3 (see Section 8.9), from relation (9.62) we have

$$q_{1n} = \frac{dF}{dx_1} = -\left(\frac{Q_3 I_{23} - Q_2 I_{22}}{I_{22} I_{33} - I_{23}^2} \right) Z_3 + \left(\frac{Q_3 I_{33} - Q_2 I_{23}}{I_{22} I_{33} - I_{23}{}^2} \right) Z_2 \tag{9.63}$$

where Z_2 and Z_3 are the first moments about x_2 and x_3 axes, respectively, of the portion of area A_n of the cross section of the beam. That is,

$$Z_2 = \iint_{A_n} x_3\, dA = \bar{x}_{3n} A_n \qquad\qquad Z_3 = \iint_{A_n} x_2\, dA = \bar{x}_{2n} A_n \tag{9.64}$$

and

$Q_i (i = 2 \text{ or } 3)$ = shearing component of the force acting on the cross section of the beam in the direction of the axis x_i.
$I_{ii} (i = 2 \text{ or } 3)$ = moment of inertia of the cross section of the beam about the axis x_i.
I_{23} = product of inertia of the cross section of the beam about the axes x_2 and x_3.
$\bar{x}_{in} (i = 2 \text{ or } 3)$ = distance of the centroid of the area A_n from the axis x_j ($j = 3$ or 2, $j \neq i$). Notice that A_n is always positive. Consequently, the sign of Z_2 and Z_3 depends on the sign of $\tilde{x}_{3n}$ and $\tilde{x}_{2n}$, respectively.

If the x_2 and x_3 axes are principal centroidal, relation (9.63) reduces to

$$q_{1n} = \frac{Q_3 Z_2}{I_2} + \frac{Q_2 Z_3}{I_3} \tag{9.65}$$

In general, the distribution of the shearing component of stress τ_{1n} on the surface $ABCD$ is not known. In the theories of mechanics of materials for beams, we do not have a rational means for establishing this distribution for beams of any cross section. However, for beams of certain cross section (see Fig. 9.11), we know a priori that the shearing component of stress τ_{1n} does not vary very much along the direction normal to the unit vector i_n. For such beams, the shearing component of stress τ_{1n} is obtained from relation (9.63) as

$$\tau_{1n} = \frac{q_{1n}}{b_s} = -\frac{Q_3 I_{23} - Q_2 I_{22}}{b_s(I_{22}I_{33} - I_{23}^2)} Z_3 + \frac{Q_3 I_{33} - Q_2 I_{23}}{b_s(I_{22}I_{33} - I_{23}^2)} Z_2 \tag{9.66}$$

A positive value of τ_{1n} or q_{1n} acting on the positive† cross section indicates that its sense is that of the unit vector i_n, that is, toward the area A_n (see Fig. 9.10b).

If the x_2 and x_3 axes are principle centroidal axes, relation (9.66) reduces to

$$\tau_{1n} = \frac{Q_3 Z_2}{b_s I_2} + \frac{Q_2 Z_3}{b_s I_3} \tag{9.67}$$

In Fig. 9.11 we show cross sections of prismatic beams for which we can employ formula (9.66) or (9.67) to compute the shearing components of stress acting on them. They include

1. Open thin-walled cross sections (see Figs. 9.11a and b). The total shearing stress acting on the particles of the boundary of a cross section is tangent to the boundary. Moreover, since the thickness of the cross section is small, the component of shearing stress normal to its boundary is negligible throughout the cross section. Furthermore, the variation of the shearing stress along the normal to the boundary is small. Thus, we assume that the shearing stress acting on the cross sections of beams having thin-walled open cross sections does not vary along its thickness and it is parallel to the tangent to its boundary. Consequently, we can use formula (9.66) or (9.67) to compute the magnitude of shearing stress acting on the cross sections of such members. Referring to Fig. 9.11a, when the formula (9.67) is employed to compute τ_{12} in the flanges of the beam, b_s is equal to t_F while when it is employed to compute τ_{13} in the web of the beam, b_s is equal to t_W.
2. Closed, thin-walled cross sections having an axis of symmetry (see Figs. 6.11c and d) when they are subjected to transverse forces whose plane contains the axis of symmetry. In this case $b_s = 2t$.
3. Solid cross sections having an axis of symmetry (say the x_3 axis) when the plane of the transverse forces contains the axis of symmetry, provided that the ratio of the width of the cross section to its depth is not large (see Fig. 9.11e and f). The shearing component of stress τ_{13} in beams of rectangular cross section subjected to transverse forces acting in the x_1x_3 plane in the direction of the x_3 axis (see Fig. 9.11e) has been computed on the basis of the theory of elasticity. It was established that its variation along the direction of the x_2 axis is small when the ratio b/h is less than 1/2 . Thus, we can use formula (9.67) to compute the

†The unit vector normal to a positive cross section is $\mathbf{i}_1$ while the unit vector normal to a negative cross section is $-\mathbf{i}_1$.

magnitude of the shearing component of stress τ_{13} acting on the cross sections of such a beam. After the shearing component of stress τ_{13} has been computed, using relation (9.67), the shearing component of stress τ_{12} can be established by recalling that the total shearing stress τ at a particle located on the perimeter of a cross section must be tangent to the perimeter; otherwise there will be a component of shearing stress normal to the boundary which requires a component of shearing stress on the traction-free lateral surface of the beam. Thus, denoting the equation of the boundary by $x_2 = f(x_3)$ and referring to Fig. 9.11f at any point of the boundary, we have

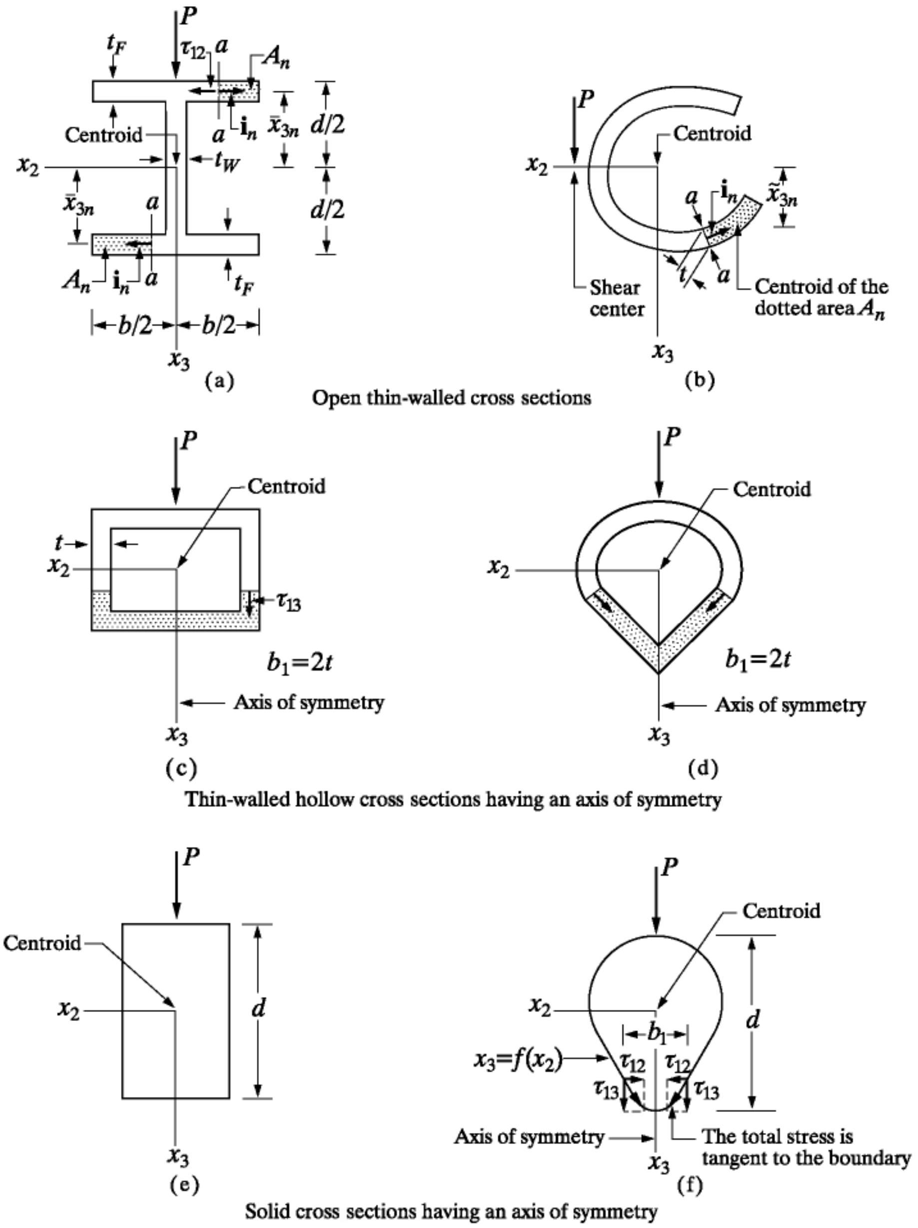


Figure 9.11 Types of cross section of beams for which formula (9.65) can be used to compute the distribution of shearing stress due to transverse forces.

$$\tau_{12} = \tau_{13}\frac{df}{dx_3} \tag{9.68}$$

Moreover, since the component of stress τ_{12} must be symmetric with respect to the x_3 axis, it must vanish on the x_3 axis. Assuming that the variation of the component of stress τ_{12} in the x_2 direction is linear, we have

$$\tau_{12} = \frac{2x_2\tau_{13}\left(\dfrac{df}{dx_3}\right)}{b_s} \tag{9.69}$$

The accuracy of the results obtained on the basis of relation (9.69) decreases as df/dx_3 increases. For example, consider a cantilever beam of solid circular cross sections, of radius R, made from an isotropic, linearly elastic material. The beam is subjected at its unsupported end to a force P_3 acting in the x_1x_3 plane in the direction of the x_3 axis. The stress distribution in such a beam has been established on the basis of the theory of elasticity†. The shearing components of stress were found to be

$$\tau_{13}^{E} = \frac{(3 + 2\nu)P_3}{8(1 + \nu)I_2}\left(R^2 - x_3^2 - \frac{1 - 2\nu}{3 + 2\nu}x_2^2\right) \tag{9.70a}$$

$$\tau_{12}^{E} = -\frac{(1 + 2\nu)P_3x_2x_3}{4(1 + \nu)I_2} \tag{9.70b}$$

Referring to relations (9.70b), we see that the assumption that τ_{12} varies linearly with x_2 is valid for beams of circular cross section. The distribution of the shearing components of stress τ_{13} on lines $x_3 = 0$ and $x_3 = R/2$ obtained on the basis of the theory of elasticity is shown in Fig. 9.12. Referring to this figure, we find that the average values of the shearing component of stress τ_{13} acting on the particles of lines AB and DE for ($\nu = 1/3$) are equal to

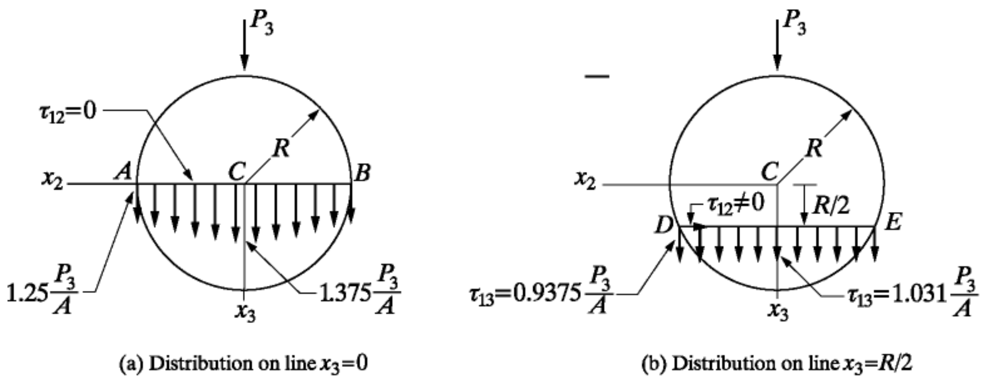


Figure 9.12 Distribution of the shearing components of stress on the cross section of a beam of solid circular cross section obtained on the basis of the theory of elasticity, for $\nu = 1/3$.

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† See Timoshenko, S.P., Goodier, J.N., *Theory of Elasticity,* McGraw-Hill, 3rd edition, New York, 1970, p. 358.

$$\left(\tau_{13}^{E}\right)_{\text{Average on line AB}} = \frac{1}{2R}\int_{-R}^{R}\tau_{13}^{E}\,dx_2 = 1.333\frac{P_3}{A} \tag{9.71a}$$

and

$$\left(\tau_{13}^{E}\right)_{\text{Average on line DE}} = \frac{1}{\sqrt{3}R}\int_{-\frac{\sqrt{3}R}{2}}^{\frac{\sqrt{3}R}{2}}\tau_{13}^{E}\,dx_2 = 1.00\frac{P_3}{A} \tag{9.71b}$$

Thus, the maximum variation of the shearing component of stress τ_{13} acting on the particles of lines AB and DE from the average stress τ_{13} , is about 6%. Moreover, using formula (9.67) and referring to the table on the inside of the back cover of the book, we obtain

$$\left(\tau_{13}^{M}\right)_{\text{on line}\,AB} = \frac{P_3\left(\dfrac{\pi R^2}{2}\right)\left(\dfrac{4R}{3\pi}\right)}{2R\left(\dfrac{\pi R^4}{8}\right)} = 1.333\frac{P_3}{A} \tag{9.72a}$$

$$\left(\tau_{13}^{M}\right)_{\text{on line}\,DE} = \frac{P_3R^2\left(\dfrac{\pi}{3} - \sin 60\cos 60\right)\left(\dfrac{2R}{3}\right)\left(\dfrac{3\sin^3 60}{\pi} - \sin 60\cos 60\right)}{2R\sin 60\left(\dfrac{\pi R^4}{4}\right)} = 1.00\frac{P_3}{A} \tag{9.72b}$$

The shearing component of stress τ_{13} acting on the particles of lines AB and DE computed on the basis of the theory of mechanics of materials is equal to the average value of the corresponding component of stress computed on the basis of the theory of elasticity (9.70).

The total shearing stress acting on any particle of the boundary of the cross section of the beam must be tangent to the boundary. At points A and B the shearing component of stress τ_{13} is tangent to the boundary, while at points D and E it is not. Thus, on line AB the shearing component of stress τ_{12} vanishes, while on line DE it does not. The boundary of the cross section of the beam under consideration is

$$x_2 = f(x_3) = (R^2 - x_3^2)^{\frac{1}{2}}$$

Therefore,

$$\frac{df}{dx_3} = -\frac{x_3}{(R^2 - x_3^2)^{1/2}} \tag{9.73}$$

Substituting relations (9.73) into (9.69), we obtain

$$\tau_{12}^{M} = \frac{2\tau_{13}\left(\dfrac{df_3}{dx_3}\right)x_2}{R\sqrt{3}} = -\frac{2\tau_{13}x_3x_2}{R\sqrt{3}(R^2 - x_3^2)^{1/2}} \tag{9.74}$$

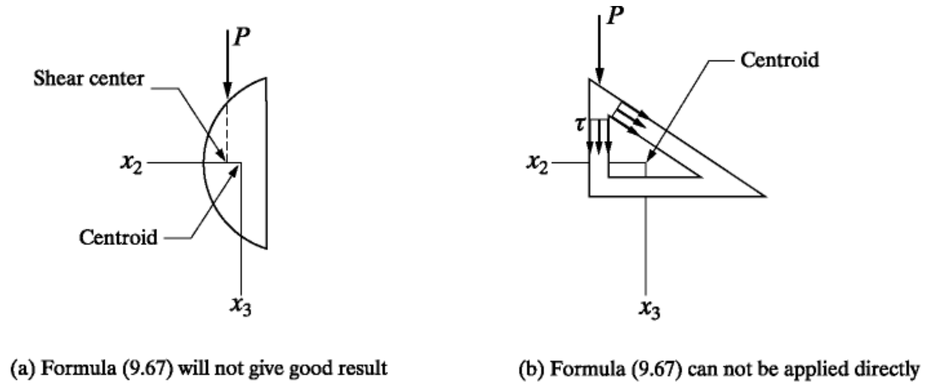


Figure 9.13 Types of cross sections of beams for which formula (9.67) will not give accurate results or cannot be applied directly.

The distribution of the shearing component of stress τ_{12} on line DE ($x_3 = R/2$) on the basis of the theory of mechanics of materials is obtained by substituting relation (9.72b) into (9.74). Thus, we get

$$\left(\tau_{12}^{M}\right)_{\text{on line } DE} = -\frac{1.00 P_3 R 2x_2}{2A\sqrt{3}R\left(\sqrt{3}\dfrac{R}{2}\right)} = -\frac{(1.00)}{3}\frac{2P_3x_2}{AR} = -0.670\frac{P_3x_2}{AR} \tag{9.75}$$

Referring to relation (9.70b), the distribution of the shearing component of stress τ_{12} on line DE obtained on the basis of the theory of elasticity is equal to

$$\left(\tau_{12}^{E}\right)_{\text{on line } DE} = -0.625\frac{P_3x_2}{AR} \tag{9.76}$$

Relation (9.66) or (9.67) does not give a satisfactory approximation of the components of shearing stress in beams having solid, unsymmetric cross sections (see Fig. 9.13a) or solid cross sections with one axis of symmetry if the plane of the external forces to which they are subjected does not contain the axis of symmetry. In this case, relations (9.66) or (9.67) gives the average value of the distribution of the shearing component of stress τ_{1n} along the distance b_s. Moreover, relation (9.66) or (9.67) cannot be employed directly to compute the total shearing stress in beams having unsymmetric, thin-walled, hollow cross sections (see Fig. 9.13b) or closed, thin-walled cross sections with one axis of symmetry if the plane of the external forces to which they are subjected does not contain the axis of symmetry. A formula for computing the shearing stress in such beams is presented in Section 12.6.

In the sequel, we compute the distribution of shearing stress in prismatic beams subjected to bending without twisting. We consider beams whose cross sections are (a) rectangular, and (b) I cross section (c) thin-walled angle. Whenever possible we compare the results with those obtained on the basis of the theory of elasticity.

Example 7 Compute the distribution of shearing stress in a beam having a rectangular cross section and loaded by forces acting in the x_1x_3 plane parallel to the x_3 axis.

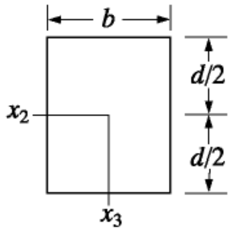


Figure a Geometry of the cross section of the beam.

Solution We assume that the shearing component of stress τ_{13} is constant along the width of the beam. Hence, referring to Fig. b, the shearing component of stress τ_{13} at any particle of line AB can be established using relation (9.67). That is, denoting by A_n the area of the portion $ABCD$ of the cross section of the beam, we have

$$\tau_{13} = \frac{Q_3}{bI_2}\iint_{A_n} x'_3 dA = \frac{Q_3}{I_2}\int_{x_3}^{d/2} x'_3 dx'_3 = \frac{Q_3}{2I_2}\left(\frac{d^2}{4} - x_3^2\right) \tag{a}$$

The above relation indicates that the distribution of the shearing component of stress τ_{13} along the x_3 axis is parabolic (see Fig. b). It is zero at $x_3 = \pm d/2$ and assumes its maximum value, at $x_3 = 0$, equal to

$$(\tau_{13})_{\max} = \frac{3Q_3}{2A} \tag{b}$$

where A is the area of the cross section of the beam.

The shearing component of stress τ_{12}, obtained on the basis of relation (9.67), is zero. This may be readily seen by considering a segment of infinitesimal length of the beam cut by two planes normal to its axis and a plane normal to the x_2 axis (see Fig. c). The resultant forces F_1 and F_2 of the normal components of stress acting on the end surfaces of this segment vanish. Consequently, the resultant shearing force dF acting on the plane $x_2 = -x_2'$ must vanish.

The shearing stress τ_{13} induces shearing deformation whose magnitude is a function of x_3. As shown in Fig. d, as a result of shear deformation, the cross sections of the beam warp. The nature of warping becomes apparent by noting that $\tau_{13} = 0$ at $x_3 = \pm d/2$. Consequently, the cross section is not distorted at $x_3 = \pm d/2$, whereas the distortion of the cross section is maximum (τ_{13} is maximum) at $x_3 = 0$. For beams whose depth is small as compared to their length, the warping is negligible. Therefore, for such beams, the assumption that plane

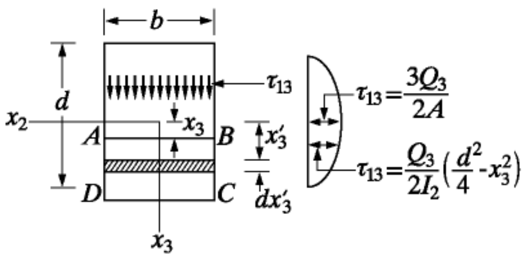


Figure b Distribution of the shearing component of stress τ_{13}.

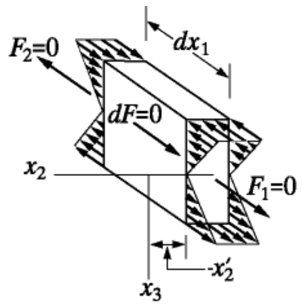


Figure c Part of a segment of the beam cut by a plane normal to the x_2 axis.

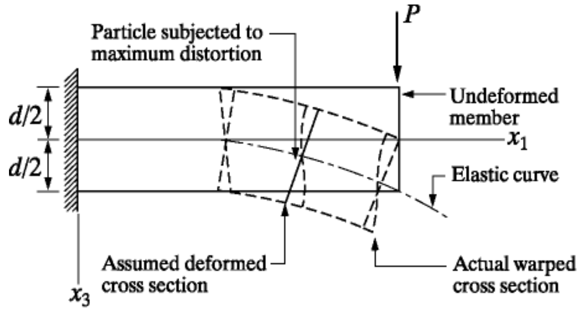


Figure d Deformation of a cantilever beam subjected to a transverse force.

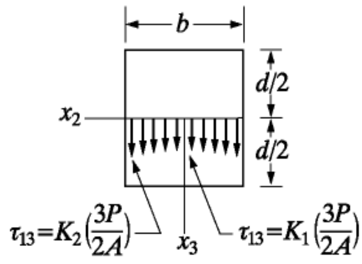


Figure e Distribution of shearing stress at $x_3 = 0$ on a cross section of a prismatic beam of rectangular cross section obtained on the basis of the theory of elasticity. (see Ibid *Timoshenko and Goodier*).

sections normal to the axis of the beam before deformation remain plane, subsequent to deformation, is a very satisfactory approximation.

The distribution of the shearing components of stress in cantilever beams of constant rectangular cross section has been obtained on the basis of the theory of elasticity. For values of the ratio b/d less than 1/2, the shearing component of stress τ_{12} has a very small magnitude, while the shearing component of stress τ_{13} is almost uniformly distributed along the x_2 axis. Consequently, for this range of the ratio b/d, the results obtained on the basis of relation (a) are in satisfactory agreement with the results obtained on the basis of the theory of elasticity (see Table a). However, for larger values of the ratio b/d, the solution obtained on the basis of the theory of elasticity yields values for the component of stress τ_{13} which vary considerably with x_2 (see Table a). Consequently, in this case, the results obtained on the basis of relation (a) represent only the average value of the shearing stress τ_{13} obtained on the basis of the theory of elasticity. At the centroid of the cross section of a member the magnitude of τ_{13} obtained on the basis of the theory of elasticity is less than that given by relation (a), whereas at $x_2 = \pm b/2$, the magnitude of τ_{13}, obtained on the basis of the theory of elasticity, is greater than that given by relation (a) (see Table a).

Table a Shearing stress coefficients for beams of rectangular cross sections.†

b/d	1/2	1	2	4
K_1††	0.983	0.940	0.856	0.805
K_2††	1.033	1.126	1.396	1.988

Example 2 Compute the distribution of the shearing components of stress for a beam having an S or a W (wide flange) cross section (see Appendix H), subjected to external forces in the x_1x_3 plane parallel to the x_3 axis (see Fig. a).

† See Timoshenko, S.P., Goodier, J.N. *Theory of Elasticity,* 3rd edition, McGraw-Hill, New York, 1970.
†† For definitions of K_1 and K_2, see Fig. e.

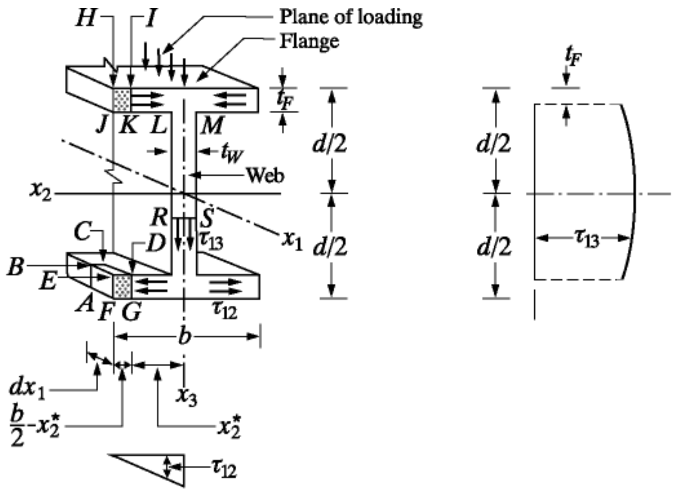


Figure a Geometry of the cross sections of the beam and distribution of the shearing components of stress.

Solution The shearing component of stress τ_{13} in the flanges of the beam under consideration is generally small and may be neglected. Moreover, in the flanges, the component of stress τ_{13} varies considerably with x_2 and thus, it cannot be computed using relation (9.67). This becomes apparent by noting that at $x_3 = \pm(d/2 - t_F)$ and $|x_2| > t_W/2$ the shearing component of stress τ_{13} must be zero because this stress must be equal to the shearing component of stress τ_{13} acting on the free surface of the flanges. However, across the junction LM ($|x_2| < t_W/2$), the shearing component of stress τ_{13} is different than zero. This indicates that at the points of the flange close to $x_3 = \pm(d/2 - t_F)$, the distribution of the shearing component of stress τ_{13} in the direction of the x_2 axis is not uniform.

The values of the shearing component of stress τ_{13} on the web, obtained on the basis of relation (9.67), are a satisfactory approximation of its actual values because on the web the shearing component of stress τ_{13} varies negligibly in the direction of the x_2 axis. In order to establish the shearing component of stress τ_{13} at a point of the web of the beam of Fig.a [say on the line RS ($x_3 = x_3^*$)], we compute the first moment of the area of the portion of the cross section of the beam below line RS. Referring Fig. a, this area consists of two rectangles, the flange of area bt_F and the portion of the web below line RS of area $t_W(d/2 - t_F - x_3^*)$. The distance of the centroid of the flange from the x_2 axis is equal to $(d - t_F)/2$ while the distance of the centroid of the portion of the web below line RS from the x_2 axis is equal to $(d/2 - t_F + x_3^*)/2$. Thus, the first moment of the area of the portion of the cross section of the beam below line RS is equal to

$$\iint_{A_n} x_3\, dA = b\, t_F\left(\frac{d}{2} - \frac{t_F}{2}\right) + \frac{t_W}{2}\left(\frac{d}{2} - t_F - x_3^*\right)\left(\frac{d}{2} - t_F + x_3^*\right) \tag{a}$$

Substituting relation (a) into (9.67) we get

$$\tau_{13} = \frac{Q_3}{2I_2 t_W}\left\{ bt_F(d - t_F) + t_W\left[\left(\frac{d}{2} - t_F\right)^2 - (x_3^*)^2\right]\right\} \tag{b}$$

That is, as shown in Fig. a, the distribution of the shearing component of stress τ_{13} on the web is parabolic. The maximum value of τ_{13} is at $x_3^* = 0$. For the I-beams used in practice, the shearing component of stress τ_{13} on the web accounts for 90% to 98% of the shearing forces Q_3.

In order to establish the shearing component of stress τ_{12} at any point of the bottom flange of the beam of Fig. a, consider a segment $ABCDEFG$ of length dx_1 and width $(b/2 - x_2^*)$ (see Fig. a). The equilibrium of this segment requires that a force dF acts on the plane CDG (see Fig. a). As discussed previously, the shearing component of stress τ_{12} acting on the flanges of the beam of Fig. a can be considered approximately as being uniformly distributed over the thickness t_F of each flange. Thus, the shearing component of stress τ_{12} acting on the bottom flange is

$$\tau_{12} = \frac{Q_3}{I_2 t_F} \iint_{A_n} x_3 dA \tag{c}$$

where A_n is the area $EDGF$. Referring to Fig. a, relation (c) may be rewritten as

$$\tau_{12} = \frac{Q_3\left(\frac{b}{2} - x_2^*\right)(d - t_F)}{2I_2} \tag{d}$$

Notice that since τ_{12} is positive, its sense is toward the area A_n. That is, referring to Fig. a, its sense is from line DG to EF.

The shearing component of stress τ_{12} at any point of line $x_2 = x_2^*$ of the top flange is equal to

$$\tau_{12} = \frac{Q_3\left(\frac{b}{2} - x_2^*\right)[-(d - t_F)]}{2I_2} \tag{e}$$

In this case, τ_{12} is negative. Consequently, its sense is away from the area A_n; that is, referring to Fig. a, its sense is from line HJ to IK. The distribution of the shearing component of stress τ_{12} is shown in Fig. a.

Example 9 Consider a cantilever beam whose cross section is the thin-walled angle with

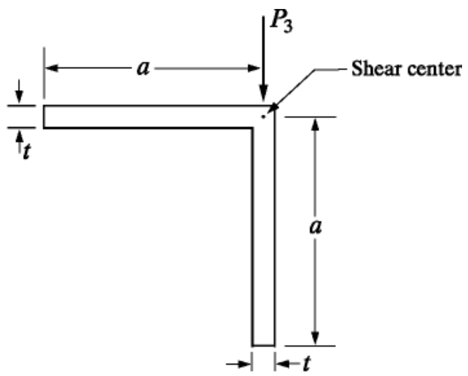


Figure a Geometry of the cross sections of the beam.

equal legs shown in Fig. a. The beam is subjected at its free end to a concentrated force P_3 passing through the shear center of its cross section. Compute the distribution of the shearing component of stress on the cross sections of the beam.

Solution Inasmuch as the legs of the angle are thin we disregard their thickness t as compared to their length a. We first locate the centroid of the cross section of the beam with respect to the system of axes x_2^*, x_3^* shown in Fig. a; referring to this figure, we have

$$\bar{x}_3 = \bar{x}_2 = \frac{(at)a}{4(at)} = \frac{a}{4}$$

The moments and the product of inertia with respect to the centroidal axes x_2, x_3 are

$$I_{22} = I_{33} = \frac{a^3 t}{12} + at\left(\frac{a}{2} - \frac{a}{4}\right)^2 + (at)\frac{a^2}{16} = \frac{5a^3 t}{24} \tag{b}$$

$$I_{23} = 2at\left(-\frac{a}{4}\right)\left(\frac{a}{2} - \frac{a}{4}\right) = -\frac{a^3 t}{8} \tag{c}$$

Using the above results, we get

$$I_{22}I_{33} - I_{23}^2 = \left(\frac{5a^3 t}{24}\right)^2 - \left(-\frac{a^3 t}{8}\right)^2 = \frac{a^6 t^2}{36} \tag{d}$$

The shearing components of stress τ_{12} and τ_{13} are computed using formula (9.66). That is, referring to Fig. ca, we have

$$\tau_{12} = \frac{Q_3 I_{33} A_n \bar{x}_{3n}}{(I_{22}I_{33} - I_{23}^2)t} - \frac{Q_3 I_{23} A_n \bar{x}_{2n}}{(I_{22}I_{33} - I_{23}^2)t} = \frac{P_3\left(\frac{5a^3 t}{24}\right)(tx_2')\left(-\frac{a}{4}\right)}{\left(\frac{a^6 t^2}{36}\right)t} - \frac{P_3\left(-\frac{a^3 t}{8}\right)(tx_2')\left(3\frac{a}{4} - \frac{x_2'}{2}\right)}{\left(\frac{a^6 t^2}{36}\right)t}$$

$$= \frac{3P_3 x_2'}{4a^3 t}(2a - 3x_2') \tag{e}$$

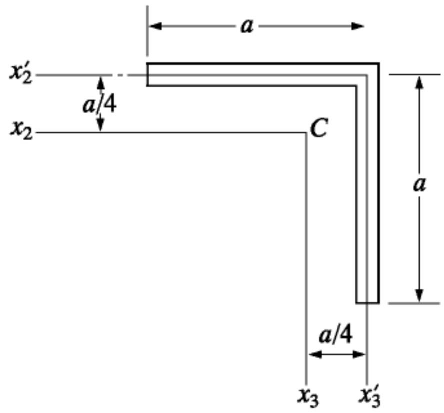


Figure b Location of the centroid of the cross section of the beam.

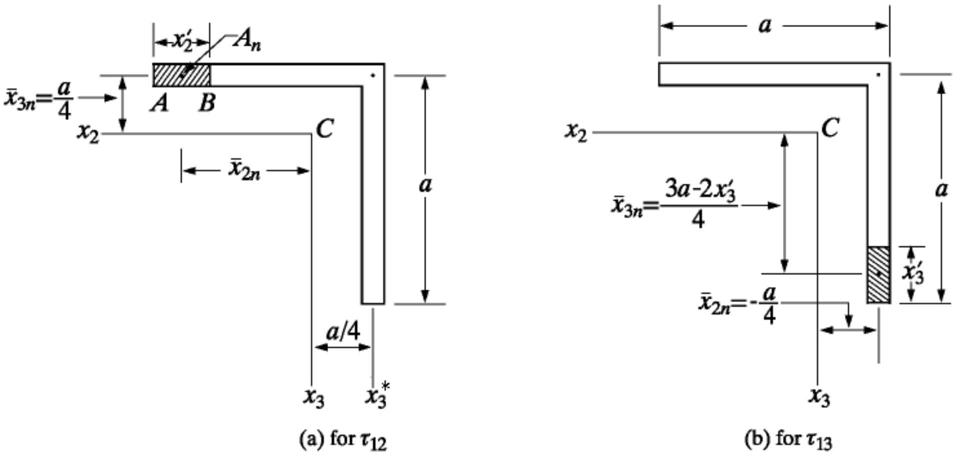


Figure c Area A_n and distances $\bar{x}_{3n}$ and $\bar{x}_{2n}$ for computing the shearing components of stress τ_{12} and τ_{13}.

Taking into account that $x_2' = 3a/4 - x_2$ the above relation reduces to

$$\tau_{12} = -\frac{3P_3(3a - 4x_2)(a - 12x_2)}{64a^3t} \tag{f}$$

The minus sign indicates that for $x_2 < a/12$ the direction of the shearing component of stress τ_{12} acting on the flange of the beam is from point A to point B (see Fig ca). From relation (f) we see that τ_{12} vanishes when $x_2 = a/12$ and $x_2 = 3a/4$. The distribution of the shearing component of stress τ_{12} is shown in Fig. d. Referring to Fig. cb, we obtain

$$\tau_{13} = \frac{Q_3 I_{33} A_n \bar{x}_{3n}}{(I_{22}I_{33} - I_{23}^2)t} - \frac{Q_3 I_{23} A_n \bar{x}_{2n}}{(I_{22}I_{33} - I_{23}^2)t}$$

$$= \frac{P_3\left(\frac{5a^3t}{24}\right)(tx_3')\left(\frac{3a - 2x_3'}{4}\right)}{\left(\frac{a^6t^2}{36}\right)t} - \frac{P_3\left(-\frac{a^3t}{8}\right)(tx_3')\left(-\frac{a}{4}\right)}{\left(\frac{a^6t^2}{36}\right)t} = \frac{3P_3x_3'}{8a^3t}(12a - 10x_3') \tag{g}$$

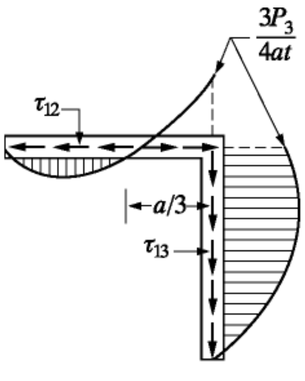


Figure d Distribution of the shearing components of stress on the cross sections of the beam.

Taking into account that $x_3' = 3a/4 - x_3$ relation (g) reduces to

$$\tau_{13} = \frac{3P_3(3a - 4x_3)(9a + 20x_3)}{64a^3t} \tag{h}$$

The distribution of the shearing component of stress τ_{13} is shown in Fig. d.

9.6 Build-Up Beams

A build-up beam is made from two or more pieces of the same material connected together to form one beam. The connections of these pieces must have the strength to transmit the shearing forces, which must be transmitted in order that the pieces act as one unit. As an example consider the plate girder whose cross section is shown in Fig. 9.14a. This girder consists of three plates welded together to form an I-beam. In Fig. 9.15 we show the free-body diagram of a segment of infinitesimal length of the bottom flange of the girder of Fig. 9.14a. If we designate the normal component of stress acting on face ABC by τ_{11}, the normal component of stress acting on face DEF in general will be $\tau_{11} + d\tau_{11}$. That is, the resultant force F_1 of the normal component of stress acting on the face ABC of the flange of the grider is different than the resultant force F_2 of the normal components of stress acting on its face DEF. Consequently, since the segment $ABCDEF$ is in equilibrium, a horizontal shearing force is exerted by the web of the beam on its flange. Referring to Section 9.5, it is clear that the value of this force per unit length is equal to the shear flow q_{13} between web and flange given by relation (9.65). For the beam of Fig. 9.14a we have

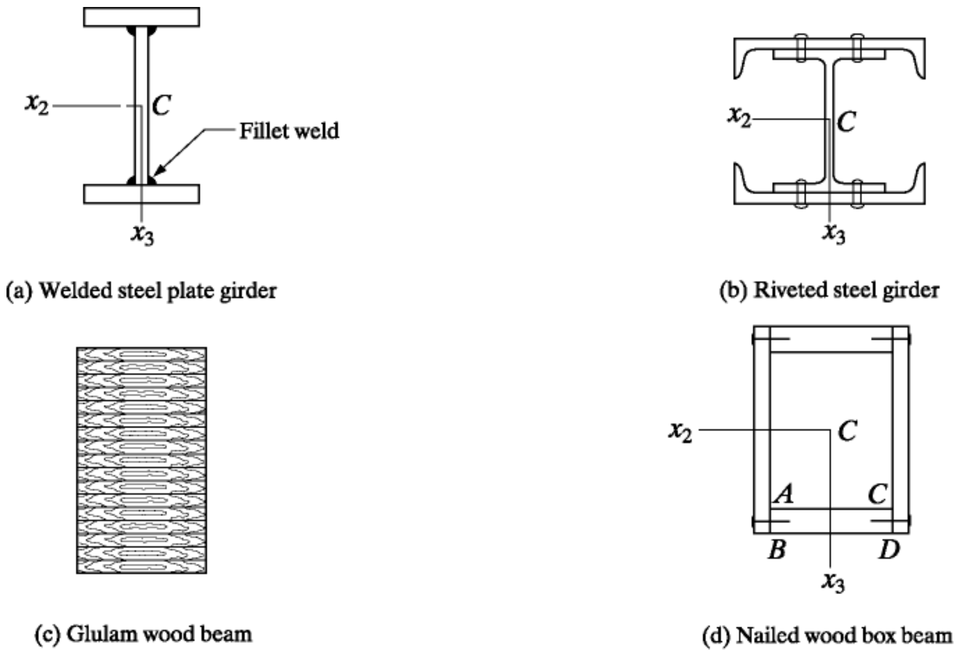


Figure 9.14 Cross sections of build-up beams.

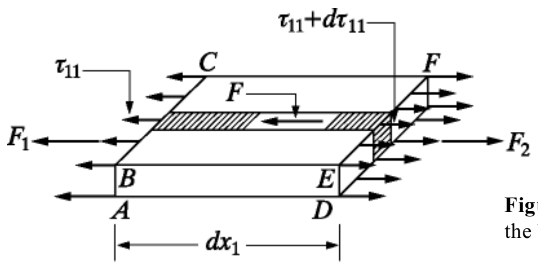


Figure 9.15 Free-body diagram of a segment of the bottom flange of the girder of Fig. 9.14a.

$$q_{13} = \frac{Q_3 A_n x_{3n}}{I_2} \tag{9.77}$$

where

A_n = area of the cross section of the bottom flange of beam.

$x_{3n} = x_3$ coordinate of the centroid of the cross section of the bottom flange.

I_2 = moment of inertia of the cross section of beam about the x_2 axis.

On the basis of the foregoing presentation, it is clear that each unit length of the weld must be able to transmit from the web of the beam to its flange a shearing force equal to the shear flow given by relation (9.77).

The connections of pieces made from steel are often welded using either a *butt weld* as shown in Fig. 9.16a or a *fillet weld* as shown in Fig. 9.16b. The strength of a butt weld is established by multiplying the area of the cross section of the thinner plate being connected by the allowable stress for the welding metal. Pressure vessels are often manufactured using butt welds. The strength per unit length of a filled weld is equal to the smallest dimension (throat) across the weld multiplied by the allowable shearing stress for the welding metal.

As a second example consider the girder whose cross section is shown in Fig. 9.14b which consists of two channels bolted on the flanges of a wide flange. In Fig. 9.17 we show the free-body diagram of a segment of length s of the bottom channel of this girder extending from center to center of two adjacent lines of bolts. In general, the resultant force F_1 of the normal component of stress τ_{11} acting on the face ABC of the segment under consideration will be different than the resultant force F_2 of the normal component of stress acting on face DEF. Thus, a horizontal shearing force F must be applied to the segment $ABCDEF$ of the channel in order to keep it in equilibrium. This force is equal to the product of the shear

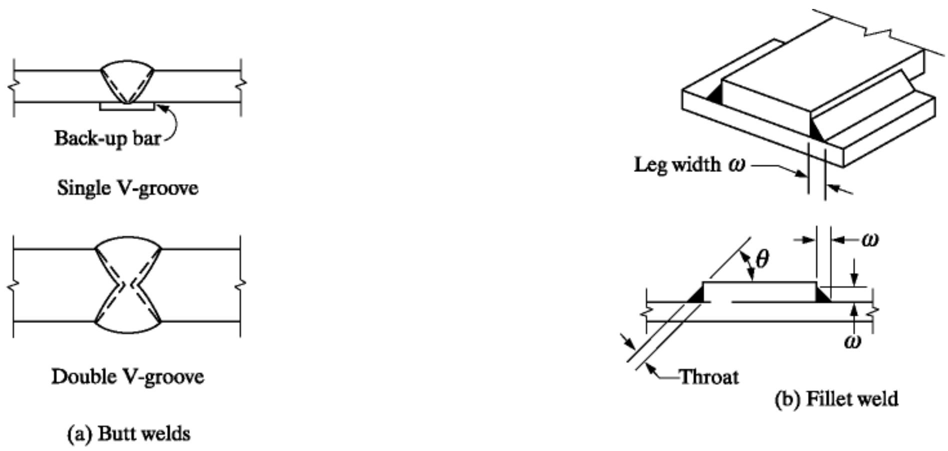


Figure 9.16 Types of welds.

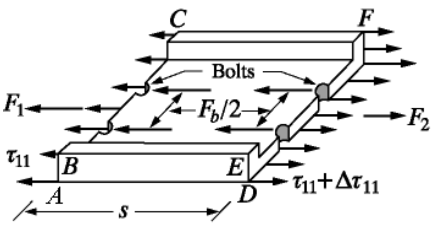


Figure 9.17 Free-body diagram of a segment of the bottom channel of the girder of Fig. 9.14b.

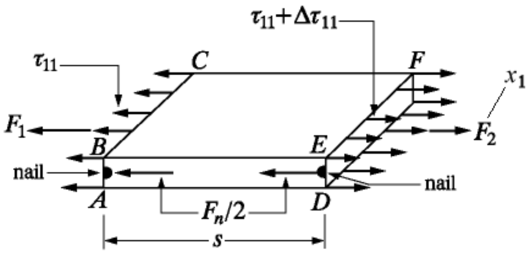


Figure 9.18 Free-body diagram of a segment of the bottom plank of the girder of Fig. 9.14c.

flow q_{13} given by relation (9.65) multiplied by the distance s between the two adjacent rows of bolts. That is,

$$F = F_2 - F_1 = q_{13}s \tag{9.78}$$

Disregarding the frictional forces between the channel and the flange of the wide- flange, the force $F = q_{13}s$ must be transmitted from the channel to the flange of the wide-flange by the four half bolts. That is, the shearing force F_b that one bolt must transmit is equal to

$$F_b = \frac{F}{2} = \frac{q_{13}s}{2} \tag{9.79}$$

As a third example consider the wood box girder whose cross section is shown in Fig. 9.14d. In Fig. 9.18 we show the free-body diagram of a segment of the bottom plank of length s extending from center to center of two adjacent rows of nails. The resultant force F_1 of the normal component of stress acting on the face ABC of this segment in general is not equal to the resultant force F_2 of the normal component of stress acting on its face DEF. Thus, a horizontal force F must act on the segment $ABCDEF$ of the girder in order to keep it in equilibrium. This force is equal to the shear flow q_{13} multiplied by the distance s between two adjacent nails. Disregarding the shearing forces transmitted between the vertical and horizontal planks due to friction, the force F must be transmitted by the four half nails. That is, the shearing force F_n that one nail must transmit is equal to

$$F_n = \frac{F}{2} = \frac{q_{13}s}{2} \tag{9.80}$$

In what follows we present two examples.

Example 10 Consider a simply supported steel girder whose cross section is shown in Fig. a. The girder consists of two C 150 ×12.2 channels connected by 12 mm-diameter bolts on the flanges of a wide-flange W 150 × 18. The girder is subjected to a concentrated force of 80 kN acting at the middle of its span. The allowable shearing stress in the bolts is 40 MPa. Compute the maximum allowable spacing of the bolts.

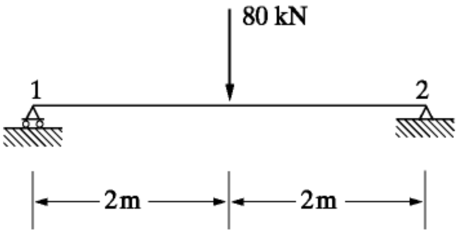
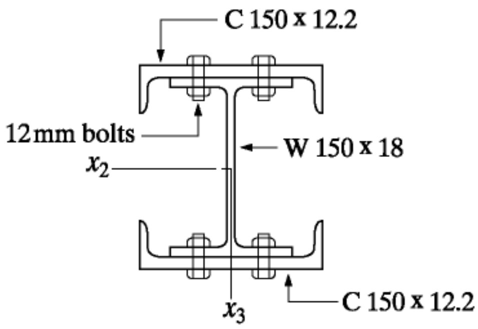


Figure a Geometry and loading of the girder.



Cross-section

Solution Referring to Appendix H and to Fig. b, we have

$$A_n = A_{channel} = 1540 \text{ mm}^2 \qquad \bar{x}_3' = 12.7 \text{ mm} \qquad I'_{channel} = 0.276(10^6) \text{ mm}^4$$
$$I_{WF} = 9.17(10^6) \text{ mm}^4 \qquad (t_W)_{channel} = 5.1 \text{ mm} \qquad h_{WF} = 153 \text{ mm} \tag{a}$$

Using quantities (a) and referring to Fig. b, we have

$$\bar{x}_{3n} = \frac{h}{2} + (t_W)_{channel} - \bar{x}_3' = 76.5 + 5.1 - 12.7 = 68.9 \text{ mm}$$

$$I_2 = I_{WF} + 2I'_{channel} + 2(A_{channel})(\bar{x}_{3n})^2 = 9.17(10^6) + 2[0.276(10^6)] + 2[1540(68.9)^2] \tag{b}$$

$$= 24.343(10^6) \text{ mm}^4$$

Substituting the first of quantities (a) and quantities (b) into relation (9.65) and using Q_3 = 40 kN, we get

$$q_{13} = \frac{Q_3 A_n \bar{x}_{3n}}{I_2} = \frac{40(1540)68.9}{24.343(10^6)} = 0.1744 \frac{\text{kN}}{\text{mm}} \tag{c}$$

Substituting result (c) into relation (9.79), we obtain

$$F_{bolt} = \frac{q_{13}s}{2} \leq \text{(allowable shearing stress for bolts) x (area of cross section of bolt)}$$
$$\leq 0.040\pi 6^2 = 4.52 \text{ kN}$$

or

$$s \leq \frac{(4.52)2}{0.1744} = 51.8 \text{ mm}$$

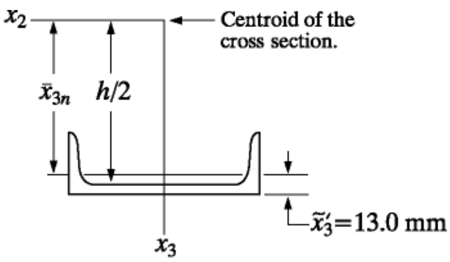


Figure b Area A_n and distance $\bar{x}_{3n}$ of the girder.

Example 11 Consider a cantilever wood box beam whose cross section is shown in Fig. a. The beam is subjected to a concentrated force of 24 kN at its unsupported end. The beam is made of four planks screwed together as shown in Fig. a. The screws have a diameter of 4 mm and are spaced at 20 mm. Compute the average shearing stress acting on the cross section of the screws.

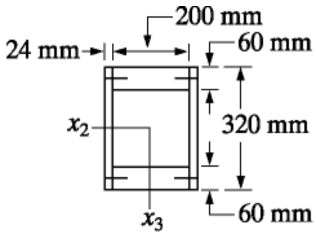


Figure a Cross section of the beam.

Solution Referring to Fig. a, we have

$$
\begin{aligned}
A_n &= 200(60) = 12{,}000 \text{ mm}^2 \\
\bar{x}_{3n} &= 130 \text{ mm} \\
I_2 &= \frac{2(24)(320^3)}{12} + \frac{2(60^3)200}{12} + 2(60)(200)(130^2) = 543.87(10^6) \text{ mm}^4
\end{aligned}
\tag{a}
$$

Substituting the quantities (a) into relation (9.65), we obtain

$$q_{13} = \frac{Q_3 A_n \bar{x}_{3n}}{I_2} = \frac{24(12{,}000)(130)}{543.87(10^6)} = 0.06884 \frac{\text{kN}}{\text{mm}}$$

$$F_{\text{screw}} = \frac{q_{13} s}{2} = \frac{0.06884(20)}{2} = 0.6884 \text{ kN}$$

$$(\tau_{\text{screw}})_{\text{average}} = \frac{F_{\text{screw}}}{A_{\text{screw}}} = \frac{0.6884}{\pi 2^2} = 0.0548 \frac{\text{kN}}{\text{mm}^2} = 54.8 \text{ MPa}$$

9.7 Location of the Shear Center of Thin-Walled Open Sections

In Section 9.5, we have established a formula for the shearing component of stress in beams subjected to bending without twisting. In order that a beam is subjected to bending without twisting, the external moments acting on it should not have a torsional component. Moreover, the external transverse forces acting on it must lie on a plane which contains the shear center of the cross sections of the beam. This is a point on the plane of any cross section of the beam which has the property that the moment of the shear flow, acting on the cross section about an axis through the shear center parallel to the x_1 axis, vanishes. It can be shown that

1. The shear center of cross sections having an axis of symmetry or a center of symmetry is located on the axis of symmetry or is the center of symmetry (see Fig. 9.19a and b).

2. The shear center of cross sections having two axes of symmetry is the intersection of these axes; that is, the shear center of such cross sections coincides with their centroid.

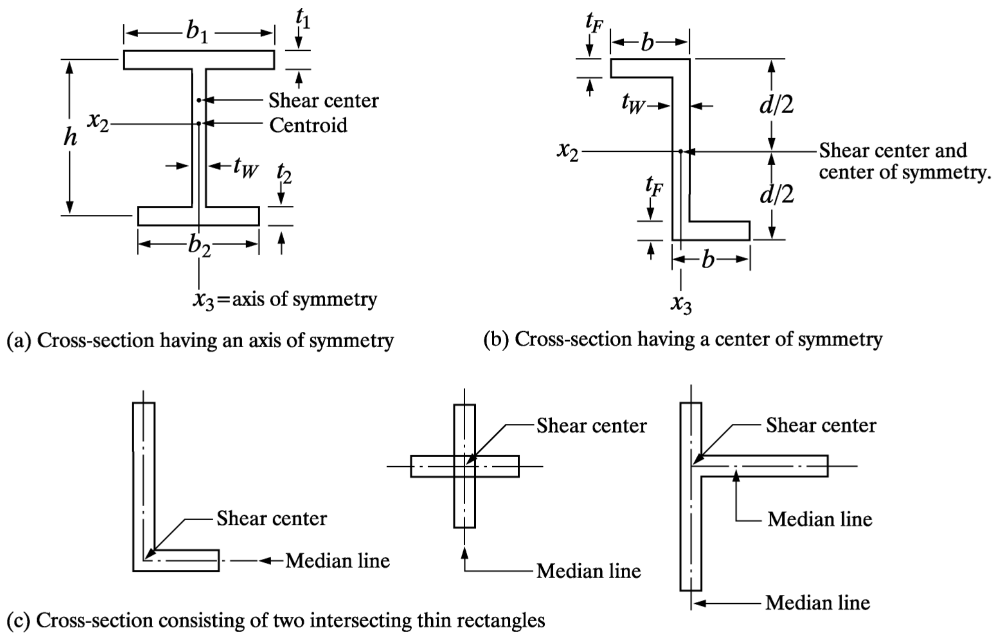


Figure 9.19 Shear centers of cross section having an axis of symmetry or a center of symmetry or consisting of two thin intersecting rectangles.

3. The shear center of cross sections consisting of two thin, intersecting rectangles, as shown in Fig. 9.19c, is the point of intersection of the median lines of the rectangles. This is so because the resultant forces of the shearing components of stress acting on the rectangular parts of a cross section intersect at that point.

When the line of action of the external transverse forces acting on a beam does not lie in a plane which contains the shear centers of its cross sections, they can be replaced by a statically equivalent system of external torsional moments and of external transverse forces whose line of action lies in a plane which contains the shear centers of the cross -sections of the beam (see Fig. 9.20). The shearing component of stress $(\tau_{1n})_Q$ due to the external transverse forces whose line of action lies in a plane which contains the shear centers of the cross sections of the beam can be established using relation (9.67). If the beam has a circular cross section the shearing component of stress $(\tau_{1n})_{M_1}$ due to the torsional moment

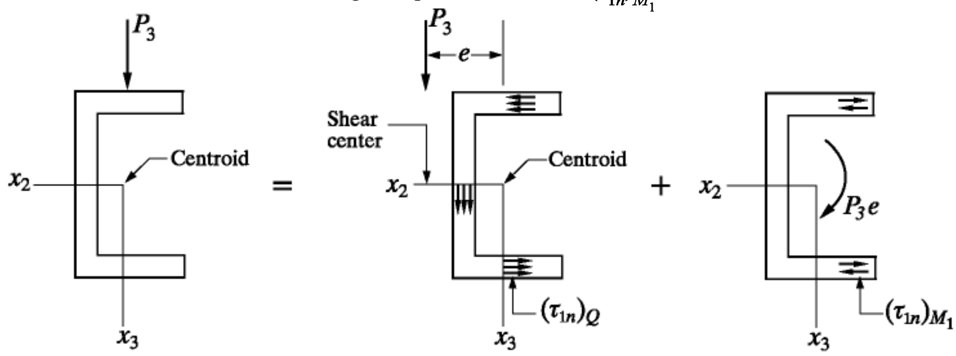
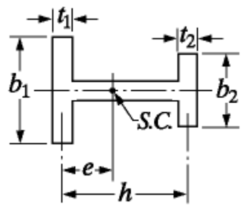
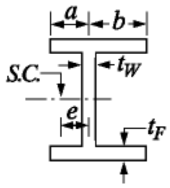
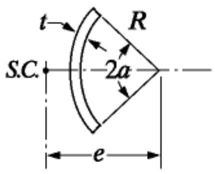
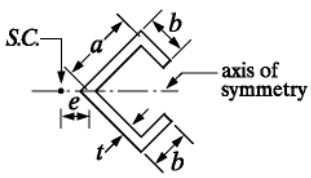
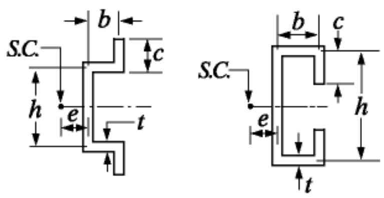


Figure 9.20 Channel subjected to transverse forces whose plane does not pass through the shear center of its cross section.

Table 9.1 Location of the shear center of thin-walled open cross sections having an axis of symmetry.

Geometry of the cross section	Distance e
	$\dfrac{t_2 b_2^3 h}{t_1 b_1^3 + t_2 b_2^3}$
	$\dfrac{3t_F(b^2 - a^2)}{6(a + b)t_F + ht_W}$ $\quad a < b$
	$\dfrac{2R(\sin\alpha - \alpha\cos\alpha)}{\alpha - \sin\alpha\cos\alpha}$
	$\dfrac{b^2 a(3a - 2b)}{\sqrt{2}(a^3 + 3ba^2 - 3b^2 a + b^3)}$
	$\dfrac{3b^2h^2 + 2bc(3h^2 - 4c^2)}{6bh^2 + h^3 + 2c(3h^2 \pm 6ch + 4c^2)}$ The + sign of the ± term refers to the figure on the left; the − sign, to the figure on the right.

can be established using relation (8.72). For beams of non-circular cross sections see Chapter 6. The shearing component of stress acting on the particles of a beam subjected to given forces is

$$\tau_{1n} = (\tau_{1n})_Q + (\tau_{1n})_{M_1} \tag{9.81}$$

The shear center of a cross section of a beam can be located by considering a segment of the beam and setting equal to zero the sum of the component along the x_1 axis of the moments of the external forces acting on this segment and of the shearing components of stress acting on the end faces of this segment. In the examples which follow, we locate the shear center of two cross sections—a channel and a thin-walled semicircular cross section.

Example 12 Locate the shear center of the thin channel shown in Fig. a.

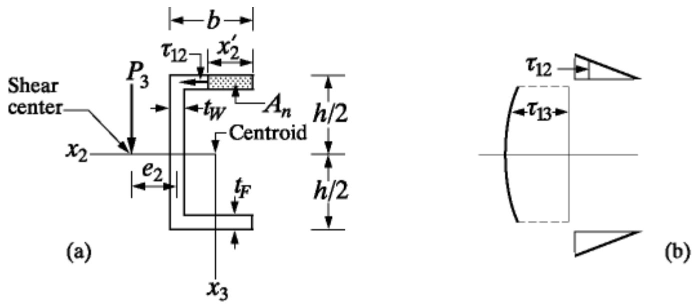


Figure a Geometry of the cross section of the channel and distribution of the shearing components of stress.

Solution The shear center of the thin channel shown in Fig. a is located on the x_2 axis because it is an axis of symmetry. In order to establish the distance e_2 of the shear center from the center line of the web of the channel, we assume that the channel is a cantilever beam subjected at its unsupported end to a force P_3, whose line of action passes through the shear center of the end cross section of the beam (see Fig. b). The component of stress τ_{13} acting on the particles of the flanges of the channel generally is small and its effect on the location of the resultant shearing force Q_3 of the shearing component of stress τ_{13} acting on the cross sections of the channel may be disregarded. That is, the shearing force Q_3 is assumed to act through the center line of the web of the channel. The force P_3 induces a component of shearing stress τ_{12} on the flanges (see Fig. a) which referring to relation (9.67)

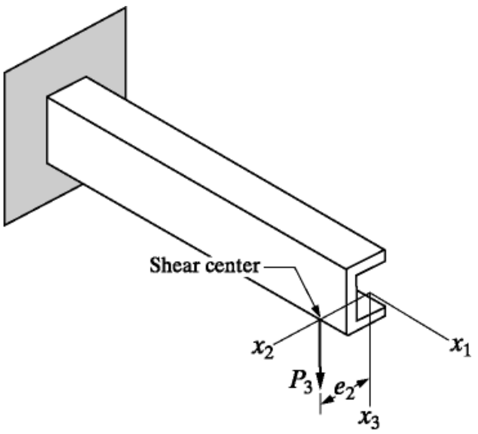


Figure b Cantilever channel subjected to a a transverse force P through its shear center.

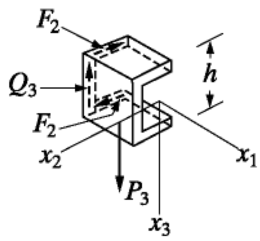


Figure c Free-body diagram of a segment of the cantilever beam of Fig. b.

is equal to

$$\tau_{12} = \frac{Q_3 h A_n}{2 I_2 t_F} = \frac{Q_3 h x_2'}{2 I_2} \tag{a}$$

As shown in Fig. a, the coordinate x_2' is measured from the edge B of the flange of the channel. Using the above relation, we find that the resultant horizontal force F_2 of the shearing component of stress τ_{12} acting on each flange, is equal to

$$F_2 = \int_0^{b-t_W} \tau_{12} t_F \, dx_2' = \frac{Q_3 t_F \, h(b - t_W)^2}{4 I_2} \tag{b}$$

Considering the equilibrium of the segment of the channel, whose free-body diagram is shown in Fig. c, we have

$$\begin{aligned} &\sum \bar{F}_v = 0 \qquad Q_3 = P_3 \\ &\sum M_1 = 0 \qquad P_3 e_2 = F_2 h = \frac{P_3 t_F \, h^2 (b - t_W)^2}{4 I_2} \end{aligned} \tag{c}$$

Thus,

$$e_2 = \frac{t_F h^2 (b - t_W)^2}{4 I_2} \tag{d}$$

Example 13 Compute the coordinates of the shear center of the thin-walled beam of constant thickness whose cross section is shown in Fig. a.

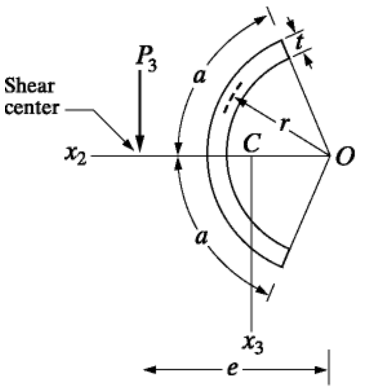


Figure a Geometry of the cross section of the beam.

Solution The x_2 axis is an axis of symmetry of the cross sections of the beam. Consequently, its shear center and its centroid are located on the x_2 axis. Therefore, we only compute the distance e of the shear center from the center O of the circles which bound the cross section of the beam. Consider a cantilever beam having the cross section of Fig. a and assume that the beam is subjected at its unsupported end to a concentrated force P_3 acting in the direction of the x_3 axis and passing through the shear center of its end cross section.

For this beam we have

$$Q_3 = P_3 \tag{a}$$

The shearing component of stress acting on the cross sections of the beam may be computed using relation (9.67). That is,

$$\tau_{1s} = \frac{Q_3}{I_2 t}\iint_{A_n} x_3\, dA \tag{b}$$

where dA is the shaded area shown in Fig. b. Referring to Fig. b, we have

$$\iint_{A_n} x_3\, dA = \int_{\varphi=\theta}^{\varphi=\alpha} (R\sin\varphi)tR\,d\varphi = -R^2 t\cos\varphi\Big|_{\varphi=\theta}^{\varphi=\alpha} = R^2 t(-\cos\alpha + \cos\theta) \tag{c}$$

Substituting relation (c) in (b), we get

$$\tau_{1s} = \frac{Q_3 R^2(-\cos\alpha + \cos\theta)}{I_2} \tag{d}$$

The moment of the shearing force acting on area dA about an axis through point O, parallel to the x_1 axis, is equal to

$$dM_1^{(o)} = (\tau_{1s} tR\,d\theta)R$$

Thus,

$$M_1^{(o)} = \int_{-\alpha}^{\alpha} \frac{Q_3 R^2(\cos\theta - \cos\alpha)R^2 t\,d\theta}{I_2} = \frac{2Q_3 R^4 t(\sin\alpha - \alpha\cos\alpha)}{I_2} \tag{e}$$

For bending without twisting, the moment of the shearing components of stress acting on a cross section about an axis parallel to the x_1 axis through point O must be equal to the moment of the external force about the same axis. That is,

$$\frac{2Q_3 R^4 t(\sin\alpha - \alpha\cos\alpha)}{I_2} = eP_3 = eQ_3 \tag{f}$$

The moment of inertia of the cross section of the beam about the x_2 axis is equal to

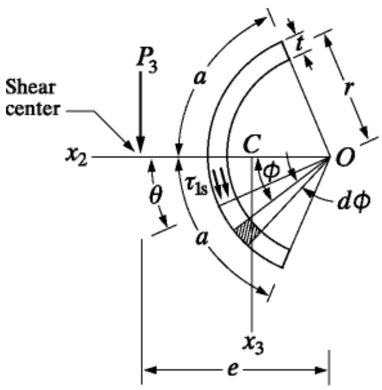


Figure b Cross section of the beam.

$$I_2 = \iint_A x_3^2\, dA = \int_{\varphi=-\alpha}^{\varphi=\alpha} R^2 \sin^2\varphi (Rt)\, d\varphi = R^3 t(\alpha - \sin\alpha\cos\alpha) \tag{g}$$

Substituting relation (g) into (f), we obtain

$$e = \frac{2R(\sin\alpha - \alpha\cos\alpha)}{\alpha - \sin\alpha\cos\alpha} \tag{h}$$

9.8 Members Whose Cross Sections Are Subjected to a Combination of Internal Actions

In the previous sections of this chapter we focus our attention to members subjected to external loads (transverse forces, bending moments and changes of temperature) which are resisted only by internal bending moments and shearing forces. In practice, however, members are often required to resist loads which subject their cross-sections to a combination of internal actions (axial centroidal forces, torsional moments, bending moments and shearing forces); such loads are called *combined loads.*

The components of stress and displacement of members subjected to combined loads are established by superimposing the corresponding quantities resulting from only one type of load at a time. As discussed in Section 3.13 superposition of the results is permitted for bodies made from linearly elastic materials when subjected to deformation whose magnitude is in the range of validity of the assumption of small deformation.

In this section we illustrate the computation of the components of stress in prismatic members subjected to combined loads. We distinguish the following cases:

1. Members subjected to eccentric axial forces.
2. Members subjected to external transverse forces acting on a plane which does not contain the shear center of their cross-sections.

9.8.1 Members Subjected to Eccentric Axial Forces

Consider a straight prismatic member subjected to equal and opposite eccentric axial forces P_1 at its end cross sections ($x_1 = 0$ and $x_1 = L$) as shown in Fig. 9.21. In order to simplify our presentation, we refer our discussion to principal centroidal axes x_2 and x_3. Taking advantage of the principle of Saint Venant, we replace the eccentric force P_1 with a statically equivalent system consisting of a centroidal axial force P_1 and two components of bending moment M_2 and M_3 equal to

$$\begin{aligned} M_2 &= P_1 e_3 \\ M_3 &= -P_1 e_2 \end{aligned} \tag{9.82}$$

The normal components of stress acting on a particle on a cross section of the member can be computed by superimposing the normal components of stress due to the axial centroidal force [given by relation (8.59)] to those due to the two components of bending moment M_2 and M_3 [given by relation (9.12b)]. That is,

$$\tau_{11} = \frac{P_1}{A} - \frac{M_3 x_2}{I_3} + \frac{M_2 x_3}{I_2} \tag{9.83}$$

where I_2 and I_3 are the moments of inertia of the cross section of the member about the x_2 and x_3 axes, respectively. A is the area of the cross section. Substituting relations (9.82) in (9.83) we get

$$\tau_1 = \frac{P_1}{A} + \frac{P_1 e_2 x_2}{I_2} + \frac{P_1 e_3 x_3}{I_3} \tag{9.84}$$

The equation of the neutral axis is obtained by setting τ_{11} equal to zero in relation (9.84). That is,

$$\frac{1}{A} + \frac{e_3 x_3}{I_2} + \frac{e_2 x_2}{I_3} = 0 \tag{9.85}$$

This is the equation of a straight line. From relation (9.85) we see that when a member is subjected to an eccentric force the neutral axis may or may not be on its cross section. Its location depends on the position of the eccentric force. When the eccentricity e_2 and e_3 of the force is small the neutral axis is not on the cross section of the member. In this case the normal component of stress acting on the cross section will not change sign. There exists a region on the cross section of a member called the *kern* which has the property that when a compressive force is applied at any one of its points, it produces compression on all the particles of the cross section. When a member is made from a material which is very weak in tension, like concrete, and is subjected to an eccentric compressive force, it is important that this force is applied at a point inside the kern of its cross section.

Consider a member of rectangular cross section (see Fig. 9.22), when a compressive force is applied at some point of the x_2 axis ($e_3 = 0$) from relation (9.85) we get

$$x_2 = -\frac{I_3}{Ae_2} \tag{9.86}$$

This is the equation of a line normal to the x_2 axis. As e_2 decreases, the neutral axis moves away from the centroid of the cross section. For certain positive value e^*_2 of the eccentricity of the applied force, the neutral axis becomes the edge $x_2 = -b/2$ of the cross section of the member. That is

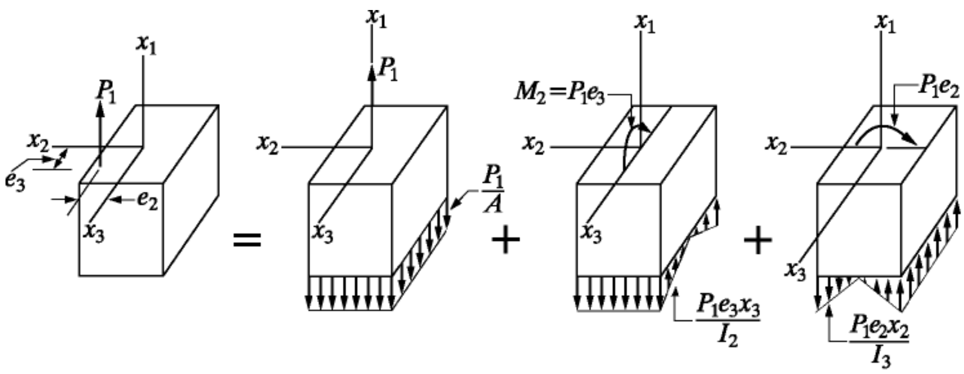


Figure 9.21 Prismatic member subjected to an eccentric axial force.

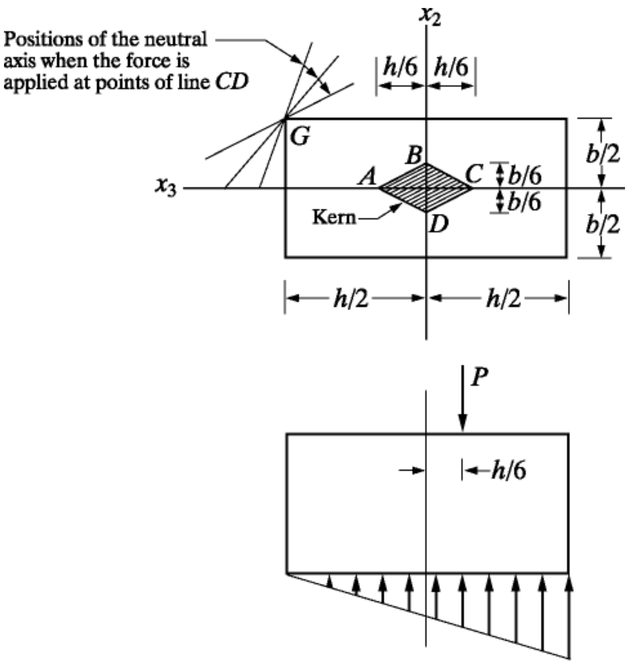


Figure 9.22 Kern of a rectangular cross section.

$$x_2 = -\frac{b}{2} = -\frac{I_3}{Ae_2^*}$$

and

$$e_2^* = \frac{2I_3}{Ab} \tag{9.87}$$

Noting that

$$I_3 = \frac{b^3h}{12}$$

$$A = bh$$

relation (9.87) gives

$$e_2^* = \frac{b}{6} \tag{9.88a}$$

Similarly it can be shown that the edge $x_3 = -h/2$ of the cross section of a member of rectangular cross section becomes its neutral axis when an eccentric force is applied on the positive x_3 axis with eccentricity equal to

$$e_3^* = \frac{h}{6} \tag{9.88b}$$

Thus as shown in Fig. 9.22, the kern of a rectangular cross section is a parallelogram $ABCD$. When the force is applied at points A, B, C or D, the neutral axis is the opposite edge of the cross section. When the force is applied at any point of line AB, the neutral axis passes through corner u of the cross section.

9.8.2 Prismatic Beams Subjected to External Transverse Forces Acting in a Plane Which Does not Contain the Shear Centers of their Cross Sections

In order to establish the stress distribution on the cross sections of beams subjected to transverse forces whose line of action is not located in a plane which contains the shear center of their cross sections, as we mention in Section 8.2, the transverse forces are replaced by an equivalent system of forces and moments consisting of

1. Distributed forces $p_2(x_1)$ and $p_3(x_1)$ and concentrated forces $P_2^{(n)}(n = 1, 2, ..., n_2)$ and $P_3^{(n)}(n = 1, 2, ... , n_3)$ acting in the direction of the x_2 and x_3 axis, respectively. The line of action of each one of these forces is located in a plane which contains the shear centers of the cross sections of the member. In Section 9.7 we present a procedure for determining the location of the shear center of thin-walled open cross sections.

2. Distributed moments $m_1(x_1)$ and concentrated moments $M_1^{(m)}(m = 1, 2, ..., m_1)$ occur about an axis parallel to the x_1 axis and pass through the shear center of the cross section of the member.

The first loading described above induces a distribution of normal (τ_{11}) and shearing (τ_{13} or τ_{12}) components of stress. The normal component of stress can be computed using formula (9.12a or 9.12b). The shearing components of stress for beams whose cross sections have the properties of one of the cross sections shown in Fig. 9.11 can be computed using relation (9.66 or 9.67). The second loading described above induces only a distribution of shearing stress. For beams of circular cross sections, this distribution of shearing stress can be computed using relations (8.72). For beams having a thin-walled open cross section, this distribution of shearing stress can be computed using relation (6.103). For beams having a thin-walled hollow cross section, this distribution of shearing stress can be computed using the procedures described in Sections 12.2 and 12.5.

The distribution of stress in beams subjected to external transverse forces acting in a plane which does not contain the shear center of their cross sections can be established by superimposing the stress distribution obtained from the two loading cases described above.

In what follows we present an example.

Example 14 A cantilever beam is made from a standard American channel C 75 x 8.9. The

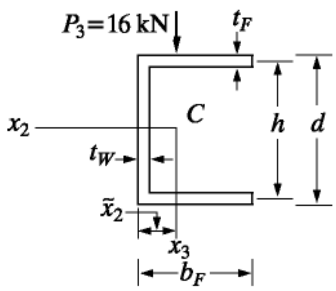


Figure a Cross section of the beam.

beam is subjected to a vertical force of 16 kN at its free end, whose line of action passes through the centroid of its cross section (see Fig. a). Compute the maximum shearing stress at cross sections sufficiently away from the fixed support of the beam, so that the effect of restraining the warping at the fixed end can be disregarded (see Section 9.11). Show on a sketch the state of stress of the particles on which the maximum shearing stress acts.

Solution Referring to the table of standard American rolled steel shapes given in Appendix H the dimensions and properties of the channel are

$$
\begin{aligned}
A &= 1{,}130 \text{ mm}^2 & t_F &= 6.9 \text{ mm} \\
d &= 76.2 \text{ mm} & t_W &= 9.0 \text{ mm} \\
h &= 76.2 - 6.9 = 69.3 \text{ mm} & \bar{x}_2 &= 11.3 \text{ mm} \\
b_F &= 40 \text{ mm} & I_2 &= 0.850(10^6) \text{ mm}^4 \\
& & e_2 &= 7.96
\end{aligned}
\tag{a}
$$

As shown in Fig. b the applied transverse force of 16 kN is statically equivalent to a transverse force of 16 kN acting through the shear center of the cross section of the beam and a torsional moment

$$M_1 = -16(7.96 + 11.3) = -308.16 \text{ kN·m} \tag{b}$$

We denote the shearing components of stress due to bending by τ_{13}^{B} or $\tau_{12}^{(B)}$ and the shearing components of stress due to torsion by τ_{13}^{T} or τ_{12}^{T}. Referring to relation (9.67) we have

$$\tau_{13}^{B} = \frac{Q_3 A_n \bar{x}_{3n}}{I_2 t_W} \tag{c}$$

$$\tau_{12}^{B} = \frac{Q_3 A_n \bar{x}_{3n}}{I_2 t_F} \tag{d}$$

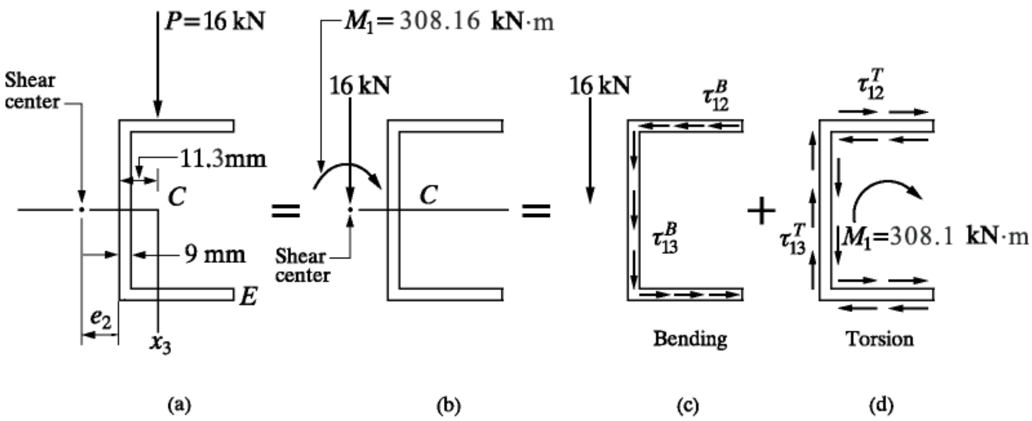


Figure b Distribution of the shearing components of stress on the cross section of the beam.

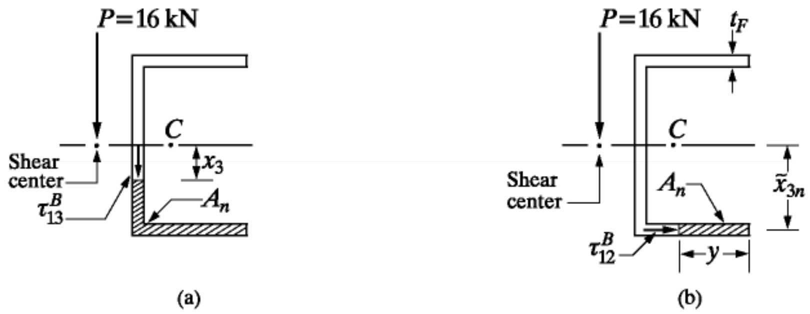


Figure c Cross section of the beam showing the area A_n and distance $\bar{x}_{3n}$.

Referring to Fig. ca and substituting relations (a) into (c) and (d), we get:

In the web

$$\tau_{13}^B = \frac{16\left[(40 - 9)6.9(38.1 - 3.45) + 9(38.1 - x_3)[x_3 + (38.1 - x_3)/2]\right]}{0.850(10^6)9}$$

$$= (29164 - 0.941176x_3^2)10^{-5}$$

In the flanges referring to Fig cb

$$\tau_{12}^B = \frac{16(6.9y)(38.1 - 3.45)}{0.850(10^6)6.9} = 0.6522(10^{-3})y \tag{e}$$

The maximum shearing stress in the web occurs at $x_3 = 0$ and in flanges at $y = b_F - t_W = 31$ mm. Thus,

$$(\tau_{13}^B)_{\max} = 29164(10^{-5})\frac{\text{kN}}{\text{mm}^2} = 29.1 \text{ MPa} \tag{f}$$

$$(\tau_{12}^B)_{\max} = 0.6522(10^{-3})(40 - 9) = 20.21(10^{-3})\frac{\text{kN}}{\text{mm}^2} = 20.21 \text{ MPa} \tag{g}$$

Moreover, referring to relation (6.103), we have

$$\tau^{(j)T} = \frac{6M_1x_{2j}}{\sum_{i=1}^{2} b_i t_i^3} \tag{h}$$

Where, referring to Fig. a, we get

$$\sum_{i=1}^{2} b_i t_i^3 = 2\left(b_F - \frac{t_W}{2}\right)t_F^3 + (h - t_F)\,t_W^3 \tag{i}$$

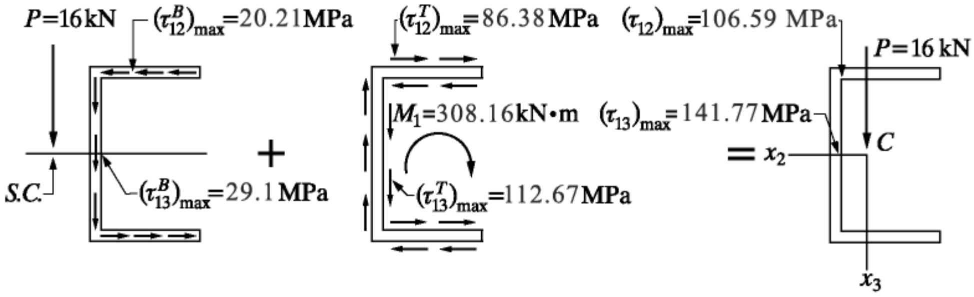
$$= 2(40 - 4.5)(6.9)^3 + (76.2 - 6.9)9^3 = 73{,}844$$

Substituting relations (b) and (i) into (k), we obtain

$$(\tau_{12}^{T})_{max} = \frac{6(308.16)(3.45)}{73{,}844} = 0.08638\ \frac{kN}{mm^2} = 86.38\ \text{MPa}$$

$$(\tau_{13}^{T})_{max} = \frac{6(308.16)(4.5)}{73{,}844} = 0.11267\ \frac{kN}{mm^2} = 112.67\ \text{MPa} \tag{m}$$

The results are shown in Fig. d.



(a) Bending (b) Torsion (c) Maximum shearing stress

Figure d Results.

9.9 Composite Beams

A prismatic composite beam consists of two or more prismatic components of different material firmly bonded together to act as a single beam. Examples of composite beams are sandwich and reinforced concrete beams (see Fig. 9.23). In this section we limit our attention to prismatic beams whose cross sections have an axis of symmetry. We chose the x_3 axis to be the axis of symmetry of the cross section of the beam at x_1 = 0. Moreover, in this section we consider only beams subjected to external forces whose line of action is in the x_1x_3 plane parallel to the x_3 axis and to external moments whose vector is normal to the x_1x_3 plane. Thus,

$$u_2(x_1) = \theta_3(x_1) = 0 \tag{9.89}$$

We choose the x_2 axis so that

$$e_{11}(x_1, x_2, 0) = 0 \tag{9.90}$$

and

$$\hat{u}_1(x_1, x_2, 0) = 0 \tag{9.91}$$

That is, the x_2 axis is the axis of zero e_{11} of the cross sections of the composite beam.

We assume that the fundamental assumptions of the theories of mechanics of materials discussed in Section 8.2 are valid for composite beams. Consequently, referring to relations (8.1), (8.5), and (8.6) and taking into account relations (9.89) and (9.91) for the beams under consideration, we have

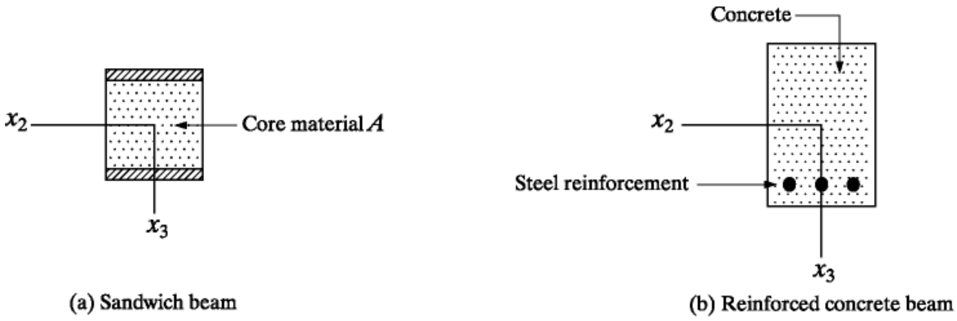


Figure 9.23 Cross sections of composite beams.

$$\tau_{22} = \tau_{33} = \tau_{23} = 0 \tag{9.92}$$

$$\hat{u}_1(x_1, x_2, x_3) = x_3\theta_2(x_1) \tag{9.93}$$

and

$$e_{11}(x_1, x_2, x_3) = x_3\frac{d\theta_2}{dx_1} \tag{9.94}$$

Consider a prismatic composite beam made from two different materials A and B having modulus of elasticity E_A and E_B, respectively. The area of the cross section of the beam consists of two parts: area A_A (area RSMN in Fig. 9.24a) made from material A and area A_B (area MNGD in Fig. 9.24a) made from material B. The beam is subjected to external actions in an environment of constant temperature. We denote by $\tau_{11}^{(A)}$ and $\tau_{11}^{(B)}$ the normal component of stress acting on the parts of the cross section of the composite beam made from material A and B, respectively. Referring to relations (8.44) and using (9.94), we have

$$\tau_{11}^{(A)}(x_1, x_2, x_3) = E_A e_{11}(x_1, x_2, x_3) = E_A x_3\frac{d\theta_2}{dx_1} \tag{9.95a}$$

$$\tau_{11}^{(B)}(x_1, x_2, x_3) = E_B e_{11}(x_1, x_2, x_3) = E_B x_3\frac{d\theta_2}{dx_1} \tag{9.95b}$$

The resultant axial force acting on a cross sections of the beam is zero. That is, using relations (9.95), we get

$$0 = \iint_{A_A}\tau_{11}^{(A)}dA + \iint_{A_B}\tau_{11}^{(B)}dA = \frac{d\theta_2}{dx_1}\left(E_A\iint_{A_A}x_3\,dA + E_B\iint_{A_B}x_3\,dA\right) \tag{9.96}$$

Thus,

$$\iint_{A_A}x_3\,dA + n\iint_{A_B}x_3\,dA = 0 \tag{9.97}$$

where

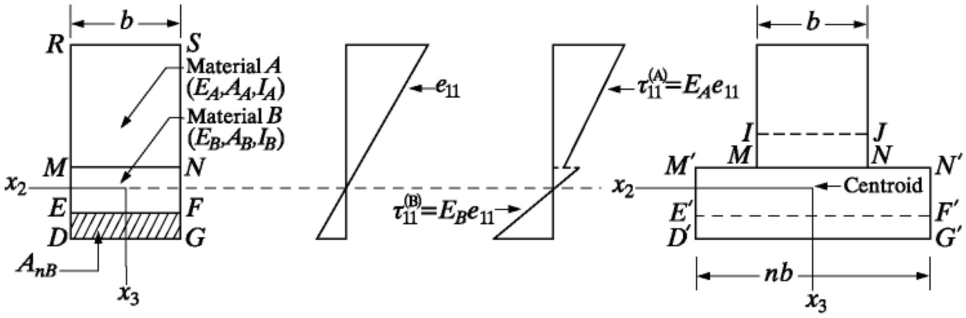


Figure 9.24 Distribution of the axial component of strain and stress on a cross section and transformed cross section of a two-material composite beam.

$$n = \frac{E_B}{E_A} \tag{9.98}$$

Relation (9.97) is a generalization of relation (9.6b) for beams made from two materials and it is used to locate the x_2 axis of the beam which is the neutral axis of its cross sections (see example at the end of this section).

The relation between the bending moment $M_2(x_1)$ and the angle of rotation $\theta_2(x_1)$ may be established by substituting relations (9.95) into (8.9e). That is,

$$M_2 = \iint_A \tau_{11} x_3 \, dA = \iint_{A_A} \tau_{11}^{(A)} x_3 \, dA + \iint_{A_B} \tau_{11}^{(B)} x_3 \, dA = \frac{d\theta_2}{dx_1}\left(E_A \iint_{A_A} x_3^2 \, dA + E_B \iint_{A_B} x_3^2 \, dA \right)$$
$$= \frac{d\theta_2}{dx_1}(E_A I_A + E_B I_B) \tag{9.99}$$

where I_A or I_B is the moment of inertia about the x_2 axis of the area of the portion of the cross section made from material A or B, respectively. Substituting relation (9.27a) into (9.99), we get

$$\frac{d^2 u_3}{dx_1^2} = -\frac{M_2}{E_A I_A + E_B I_B} \tag{9.100}$$

For any statically determinate beam, $M_2(x_1)$ can be established by considering the equilibrium of appropriate segments of the beam. The expressions for $M_2(x_1)$ can be substituted in relation (9.100) and the resulting differential equations can be solved to obtain the deflection of the statically determinate composite beam, following the procedure described in Section 9.2.

Differenting relation (9.100) twice and using relation (8.23), we obtain

$$\frac{d^2}{dx_1^2}\left[(E_A I_A + E_B I_B)\frac{d^2 u_3}{dx_1^2}\right] = p_3 + \frac{dm_2}{dx_1} \tag{9.101}$$

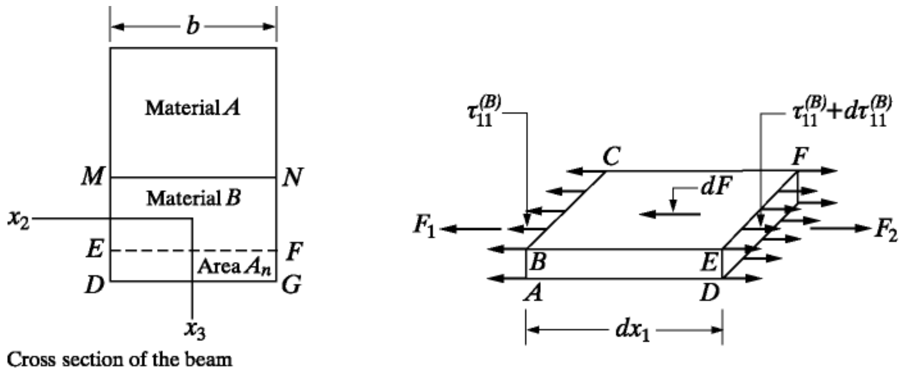


Figure 9.25 Free-body diagram of an infinitesimal segment of the beam made form material *B*.

This equation can be solved to obtain the deflection of any composite statically determinate or indeterminate beam following the procedure described in Section 9.2.

Using relation (9.99) to eliminate $d\theta_2/dx_1$ from relation (9.95), we get

$$\tau_{11}^{(A)}(x_1, x_2, x_3) = \frac{E_A M_2 x_3}{E_A I_A + E_B I_B} = \frac{M_2 x_3}{I_A + nI_B} \tag{9.102a}$$

$$\tau_{11}^{(B)}(x_1, x_2, x_3) = \frac{E_B M_2 x_3}{E_A I_A + E_B I_B} = \frac{nM_2 x_3}{I_A + nI_B} \tag{9.102b}$$

In order to compute the shearing component of stress τ_{13}^{B} acting at any particle of material B, we consider the free-body diagram of the segment of infinitesimal length of the beam made only from material B, shown in Fig. 9.25b. Referring to this figure and using relation (9.102b), the resultant forces F_1 and F_2 of the distribution of the normal component of stress on faces ABC and $DEFG$ of the segment under consideration are

$$F_1 = \iint_{A_n} \tau_{11}^{(B)} dA = \frac{nM_2}{I_A + nI_B} \iint_{A_n} x_3 dA \tag{9.103a}$$

$$F_2 = \iint_{A_n} (\tau_{11}^{(B)} + d\tau_{11}^{(B)}) dA = \frac{n(M_2 + dM_2)}{I_A + nI_B} \iint_{A_n} x_3 dA \tag{9.103b}$$

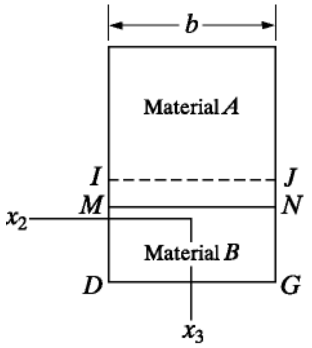
where

$$\text{Area } ABC = \text{Area } DEFG = A_n$$

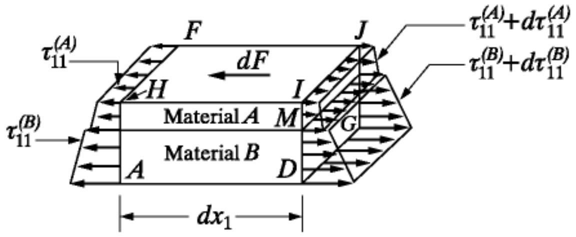
In order that the segment under consideration be in equilibrium, a shearing force dF must act on its surface $BEFC$. This force is equal to

$$dF = F_2 - F_1 = \frac{ndM_2 \left(\iint_{A_n} x_3 dA \right)}{I_A + nI_B} \tag{9.104}$$

Referring to relation (9.104) and using relation (8.21), the shear flow $q_{13}^{(B)}$ on the portion of the cross section made from material B is equal to



(a) Cross section of the beam



(b) Segment of the beam

Figure 9.26 Free-body diagram of a segment of infinitesimal length of the composite beam.

$$q_{13}^{(B)} = \frac{dF}{dx_1} = \frac{nQ_3 \iint_{A_n} x_3 \, dA}{I_A + nI_B} \tag{9.105}$$

For the beam under consideration, as discussed in Section 9.5, the shearing component of stress $\tau_{13}^{(B)}$ does not vary much in the x_2 direction. Consequently,

$$\tau_{13}^{(B)} = \frac{nQ_3 \iint_{A_n} x_3 \, dA}{b(I_A + nI_B)} \tag{9.106}$$

In order to compute the shearing component of stress acting at any particle of the portion of the beam made from material A we consider the free-body diagram of the segment of infinitesimal length shown in Fig. 9.26b. This segment is cut from the beam by two planes normal to its axis and by a plane parallel to its axis and normal to the x_1x_3 plane. Referring to Fig. 9.26b, the resultant forces F_1 and F_2 of the stress distribution on faces AHF and $DIJG$ of the segment of the beam under consideration are

$$F_1 = \iint_{A_B} \tau_{11}^{(B)} dA + \iint_{A_{nA}} \tau_{11}^{(A)} dA \tag{9.107a}$$

$$F_2 = \iint_{A_B} [\tau_{11}^{(B)} + d\tau_{11}^{(B)}] dA + \iint_{A_{nA}} [\tau_{11}^{(A)} + d\tau_{11}^{(A)}] dA \tag{9.107b}$$

where A_B or A_{nA} is the part of the end surfaces of the segment under consideration which is made from material B or A, respectively. In order that the segment under consideration is in equilibrium, a shearing force dF must act on its surface $HFJI$. Using relations (9.102) and (9.107) this force is equal to

$$dF = F_2 - F_1 = \iint_{A_B} d\tau_{11}^{(B)} dA + \iint_{A_{nA}} d\tau_{11}^{(A)} dA = \frac{dM_2}{I_A + nI_B}\left(n \iint_{A_B} x_3 \, dA + \iint_{A_{nA}} x_3 \, dA \right) \tag{9.108}$$

Using relation (8.21) the shear flow $q_{13}^{(A)}$ on the portion of the cross section made from material A is equal to

$$q_{13}^{(A)} = \frac{dF}{dx_1} = \frac{Q_3}{I_A + nI_B}\left(n\iint_{A_B} x_3 dA + \iint_{A_{nA}} x_3 dA \right) \tag{9.109}$$

For the beam under consideration the shearing component of stress does not vary much in the x_2 direction. Consequently,

$$\tau_{13}^{(A)} = \frac{Q_3}{b(I_A + nI_B)}\left(n\iint_{A_B} x_3 dA + \iint_{A_{nA}} x_3 dA \right) \tag{9.110}$$

Consider an auxiliary beam made from material A whose cross section consists of two parts. The one part has a geometry identical to that of the portion of the cross section made from material A of the real beam while the other has a depth equal to that of the portion of the cross section made from material B, a width n times that of the portion of the cross section of the real beam made from material B and, hence, an area equal to nA_B (see Fig. 9.24d). Referring to Fig. 9.24a and 9.24d we see that the x_3 coordinate of the centroid of the portion $DMNG$ of the cross section of the actual beam is equal to the x_3 coordinate of the portion $D'M'N'G'$ of the cross section of the auxiliary beam. Thus, referring to relation (C.1) of Appendix C for the cross section of the auxiliary beam, we have

$$\iint_{nA_B} x_3 dA = (nA_B)\bar{x}_3 = n\iint_{A_B} x_3 dA \tag{9.111}$$

We call the cross section of the auxiliary beam the *transformed cross section* of the beam. In order to locate the centroid of the transformed cross section we set its first moment about its centroidal axis x_2 equal to zero. That is, using relation (9.111) we have

$$\iint_{A_A} x_3 dA + \iint_{nA_B} x_3 dA = \iint_{A_A} x_3 dA + n\iint_{A_B} x_3 dA = 0 \tag{9.112}$$

Thus, referring to relation (9.97), we see that the centroidal axis x_2 of the cross section of the auxiliary beam is the axis of zero normal component of strain e_{11} of the composite beam, which we have denoted by x_2. The moment of inertia of the cross section of the auxiliary beam about its centroidal axis x_2 is equal to

$$(I_2)_{\text{aux}} = I_A + nI_B \tag{9.113}$$

where I_A or I_B is the moment of inertia about the x_2 axis of the area of the portion of the cross section of the actual beam made from material A or B, respectively. The distribution of the normal component of stress on the cross sections of the auxiliary beam is obtained by substituting relation (9.113) into (9.12b). That is,

$$(\tau_{11})_{\text{aux}} = \frac{M_2 x_3}{I_A + nI_B} \tag{9.114}$$

Comparing relation (9.114) with relations (9.102), we find

$$\tau_{11}^{(A)} = (\tau_{11})_{\text{aux}} \qquad \tau_{11}^{(B)} = n(\tau_{11})_{\text{aux}} \tag{9.115}$$

The distribution of the shearing component of stress on the cross sections of the auxiliary beam may be established by substituting relation (9.113) into (9.67). When relation (9.67) is used to compute the shearing stress acting on a particle of the portion of the cross section made from material A, referring to Fig. 9.24d, we see that

$$A_n = nA_B + A_{nA} \tag{9.116}$$

where, for a particle on a line IJ (see Fig. 9.24d), we have

$$nA_B = \text{area } D'G'M'N' \tag{9.117}$$

$$A_{nA} = \text{area } MNJI \tag{9.118}$$

When relation (9.67) is used to compute the shearing stress acting on a particle of the portion of the cross section made from material B, referring to Fig. 9.24d, we see that

$$A_n = nA_{nB}$$

where, for a particle on line $E'F'$

$$nA_{nB} = \text{area } D'G'F'E'$$

Thus, referring to relation (9.111), we have

$$[\tau_{13}^{(A)}]_{\text{aux}} = \frac{Q_3\left(n\iint_{A_B} x_3 dA + \iint_{A_{nA}} x_3 dA\right)}{b(I_A + nI_B)} \tag{9.119a}$$

$$[\tau_{13}^{(B)}]_{\text{aux}} = \frac{Q_3\iint_{nA_{nB}} x_3 dA}{nb(I_A + nI_B)} = \frac{Q_3\iint_{A_{nB}} x_3 dA}{b(I_A + nI_B)} \tag{9.119b}$$

Referring to Fig. 9.24d and to relations (9.106), (9.110) and (9.119), we see that

$$\tau_{13}^{(A)} = [\tau_{13}^{(A)}]_{\text{aux}} \qquad \tau_{13}^{(B)} = n[\tau_{13}^{(B)}]_{\text{aux}} \tag{9.120}$$

Referring to relation (9.32a), we have

$$M_2 = -E_A(I_2)_{\text{aux}}\frac{d^2(u_3)_{\text{aux}}}{dx_1^2} \tag{9.121}$$

Differentiating twice relation (9.121) and using (8.23), we obtain

$$\frac{d^2}{dx_1^2}\left[E_A(I_2)_{\text{aux}}\left(\frac{d^2(u_3)_{\text{aux}}}{dx_1^2}\right)\right] = p_3 + \frac{dm_2}{dx_1} \tag{9.122}$$

Comparing relation (9.100) with (9.121) and (9.101) with (9.122) and taking into account relation (9.113), we see that

$$u_3(x_1) = (u_3)_{\text{aux}} \tag{9.123}$$

Consider a prismatic beam, consisting of two or more layers of different material, whose cross section has an axis of symmetry. We choose as the x_3 axis the axis of symmetry of the cross section of the left end surface of the beam and as the x_2 axis the

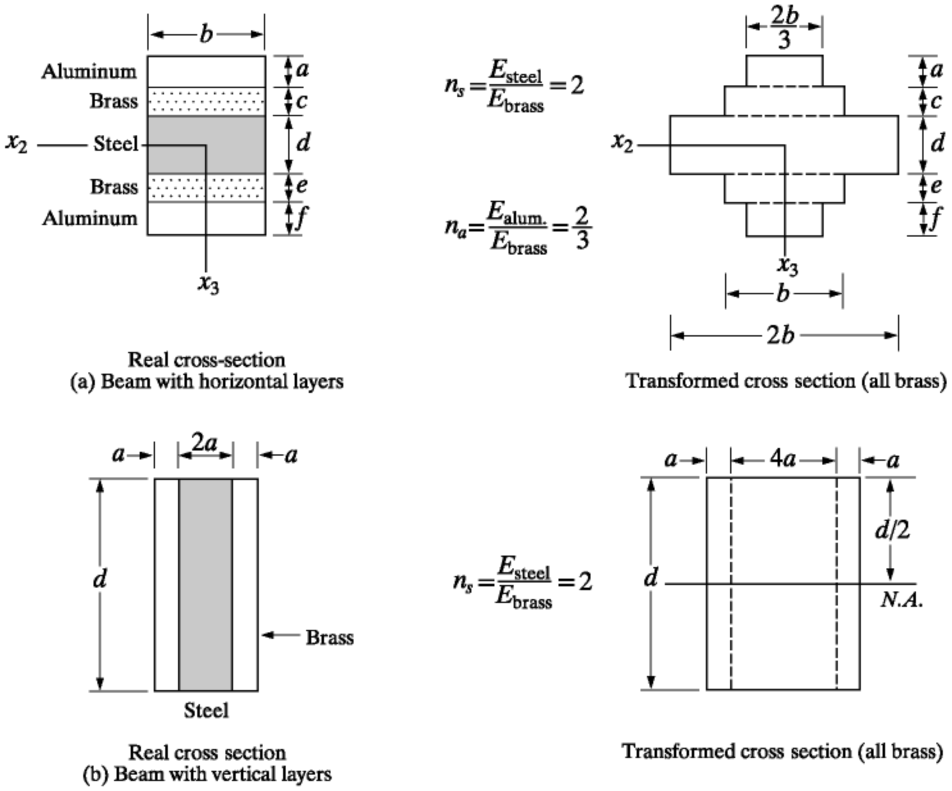


Figure 9.27 Real and transformed cross sections of a composite beam.

neutral axis of this cross section. The beam is subjected to external transverse forces acting in a plane which contains the axis of symmetry of its cross sections and to bending moments whose vector is normal to the x_1x_3 plane. In Fig. 9.27a we show the cross section of a beam consisting of five horizontal layers of different materials, while in Fig. 9.27b we show the cross section of a beam consisting of three vertical layers of different materials. Consider an auxiliary beam of the same length and support conditions and subjected to the same external actions as the real beam. The cross section of this auxiliary beam is called the *transformed cross section* and consists of the same number of layers as that of the real beam (see Fig. 9.27). However, all its layers are made from the same material (say that of layer A) and each layer has a depth equal to that of the corresponding layer of the real beam and a width equal to

$$(b_i)_{\text{aux}} = \left(\frac{E_i}{E_A}\right)(b_i)_{\text{real}} \qquad i = A, B, C, \ldots$$

where b_i is the width of layer i. It is clear that the components of stress acting on the cross sections of the auxiliary beam and its deflection can be computed using relations (9.12b), (9.34b) and (9.67) . On the basis of the foregoing presentation the components of stress acting on the cross sections of the real beam and its deflection can be established from the corresponding quantities of the auxiliary beam on the basis of relations (9.115), (9.120) and (9.123). This method is known as the transformed cross section method and it involves the following steps:

STEP 1 The auxiliary beam is formed and its centroid is located. The x_2 axis of the real beam is chosen to be at the same distance from its top and bottom surfaces as the x_2 centroidal axis of the auxiliary beam (see Fig. 9.27).

STEP 2 The components of stress acting on the cross sections of the auxiliary beam are computed using relations (9.12b) and (9.67). The deflection of the auxiliary beam is computed using relation (9.34b).

STEP 3 The components of stress and the deflection of the real beam are obtained from those of the auxiliary beam on the basis of the following relations

$$[\tau_{11}^{(i)}(x_1, x_3)]_{\text{real}} = n_i[\tau_{11}^{(i)}(x_1, x_3)]_{\text{aux}}$$

$$[\tau_{13}^{(i)}(x_1, x_3)]_{\text{real}} = n_i[\tau_{13}^{(i)}(x_1, x_3)]_{\text{aux}} \qquad i = A, B, C, \dots \tag{9.124}$$

where

$$[u_3(x_1)]_{\text{real}} = [u_3(x_1]_{\text{aux}}$$

$$n_i = \frac{E_i}{E_A} \tag{9.125}$$

The transformed cross section method can be used to analyze beams consisting of horizontal or vertical layers. We apply it in Section 9.9.2.

In what follows we present an example.

Example 15 Consider a 2 m long, simply supported, composite beam whose cross section is shown in Fig. a. The upper part of the beam is made of wood (E_w = 10 GPa) while the lower part is a strap of steel (E_s = 200 GPa). The beam is subjected to a transverse force of 40 kN at the midpoint of its span acting in the direction of the x_3 axis. Compute

(a) The normal component of stress in the wood and the steel
(b) The required strength of the glue
(c) The deflection of the beam

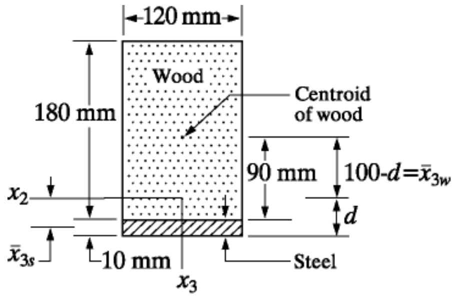


Figure a Geometry of the cross section of the composite beam.

Solution

STEP 1 We locate the x_2 axis of the cross section of the beam. We assume that the x_2 axis lies in the wood at a distance d from the bottom surface of the beam as shown in Fig. a. Referring to Fig. a, we have

$$\iint_{A_w} x_3 \, dA = A_w \bar{x}_{3w} = -(120)(180)(100 - d) = -216(10^4) + 216(10^2)d$$

$$\iint_{A_s} x_3 \, dA = A_s \bar{x}_{3s} = (120)(10)(d - 5) = 1200d - 6000 \tag{a}$$

$$n = \frac{200}{10} = 20 \tag{b}$$

Substituting relations (a) and (b) into (9.97), we get

$$\iint_{A_w} x_3 dA + n \iint_{A_s} x_3 dA = -216(10^4) + 216(10^2)d + 20(1200d - 6000) = 0$$

or

$$d = 50 \text{ mm} \tag{c}$$

Referring to Fig. a, the moments of inertia of the cross section of the wood and of the steel about the x_2 axis are

$$I_w = \frac{180^3(120)}{12} + 180(120)(50^2) = 112.32(10^6) \text{ mm}^4$$

$$I_s = \frac{(120)10^3}{12} + 120(10)(45)^2 = 244(10^4) \text{ mm}^4 \tag{d}$$

Thus,

$$E_w I_w + E_s I_s = 10(112.32)(10^6) + 200(244)(10^4) = 1611.2(10^6) \text{ kN}\cdot\text{mm}^2 \tag{e}$$

Substituting relation (e) into (9.102) and noting that $(M_2)_{max} = 20 \text{ kN} \cdot \text{m}$, we get

$$\tau_{11}^{(w)} = \frac{E_w M_2 x_3}{E_w I_w + E_s I_s} = \frac{10 M_2 x_3}{1611.2(10^6)} = \frac{10(20)(10^3)x_3}{1611.2(10^6)} = 1.24(10^{-4})x_3 \text{ (kN/mm}^2) \tag{f}$$

$$\tau_{11}^{(s)} = \frac{E_s M_2 x_3}{E_w I_w + E_s I_s} = \frac{200 M_2 x_3}{1611.2(10^6)} = \frac{200(20)(10^3)x_3}{1611.2(10^6)} = 2.48(10^{-3})x_3 \text{ (kN/mm}^2) \tag{g}$$

Noting that $(\tau_{11}^{(w)})_{max}$ occurs at $x_3 = -140$ mm, while $(\tau_{11}^{(s)})_{max}$ occurs at $x_3 = 50$ mm, we have

$$(\tau_{11}^{(w)})_{max} = 1.24(10^{-4})(-140) = -1.736(10^{-2})\frac{\text{kN}}{\text{mm}^2} = 17.36 \text{ MPa compression} \tag{h}$$

$$(\tau_{11}^{(s)})_{max} = 2.48(10^{-3})(50) = 1.24(10^{-1})\frac{\text{kN}}{\text{mm}^2} = 124 \text{ MPa tension} \tag{i}$$

The required strength of the glue is equal to the value of the shearing stress $\tau_{13}^{(G)}$ acting on the particles located between the steel and the wood. That is, referring to relation (9.119a) and to Fig. a, noting that Q_3 =20 kN and using relation (e), we have

$$\tau_{13}^{(G)} = \frac{E_s Q_3 \iint_{A_s} x_3 dA}{b(E_w I_w + E_s I_s)} = \frac{200(20)(10)(120)(45)}{(120)1611.2(10^6)} = 1.117(10^{-3})\frac{\text{kN}}{\text{mm}^2} = 1.117 \text{ MPa}$$

In order to obtain the deflection of the statically determinate beam, we compute the moment $M_2(x_1)$ acting on its cross sections. That is,

$$M_2 = 20x_1 \qquad \text{for} \quad 0 \leq x_1 < 1$$

$$M_2 = 20x_1 - 40(x_1 - 1) \qquad \text{for} \quad 1 \leq x_1 \leq 2$$

or

$$M_2 = 20x_1 - 40(x_1 - 1)\,\Delta(x_1 - 1) \tag{j}$$

Substituting relation (j) into (9.100), we get

$$\left(E_s I_s + E_w I_w\right)\frac{d^2 u_3}{dx_1^2} = -20x_1 + 40(x - 1)\Delta(x_1 - 1) \tag{k}$$

Integrating relation (k) twice, we obtain

$$\left(E_s I_s + E_w I_w\right)\frac{du_3}{dx_1} = -10x_1^2 + 20(x_1 - 1)^2\Delta(x_1 - 1) + C_1 \tag{l}$$

$$\left(E_s I_s + E_w I_w\right)u_3 = \frac{-10x_1^3}{3} + \frac{20(x_1 - 1)^3\Delta(x_1 - 1)}{3} + C_1 x_1 + C_2 \tag{m}$$

The constants C_1 and C_2 are evaluated from the essential boundary conditions of the beam. That is,

$$u_3(0) = 0 \qquad\qquad u_3(2) = 0 \tag{n}$$

Substituting relation (m) into relations (n) and solving the resulting equation, we get

$$C_1 = 10 \qquad\qquad C_2 = 0 \tag{o}$$

Substituting the values of the constants (o) into relation (m), we get

$$u_3(x_1) = \frac{1}{1611.2}\left[-\frac{10x_1^3}{3} + 10x_1 + \frac{20(x_1 - 1)^3\Delta(x_1 - 1)}{3}\right] \quad \textbf{(meters)}$$

9.9.1 Sandwich Beams

A sandwich beam consists of two thin layers of the same materials called the *faces* bonded to a thick core (see Fig. 9.28). The faces are usually made from a material of high strength while the core is made from a light material of low strength. Thus, although sandwich beams can be analyzed as described in the previous section, usually their analysis is simplified by assuming that the particles of the faces carry all the normal component of stress, while the particles of the core carry only the shearing component of stress. On the basis of this assumption we have

$$\tau_{11}^F = \frac{M_2 h}{2I_2^F} \tag{9.126a}$$

and

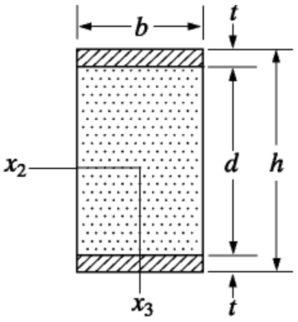


Figure 9.28 Cross section of a sandwich beam.

$$\left(\tau_{13}^{C}\right)_{\max} = \frac{3Q_3}{2bd} \tag{9.126b}$$

where I_2^F is the moment of inertia of the two faces about the x_2 axis. That is,

$$I_2^F = 2\left[\frac{b(h-d)^3}{8(12)} + \frac{b(h-d)}{2}\left(\frac{d}{2} + \frac{h-d}{4}\right)^2\right] = \frac{b}{12}\left(h^3 - d^3\right) \tag{9.127}$$

9.9.2 Reinforced Concrete Beams

Concrete is a material which is very weak in tension but able to carry considerable compression. The maximum allowable stress in compression for concrete could be as much as 100 times larger than that for tension. For this reason, when a concrete beam is subjected to a positive bending moment, it is reinforced by steel rods placed a small distance above its bottom surface. The lower part of the concrete of such a beam cracks and the steel reinforcement carries all the tensile stresses.

The transformed cross section of a reinforced concrete beam is obtained by using only the portion of the cross section of the concrete above the neutral axis and replacing the steel by an equivalent cross section of area nA_S where n is the ratio of the modulus of elasticity of steel to concrete. That is,

$$n = \frac{E_s}{E_c} \tag{9.128}$$

In what follows we present an example.

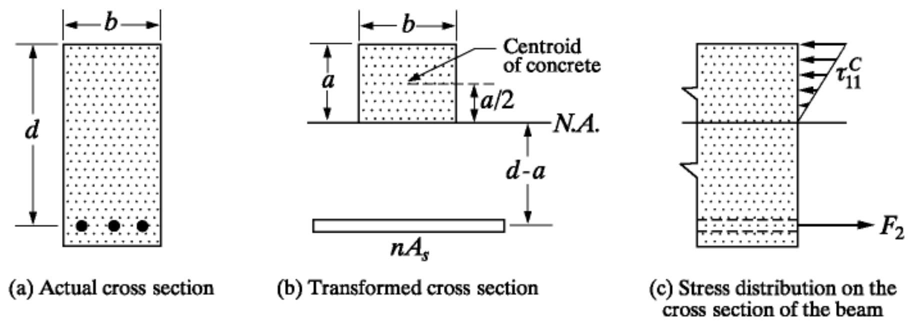


Figure 9.29 Actual and transformed cross sections of a reinforced concrete beam.

Example 16 Consider a 4 m long simply supported concrete beam whose cross section is shown in Fig. a. The modulus of elasticity of concrete is 20 GPa while that of steel is 200 GPa. The beam is subjected to a uniform load of 10 kN/m including its weight. Compute the maximum value of the normal component of stress acting on the particles of the concrete and of the steel.

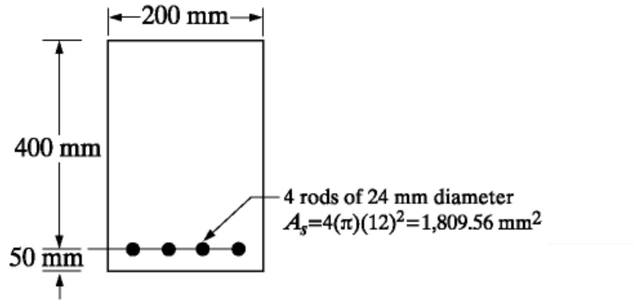


Figure a Geometry of the cross-section of the beam.

Solution For the beam under consideration we have

$$n = \frac{200}{20} = 10$$

The transformed cross section of the beam is shown in Fig. b. The location of the centroidal axis of the transformed section can be established using relation (C.1) of Appendix C. That is,

$$a = \frac{\dfrac{aba}{2} + nA_s d}{ab + nA_s} = \frac{100a^2 + 18{,}095.6(400)}{200a + 18{,}095.6} \tag{a}$$

or

$$a^2 + 180.956a - 72{,}382.4 = 0 \tag{b}$$

Thus,

$$a = \frac{-180.956 \pm \sqrt{(180.956)^2 + 4(72{,}382.4)}}{2} = 193.37\text{ mm}$$

and

$$I_2 = \frac{a^3 b}{12} + ab\left(\frac{a}{2}\right)^2 + nA_S(d-a)^2 = \frac{a^3 b}{3} + nA_n(d-a)^2$$

$$= \frac{(193.37)^3(200)}{3} + 10(1{,}809.56)(206.63)^2 = 1{,}254{,}641{,}138\ \text{mm}^4$$

The maximum moment occurs at $x_2 = L/2$ and it is equal to

$$(M_2)_{\text{max.}} = \frac{p_3 L^2}{8} = \frac{10(4)^2}{8} = 20\ \text{kN·m}$$

The maximum normal components of stress acting on the cross sections of the concrete and the steel rods of the beam are

$$(\tau_{11}^S)_{\substack{\text{real}\\ \text{max}}} = n(\tau_{11}^S)_{\substack{\text{aux}\\ \text{max}}} = \frac{10[20{,}000(400-193.37)]}{1{,}254{,}641{,}138} = 0.03294\,\frac{\text{kN}}{\text{mm}^2} = 32.94\ \text{MPa}$$

$$(\tau_{11}^{C})_{\substack{\text{real}\\ \text{max}}} = (\tau_{11}^{C})_{\substack{\text{aux}\\ \text{max}}} = \frac{(M_2)_{\text{max}}a}{I_2} = \frac{20{,}000(193.37)}{1{,}254{,}641{,}138} = 3.0824(10^{-3})\frac{\text{kN}}{\text{mm}^2} = 3.08\ \text{MPa}$$

The results are shown in Fig. c.

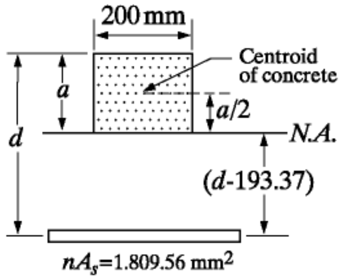


Figure b Transformed cross section of the beam.

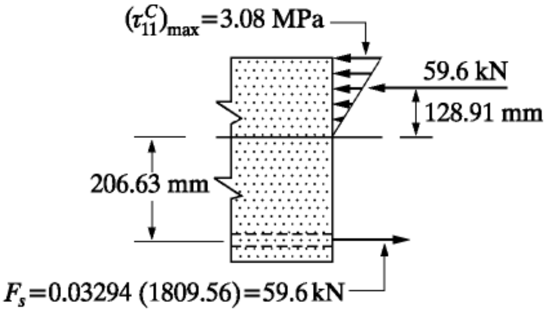


Figure c Results.

9.10 Prismatic Beams on Elastic Foundation

In the previous sections of this chapter we consider prismatic beams supported at discrete locations along their length. In this section we consider prismatic beams resting on an elastic foundation originally at a stress-free, strain-free reference state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, these beams are subjected to external actions along their length and reach a second state of mechanical and thermal equilibrium at the uniform temperature T_o. We assume that the foundation resists the loads transmitted to it by the beams as a linearly elastic body. That is, the traction exerted at any point of a beam by the foundation is related linearly to its deflection at that point. This type of foundation is known as the *Winkler foundation*. It represents an idealization which approximates closely many cases encountered in practice, provided that the deflection of the beam is not large. Examples of such cases are railroad tracks and concrete footings. One may think of the Winkler foundation as consisting of a number of linear springs of constant stiffness k attached next to each other along the entire length of a beam. The stiffness k also known as the *modulus of the foundation* is given in units of force per unit length of the beam, per unit deflection. In order to simplify our presentation we limit our attention to beams whose cross sections have an axis of symmetry which is normal to the plane of the foundation. Moreover, we assume that the vector of the external moments acting on these beams is normal to the plane specified by their axis and the axis of symmetry of their cross sections, while the line of action of the external forces acting on these beams lie in this plane and is parallel to the axis of symmetry of its cross sections. The force exerted by the foundation per unit length of the beam is equal to $-ku_3$. Thus the total load on the beam is equal to

$$\text{load on beam} = \text{given load} + \text{force exerted by the foundation} = p_3(x_1) - ku_3 \tag{9.129}$$

Using relation (9.129) for prismatic beams resting on an elastic foundation the displacement equation of equilibrium (9.34b) becomes

$$\frac{d^4u_3}{dx_1^4} + 4\beta^4 u_3 = \frac{p_3}{EI_2} \tag{9.130}$$

where

$$\beta = \sqrt[4]{\frac{k}{4EI_2}} \tag{9.131}$$

The solution of equation (9.130) may be expressed as the sum of the solution $u_3^{(h)}$ of its homogeneous part and a particular solution $u_3^{(P)}$. That is,

$$u_3(x_1) = u_3^{(h)} + u_3^{(P)} = e^{\beta x_1}(C_1 \cos \beta x_1 + C_2 \sin \beta x_1) + e^{-\beta x_1}(C_3 \cos \beta x_1 + C_4 \sin \beta x_1) + u_3^{(P)} \tag{9.132}$$

The particular solution $u_3^{(P)}$ of equation (9.132) is obtained from the given loading of the beam and is added to $u_3^{(h)}$ to give an expression for the deflection of the beam, involving the constants C_i (i = 1, 2, 3, 4) which are evaluated by requiring that the solution (9.132) satisfies the boundary conditions of the beam. This is done easily for infinitely long beams by taking into account that their deflection and their rotation at infinity must remain bounded. Consequently, the constants C_1 and C_2 must vanish. However, for beams of finite length the evaluation of the constants requires very lengthy calculations†. In the next section we apply the method of finite differences to obtain approximate solutions of the boundary value problem for computing the deflection of a prismatic beam of finite length resting on an elastic foundation.

9.10.1 Computation of the Deflection and the Internal Actions of Beams of Finite Length on Elastic Foundation Using the Method of Finite Differences

The analytical solution of practical problems involving beams of finite length on elastic foundation, although conceptually straightforward, it is very time consuming because it involves lengthy algebraic calculations. For this reason such problems are often solved using a numerical method. In this section we present an example of obtaining numerical solutions for a beam of finite length, using the method of finite differences discribed in Section D.3 of Appendix D.

Example 17 Determine the deflection and the internal moments of a prismatic beam of length L = 8 m, moment of inertia I_2 = 240(10^6) mm^4, fixed at its one end, simply supported at its other end and resting on an elastic foundation of modulus k = 24 MPa. The beam is made from an isotropic, linearly elastic material and is subjected to a uniformly distributed force of 20 kN/m.

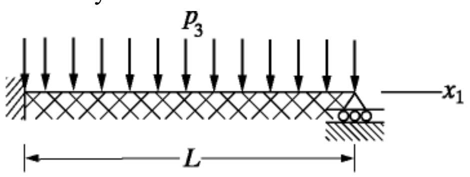


Figure a Geometry and loading of the beam.

Solution The boundary conditions for the beam of Fig. a are

†For a detailed presentation of solutions to problems of beams of finite length on elastic foundation see
Hetényi, M., *Beams on Elastic Foundations,* McGraw-Hill, New York, 1960.
Ting, B.Y., Finite beams on elastic foundation with restraints, *Journal of Structural Division Proceedings of ASCE,* 108, No. ST3, March 1982, p. 611-621.
Faupel, J.H., Fisher F.E., *Engineering Design,* 2nd edition, John Wiley & Sons, New York, 1981, Chapter 3.

$$u_3(0) = 0 \qquad \left.\frac{du_3}{dx_1}\right|_{x_1 = 0} = 0$$
$$u_3(L) = 0 \qquad \left.\frac{d^2u_3}{dx_1^2}\right|_{x_1 = L} = 0 \tag{a}$$

STEP 1 As shown in Fig. b we subdivide the length of the beam into eight equal intervals of length $h = L/8$ by nine pivotal points (k = 0, 1, 2, 3, ..., 8).

STEP 2 We replace the derivatives appearing in the differential equation (9.130) by their central difference approximation. That is, referring to Fig. D.3 of Appendix D, for the k^{th} pivotal point, we have

$$u_{k-2} - 4u_{k-1} + (6 + 4\beta^4h^4)u_k - 4u_{k+1} + u_{k+2} = \left(\frac{p_3h^4}{EI_2}\right)_k \tag{b}$$

When the beam is subdivided into eight equal intervals ($h = L/8$), from relation (9.131), we obtain

$$4\beta^4h^4 = \frac{kh^4}{EI_2} = \frac{kL^4}{8^4EI_2} = \frac{24(10^{-3})8^4(10^{12})}{8^4(200)(240)10^6} = 0.5 \tag{c}$$

We apply relation (b) to each pivotal point of the beam. Referring to Fig. b, we have

$$\begin{aligned}
u_{-2} - 4u_{-1} + (6 + 4\beta^4h^4)u_0 - 4u_1 + u_2 &= \left(\frac{p_3h^4}{EI_2}\right)_0 \\
u_{-1} - 4u_0 + (6 + 4\beta^4h^4)u_1 - 4u_2 + u_3 &= \left(\frac{p_3h^4}{EI_2}\right)_1 \\
u_0 - 4u_1 + (6 + 4\beta^4h^4)u_2 - 4u_3 + u_4 &= \left(\frac{p_3h^4}{EI_2}\right)_2 \\
u_1 - 4u_2 + (6 + 4\beta^4h^4)u_3 - 4u_4 + u_5 &= \left(\frac{p_3h^4}{EI_2}\right)_3 \\
u_2 - 4u_3 + (6 + 4\beta^4h^4)u_4 - 4u_5 + u_6 &= \left(\frac{p_3h^4}{EI_2}\right)_4 \\
u_3 - 4u_4 + (6 + 4\beta^4h^4)u_5 - 4u_6 + u_7 &= \left(\frac{p_3h^4}{EI_2}\right)_5 \\
u_4 - 4u_5 + (6 + 4\beta^4h^4)u_6 - 4u_7 + u_8 &= \left(\frac{p_3h^4}{EI_2}\right)_6
\end{aligned} \tag{d}$$

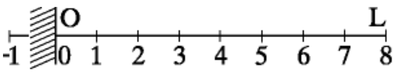


Figure b Beam subdivided into eight equal intervals.

$$
\begin{aligned}
&u_5 - 4u_6 + (6 + 4\beta^4 h^4)u_7 - 4u_8 + u_9 = \left(\frac{p_3 h^4}{EI_2}\right)_7 \\
&u_6 - 4u_7 + (6 + 4\beta^4 h^4)u_8 - 4u_9 + u_{10} = \left(\frac{p_3 h^4}{EI_2}\right)_8
\end{aligned}
\tag{d}
$$

STEP 3 We replace the derivatives appearing in the boundary conditions of the beam (a) by their central difference approximations [see Fig. D.3 of Appendix D]. That is,

$$
\begin{aligned}
&u_0 = 0 \\
&-u_{-1} + u_1 = 0 \qquad \text{or} \qquad u_{-1} = u_1 \\
&u_8 = 0 \\
&u_7 - 2u_8 + u_9 = 0 \qquad \text{or} \qquad u_9 = -u_7
\end{aligned}
\tag{e}
$$

STEP 4 We use the relations obtained in step 3 to eliminate u_{-1} and u_9 from relations (d). Using relation (c), and disregarding the first and last of equations (d), we get

$$
\begin{bmatrix}
7.5 & -4 & 1 & 0 & 0 & 0 & 0 \\
-4 & 6.5 & -4 & 1 & 0 & 0 & 0 \\
1 & -4 & 6.5 & -4 & 1 & 0 & 0 \\
0 & 1 & -4 & 6.5 & -4 & 1 & 0 \\
0 & 0 & 1 & -4 & 6.5 & -4 & 1 \\
0 & 0 & 0 & 1 & -4 & 6.5 & -4 \\
0 & 0 & 0 & 0 & 1 & -4 & 5.5
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix}
=
\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}
\left\{\frac{p_3 h^4}{EI_2}\right\}
\tag{f}
$$

where p_3 is in kN/m. E is in kN/m^2 I_2 is in m^4 and u_i is in meters. From relation (f), we obtain

$$
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{Bmatrix}
=
\begin{Bmatrix} 0.6047 \\ 1.3523 \\ 1.87240 \\ 2.1248 \\ 2.1230 \\ 1.8242 \\ 1.1235 \end{Bmatrix}
\left\{\frac{p_3}{EI_2}\right\}
\tag{g}
$$

Referring to relation (9.32a) the moment acting on the cross sections of the beam is given as

$$\frac{M_2}{EI_2} = -\frac{d^2u_3}{dx_1^2} \tag{h}$$

We replace the derivatives appearing in relation (h) by their central difference approximations [see Fig. D.3 of Appendix D]. That is,

$$\left(\frac{M_2}{EI_2}\right)_k = -\frac{1}{h^2}(u_{k-1} - 2u_k + u_{k+1}) \tag{i}$$

We obtain the values of the moment by substituting the values the deflection given by relation (g) into relation (i) and taking into account relations (e). For example,

$$\begin{aligned}\left(\frac{M_2}{EI_2}\right)_0 &= -\frac{1}{h^2}(u_{-1} - 2u_0 + u_1) = -\frac{2u_1}{h^2} = -\frac{1.210p_3}{EI_2}\\ \left(\frac{M_2}{EI_2}\right)_4 &= -\frac{1}{h^2}(u_3 - 2u_4 + u_5) = \frac{0.255p_3}{EI_2}\end{aligned} \tag{j}$$

Table a Results.

x_1 (m)	*Deflection* x EI_2/p_3		
	8 Equal Spaces	16 Equal Spaces	32 Equal Spaces
1	0.6047	0.5137	0.4912
2	1.3523	1.2825	1.2715
3	1.8740	1.8513	1.8594
4	2.1248	2.1371	2.1587
5	2.1230	2.1490	2.1747
6	1.8242	1.8443	1.8650
7	1.1235	1.1259	1.1353

9.11 Effect of Restraining the Warping of One Cross Section of a Prismatic Member Subjected to Torsional Moments at Its Ends

Our presentation in Chapter 6 has been restricted to prismatic members subjected to equal and opposite torsional moments at their ends in a way that all their cross sections are free to warp. We have found that the length of the longitudinal fibers of such

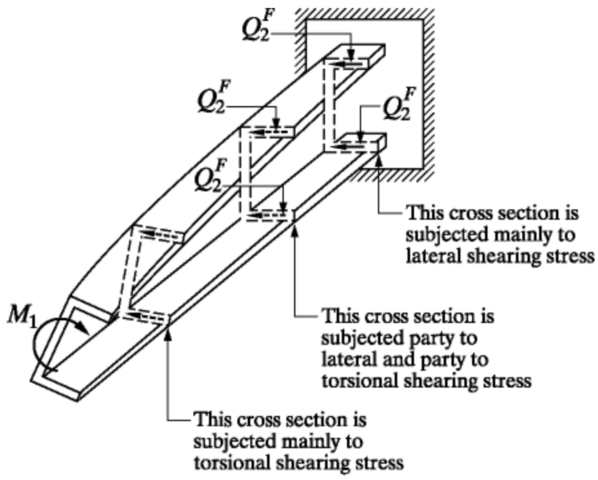


Figure 9.30 Distribution of shearing stress on the cross section of a cantilever channel subjected to a torsional moment at its free end.

members does not change. Consequently, the normal component of stress τ_{11} acting on their cross sections is zero. Moreover, plane sections normal to the axis of such members do not remain plane after deformation; they warp. Only plane sections normal to the axis of prismatic members of circular (solid or hollow) cross sections and of non-circular thin-walled multiply connected cross sections of certain geometry† remain plane subsequent to deformation; that is, they do not warp. Furthermore, a material straight line located prior to deformation on a plane normal to the axis of such members does not always remain straight subsequent to deformation. It can become a curve whose projection on a plane normal to the axis of the member is a straight line which, subsequent to deformation, rotated by an angle θ_1 about an axis normal to the cross sections of the member and passing through their center of twist. Every line of a plane normal to the axis of a member prior to deformation rotates by the same angle θ_1 about this axis subsequent to deformation.

In practice, however, one or more cross sections of a member subjected to torsional moments are usually restrained from warping (see Fig. 9.30). Moreover, when a member is subjected to torsional moments which vary along its axis, the warping of its cross sections also varies along its axis. Consequently, the cross sections of such a member are not free to warp because they are restrained by the adjacent cross sections which warp differently. Due to the restraint of the warping of one or more cross sections of a member the length of its longitudinal fibers changes. Consequently, a normal component of stress must act on particles located on cross sections at or near those which are restrained from warping. For members having thin-walled open cross sections this component of stress is large and must be taken into account. Moreover, the effect of restraining the warping of a cross section of such members on the angle of rotation of their cross sections about their center of twist may not be negligible. For members having other types of cross sections the effect of restraining the warping of one of their cross sections on the values of the normal component of stress and on the angle of rotation about their center of twist is small and it is neglected. For example, the effect of restraining the warping of a cross section of members of thin-walled hollow tubular cross sections is discussed† by Von Karman and Chien (1946) and by Smith *et al.,* (1970) who concluded that this effect is small.

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† See Von Karman, T., Chien, W. Z., Torsion with variable twist, *Journal of Aerosol Science,* 13(10), 1946 p. 503-510.
Smith, F.A., Thomas, F.M., Smith, J.O., Torsion analysis of wavy box beams in structures, *Journal Structural Division of ASCE,* 96(553), 1970, p. 613-635.

In this section we investigate the effect on the components of stress and the angle of twist of restraining the warping of a cross section of members having thin-walled open cross sections.

Consider a cantilever channel subjected at its unsupported end to a distribution of shearing traction which is statically equivalent to a torsional moment M_1 (see Fig. 9.30). The cross section of this member adjacent to its fixed support is completely prevented from warping. This is accomplished by a distribution of normal components of stress τ_{11} acting on the flanges of the member whose magnitude may not be negligible. Moreover, since the cross section adjacent to the fixed support cannot twist, the torsional moment is resisted by a distribution of the transverse shearing components of stress τ_{12} acting on the flanges of the member. The resultant of these shearing components of stress is denoted in Fig. 9.30 by Q_2^F. The warping of cross sections of the member close to its unsupported end is not restrained. The normal component of stress acting on these cross sections is negligible. At any other cross section of the member the value of the normal component of stress τ_{11} acting on its flanges is somewhere between the value of zero at the unsupported end and the maximum value at the fixed end of the member. Consequently, it is a function of the axial coordinate x_1. Moreover, the amount of warping of any cross section is between zero at the fixed end and a maximum at the unsupported end of the member. Furthermore, the value of the shearing stress acting on a particle of the flanges of the member is the algebraic sum of the shearing stress due to bending of the flanges and the shearing stress due to twisting. The stress distribution acting on any cross section of a member is statically equivalent to the torsional moment M_1. That is, the distribution of the normal component of stress τ_{11} is statically equivalent to zero resultant force and moment, while the distribution of the shearing component of stress due to bending of the flanges is statically equivalent to zero resultant force and to a resultant moment $M_1^B \mathbf{i}_1$. According to the principle of Saint Venant, the effect of such statically equivalent to zero stress distribution should decay rapidly at cross sections away from the fixed end of the member. However, for members having thin-walled open cross sections, the stress distribution produced by restraining the warping of a cross section not only could be large but diminishes rather slowly at cross sections away from their restrained cross section.

In what follows, we use the theories of mechanics of materials presented in Chapters 8 and 9, to establish the normal component of stress acting on the cross sections as well as the angle of twist per unit length of the cross sections of cantilever members having a wide flange cross section when they are subjected to a torsional moment at their unsupported end.

Example 18 Consider the cantilever beam of constant wide flange cross section shown in Fig. a. The beam is subjected to a torsional moment M_1^* at its unsupported end as well as to a constant distributed torsional moment m_1 given in units of moment per unit of length

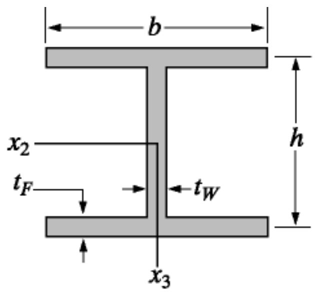


Figure a Cross section of the beam.

Solution The center of twist of each cross section of this beam coincides with its centroid. Thus, the centroids of its cross sections do not translate and, consequently, the web of the beam is not subjected to bending.

At a small distance from the fixed support of the beam, the torsional moment $M_1(x_1)$ acting on its cross sections may be considered as made up of two parts. The one part $M_1^B(x_1)$ is the moment which bends the flanges of the beam. It is equal to the moment of the resultant forces of the shearing stresses acting on the cross sections of the flanges of the beam. The resultant force of the shearing stresses acting on the cross section of the top flange is equal and opposite to that of the bottom flange. In Fig. b we denote these resultant forces by $Q_2^F(x_1)$; referring to this figure, we have

$$M_1^B = -Q_2^F h \tag{a}$$

The second part, $M_1^T(x_1)$, of the torsional moment is the moment which twists the cross sections of the beam and may be approximated by relation (8.69). That is,

$$M_1^T = R_C G \frac{d\theta_1}{dx_1} = R_C G \alpha \tag{b}$$

Where the angle of twist per unit length α Iis a function of x_1. The torsional constant R_C is given by relation (6.100) as

$$R_C = \frac{2t_F^3 b + t_W^3 h}{3} \tag{c}$$

From relations (a) and (b), we find that the total torsional moment $M_1(x_1)$ acting on a cross section of the beam is equal to

$$M_1(x_1) = M_1^B + M_1^T = -Q_2^F h + R_C G \frac{d\theta_1}{dx_1} \tag{d}$$

In order to determine the shearing force $Q_2^F(x_1)$, we consider the bending of the flanges. We assume that each flange bends in its own plane as a cantilever beam. Thus, referring to relations (8.20) and (9.32b) for the bottom flange of the beam, we have

$$Q_2^F(x_1) = -\frac{dM_3^F}{dx_1} = -EI_3^F \frac{d^3 u_2^F}{dx_1^3} \tag{e}$$

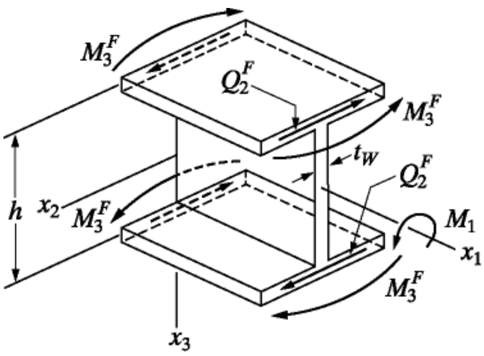


Figure b Internal actions acting on the cross sections of the beam as a result of restraining the warping of a cross section.

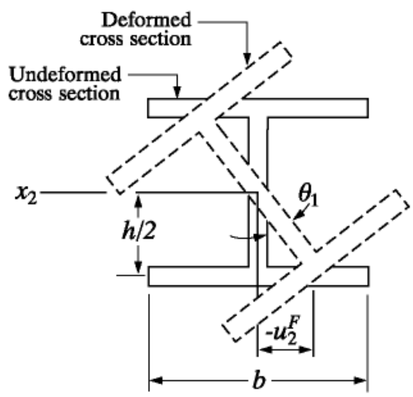


Figure c Deformed configuration of a typical cross section of the beam.

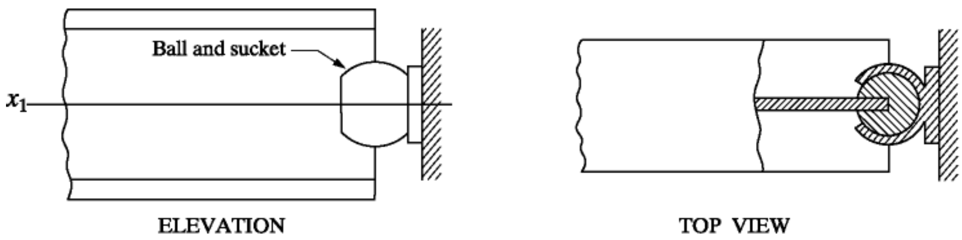


Figure d Detail of simply supported end of a beam.

where I_3^F is the moment of inertia of the cross section of the bottom flange about the x_3 axis and $u_2^F(x_1)$ is the component of translation of the centroid of the bottom flange in the x_2 direction. Referring to Fig. c, this component of translation is related to the angle of twist $\theta_1(x_1)$ by the following relation:

$$u_2^F = -\frac{h\theta_1}{2} \tag{f}$$

Referring to relation (9.27b) and using relation (f), the rotation of the bottom flange of the beam about the x_3 axis is equal to

$$\theta_3^F = \frac{du_2^F}{dx_1} = -\frac{h}{2}\frac{d\theta_1}{dx_1} \tag{g}$$

Substituting relation (f) into (e), we get

$$Q_2^F(x_1) = \frac{EI_3^F h}{2}\frac{d^3\theta_1}{dx_1^3} \tag{h}$$

Moreover, substituting relation (h) into (d), we obtain

$$M_1(x_1) = -E\Gamma\frac{d^3\theta_1}{dx_1^3} + R_C G\frac{d\theta_1}{dx_1} \tag{i}$$

where

$$\Gamma = \frac{I_3^F h^2}{2} \tag{j}$$

Referring to relation (9.32b) and using relations (f), (j) and (8.69), the bending moment M_3^F acting on the cross sections of each flange is equal to

$$M_3^F = EI_3^F\frac{d^2u_2^F}{dx_1^2} = -\frac{E}{h}\frac{\Gamma d^2\theta_1}{dx_1^2} = -\frac{E\Gamma}{h}\frac{d\alpha}{dx_1} \tag{k}$$

Referring to relations (9.12b) and (k) we see that the normal component of stress acting on the cross sections of the bottom flange is equal to

$$\tau_{11}^{F} = -\frac{M_3^F x_2}{I_3^F} = \frac{hEx_2}{2}\frac{d^2\theta_1}{dx_1^2} \tag{l}$$

Differentiating relation (i) with respect to x_1 and using relation (8.19), we have

$$\frac{d^4\theta_1}{dx_1^4} - k^2\frac{d^2\theta_1}{dx_1^2} = \frac{m_1}{E\Gamma} \tag{m}$$

where

$$k^2 = \frac{R_C\, G}{E\Gamma} \tag{n}$$

The general solution of equation (m) has the following form:

$$\theta_1(x_1) = A\cos h(kx_1) + B\sin h(kx_1) + Cx_1 + D + \begin{matrix}\text{particular}\\ \text{solution}\end{matrix} \tag{o}$$

For the beam of Fig. a, subjected to m_1 = constant, we have

$$\theta_1(x_1) = A\cos h(kx_1) + B\sin h(kx_1) + Cx_1 + D + \frac{m_1 x_1^2}{2k^2 E\Gamma} \tag{p}$$

The constants A, B, C and D are evaluated from the boundary conditions of the beam. The following three types of support conditions for beams subjected to torsional moments are often encountered in practice.

1. Fixed end

The angle of twist θ_1 and the rotation θ_3^F of the supported end must vanish. That is, referring to relation (g), the boundary conditions are

$$\theta_1 = 0 \qquad\qquad \frac{d\theta_1}{dx_1} = 0 \tag{q}$$

2. Unsupported end

3. Ball and socket (simply) supported end (see Fig. d)

The unsupported end and the simply supported end of beams are not subjected to normal components of stress and are free to rotate. When such ends are subjected to a torsional moment M_1^*, referring to relations (l) and (i), the boundary conditions are

$$\frac{d^2\theta_1}{dx_1^2} = 0 \qquad\qquad E\Gamma\frac{d^3\theta_1}{dx_1^3} - R_cG\frac{d\theta_1}{dx_1} = -M_1^* \tag{r}$$

On the basis of the foregoing, the boundary conditions for a cantilever beam subjected to a uniformly distributed torsional moment along its length and to a concentrated torsional moment M_1^* at its unsupported end, referring to relations (q) and (r), are

$$\theta_1(0) = 0 \qquad\qquad \left.\frac{d\theta_1}{dx_1}\right|_{x_1=0} = 0$$

$$\left.\frac{d^2\theta_1}{dx_1^2}\right|_{x_1=L} = 0 \qquad \left[E\Gamma\frac{d^3\theta_1}{dx_1^3} - R_c G\frac{d\theta_1}{dx_1}\right]_{x_1=L} = -M_1^*$$

Substituting the solution (p) into the above relations, we obtain

$$A + D = 0 \tag{t}$$

$$Bk + C = 0 \tag{u}$$

$$Ak^2 \cos h(kL) + Bk^2 \sin h(kL) + \frac{m_1}{k^2 E\Gamma} = 0 \tag{v}$$

$$(E\Gamma k^3 - R_c Gk)\big[(A \sin h(kL) + B \cos h(kL)\big] - R_c GC - \frac{m_1 L R_c G}{k^2 E\Gamma} = -M_1^* \tag{w}$$

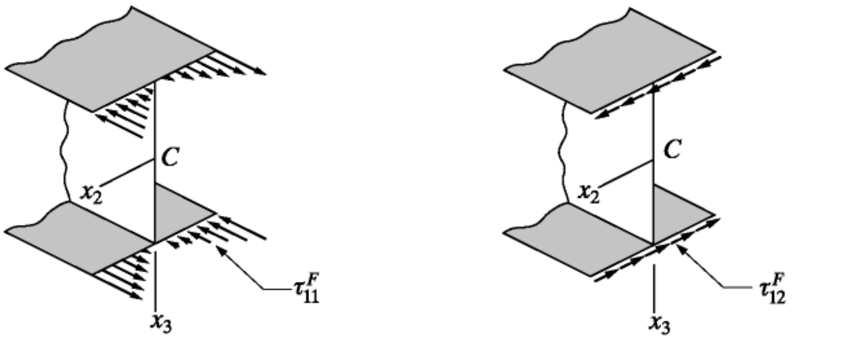
Using relations (t) to (w) and noting that $E\Gamma k^2 - R_c G = 0$ [see relation (n)], we obtain

$$A = -\frac{m_1}{k^4 E\Gamma \cos h(kL)} - B \tan h(kL) \qquad B = -\frac{M_1^* - m_1 L}{R_c Gk} \qquad C = -Bk \qquad D = -A \tag{x}$$

Substituting the values of the constants (x) into relation (p), we get the angle of twist $\theta_1(x_1)$. The angle of twist per unit length α (x_1) is equal to

$$\begin{aligned}\alpha(x_1) &= \frac{d\theta}{dx_1} = Ak \sin h(kx_1) + Bk \cos h(kx_1) + C + \frac{m_1 x_1}{k^2 E\Gamma}\\ &= kB\left[\frac{\cos h[k(L-x_1)]}{\cos h(kL)} - 1\right] + \frac{m_1 x_1}{k^2 E\Gamma} - \frac{m_1 \sin h(kx_1)}{k^3 E\Gamma \cos h(kL)}\\ &= \frac{M_1^*}{R_c G}\left[1 - \frac{\cos h[k(L-x_1)]}{\cos h(kL)}\right] - \frac{m_1 L}{R_c G}\left[1 - \frac{x_1}{L} - \frac{kL \cos h[k(L-x_1)] - \sin h(kx_1)}{kL \cos h(kL)}\right]\end{aligned} \tag{y}$$

Cantilever beam subjected only to a torsional moment at its unsupported end



(a) Normal component of stress τ_{11}^F (b) Shearing component of stress τ_{11}^F

Figure e Components of stress acting on cross sections of the beam in the vicinity of its fixed end.

From relation (y), we get

$$\alpha(x_1) = \frac{M_1^*}{R_c G}\left[1 - \frac{\cosh[k(L-x_1)]}{\cosh(kL)}\right] \tag{z}$$

Substituting relation (z) into (b) and (k), we get

$$\begin{aligned} M_1^T(x_1) &= M_1^*\left[1 - \frac{\cosh[k(L-x_1)]}{\cosh(kL)}\right] \\ M_3^F(x_1) &= -\frac{M_1^*}{kh}\left[\frac{\sinh[k(L-x_1)]}{\cosh(kL)}\right] \end{aligned} \tag{za}$$

Thus, the maximum bending moment acting on the cross sections of the flanges of the beam under consideration is equal to

$$M_3^F(0) = -\frac{M_1^* \tanh(kL)}{kh} \tag{zc}$$

For $kL > 2.5$, tan h(kL) may be considered approximately equal to unity. This occurs for relatively long beams. For such beams relation (zc) may be approximated as

$$M_3^F(0) \approx -\frac{M_1^*}{kh} \tag{zd}$$

Substituting relation (zd) into (l), noting that the maximum tensile stress in the bottom flange τ_{11}^F occurs at $x_1 = 0$ and $x_2 = b/2$, and using relation (n), we get

$$(\tau_{11}^F)_{max} = \frac{M_1^* b}{2khI_3^F} = \frac{\sqrt{6}\, M_1^*}{2(\sqrt{R_C b t_F})}\sqrt{\frac{E}{G}} = \frac{\sqrt{3(1 + \nu}\, M_1^*}{\sqrt{R_C b t_F}} \tag{ze}$$

For the beam under consderation, substituting relations (zf) and (c) into (ze) and using $h = 2b = 400$ mm, $t_F = 2t_W = 2t = 12$ mm, and $M_1^* = 2$ kN·m and referring to Fig. a, we get

$$(\tau_{11}^F)_{max} = \frac{\sqrt{6}\, M_1^* \sqrt{2(1 + \nu)}}{2\left[\sqrt{\dfrac{(2t_F^3 b + t_W^3 h) b t_F}{3}}\right]} = \frac{\sqrt{2} M_1^*}{\sqrt{6}\, t^2 b} = \frac{\sqrt{2}(2{,}000)}{\sqrt{6}\,(6)^2(200)} = 0.1604\,\frac{\text{kN}}{\text{mm}^2} = 160.4 \text{ MPa} \tag{zf}$$

Referring to relation (a), the shearing force $Q_2^F(x_1)$ acting on the cross section of the bottom flange at $x_1 = 0$ ($M_1^T = 0$ and $M_1^B = M_1^*$) is equal to

$$Q_2^F(0) = -\frac{M_1^B}{h} = -\frac{M_1^*}{h} = -\frac{2{,}000}{400} = -5 \text{ kN} \tag{zg}$$

Referring to relation (b) of Example 7 of Section 9.5, the maximum shearing stress acting on the cross sections of the bottom flange of the beam at the fixed end ($x_1 = 0$) of the beam is equal to

$$(\tau_{12}^F)_{\max} = \frac{3Q_2^F(0)}{2A_F} = \frac{3Q_2^F(0)}{2t_F b} = -\frac{3(5)}{2(12)(200)} = -0.00313\,\frac{\text{kN}}{\text{mm}^2} = -3.13 \text{ MPa} \qquad \text{(zh)}$$

Moreover, referring to relation (6.103) the maximum shearing stress due to torsion acting on cross sections sufficiently removed from the fixed end of the beam is equal to

$$(\tau_{12})_{\max} = \frac{3M_1^* t_F}{\sum_{i=1}^{3} b_i t_i^3} = \frac{3(2{,}000)(12)}{2(200)(12)^3 + 400(6)^3} = 0.0926\,\frac{\text{kN}}{\text{mm}^2} = 92.6 \text{ MPa} \qquad \text{(zi)}$$

From the results of this example we may deduce that when a relatively long cantilever beam is subjected to a torsional moment at its unsupported end, the magnitude of the normal component of stress τ_{11}^F acting on the cross section of its flanges in the vicinity of its fixed end is larger than the magnitude of the maximum shearing stress acting on its cross sections in the vicinity of its unsupported end. However, taking into account that for many materials the allowable stress in shear is less than that in tension or compression, it is probable that the maximum shearing stress is the critical design stress for beams made from such materials. Nevertheless, in the design of beams subjected to torsional moments the normal component of stress acting on their cross sections should not be disregarded.

Comments

Referring to Fig. b it can be seen that in this example the normal component of stress τ_{11}^F and the normal component of strain e_{11}^F of the particles of the flange vanish at $x_2 = 0$. Consequently the normal component of strain e_{11}^W of the particles of the web at its interfaces with the flanges must also vanish. That is, the web is not subjected to normal component of stress. However, this is not the case for channels and for z-members subjected to torsional moments and having a cross section restrained from warping. As shown in Fig. d the cross section of a channel is subjected to a distribution

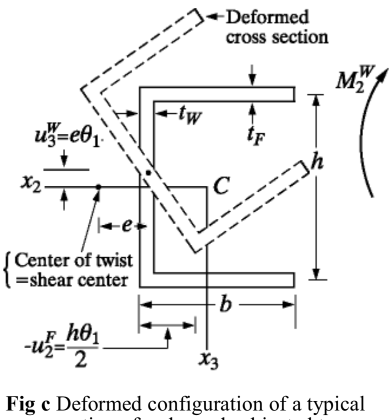


Fig c Deformed configuration of a typical cross section of a channel subjected to a torsional moment at its unsupported length.

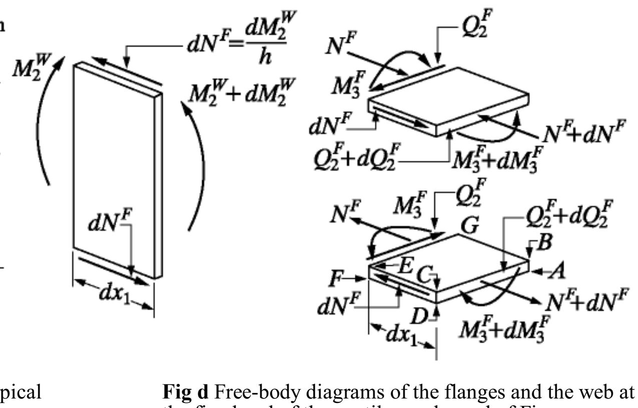


Fig d Free-body diagrams of the flanges and the web at the fixed end of the cantilever channel of Fig c.

of normal component of stress τ_{11}^{W} which is statically equivalent to a moment M_2^W about the x_2 axis. Moreover, as shown in Fig. f the cross section of the flanges of a z-member is subjected to a distribution of normal component of stress τ_{11}^{F} which is statically equivalent to an axial centroidal force and a moment, the magnitude of which can be established from the following compatibility of strains relation:

$$e_{11}^{F}\Big|_{x_2=0} = \frac{1}{E}\left(\frac{N^F}{bt_F} + \frac{M_3^F b}{2I_3^F}\right) = e_{11}^{W} = -\frac{2N^F}{Et_W h} \tag{zj}$$

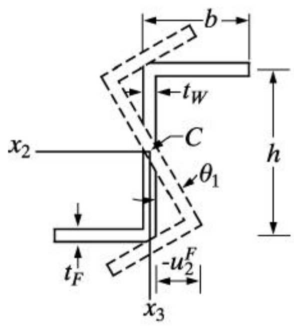


Fig e Deformed configuration of a typical cross section of a z-section subjected to a torsional moment at its unsupported length.

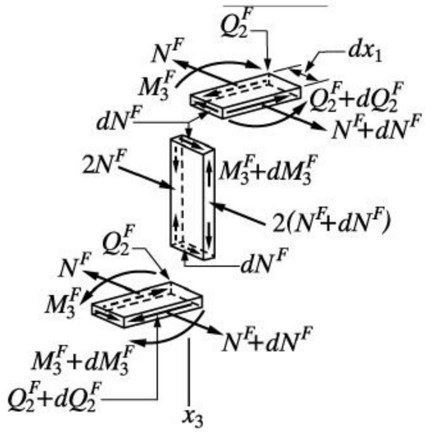


Fig f Free-body diagrams of the flanges and the web at the fixed-end of the cantilever z-section of Fig c.

9.12 Problems

Section 9.1

1. Consider a 2 m long cantilever beam made from an American steel W 460x74 wide flange section. The beam is subjected to a uniformly distributed force of 10 kN/m as shown in Fig. 6P1. Plot the distribution of the normal component of stress on the cross section of this beam at its fixed end and locate the neutral axis of its cross sections.

Ans. $\tau_{11} = 0.21x_2 - 0.11856x_2$ MPa

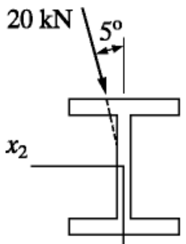


Figure 9P1

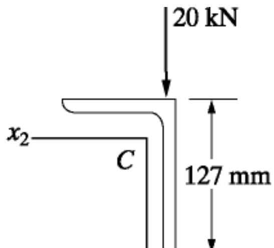


Figure 9P2

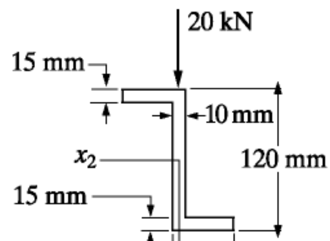


Figure 9P3

2. A 1 m long cantilever beam made from an American rolled steel angle L 127 x 76 x 9.5 is subjected to a concentrated force of 20 kN at its unsupported end as shown in Fig. 9P2. The plane of the loading of the beam contains the shear center of its cross sections. Plot the distribution of the normal component of stress on the cross section of this beam at its fixed end and locate the neutral axis of its cross sections.

Ans. $\tau_{11} = 10^{-6}(10^{-3} - x_1)(7.02x_2 + 1.65x_3)$

3. and **4**. Consider a 2 m long cantilever beam whose cross section is shown in Fig. 9P3 and is subjected to a concentrated force P_3 = 20 kN at its unsupported end. Plot the distribution of the normal component stress acting on the cross section of beam at its fixed end. Repeat with beam whose cross section is shown in Fig. 9P4.

Ans. 3 $\tau_{11} = -(2 - x_1)(10.125x_2 + 7.031x_3)10^{-3}\dfrac{\text{N}}{\text{m}^2}$ $\quad x_1, x_2, x_3$ in m

Ans. 4 $\tau_{11} = (2 - x_1)(3.895x_2 - 5.003x_3)10^{-3}\dfrac{\text{N}}{\text{mm}^2}$ $\quad x_1, x_2, x_3$ in m

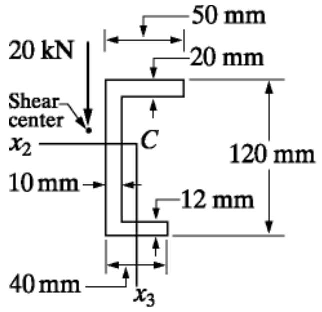


Figure 9P4

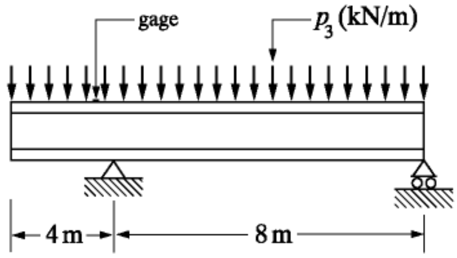


Figure 9P5

5. The strain gage attached to the top of the top flange of the standard W 460 x 113 steel rolled beam shown in Fig. 9P5 measures a strain of e_{11} = 0.0004 when the beam is subjected to the forces shown in this figure. The modulus of elasticity for steel is E = 200 GPa. Determine the magnitude of the uniformly distributed load p_3.

Ans. $p_3 = 23.93\ \dfrac{\text{kN}}{\text{m}}$

6. The strain gage attached to the top of the top flange of the wide flange steel beam shown in Fig. 9P6 measures a strain of $e_{11} = -0.004$ when the beam is subjected to the forces shown in this figure. The modulus of elasticity for steel is E = 200 GPa. Determine the depth d of the beam. Ans. d = 701 mm

7. The strain gage attached to the top of the top flange of the beam shown in Fig. 9P7 measures a strain of e_{11} = 0.0006 when the beam is subjected to the forces shown in this figure. The modulus of elasticity for steel is E = 200 GPa. Determine the magnitude of the uniformly distributed forces p_3 and plot the distribution of the normal component of stress acting on the cross section of the beam at x_1 = 3 m.

Ans. 1.087 kN/m downward

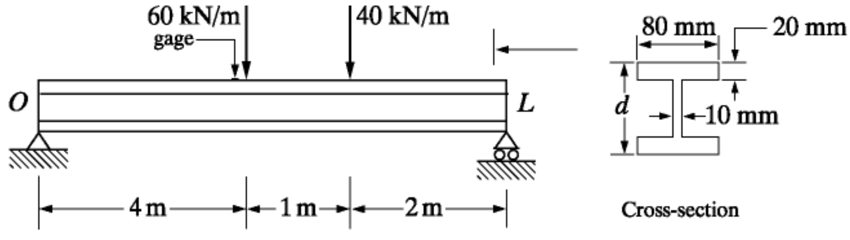


Figure 9P6

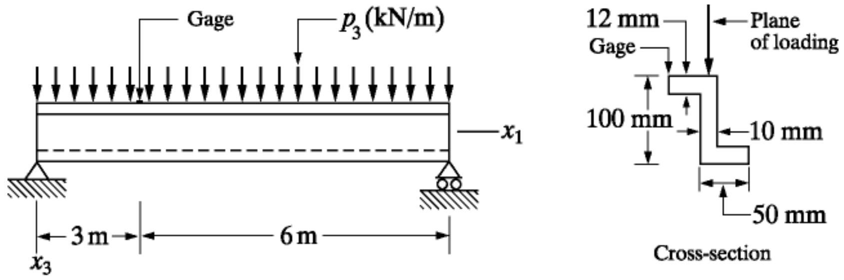


Figure 9P7

Section 9.2

8. Using the classical theory of beams compute the components of translation $u_2(x_1)$ and $u_3(x_1)$ of the cantilever beam of Z cross section subjected to an external bending moment, as shown in Fig. 9P8. Moreover, compute the maximum value of the normal component of stress acting on the cross sections of the beam. The beam is made from an isotropic, linearly elastic material (E = 200 GPa).

Ans. $u_2(x_1) = -58.9(10^{-6})x_1^2,\ u_3(x_1) = -57.9(10^{-6})x_1^2,\ (\tau_{11})_{\max} = 135.4$ MPa

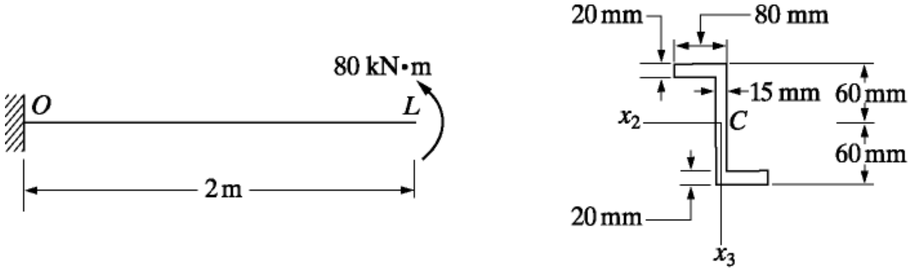


Figure 9P8 cross section

9. to **11**. Using the classical theory of beams, compute the equation of the elastic curve of the beam subjected to bending about the x_2 axis without twisting, due to the external actions shown in Fig. 9P9. The x_2 and x_3 axes are principal centroidal. The beam is made from an isotropic, linearly elastic material of modulus of elasticity is E = 200 GPa. The moment of inertia of the beam about its x_2 axis is $I_2 = 179(10^6)$ mm^4. Do not use functions of discontinuity. Repeat with the beam of Fig. 9P10 to 9P11.

Ans. 9 $$EI_2u_3(x_1) = \frac{P_3^{(1)}(L-a)x_1}{6L}\left(-x_1^2 + 2aL - a^2\right) \qquad 0 \le x_1 \le a$$

$$EI_2u_3(x_1) = \frac{P_3^{(1)}}{6L}\left[(L-a)x_1(-x_1^2 + 2La - a^2) + L(x_1 - a)^3\right] \qquad a \le x_1 \le L$$

Ans. 10 $$EI_2u_3(x_1) = \frac{P_3^L x_1^2(3L - x_1)}{6} + \frac{M_2^{(1)}x_1^2}{2} \qquad 0 \le x_1 \le b$$

$$EI_2u_3(x_1) = \frac{P_3^L x_1^2(3L - x_1)}{6} + \frac{M_2^{(1)}b}{2}(2x_1 - b) \qquad b \le x_1 \le L$$

12. Consider the cantilever beam subjected to the forces shown in Fig. 9P12 and to a temperature difference of 20° C. That is, the temperature of its top surface is 20°C while the temperature of its bottom surface is 0°C. The beam has a constant cross section [I_2 = $200(10^6)$ mm^4 $h^{(3)}$ = 200 mm (see Section 4.7)] and it is made from a homogeneous, isotropic, linearly elastic material (E = 200 GPa, α = 10^{-5}/°C). The x_3 and x_2 axes are principal centroidal of the cross sections of the beam. Moreover, the plane of the external forces is parallel to the x_3 axis and contains the shear center of the cross sections of the beam. Compute the deflection of the beam $u_3(x_1)$ and its internal bending moment $M_2(x_1)$ and shearing force $Q_3(x_1)$.

Ans. 12 $EI_2 u_3(x_1) = -\frac{M_2^L x_1^2}{2} + \frac{p_3 x_1^2 (L-c)}{12 EI_2}(3L + 3c - 2x_1) + \frac{EI_2 x_1^2}{2}(10^{-6}) \qquad 0 \le x_1 \le c$

$$EI_2 u_3(x_1) = -\frac{M_2^L x_1^2}{2} + \frac{p_3}{24}(c^4 - 4c^3 x_1 + 6L^2 x_1^2 - 4L x_1^3 + x_1^4) + \frac{EI_2 x_1^2}{2}(10^{-6}) \qquad c \le x_1 \le L$$

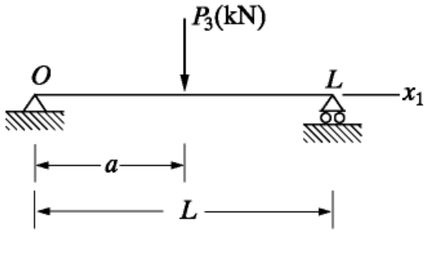


Figure 9P9

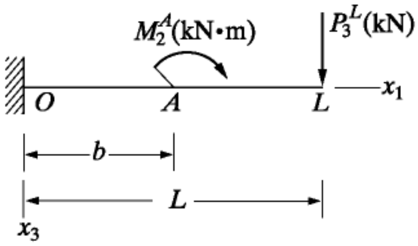


Figure 9P10

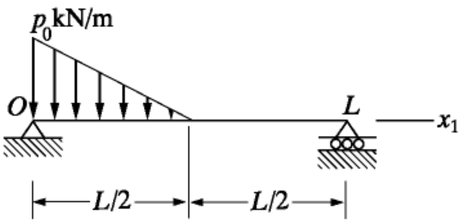


Figure 9P11

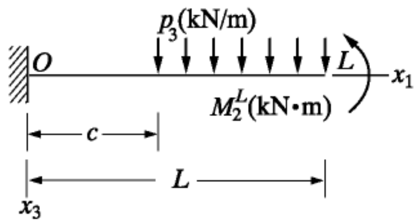


Figure 9P12

13. and **14**. Using the classical theory of beams, compute the components of translation $u_2(x_1)$ and $u_3(x_1)$ of the simply supported beam subjected to the external forces shown in Fig. 9P13. Moreover, compute the maximum value of the normal component of stress acting on any cross sections of the beam. The beam is a standard steel (E = 200 GPa) rolled angle L 64 × 51 × 6.5. Repeat with the beams of Fig. 9P14.

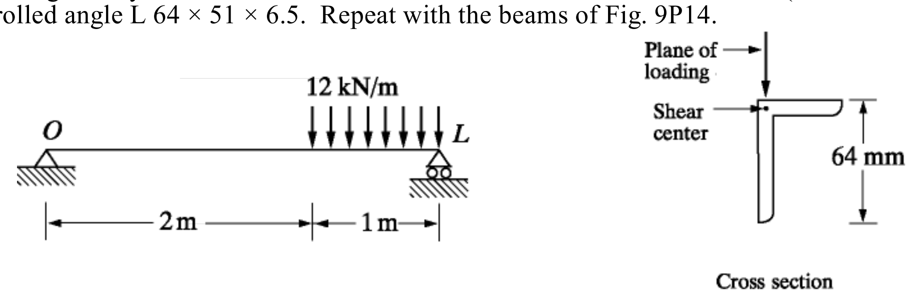
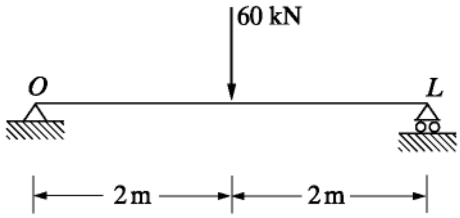


Figure 9P13

Ans. 13
$$\left.\begin{aligned} u_3(x_1) &= \frac{I_{33}(-x_1^3 + 5x_1)}{3E\,(I_{22} + I_{33} - I_{23}^2)} \\ u_2(x_1) &= \frac{I_{23} x_1 (x_1^2 + 1)}{3E\,(I_{22} I_{33} - I_{23}^2)} \end{aligned}\right\} 0 \le x_1 \le 2$$

Ans. 14
$$u_3(x_1) = 8.665\,[60x_1 - 5x_1^3 + 10(x_1 - 2)^3 \Delta(x_1 - 2)]$$
$$u_2(x_1) = 3.4\,[5x_1^3 - 60x_1 - 10(x_1 - 2)^3 \Delta(x_1 - 2)]$$



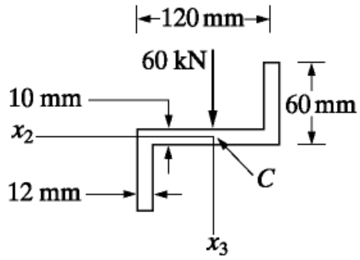
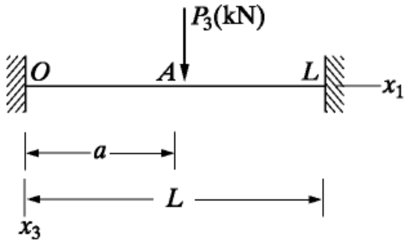


Figure 9P14

Cross section

Section 9.3

15. to **18**. Using the classical theory of beams and functions of discontinuity compute the equation of the elastic curve of the beam subjected to bending about the x_2 axis without twisting due to the external actions shown in Fig. 9P9. The x_2 and x_3 axes are principal centroidal. The beam is made from an isotropic, linearly elastic material of modulus of elasticity $E = 200$ GPa. The moment of inertia of the beam about its x_2 axis is $I_2 = 179$ (10^6) mm^4. Repeat with the beam of Figs. 9P10 to 9P12. Ans. See problems 9 to 12



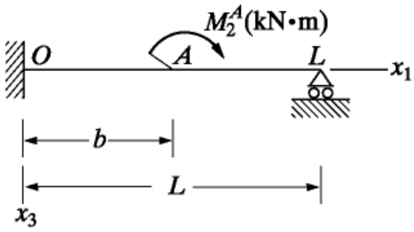


Figure 9P19

Figure 9P20

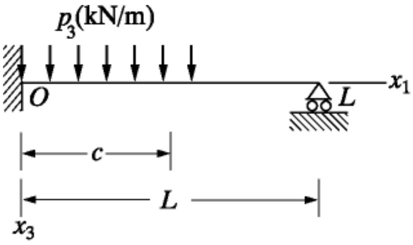
19. to **22**. Using the classical theory of beams and functions of discontinuity, compute the reactions of the prismatic beam subjected to the external forces shown in Fig. 9P19. The x_2 and x_3 axes are principal centroidal. Plot the shear and moment diagrams for the beam. Repeat with the beam of Figs. 9P20 to 9P22.

Ans. 19 $M_2^0 = -\dfrac{P_3 a(L-a)^2}{L^2}$, $M_2^L = -\dfrac{P_3(L-a)a^2}{L^2}$

Ans. 20 $M_2^0 = \dfrac{M_2}{2}\left[1 - \dfrac{3(L-b)^2}{L^2}\right]$

Ans. 21 $M_2^0 = -\dfrac{p_3 c^2}{2}\left(1 - \dfrac{c}{L} + \dfrac{c^2}{4L^2}\right)$

Ans. 22 $R_3^0 = \dfrac{p_3(L-c)^3(L+c)}{2L^3}$, $M_2^0 = \dfrac{p_3(L-c)^3(L+3c)}{12L^2}$



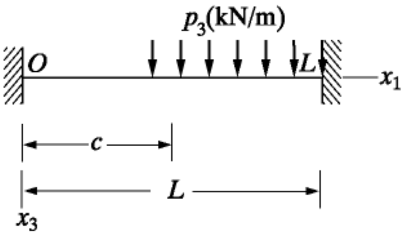


Figure 9P21

Figure 9P22

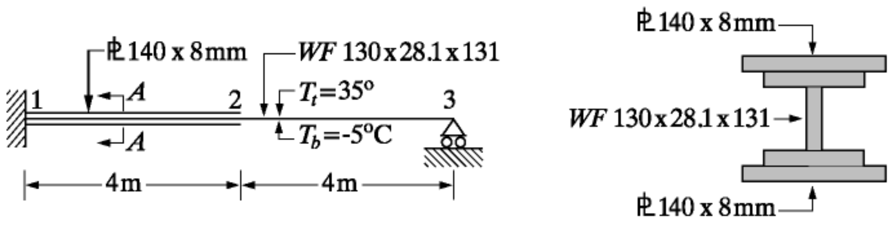


Figure 9P23

23. to **25**. Using the classical theory of beams, compute the reactions of the beam whose cross section is shown in Fig. 9P23 resulting from the loading shown in this figure and plot its shear and moment diagrams. The temperature during construction was $T_o = 20$°C. The beam is made from steel ($E = 200$ GPa, $\alpha = 10^{-5}$/°C). Repeat with the beams of Figs. 9P24 and 9P25.

Ans. 24 $M_2(x_1) = -\dfrac{EI_2 \alpha \delta T_3}{h^{(3)}} = 9.866$ kN·m Ans. 25 $R_3^{(0)} = 1.4375 p_3$, $M_3^{(0)} = -3.497 p_3$

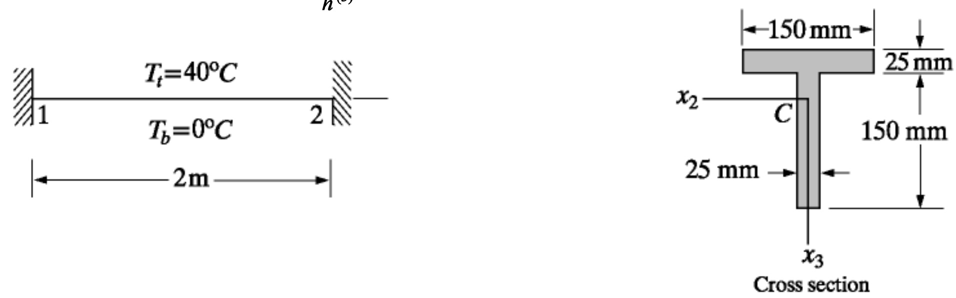


Figure 9P24

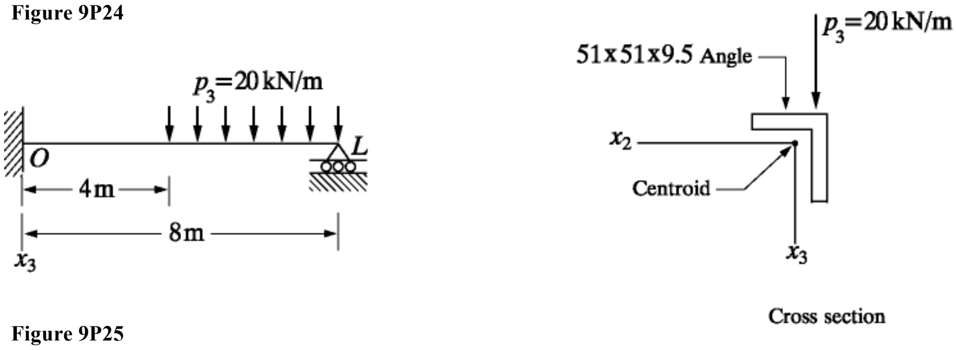


Figure 9P25

Section 9.4

26. to **29**. Using the Timoshenko theory of beams, compute the components of translation $u_3(x_1)$ and of rotation $\theta_2(x_1)$ of the beam of rectangular cross section (depth = 200 mm, width = 120 mm) subjected to the external actions shown in Fig. 9P26. The beam is made from an isotropic, linearly elastic material ($E = 200$ GPa, $G = 75$GPa). The x_2 and x_3 axes are principal centroidal axes [$I_2 = 80(10^6)$ mm^4]. Repeat with the beams of Figs. 9P27 to 9P29.

Ans. 26 $u_3(x_1) = 0.02(10^{-6})x_1 + 0.745(10^{-3})x_1^2 - 0.03125(10^{-3})x_1^4$ (m)

$\theta_2(x_1) = 0.125(10^{-10})x_1^3 - 1.5(10^{-10})x_1$

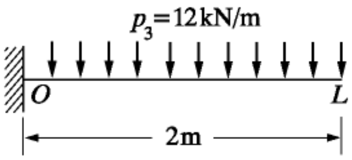


Figure 9P26

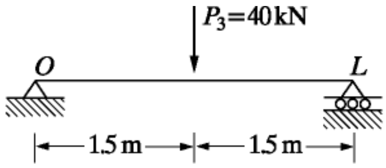


Figure 9P27

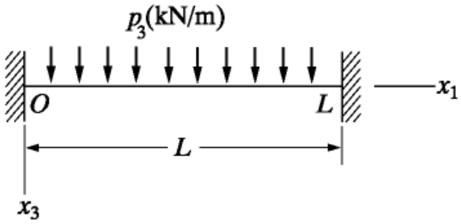


Figure 9P28

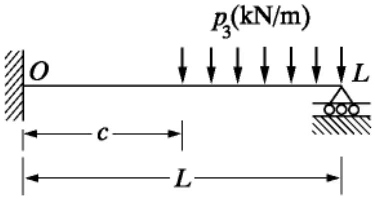


Figure 9P29

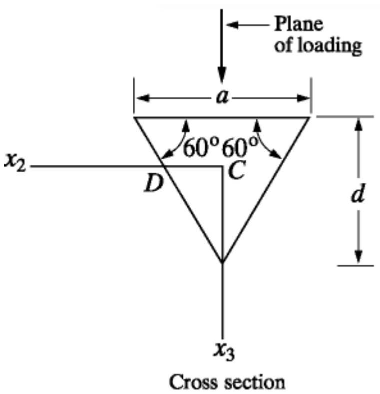
Section 9.5

30. Consider a cantilever beam whose cross section is an equilateral triangle shown in Fig. 9P30 with a = 200 mm. The beam is subjected at its free end to a force of 120 kN acting in the x_1x_3 plane in the direction of the x_3 axis. Compute the shearing components of stress τ_{13} and τ_{12} acting on the particles of the cross section of the beam. Evaluate the results for the particles of line CD of the beam.

Ans. Resulting shearing stress at point D = 10.67 MPa

31. Establish formulas for the shearing stress in the flanges and in the web of a thin-walled beam having the Z cross section shown in Fig. 9P31. The beam is subjected to a shearing force Q_3 (Q_2 = 0). Disregard the thickness t_F and t_W compared to b and d, respectively.

Ans. $\tau_{13} = \dfrac{Q_3}{2I_2 t_W}\left[bt_F d + t_W\left(\dfrac{d^2}{4} - x_3^2\right)\right]$, $\tau_{12} = \dfrac{Q_3 d}{2I_2}(b - x_2)$



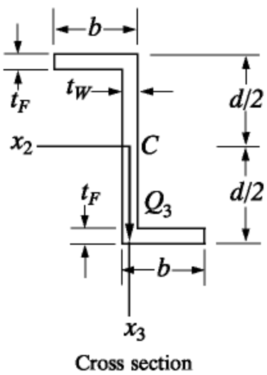


Figure 9P30

Figure 9P31

32. Derive formulas for the shearing stress in the flanges and the web of the thin-walled box-beam whose cross section is shown in Fig. 9P32. Disregard the thickness t_F and t_W compared to b and h.

Ans. $\tau_{13} = \dfrac{Q_3}{8I_2 t_W}\left(2bt_F b + t_W d^2 - 4t_W x_3^2\right) \qquad \tau_{12} = \dfrac{Q_3 x_2 (d - t_F)}{2I_2}$

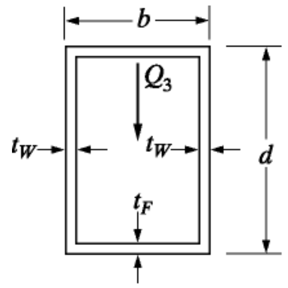


Figure 9P32

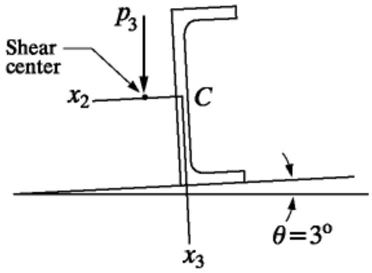


Figure 9P33

33. A rolled steel channel C 200 × 20.5 is used as a 4 m simply supported beam on an inclined plane, as shown in Fig. 9P33. Determine the maximum tensile stress acting on the cross sections of the beam when subjected to a uniformly distributed force of 2kN/m including its weight. Compare the results with those obtained when $\theta = 0$. The plane of loading of the beam contains the shear centers of its cross sections and it is vertical. Compute the maximum value of the shearing component of stress.

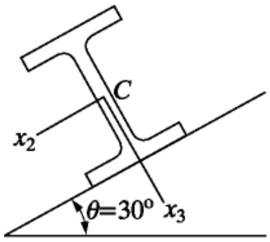


Figure 9P34

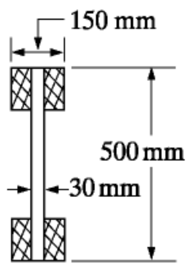


Figure 9P35

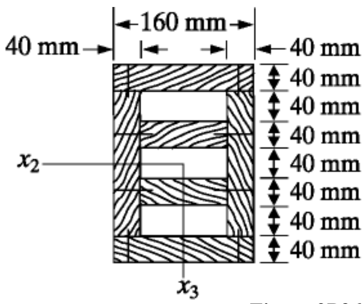


Figure 9P36

34. A wide flange beam W 130 × 28.1 of length 3 m is simply supported on an inclined position as shown in Fig. 9P34. Compute the maximum values of the normal and shearing components of stress due to its own weight and indicate on a sketch the location of the particles on which they act. Locate the neutral axis of the cross sections of the beam.

Ans. 34 $(\tau_{13})_{max} = 0.44787$ MPA, $(\tau_{12})_{max} = 0.11081$ MPa, $(\tau_{11})_{max} = 4.219$ MPa

Section 9.6

35. A 2 m long simply supported beam is made up from four 60 × 120 mm rectangular wood planks glued to a 30 × 500 mm wood plank, as shown in Fig. 9P35. The beam is subjected to a concentrated force P_3 at the middle of its span. The allowable normal and shearing components of stress in the wood is 12 MPa and 1 MPa, respectively. The allowable value of the average shearing stress in the glue is 0.5 MPa. Determine the maximum value of P_3.

Ans. $(P_3)_{max} = 22.6$ kN

36. A simply supported beam is made from four 40 mm wood planks nailed together as shown in Fig. 9P36. The beam is subjected to a concentrated force at the middle of its span. Disregard the effect of the weight of the beam. The nails are spaced every 20 mm. If the allowable shearing force per nail is 10 N, compute the maximum allowable value of the applied force.

Ans. $(P_3)_{allowable} = 653.6$ N

37. Two W 100 × 19.3 beams are connected together with bolts to form one beam, whose cross section is shown in Fig. 9P37. The beam is simply supported and is subjected to a concentrated force of 120 kN at the middle of its span. The allowable force in shear for each bolt is 40 kN. Assuming that the spacing of the bolts is to be the same throughout the length of the beam, compute the maximum allowable spacing of the bolts.

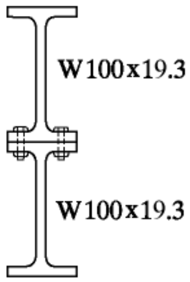


Figure 9P37

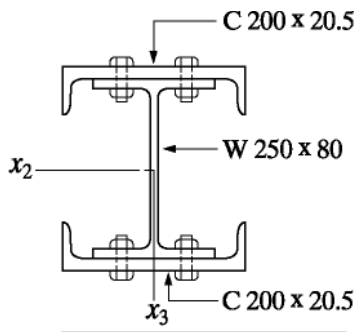


Figure 9P38

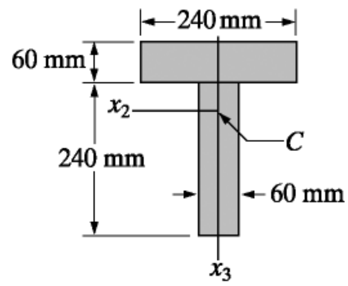


Figure 9P39

38. A cantilever beam is made with a standard wide-flange and two standard channels bolted together as shown in Fig. 9P38. Each bolt can carry a maximum allowable shearing force of 20 kN. The beam is subjected at its unsupported end to a concentrated force of 80 kN. Compute the maximum allowable spacing of the bolts.

Ans. $s \leq 260.7$ mm

39. A cantilever T-beam is made from two wood planks nailed together as shown in Fig. 9P39. The beam is subjected at its unsupported end to a concentrated force of 4 kN. The allowable force that each nail can carry is 1.2 kN. Compute the spacing of the nails.

Ans. $s \leq 130.8$ mm

Section 9.7

40. to 45. Derive a formula for the distance e which locates the shear center of the cross section of the thin-walled beam shown in Fig. 9P40. Disregard t_W and t_F as compared to h, b or a. Repeat with the cross sections of Figs 9P41 to 9P45.

Ans. 40 $e = \dfrac{7\sqrt{3}a}{18}$, Ans. 41 $e = \dfrac{a\sqrt{3}}{9}$, Ans. 42 $e = \dfrac{2(2r^2 + \pi b r + b^2)}{\pi r + 4b}$

Ans. 43 $e = \dfrac{b(6h^2 a + 3h^2 b - 8a^3)}{6h^2 b + 8a^3 + 6ah^2 - 12a^2 h + h^3}$

Ans. 44 $e = \dfrac{b[3bh^2 + 6ah^2 - 8a^3]}{6bh^2 + h^3 + 6ah^2 + 12a^2 h + 8a^3}$, Ans. 45 $e = h\left(1 - \dfrac{t_1 b_1^3}{12 I_2}\right) = \dfrac{h t_2 b_2^3}{t_1 b_1^3 + t_2 b_2^3}$

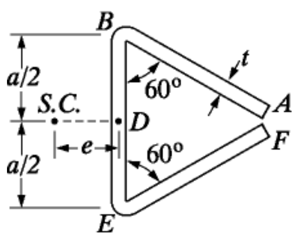


Figure 9P40

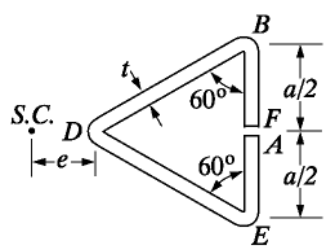


Figure 9P41

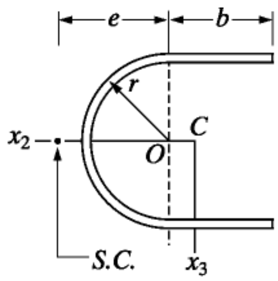


Figure 9P42

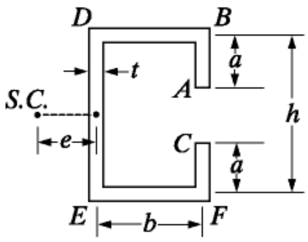


Figure 9P43

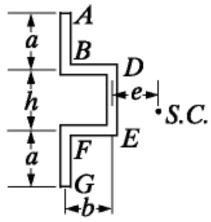


Figure 9P44

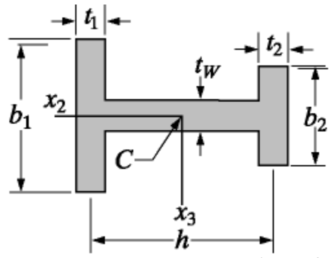


Figure 9P45

Section 9.8

46. A member of a structure consists of two steel rods welded to a 10 mm thick steel plate and is subjected to the forces shown in Fig. 9P46. Assuming that the rods and the plate are sufficiently longer than the dimensions of the cross sections of the member, determine the largest tensile stress acting on the cross sections of the rods and the plate.

Ans. $\tau_{11}^{(A)} = (\tau_{11})_{max} = 188.51$ MPa

47. The inner and outer diameters of the steel pipe of Fig. 9P47 are 30 mm and 40 mm, respectively. The pipe is fixed at its end B. Compute the value of the forces $P^{(1)}$ and $P^{(2)}$ acting on the pipe so that the normal components of stress acting on the particles at points A (0.2 m, 0, −0.02 m) and B (0, 0.02 m, 0) are equal to $\tau_{11}^{(A)} = 60$ MPa compression and $\tau_{11}^{(B)} = 120$ MPa tension.

Ans. $P^{(1)} = 33.33$ KN $P^{(2)} = 38.795$ KN

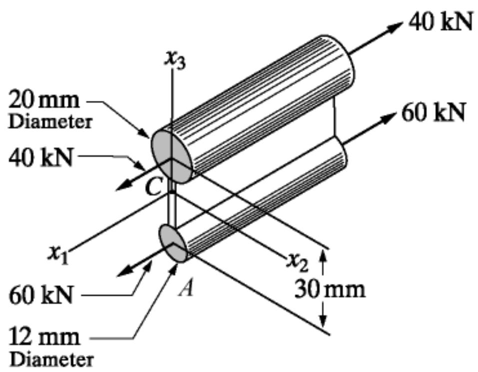


Figure 9P46

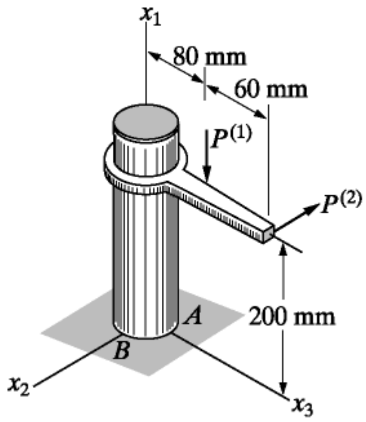


Figure 9P47

48. Consider member 1, 2, 3 shown in Fig. 9P48 subjected at its end 3 to an axial centroidal force of 40 kN. The cross section of the member is 80 mm by 80 mm. Compute the maximum normal component of stress acting on the cross section AA.

Ans. $(\tau_{11}^{(A)})_{max} = 22.48$ MPa

49. The structural tubing shown in Fig. 9P49 has an uniform thickness of 6 mm. Compute the normal component of stress at point A. Ans. $\tau_{11}^{(A)} = 22.7$ MPa compression

50. A bar AB is welded on the unsupported end surface of a cantilever beam of solid circular cross section of radius 24 mm. Two forces are applied to the end B of the bar as shown in Fig. 9P50. Establish the components of stress acting on the particles of the beam which are located at points D and E.

Ans. $\tau_{11}^{(D)} = -275$ MPa, $\tau_{11}^{(E)} = 56.40$ MPa $\tau_{13}^{(E)} = 39.81$ MPa ↓, $\tau_{12}^{(D)} = 66.35$ MPa ←

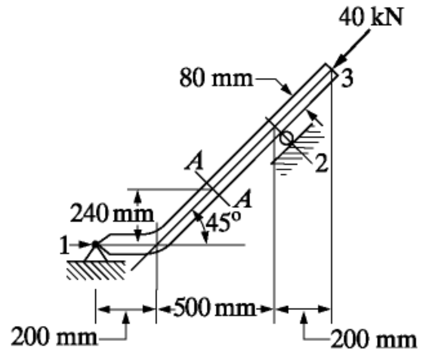


Figure 9P48

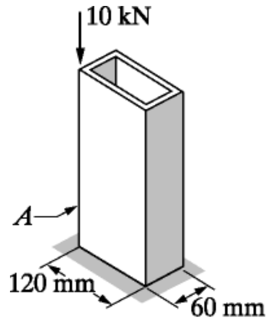


Figure 9P49

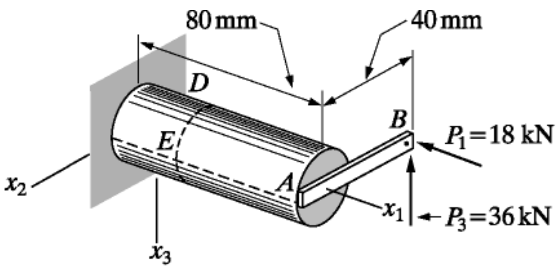


Figure 9P50

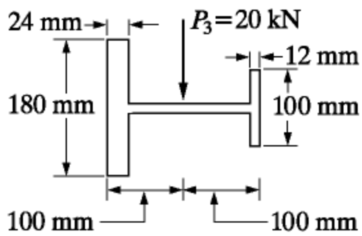


Figure 9P51

51. and 52. Consider a cantilever beam whose cross section is shown in Fig. 9P51 subjected to a concentrated force P_3 at its unsupported end. The beam is made from an isotropic linearly elastic material. Determine the maximum shearing stress at cross sections sufficiently removed from the fixed end of beam so that the effect of restraining the warping of the cross section of the beam can be disregard. Repeat with the beam whose cross section is shown in Fig. 9P52.

Ans. 51 $e = 14.37$ mm, $(\tau_{13})_{max} = 42$ MPa Ans. 52 $(\tau_{13})_{max} = 183.32$ MPa

Section 9.9

53. Consider the 2 m long simply supported beam made of two steel ($E_S = 200$ GPa) flanges connected to an aluminum alloy web ($E_A = 75$ GPa) as shown in Fig. 9P53. The beam is subjected to a concentrated force of 20 kN at the middle of its span. Compute the distribution of the normal and shearing components of stress acting on the cross section of the beam at $x_1 = 1.0$ m. Establish a formula for the deflection of the beam.

54. Consider a simply supported beam of length 6 m made of three vertical layers as shown in Fig. 9P54. The modulus of elasticity of steel is $E_S = 200$ GPa while that of the aluminum alloy is $E_a = 75$ GPa. The beam is subjected to a concentrated force of 40 kN at the middle point of its span. Compute the normal and shearing components of stress acting on the cross section of the beam.

Ans. $\tau_{13}^{(A)} = 1.02$ MPa, $\tau_{11}^{(A)} = 61.3$ MPa in aluminum

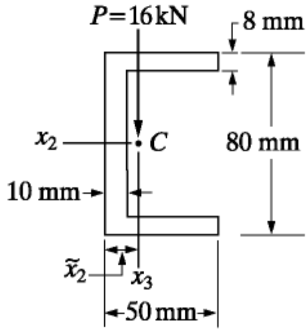


Figure 9P52

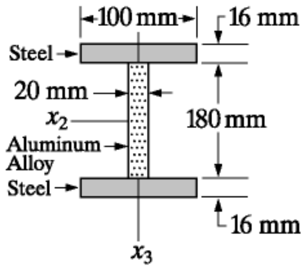


Figure 9P53

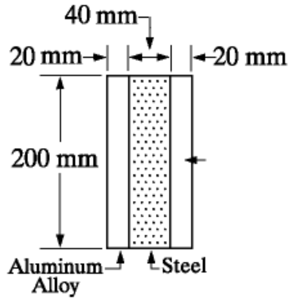


Figure 9P54

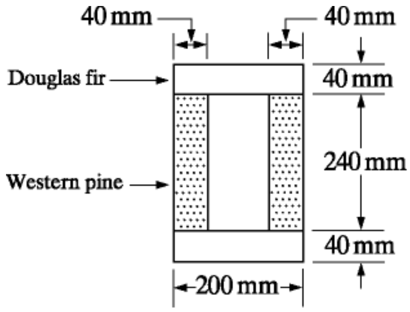


Figure 9P55

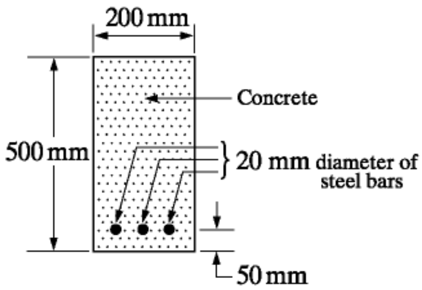


Figure 9P56

55. The webs of the 6 m long hollow simply supported box beam, whose cross section is shown in Fig. 9P55, are made of western pine (E_p = 10 GPa), while the beam flanges are made of Douglas fir (E_d = 13 GPa). The beam is subjected to a concentrated force at the middle point of its span P_3= 10 kN. Compute the maximum normal and shearing components of stress acting on the cross section of the beam just to the left of the concentrated force. Compute the maximum deflection of the beam.

Ans. $(\tau_{11}^{\text{Doug. fir}})_{max} = 5.5$ MPa, $(\tau_{11}^{\text{West. pine}})_{max} = 5.37$ MPa, $(\tau_{13}^{\text{West. pine}})_{max} = 0.2681$MPa

56. The cross section of a 2 m reinforced concrete simply supported beam is shown in Fig. 9P56. The beam is subjected to a concentrated force of 4 kN at the middle point of its span. Establish the distribution of the normal and shearing components of stress acting on the cross section of the beam just to the left of the concentrated force (E_s = 200 GPa and E_c = 20GPa).

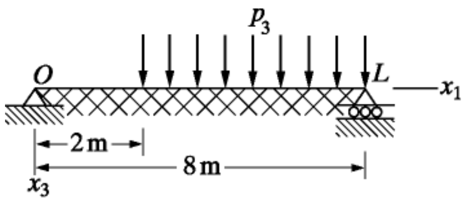


Figure 9P57

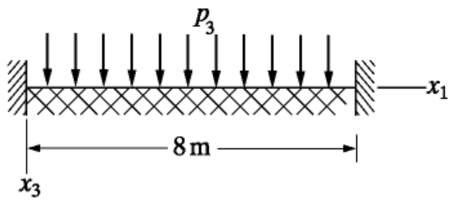


Figure 9P58

57. and **58.** Using the method of finite differences, determine the deflection and the internal moments of the prismatic beam [$I_s = 200(10^6)$ mm^4, E = 200 GPa) having the end supports and subjected to the external forces shown in Fig. 9P57. The beam bends at about the x_2 axis without twisting and rests on an elastic foundation of modulus k = 20

MPa. The x_2 and x_3 axes are principal centroidal. Subdivide the beam into ten equal segments. Repeat with the beam of Fig. 9P58.

59. An I-beam (with of flange b = 120mm, depth of cross section h = 240mm, t_W = 20 mm, t_F = 16 mm) is made from an aluminum alloy (E = 75 Gpa, v = 1/3) and having a length of 1.5 m. The beam is fixed at its one end and is connected rigidly to a thick plate at its other end, which prevents this end from warping. A torsional moment of 80 kN·m is applied to the beam by two equal and opposite forces acting on the plate in a plane perpendicular to axis of the beam. Compute the angle of twist of the plate and the distribution of the normal and of the shearing components of stress acting on the cross section of the beam at its ends.

Chapter 10

Non-Prismatic Members — Stress Concentrations

10.1 Computation of the Components of Displacement and Stress of Non-Prismatic Members

The following equations and formulas, established in Chapters 8 and 9 for prismatic line members are expected to give reasonably accurate values for the components of displacement and stress, of tapered members provided that the rate of change (dA/dx_1) of the area of their cross sections is not very large. The error increases as dA/dx_1 increases.

1. Equations (8.61) and formula (8.59) are used to compute the component of translation $u_1(x_1)$ and the components of stress, respectively, of symmetrically (see Fig. 10.1a) tapered members subjected to axial centroidal forces.
2. Equations (8.74) and formula (8.72) are used to establish the angle of twist $\theta_1(x_1)$ and the component of stress $\tau_{1\theta}(x_1)$, respectively, of symmetrically tapered members of circular cross sections subjected to torsional moments.

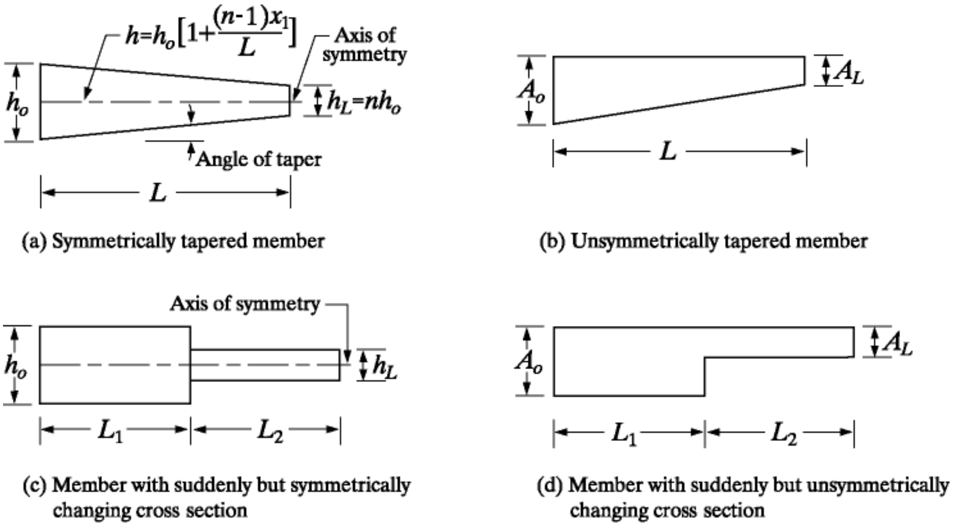


Figure 10.1 Members of variable cross section.

3. Equations (9.33) and formula (9.12) are used to compute the transverse components of translation (deflection) and the normal component of stress of tapered beams subjected to bending without twisting. The distribution of the shearing components of stress $\hat{o}_{12}$ and $\hat{o}_{13}$ on the cross sections of tapered beams, subjected to bending without twisting, could differ considerably from that of the corresponding components of stress acting on the cross sections of prismatic beams (see Section 10.2.1). Thus, formula (9.66) cannot be used to compute the shearing components of stress acting on the cross sections of non-prismatic beams.

10.2 Stresses in Symmetrically Tapered Beams

In this section we derive formulas for computing the shearing components of stress acting on the cross sections of symmetrically tapered beams (see Fig. 10.1a) subjected to bending without twisting. We assume that the rate of change of the cross sectional area (dA/dx_1) of the beams under consideration is small. For such beams the error in the value of the normal component of stress $\hat{o}_{11}$ acting on their cross sections obtained on the basis of relation (9.12a) is small. For example, the error in the maximum normal component of stress acting on the cross sections of a symmetrically tapered cantilever beam of rectangular cross sections having an angle of taper of 15° is 5%. Notice, however, that the maximum normal component of stress does not necessarily act on the cross section of maximum moment as in the case of prismatic beams. This is so because the value of the maximum normal component of stress depends not only on the value of the moment but also on the value of the moment of inertia of the cross section of the beam both of which are functions of x_1. For example, consider the tapered beam of rectangular cross section of constant width b shown in Fig. 10.1a. Referring to relation (9.12b) and denoting by n the ratio h_L/h_o, the normal component of stress acting on its cross sections is equal to

$$\tau_{11}(x_1, x_2, x_3) = -\frac{P_3(L - x_1)x_3}{I_2} = -\frac{12L^3P_3(L - x_1)x_3}{bh_o^3[L + (n - 1)x_1]^3} \tag{10.1}$$

The maximum normal component of stress at any cross section of the beam occurs at $x_3 = h/2 = h_o[1 + (n - 1)x_1/L]/2$ and it is equal to

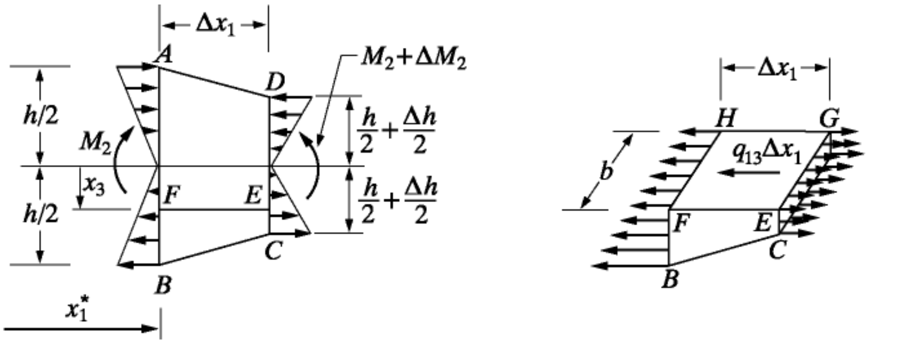
$$\tau_{11}(x_1, x_2, h/2) = -\frac{6L^2P_3(L - x_1)}{b\,h_o^2[L + (n - 1)x_1]^2} \tag{10.2}$$

Differentiating relation (10.2) with respect to x_1 and equating the derivative of τ_{11} to zero, we find the following value of x_1 for which τ_{11} is maximum

$$x_1 = \left(\frac{2n - 1}{n - 1}\right) L \tag{10.3}$$

This relation indicates that when $n \geq 0.5$ the maximum normal component of stress acts on the cross section at the fixed end, while when $0.5 > n$ the location of the cross section on which the maximum normal component of stress acts is obtained from relation (10.3).

In what follows we derive a formula for the shear flow in symmetrically tapered beams of arbitrary cross sections, when they are subjected to bending without twisting. A



(a) Segment of the beam (b) Portion of the segment of Fig. 10.2a

Figure 10.2 Free-body diagrams of a segment of length Δx_1 of a non-prismatic beam and a portion of this segment.

segment *ABCD* of length Δx_1 of such a beam is shown in Fig. 10.2a. The segment is cut by two planes ($x_1 = x_1^*$ and $x_1 = x_1^* + \Delta x_1$) normal to the axis of the beam. Moreover, the free-body diagram of portion *BCEFGH* of the segment *ABCD* is shown in Fig. 10.2b. This portion is cut from the segment *ABCD* by a plane normal to the x_3 axis. Referring to Fig. 10.2b from the equilibrium of portion *BCEFGH,* we have

$$q_{13}\Delta x_1 = \iint_{A_{CEG}} (\tau_{11} + \Delta\tau_{11})\,dA - \iint_{A_{BFH}} (\tau_{11})\,dA \tag{10.4}$$

Assuming that the rate of change of the area of the cross sections of the beam (dA/dx_1) is not large, formula (9.12a) can be use to compute the normal component of stress acting on its cross sections. Substituting relation (9.12a) into (10.4), we obtain

$$\begin{aligned}
q_{13}\Delta x_1 = &- \left[\frac{M_2(x_1^* + \Delta x_1)I_{23}(x_1^* + \Delta x_1) + M_3(x_1^* + \Delta x_1)I_{22}(x_1^* + \Delta x_1)}{I_{22}(x_1^* + \Delta x_1)I_{33}(x_1^* + \Delta x_1) - I_{23}^2(x_1^* + \Delta x_1)}\right] \iint_{A_{CEG}} x_2\,dA \\
&+ \left[\frac{M_2(x_1^*)I_{23}(x_1^*) + M_3(x_1^*)I_{22}(x_1^*)}{I_{22}(x_1^*)I_{33}(x_1^*) - I_{23}^2(x_1^*)}\right] \iint_{A_{BFH}} x_2\,dA \\
&+ \left[\frac{M_2(x_1^* + \Delta x_1)I_{33}(x_1^* + \Delta x_1) + M_3(x_1^* + \Delta x_1)I_{23}(x_1^* + \Delta x_1)}{I_{22}(x_1^* + \Delta x_1)I_{33}(x_1^* + \Delta x_1) - I_{23}^2(x_1^* + \Delta x_1)}\right] \iint_{A_{CEG}} x_3\,dA \\
&- \left[\frac{M_2(x_1^*)I_{33}(x_1^*) + M_3(x_1^*)I_{23}(x_1^*)}{I_{22}(x_1^*)I_{33}(x_1^*) - I_{23}^2(x_1^*)}\right] \iint_{A_{BFH}} x_3\,dA
\end{aligned} \tag{10.5}$$

The above relation may be used to compute the average value of the shear flow over a small length of the beam extending from x_1^* to $x_1^* + \Delta x_1$ by substituting in it the numerical values of $M_2(x_1^*)$, $M_3(x_1^*)$, $I_{22}(x_1^*)$, $I_{33}(x_1^*)$, $I_{23}(x_1^*)$, $M_2(x_1^* + \Delta x_1)$,

$M_3(x_1^* + \Delta x_1)$, $I_{22}(x_1^* + \Delta x_1)$, $I_{33}(x_1^* + \Delta x_1)$, $I_{23}(x_1^* + \Delta x_1)$, and Δx_1. This average value of the shear flow is an approximation to the shear flow at $x_1 = x_1^*$. The smaller the distance Δx_1 the more accurate the results will be. However, if Δx_1 is very small, considerable numerical accuracy is required because in this case the right side of relation (10.5) represents the sum of the differences of two terms which are almost equal.

10.2.1 Shearing Stresses in Symmetrically† Tapered Beams of Rectangular Cross Sections

Consider a symmetrically tapered beam of rectangular cross section of constant width b and varying depth h. We assume that h is larger than b and that dh/dx_1 is small. The beam is subjected to bending about the principal axis x_2 ($M_3 = 0$). For such a beam the components of stress τ_{11} and τ_{13} can be assumed constant in the direction of the x_2 axis while the shearing component of stress τ_{12} can be disregarded. Moreover, taking into account that the x_2 and x_3 axes are principle centroidal, relation (10.5) reduces to

$$q_{13}\Delta x_1 = \frac{M_2(x_1^* + \Delta x_1)}{I_2(x_1^* + \Delta x_1)} \iint_{A_{CEG}} x_3 \, dA - \frac{M_2(x_1^*)}{I_2(x_1^*)} \iint_{A_{BFH}} x_3 \, dA \tag{10.6}$$

Denoting by h and $h + \Delta h$ the depth of the cross sections at x_1^* and $x_1^* + \Delta x_1$, respectively, (see Fig. 10.2a) and disregarding terms involving $(\Delta h)^2$ and $(\Delta h)^3$, we have

$$\iint_{A_{BFH}} x_3 \, dA = (A_{BFH})\bar{x}_{3n} = \frac{b}{2}\left(\frac{h}{2} - x_3\right)\left(x_3 + \frac{h}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - x_3^2\right) \tag{10.7a}$$

$$\begin{aligned}\iint_{A_{CEG}} x_3 \, dA &= \frac{b}{2}\left[\frac{(h + \Delta h)^2}{4} - x_3^2\right] = \frac{b}{2}\left(\frac{h^2}{4} - x_3^2\right) + \frac{b}{2}\left(\frac{(\Delta h)^2 + 2h\Delta h}{4}\right) \\ &\approx \iint_{A_{BFH}} x_3 \, dA + \frac{b(\Delta h)h}{4}\end{aligned} \tag{10.7b}$$

$$I_2(x_1^*) = \frac{bh^3}{12} \tag{10.8a}$$

$$I_2(x_1^* + \Delta x_1) = \frac{b(h + \Delta h)^3}{12} \approx I_2(x_1^*) + \frac{b(\Delta h)h^2}{4} \tag{10.8b}$$

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†A derivation of a formula for the shearing component of stress acting on the cross sections of unsymmetrically non-prismatic tapered beams of rectangular cross section is available in the following reference :

Oden, T.J., Ripperger, A.E., *Mechanics of Elastic Structures,* 2nd edition, McGraw-Hill, New York, 1981, p. 112.

Substituting relations (10.7b) and (10.8b) into (10.6) using (10.7a) and multiplying both sides by $I_2 - b(\Delta h)h^2/4$, we obtain

$$q_{13}I_2\Delta x_1 - q_{13}\frac{b(\Delta h)h^2\Delta x_1}{4} = M_2(x_1^* + \Delta x_1)\iint_{A_{BFH}} x_3 dA - M_2(x_1^*)\iint_{A_{BFH}} x_3 dA$$

$$+ M_2(x_1^* + \Delta x_1)\frac{b(\Delta h)h}{4} - M_2(x_1^*)\frac{b}{2}\left(\frac{h^2}{4} - x_3^2\right)\frac{b(\Delta h)h^2}{4I_2(x_1^*)}$$

$$= \Delta M\iint_{A_{BFH}} x_3 dA - \frac{M_2(x_1^*)h(\Delta h)b}{4}\left[1 - \frac{b}{2}\left(\frac{h^2}{4} - x_3^2\right)\frac{h}{I_2(x_1^*)}\right]$$

$$+ \frac{\Delta M b(\Delta h)h}{4} \tag{10.9}$$

Dividing relation (10.9) by Δx_1, taking the limit as Δx_1 goes to zero, and noting that in the limit $\Delta M/\Delta x_1$ is equal to Q_3 while $\Delta h/\Delta x_1$ is equal to dh/dx_1, we obtain

$$q_{13} = \frac{Q_3 b\left(\frac{h^2}{4} - x_3^2\right)}{2I_2} + \left(\frac{M_2 h b}{4I_2}\right)\left[1 - \frac{bh}{2I_2}\left(\frac{h^2}{4} - x_3^2\right)\right]\frac{dh}{dx_1} \tag{10.10}$$

This relation is valid for beams of rectangular cross sections of constant width b and symmetrically varying depth h provided that dh/dx_1 is small.

In what follows we present an example.

Example 1 Derive a formula for the shearing stress τ_{13} acting on the cross sections of the tapered cantilever beam of rectangular cross sections of contant width b subjected to a force P_3 at its unsupported end (see Fig. a).

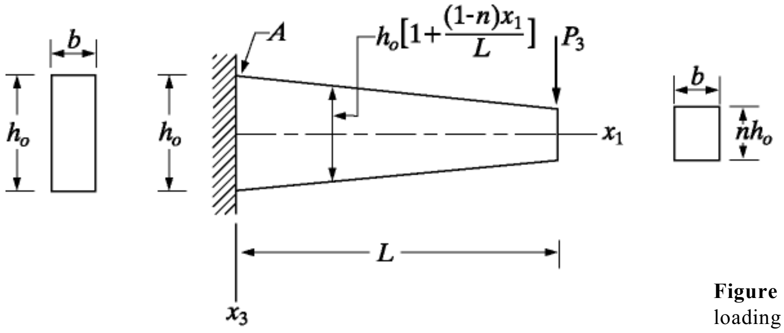


Figure a Geometry and loading of the beam.

Solution For the beam of Fig. a we have

$$Q_3 = P_3 \qquad I_2 = \frac{bh_o^3}{12}\left[1 + \frac{(n-1)x_1}{L}\right]^3 \qquad \frac{dh}{dx_1} = \frac{h_o(n-1)}{L} \tag{a}$$

The shearing component of stress τ_{13} can be established by substituting relations (a) into (10.10). That is,

$$\tau_{13} = \frac{q_{13}}{b} = \frac{6P_3L^3\left[\dfrac{h_o^2}{4L^2}[L+(n-1)x_1]^2 - x_3^2\right]}{bh_o^3[L+(n-1)x_1]^3}$$
$$- \frac{3P_3(L-x_1)L(n-1)}{bh_o[L+(n-1)x_1]^2}\left[1 - \frac{6L^2}{h_o^2[L+(n-1)x_1]^2}\left[\frac{h_o^2[L+(n-1)x_1]^2}{4L^2} - x_3^2\right]\right] \tag{b}$$

From relation (b) we see that the distribution of the shearing component of stress is a function of x_1. For example, consider a tapered beam with $n = 0.5$. For this beam at $x_1 = 0$ relation (b) gives

$$\tau_{13}(0,\, x_3) = \frac{3P_3}{4bh_o}\left[1 + 4\left(\frac{x_3}{h_o}\right)^2\right] \tag{c}$$

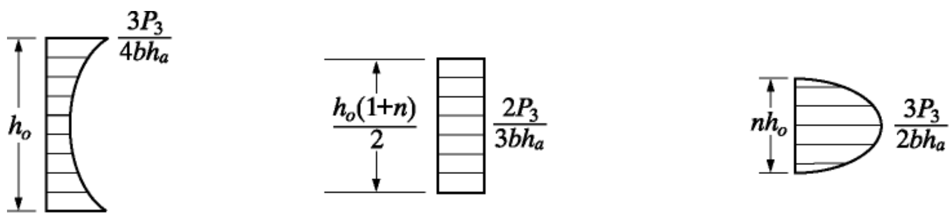
while, at $x_1 = L/2$ relation (b) gives

$$\tau_{13}\left(\frac{L}{2},\, x_3\right) = \frac{4P_3}{3bh_o} \tag{d}$$

and at $x_1 = L^-$ (just to the left of the force P_3) relation (b) gives

$$\tau_{13}(L,\, x_3) = \frac{3P_3(h_o^2 - 16x_3^2)}{bh_o^3} \tag{e}$$

The results (c) to (e) are plotted in Fig. b. Notice that the maximum normal component of stress acting on the cross section of a beam with $n = 0.5$ occurs at $x_1 = 0$, $x_3 = h/2$. However, at these points, the shearing component of stress does not vanish. Thus, the maximum normal component of stress occurs on a plane which is not normal to the axis of the beam.



(a) At $x_1=0$ (b) At $x_1=L/2$ (c) At $x_1=L$

Figure b Distribution of shearing stress on cross sections of the beam of Fig. a with $n = 0.5$.

10.3 Stress Concentrations

The conditions which increase the magnitude of the components of stress acting on some particles of line members are called *stress raisers*. They include the following:

1. Cracks which may have resulted during fabrication or erection of a structure
2. Abrupt changes in the distribution of external tractions of high intensity acting on the members of a structure
3. Abrupt or sudden changes of the cross sections of members of a structure
4. Abrupt changes of the mechanical properties of the materials from which the members of a structure are made

Often, as a result of the existance of stress raisers, large stresses are developed only in a small portion of the volume of a member. They are called *localized stresses or stress concentrations*. In this section we limit our attention to stress concentrations resulting from abrupt or sudden changes of the geometry of the cross sections of line members. In the neighborhood of such changes the state of stress is triaxial; that is, some of the components of stress which were negligible in prismatic members are not negligible in the neighborhood of a sudden or an abrupt change of the cross sections of a member. Thus, the values of the components of stress in the neighborhood of an abrupt or sudden change of the cross sections of a member cannot be computed using the theories of mechanics of materials. However, they can be established experimentally using the method of photoelasticity or the technique of Moiree or with the aid of a computer, on the basis of the theory of elasticity using a numerical method such as the finite element method. This approach requires considerable computer time.

The maximum stress due to geometric irregularities is usually expressed as

$$(\tau)_{\max} = K(\text{nominal stress}) \tag{10.11}$$

where K is called the stress concentration factor†. Its value depends on the geometry of the neighborhood of the abrupt or sudden change of the cross sections of the member. The nominal stress is the maximum stress acting on the cross section of an auxiliary prismatic member whose cross sections are identical to the smallest cross section in the neigborhood of stress concentration of the actual member. For example, the maximum value of the normal component of stress τ_{11} acting on the cross sections of a member located in the neighborhood of an abrupt or sudden change of its cross sections, when subjected to tractions on its surfaces which are statically equivalent to axial centroidal forces, acting along its length and/or at its ends is expressed as

$$(\tau_{11})_{\max} = K\frac{P_1}{A} \tag{10.12}$$

where A is the net area of the smallest cross section of the member. K is the *stress*

† For formulas for stress concentration in members having abrupt or sudden changes of their cross sections see:
Neuber, H., *Kerbspannungslehre: Grundlagen fur Genaue Spannungsrechnung,* Springer, Berlin, 1937 (Translation '74, David Taylor Model Basin, U.S. Navy Washington, D.C. 1945).
Savin, G.N., *Stress Concentrations Around Holes,* Pergamon Press, New York, 1961.
Peterson, R.E., *Stress Concentration Factors for Design,* John Wiley and Sons, New York, 1974.
Rooke, D.P. and Cartwright, D.J., *Compendium of Stress Intensity Factors,* Her Majesty's Stationary Office, London, 1976.

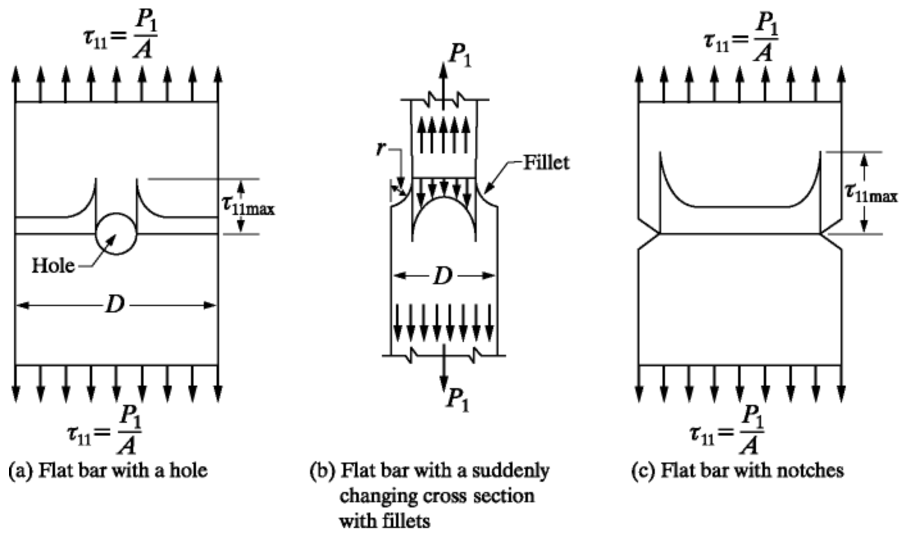


Figure 10.3 Stress concentrations in flat bars having varying cross sections subjected to equal and opposite axial centroidal tensile forces on their ends.

concentration factor. The distributions of the axial component of stress on the smallest cross section of a member with a hole, symmetric notches and suddenly changing cross sections with fillets are shown in Fig. 10.3. In Fig. 10.4 the stress concentration factors for flat bars with a hole and for flat bars with suddenly changing cross sections with fillets are plotted as functions of the geometry of the bar. These stress concentration factors have been obtained using photoelastic methods. They are valid only for values of the components of stress for which the response of the material is linearly elastic.

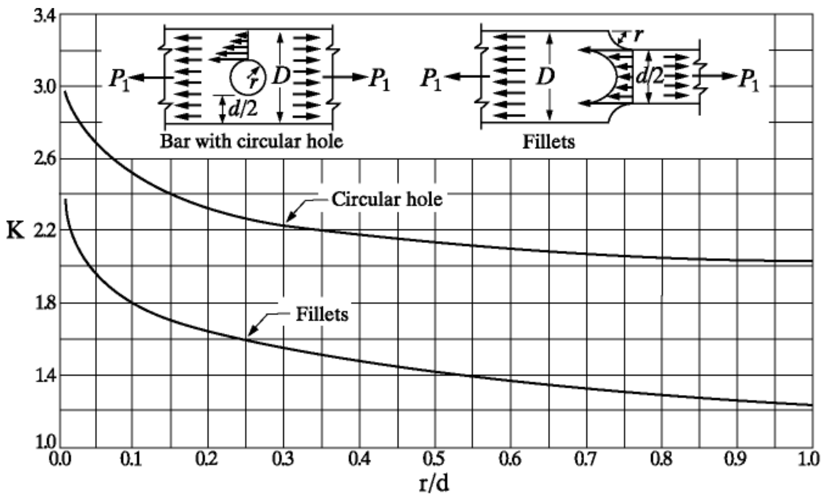
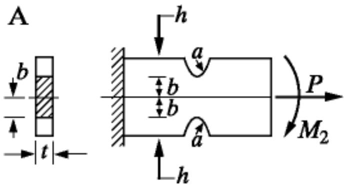
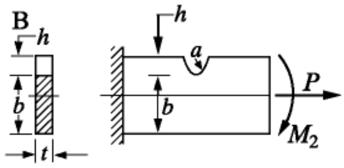
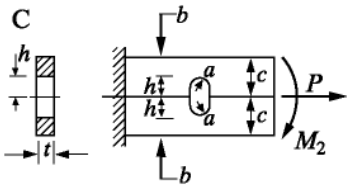
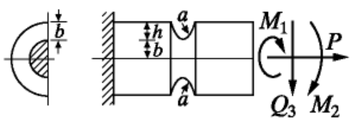


Figure 10.4 Stress concentration factors for flat members of constant thickness in tension, obtained using photoelastic methods.†

† Taken from Frocht, M.M., *Factors of Stress Concentration Photoelastically Determined,* Trans. Am. Soc. Mech. Eng. 57, p.A-67, 1935.

Table 10.1 Data for Neuber's diagram.

Type of Change of Section	Type of Loading	Normal Component of Stress	Scale for $\sqrt{h/a}$	Curve for Type of Section
A 	Tension	$\dfrac{KP}{2bt}$	f	1
	Bending	$\dfrac{3KM_2}{2b^2t}$	f	2
B 	Tension	$\dfrac{KP}{bt}$	f	3
	Bending	$\dfrac{6KM_2}{b^2t}$	f	4
C 	Tension	$\dfrac{KP}{2bt}$	f	5
	Bending	$\dfrac{3KM_2h}{2t(c^3-h^3)}$	e	5
D 	Tension	$\dfrac{KP}{\pi b^2}$	f	6
	Bending	$\dfrac{4KM_2}{\pi b^3}$	f	7
	Direct shear	$\dfrac{1.23KQ_3}{\pi b^2}$	e	8
	Torsional shear	$\dfrac{2KM_1}{\pi b^3}$	e	9

As shown in Example 3 of Section 7.7 when a plate of infinite width with a circular hole is subjected to axial centroidal tensile forces, the stress concentration factor is 3. However, this result cannot be used for plates of finite width, except if the ratio of the width D of the plate to the radius r of the hole is large ($D/r > 25$). An empirical formula for the stress

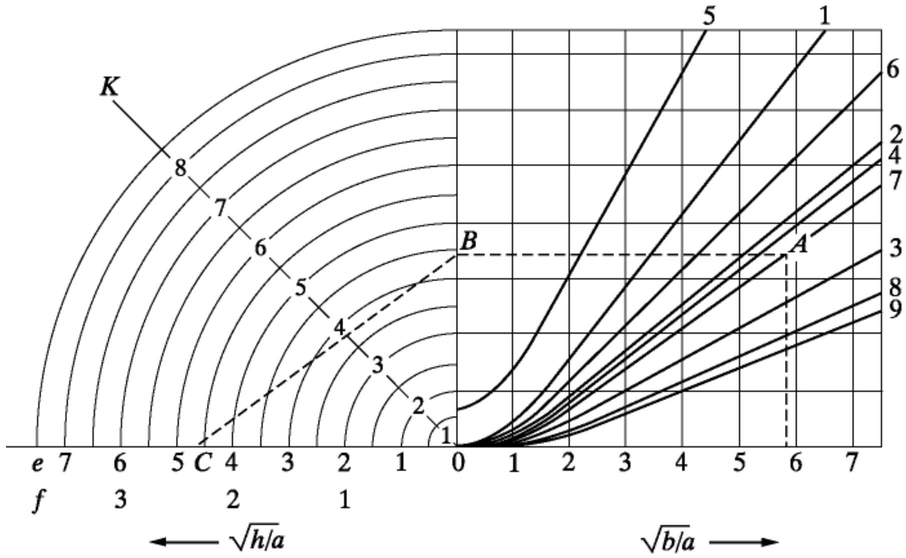


Figure 10.5 Neuber's diagram.

concentration factor K for flat members of finite width with a hole subjected to axial centroidal tensile forces is

$$K = \frac{\dfrac{3D}{r} - 1}{\dfrac{D}{r} + 0.3} \tag{10.13}$$

Approximate values of the stress concentration factor K for the members with grooves whose geometry and loading is shown in Table 7.1, may be obtained using Neuber's diagram† shown in Fig. 10.5. In order to demonstrate how to use Neuber's diagram, consider a cantilever beam of circular cross section of radius 300 mm with a circular groove of radius a = 8 mm and depth h = 44 mm. The beam is subjected to a bending moment M_2 at its unsupported end. Referring to Fig. D of Table 7.1, we have

$$b = 300 - 44 = 256 \qquad \sqrt{\frac{b}{a}} = \sqrt{\frac{256}{8}} = 5.657 \qquad \sqrt{\frac{h}{a}} = \sqrt{\frac{44}{8}} = 2.345$$

As can be seen from the fifth column of Table 7.1, curve 7 corresponds to the beam under consideration; we enter the diagram with $\sqrt{b/a} = 5.657$ and we draw a line perpendicular to its abscissa axis at that point, until it intersects curve 7. We denote the point of intersection by A. We draw from point A a line parallel to the abscissa axis of the diagram until it meets the ordinate axis at a point which we designate by B. Subsequently, we locate the point of the left abscissa axis representing $\sqrt{h/a} = 2.345$ according to scale f which is obtained from the fourth column of Table 7.1. We designate this point as C. We connect points B and C with a straight line which intersects the OK axis. We read the

† Neuber, H., *Kerbspannungslehre: Grundlagen fur Genaue Spannungsrechnung,* Springer, Berlin, 1937.

value of theconcentration factor $K \approx 3.7$ on the OK axis. Thus, referring to Table 7.1 the maximum value of the normal component of stress is equal to

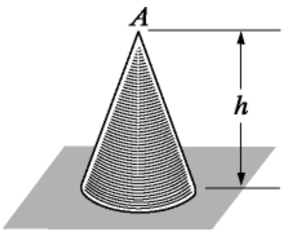
$$(\tau_{11})_{\max} = \frac{4KM_2}{\pi b^3} = \frac{14.8M_2}{\pi(256)^3} = 0.2808(10^{-6})M_2$$

where $(\tau_{11})_{\max}$ is in kN/mm^2 when M_2 is in kN·mm.

10.4 Problems

1. Consider the circular cone of height h and maximum radius R resting on a horizontal surface as shown in Fig. 10P1. The cone is made from an isotropic, linearly elastic material of modulus of elasticity E and mass density ñ. Determine the displacement of the apex A of the cone due to its own weight.

Ans. $u_1^A = \frac{\rho g h^2}{6E}$



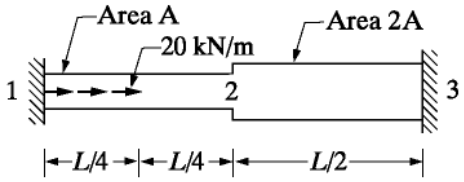


Figure 10P1 **Figure 10P2**

2. A member with a suddenly changing cross section is made from an isotropic, linearly elastic material of modulus of elasticity $E = 200$ GPa and coefficient of linear thermal expansion á = $10^{-5}/^{\circ}$C. The member is subjected to the axial centroidal forces shown in Fig. 10P2 and to an increase of temperature $\Delta T_c = 20^{\circ}$C. Compute the reactions of the member and draw its axial force diagram.

Ans. $R_1^0 = 0.33$ kN→ $R_1^L = 6.33$ kN←

3. Consider the fixed at both ends member of suddenly changing cross section shown in Fig. 10P3. The area of the cross section of the left part of the member is A_1 while that of the right part is A_2, where $A_2 > A_1$. The member is made from an isotropic, linearly elastic material with modulus of elasticity E and coefficient of thermal expansion á. The member is subjected to an increase of temperature ΔT_c. Derive formulas for the normal component of stress $\tau_{11}(x_1)$ and the axial component of translation $u_1(x_1)$.

Ans. $\tau_{11}^{(1)} = -\frac{EA_2H_1L}{A_2a + A_1(L-a)}$ $\quad u_1^{(1)} = -\frac{H_1x_1(L-a)(A_2-A_1)}{A_2a + A_1(L-a)}$

$\tau_{11}^{(2)} = -\frac{EA_1H_1L}{A_2a + A_1(L-a)}$ $\quad u_1^{(2)} = -\frac{H_1a(L-x_1)(A_2-A_1)}{A_2a + A_1(L-a)}$

4. The tapered member of constant thickness t shown in Fig. 10P4 is subjected to an axial centroidal force P_1^L at its unsupported end and to uniformly distributed axial centroidal forces p_1 along its length. The member is made from an isotropic, linearly elastic material of modulus of elasticity E. Determine the elongation of the member.

Ans. $u_1(x_1) = \dfrac{p_1 L^2}{EA_0(n-1)^2}\left[n\ln\left[1+\dfrac{(n-1)x_1}{L}\right]-\dfrac{(n-1)x_1}{L}\right]+\dfrac{P_1^L L}{EA_0(n-1)}\ln\left[1+\dfrac{(n-1)x_1}{L}\right]$

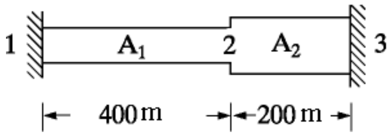


Figure 10P3

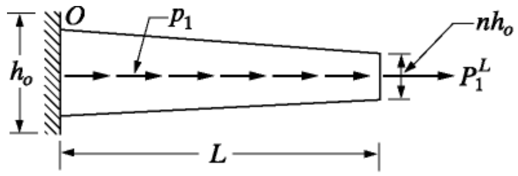


Figure 10P4

Chapter 11

Planar Curved Beams

11.1 Introduction

In this chapter we establish the stress field of curved beams, of a constant cross section whose axis (the locus of the centroids of their cross sections) lies in one plane. We denote by s the distance measured along the axis of the beam from a chosen reference point on its axis. The beams are subjected to a general loading. We call such beams *planar curved beams*.

When the radius of curvature R of the axis of a planar curved beam is large as compared to the depth h of its cross sections ($R/h > 5$), the components of stress acting on its cross sections may be approximated by assuming that the beam is straight ($R \rightarrow \infty$). Consequently, in this section we focus our attention on planar curved beams of constant cross section of small R/h ratio *($R/h < 5$)*.

11.2 Derivation of the Equations of Equilibrium for a Segment of Infinitesimal Length of a Planar Curved Beam

Consider a segment of length Δs of a planar curved beam in the x_1x_3 plane, cut by two

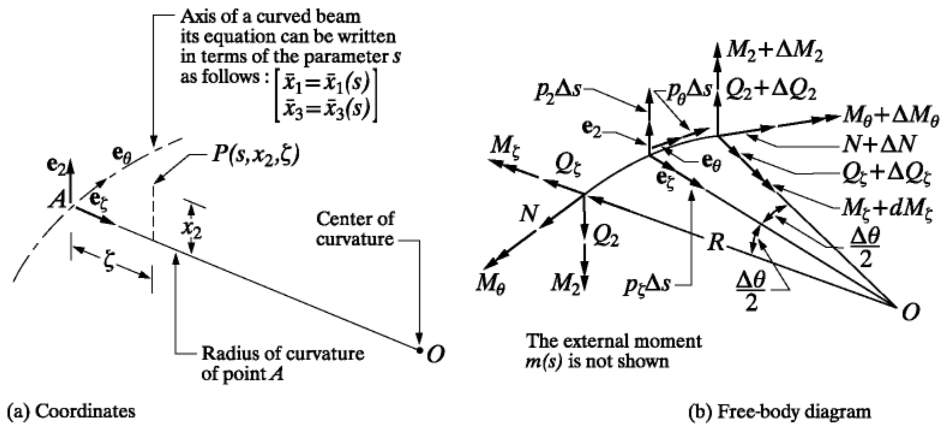


Figure 11.1 Coordinates and free-body diagram of a segment of a planar curved beam.

planes normal to its axis. We denote by $\mathbf{p}(s)$ the distributed forces and by $\mathbf{m}(s)$ the distributed moments acting on this beam. Both $\mathbf{p}(s)$ and $\mathbf{m}(s)$ are taken per unit length of the axis of the beam. These actions can be decomposed into three components, a circumferential p_θ or m_θ, a radial p_ζ or m_ζ, and a component normal to the plane of the beam p_2 or m_2. The free-body diagram of this segment of the beam is shown in Fig 11.1b. The circumferential, the normal to the plane of the beam and the radial components of the force and of the moment acting on the left cross section of this segment of the beam are denoted by N, Q_2, Q_ζ and M_θ, M_2, M_ζ, respectively. The components of the force and moment acting on the right cross section of this segment of the beam are denoted by $N + \Delta N$, $Q_2 + \Delta Q_2$, $Q_\zeta + \Delta Q_\zeta$ and $M_\theta + \Delta M_\theta$, $M_2 + \Delta M_2$, $M_\zeta + \Delta M_\zeta$. That is, the terms ΔN, ΔQ_2, ΔQ_ζ and ΔM_θ, ΔM_2, ΔM_ζ represent the change in N, Q_2, Q_ζ and M_θ, M_2, M_ζ, respectively. All internal actions are assumed positive, as shown in Fig 11.1b; referring to this figure, the equilibrium of the segment of the beam under consideration requires that

$$
\begin{aligned}
\sum F_\theta &= -N\cos\left(\frac{\Delta\theta}{2}\right) + (N+\Delta N)\cos\left(\frac{\Delta\theta}{2}\right) - Q_\zeta\sin\left(\frac{\Delta\theta}{2}\right) - (Q_\zeta+\Delta Q_\zeta)\sin\left(\frac{\Delta\theta}{2}\right) + p_\theta\Delta s = 0 \\
\sum F_2 &= -Q_2 + (Q_2+\Delta Q_2) + p_2\Delta s = 0 \\
\sum F_\zeta &= N\sin\left(\frac{\Delta\theta}{2}\right) + (N+\Delta N)\sin\left(\frac{\Delta\theta}{2}\right) - Q_\zeta\cos\left(\frac{\Delta\theta}{2}\right) + (Q_\zeta+\Delta Q_\zeta)\cos\left(\frac{\Delta\theta}{2}\right) + p_\zeta\Delta s = 0
\end{aligned}
\tag{11.1}
$$

and

$$
\begin{aligned}
\sum M_\theta^{(O)} &= -M_\theta\cos\left(\frac{\Delta\theta}{2}\right) + (M_\theta+\Delta M_\theta)\cos\left(\frac{\Delta\theta}{2}\right) + m_\theta\Delta s - M_\zeta\sin\left(\frac{\Delta\theta}{2}\right) \\
&\quad - (M_\zeta+\Delta M_\zeta)\sin\left(\frac{\Delta\theta}{2}\right) - Q_2R\cos\left(\frac{\Delta\theta}{2}\right) + (Q_2+\Delta Q_2)R\cos\left(\frac{\Delta\theta}{2}\right) + Rp_2\Delta s = 0 \\
\sum M_2^{(O)} &= -M_2 + (M_2+\Delta M_2) + m_2\Delta s + R\left[N-(N+\Delta N)\right] - Rp_\theta\Delta s = 0 \\
\sum M_\zeta^{(O)} &= -M_\zeta\cos\left(\frac{\Delta\theta}{2}\right) + (M_\zeta+\Delta M_\zeta)\cos\left(\frac{\Delta\theta}{2}\right) + m_\zeta\Delta s + M_\theta\sin\left(\frac{\Delta\theta}{2}\right) \\
&\quad + (M_\theta+\Delta M_\theta)\sin\left(\frac{\Delta\theta}{2}\right) + Q_2R\sin\left(\frac{\Delta\theta}{2}\right) + (Q_2+\Delta Q_2)R\sin\left(\frac{\Delta\theta}{2}\right)R = 0
\end{aligned}
\tag{11.2}
$$

Notice that

$$
\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s} = \frac{1}{R} \tag{11.3}
$$

and

$$
\lim_{\Delta\theta\to 0}\left[\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\left(\frac{\Delta\theta}{2}\right)}\right] = 1 \tag{11.4}
$$

where $R(s)$ is the radius of curvature of the axis of the curved beam. Dividing relations (11.1) and (11.2) by Δs, taking the limit as $\Delta s \to 0$ and $\Delta\theta \to 0$, using relations (11.3) and

(11.4) and omitting† the distributed moments m_2 and m_ζ, we obtain

$$\frac{dN}{ds} - \frac{Q_\zeta}{R} + p_\theta = 0 \tag{11.5a}$$

$$\frac{dQ_2}{ds} + p_2 = 0 \tag{11.5b}$$

$$\frac{N}{R} + \frac{dQ_\zeta}{ds} + p_\zeta = 0 \tag{11.5c}$$

and

$$\frac{dM_\theta}{ds} - \frac{M_\zeta}{R} + m_\theta + R\left(\frac{dQ_2}{ds} + p_2\right) = 0 \tag{11.6a}$$

$$\frac{dM_2}{ds} - R\left(\frac{dN}{ds} + p_\theta\right) = 0 \tag{11.6b}$$

$$\frac{dM_\zeta}{ds} + \frac{M_\theta}{R} + Q_2 = 0 \tag{11.6c}$$

Using relations (11.5), relations (11.6) become

$$\frac{dM_\theta}{ds} - \frac{M_\zeta}{R} + m_\theta = 0 \tag{11.7a}$$

$$\frac{dM_2}{ds} - Q_\zeta = 0 \tag{11.7b}$$

$$\frac{dM_\zeta}{ds} + \frac{M_\theta}{R} + Q_2 = 0 \tag{11.7c}$$

Relations (11.5) and (11.7) are the equilibrium equations for planar curved beams. For straight beams ($R \to \infty$, $s \to x_1$, subscripts $\zeta \to 3$ and $\theta \to 1$), these equations yield the equations of equilibrium (8.16) to (8.23) obtained in Section 8.9. From equations (11.5a), (11.5c) and (11.7b), we obtain the following equation for the moment M_2:

$$\frac{d^3M_2}{ds^3} + \frac{R'}{R}\frac{d^2M_2}{ds^2} + \frac{1}{R^2}\frac{dM_2}{ds} + \frac{R'}{R}p_\zeta + \frac{dp_\zeta}{ds} - \frac{p_\theta}{R} = 0$$

where

$$R' = \frac{dR}{ds}$$

Moreover, from relations (11.5b), (11.7a) and (11.7c) we get the following coupled equations for M_ζ and M_θ:

$$\frac{dM_\theta}{ds} - \frac{M_\zeta}{R} + m_\theta = 0 \tag{11.8a}$$

†In practice it is difficult to apply distributed moments m_2 and m_ζ; for this reason they are usually omitted.

$$\frac{d^2 M_\zeta}{ds^2} - \frac{R' M_\theta}{R^2} + \frac{1}{R}\frac{dM_\theta}{ds} - p_2 = 0 \tag{11.8b}$$

When a straight beam is subjected to equal and opposite bending moments M_2 or M_3 at its ends, its cross sections do not twist. Moreover, when a straight beam is subjected to equal and opposite torsional moments M_1 at its ends, its axis does not bend. However, as can be seen from relations (11.8), the bending about the radial axis of planar curved beams and their twisting are coupled. That is, when a cantilever prismatic planar curved beam is subjected to a moment M_ζ (see Fig. 11.1b) at its unsupported end, its cross sections twist. Furthermore, when a cantilever planar curved beam is subjected to a torsional moment M_θ at its unsupported end, its axis bends.

11.3 Computation of the Circumferential Component of Stress Acting on the Cross Sections of Planar Curved Beams Subjected to Bending without Twisting

We assume that the assumptions made for prismatic straight beams (see Section 8.2) are valid for planar curved beams of constant cross sections. However, the assumption that plane sections normal to the axis of the beam before deformation remain plane after deformation (Bernoulli assumption) is valid only for the effect of bending. Thus, referring to Fig. 11.1a we express the circumferential $\hat{u}_\theta^B$, the normal to the plane of the beam $\hat{u}_2^B$ and the radial $\hat{u}_\zeta^B$ components of displacement of a point (s, x_2, ζ) of a planar curved beam, due to bending only, as

$$\hat{u}_\theta^B(s, x_2, \zeta) = u_\theta^B(s) + \zeta\theta_2(s) - x_2\theta_\zeta(s) \tag{11.9a}$$

$$\hat{u}_2^B(s, x_2, \zeta) = u_2^B(s) \tag{11.9b}$$

$$\hat{u}_\zeta^B(s, x_2, \zeta) = u_\zeta^B(s) \tag{11.9c}$$

where s is zero at the cross section at the left end of the beam. The symbols $u_\theta^B(s)$ $u_2^B(s)$ and $u_\zeta^B(s)$ represent the circumferential, the normal to the plane of the beam and the radial components, respectively, of translation of the points of the axis of the beam due to bending only.

In Fig. 11.2, we denote by Δs the length of the arc of the segment of the beam measured along its axis. Moreover, we denote by Δs_ζ the length of the arc of radius $R-\zeta$ Thus, referring to Fig. 11.2 we have

$$\frac{\Delta s}{R} = \frac{\Delta s_\zeta}{R - \zeta} \tag{11.10}$$

In the limit as $\Delta s \to 0$, relation (11.10) reduces to

$$\frac{ds}{ds_\zeta} = \frac{1}{1 - \dfrac{\zeta}{R}} \tag{11.11}$$

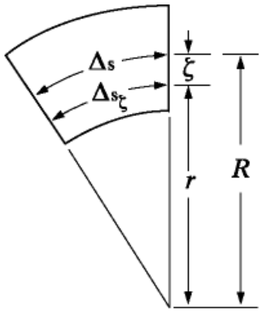


Figure 11.2 Geometry of a segment of a planar curved beam.

Comparing Fig 2.16 with Fig. 11.1a we see that $r = R - \zeta$, $dr \to -d\zeta$, $x_1 \to x_2$, $rd\theta \to -ds_\zeta$, $u_1 \to u_2$, $u_r \to -u_\zeta$, $u_\theta \to -u_\theta$, $e_{1r} \to -e_{2\zeta}$, $e_{r\theta} \to e_{\zeta\theta}$, $e_{1\theta} \to -e_{2\theta}$. Thus, the strain–displacement relations (2.83) can be rewritten as

$$e_{22}(s,\ x_2,\ \zeta) = \frac{\partial \hat{u}_2}{\partial x_2} \tag{11.12a}$$

$$e_{\zeta\zeta}(s,\ x_2,\ \zeta) = \frac{\partial \hat{u}_\zeta}{\partial \zeta} \tag{11.12b}$$

$$e_{\theta\theta}(s,\ x_2,\ \zeta) = \frac{\partial \hat{u}_\theta}{\partial s_\zeta} - \frac{\hat{u}_\zeta}{R - \zeta} \tag{11.12c}$$

$$2e_{2\zeta}(s,\ x_2,\ \zeta) = \frac{\partial \hat{u}_\zeta}{\partial x_2} + \frac{\partial \hat{u}_2}{\partial \zeta} \tag{11.12d}$$

$$2e_{2\theta}(s, x_2, \zeta) = \frac{\partial \hat{u}_\theta}{\partial x_2} + \frac{\partial \hat{u}_2}{\partial s_\zeta} \tag{11.12e}$$

$$2e_{\zeta\theta}(s,\ x_2,\ \zeta) = \frac{\partial \hat{u}_\theta}{\partial \zeta} + \frac{\hat{u}_\theta}{R - \zeta} + \frac{\partial \hat{u}_\zeta}{\partial s_\zeta} \tag{11.12f}$$

In Section 6.2 we show that, when a straight prismatic member is subjected to equal and opposite torsional moments at its ends in a way that its cross sections are free to warp, the axial component of strain e_{11} vanishes. *With this in mind we assume that when a planar curved beam is subjected to torsional moments, the circumferential component of strain* $e^T_{\theta\theta}$ *is negligible*. Thus, substituting relations (11.9) into (11.12c), we get

$$e_{\theta\theta} = e^B_{\theta\theta} = \frac{\partial \hat{u}^B_\theta}{\partial s_\zeta} - \frac{\hat{u}^B_\zeta}{R - \zeta} = \frac{a + \zeta b + x_2 c}{1 - \dfrac{\zeta}{R}} \tag{11.13}$$

where *a, b* and *c* are functions of *s* given as

$$a(s) = \frac{du_\theta^B}{ds} - \frac{u_\zeta^B}{R} \qquad b(s) = \frac{d\theta_2}{ds} \qquad c(s) = -\frac{d\theta_\zeta}{ds} \tag{11.14}$$

For planar curved beams with small R/h ratio (<10), the variation of the circumferential component of strain $e_{\theta\theta}$ given by relation (11.13) becomes considerably different than that for prismatic straight beams.

As in the theories for straight beams, we assume that the components of stress τ_{22}, $\tau_{\zeta\zeta}$ *and* $\tau_{2\zeta}$ *in planar curved beams of constant cross section are negligible compared to* $\tau_{\theta\theta}$ (see Section 8.2). Thus, the first of the stress–strain relations (3.48) for curved beams made from an isotropic, linearly elastic material reduces to

$$\tau_{\theta\theta} = Ee_{\theta\theta} \tag{11.15}$$

Substituting relation (11.13) into (11.15), we get

$$\tau_{\theta\theta} = \frac{E}{1 - \dfrac{\zeta}{R}}(a + \zeta b + x_2 c) \tag{11.16}$$

Substituting relation (11.16) into relations (8.9a), (8.9e) and (8.9f) with subscripts $1 \rightarrow \theta$, $3 \rightarrow \zeta$ and coordinate $x_3 \rightarrow \zeta$, we obtain

$$\begin{aligned}
N &= \iint_A \tau_{\theta\theta}\, dA = E\left[a\iint_A \frac{dA}{1 - \dfrac{\zeta}{R}} + b\iint_A \frac{\zeta\, dA}{1 - \dfrac{\zeta}{R}} + c\iint_A \frac{x_2\, dA}{1 - \dfrac{\zeta}{R}}\right] \\
M_2 &= \iint_A \tau_{\theta\theta}\,\zeta\, dA = E\left[a\iint_A \frac{\zeta\, dA}{1 - \dfrac{\zeta}{R}} + bJ_{22} + cJ_{2\zeta}\right] \\
M_\zeta &= -\iint_A \tau_{\theta\theta} x_2\, dA = -E\left[a\iint_A \frac{x_2\, dA}{1 - \dfrac{\zeta}{R}} + bJ_{2\zeta} + cJ_{\zeta\zeta}\right]
\end{aligned} \tag{11.17}$$

where

$$J_{22} = \iint_A \frac{\zeta^2\, dA}{1 - \dfrac{\zeta}{R}} \qquad J_{\zeta\zeta} = \iint_A \frac{x_2^2\, dA}{1 - \dfrac{\zeta}{R}} \qquad J_{2\zeta} = \iint_A \frac{x_2 \zeta\, dA}{1 - \dfrac{\zeta}{R}} \tag{11.18}$$

In obtaining relations (11.17), we took into account that, since the axes x_2 and ζ are centroidal, we have

$$\iint_A x_2\, dA = \iint_A \zeta\, dA = 0 \tag{11.19}$$

After some calculations and using relations (11.18) and (11.19), it can be shown that the following relations are valid:

$$\iint_A \frac{dA}{1-\frac{\zeta}{R}} = \iint_A dA + \frac{1}{R}\iint_A \zeta dA + \frac{1}{R^2}\iint_A \frac{\zeta^2 dA}{1-\frac{\zeta}{R}} = A + \frac{J_{22}}{R^2}$$

$$\iint_A \frac{x_2 dA}{1-\frac{\zeta}{R}} = \iint_A x_2 dA + \frac{1}{R}\iint_A \frac{x_2\zeta dA}{1-\frac{\zeta}{R}} = \frac{J_{2\zeta}}{R} \tag{11.20}$$

$$\iint_A \frac{\zeta dA}{1-\frac{\zeta}{R}} = \iint_A \zeta dA + \frac{1}{R}\iint_A \frac{\zeta^2 dA}{1-\frac{\zeta}{R}} = \frac{J_{22}}{R}$$

Substituting relations (11.20) into (11.17), we obtain

$$\frac{N}{E} = a\left(A + \frac{J_{22}}{R^2}\right) + \frac{bJ_{22}}{R} + \frac{cJ_{2\zeta}}{R}$$

$$\frac{M_2}{E} = \frac{aJ_{22}}{R} + bJ_{22} + cJ_{2\zeta} \tag{11.21}$$

$$\frac{M_\zeta}{E} = -\frac{aJ_{2\zeta}}{R} - bJ_{2\zeta} - cJ_{\zeta\zeta}$$

Solving equations (11.21) for *a, b* and *c*, we get

$$Ea = \frac{N}{A} - \frac{M_2}{AR}$$

$$Eb = \frac{M_2 J_{\zeta\zeta} + M_\zeta J_{2\zeta}}{J_{22}J_{\zeta\zeta} - J_{2\zeta}^2} - \frac{N}{RA} + \frac{M_2}{AR^2} \tag{11.22}$$

$$Ec = -\frac{M_\zeta J_{22} + M_{22}J_{2\zeta}}{J_{22}J_{\zeta\zeta} - J_{2\zeta}^2}$$

Substituting relations (11.22) into (11.16), we have

$$\tau_{\theta\theta} = \frac{N}{A} - \frac{M_2}{RA} + \frac{M_2 J_{\zeta\zeta} + M_\zeta J_{2\zeta}}{J_{22}J_{\zeta\zeta} - J_{2\zeta}^2}\left(\frac{\zeta}{1-\frac{\zeta}{R}}\right) - \frac{M_\zeta J_{22} + M_2 J_{2\zeta}}{J_{22}J_{\zeta\zeta} - J_{2\zeta}^2}\left(\frac{x_2}{1-\frac{\zeta}{R}}\right) \tag{11.23}$$

where

A = area of the cross section of the beam.
R = the distance from the center of curvature of the curved beam to the centroid of its cross section.

The first two terms on the right side of relation (11.23) represent a uniformly distributed normal (circumferential) component of stress acting on the cross sections of the beam. Notice that in case $N = 0$, a normal component of stress exists at the centroid $(\zeta = 0,\ x_2 = 0)$ of the beam. Moreover, notice that, as R becomes very large, relation (11.23) reduces to relation (9.12a) which gives the normal (axial) component of stress acting on the cross sections of straight beams.

When relation (11.23) is used to calculate the circumferential component of stress acting on the particles of curved beams having thin flanges, it does not give good results. Moreover, the error is not on the safe side. Consider the segment of infinitesimal length of a cantilever beam cut by two radial planes AB and CD shown in Fig.11.3a, subjected to equal and opposite positive bending moments on its end faces. Referring to this figure, we see that the resultant force of the circumferential stresses acting on the end cross sections of each flange of the segment under consideration has a component in the radial direction. These radial components cause the flanges of the beam to deflect radially as shown in Fig. 11.3b and thus distort its cross section. This decreases the circumferential stress $\tau_{\theta\theta}$ at the tips of the flanges and increases it at the portion near the web (see Fig. 11.3c). Thus, for beams with thin flanges, either the resulting circumferential components

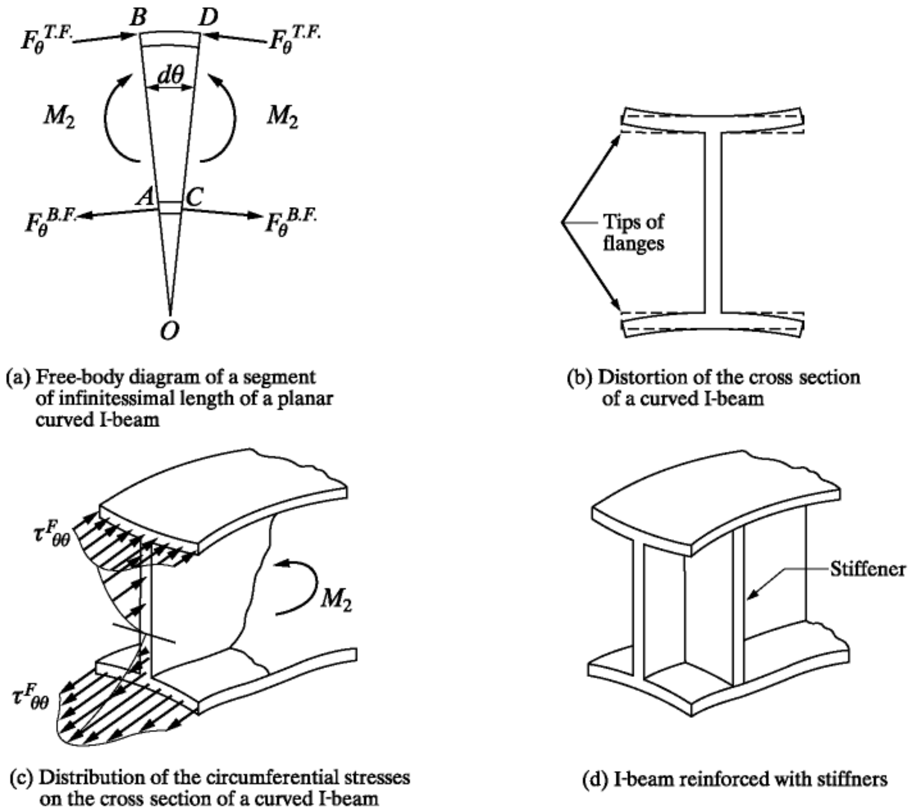


Figure 11.3 Effect of the radial forces acting the cross sections of the flanges of a curved I-beam.

of stress obtained by using relation (11.23) must be corrected†, or the distortion of the cross sections of the beam must be prevented by welding stiffeners on them (see Fig. 11.3d). In the latter case, relation (11.23) gives good results.

When the ζ axis is an axis of symmetry of the cross sections of the beam, referring to the last of relations (11.18), we see that $J_{2\zeta}$ vanishes. Thus, relation (11.23) reduces to

$$\tau_{\theta\theta} = \frac{N}{A} - \frac{M_2}{RA} + \frac{M_2}{J_{22}}\left(\frac{\zeta}{1 - \frac{\zeta}{R}}\right) - \frac{M_\zeta}{J_{\zeta\zeta}}\left(\frac{x_2}{1 - \frac{\zeta}{R}}\right) \tag{11.24}$$

When the ζ axis is an axis of symmetry of the cross section of the beam and, moreover, $M_\zeta = 0$, relation (11.24) reduces to

$$\tau_{\theta\theta} = \frac{N}{A} + M_2\left[-\frac{1}{RA} + \frac{\zeta}{J_{22}\left(1 - \frac{\zeta}{R}\right)}\right] \tag{11.25}$$

Referring to Fig. 11.2, we have

$$r = R - \zeta \tag{11.26}$$

Moreover, we define

$$A_m = \iint_A \frac{dA}{r} \tag{11.27}$$

Furthermore, referring to the first of relations (11.18), using relations (11.26) and (11.27) and noting that $\iint_A \zeta dA$ is equal to zero, we obtain

$$J_{22} = \iint_A \frac{\zeta^2 dA}{1 - \frac{\zeta}{R}} = R\iint_A \frac{(R-r)^2}{r} dA = R^2(RA_m - A) \tag{11.28}$$

Substituting relation (11.28) into (11.25) and using (11.26), we obtain the following formula for the circumferential component of stress of a curved beam whose cross sections are symmetric about the ζ axis

$$\tau_{\theta\theta} = \frac{N}{A} + \frac{M_2\left(\frac{A}{r} - A_m\right)}{A(RA_m - A)} \tag{11.29}$$

---

†A way for correcting the values of the circumferential components of stress acting on the cross section of planar curved beams with thin flanges was proposed by professor Hans Bleich "Die Spannungsverteilung in den Gurtungen gekrummter stabemit T und I-formigen Querschnitt" Der stahlblau Beilage zur Zeitschriff, Die Bautechnik, 6(1) 1933 p. 3-6.

Table 11.1 Formulas for A_m, A and R for curved beams whose cross sections are symmetric about their ζ axis.

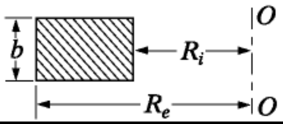
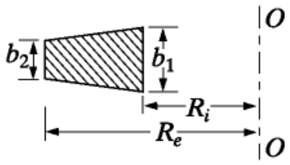
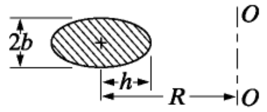
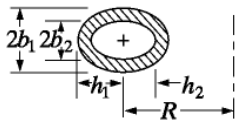
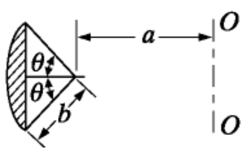
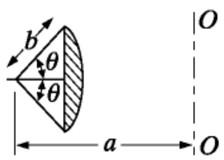
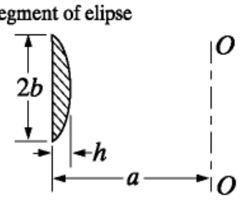
	$A = b(R_e - R_i) \quad R = \dfrac{R_e + R_i}{2} \quad A_m = b\ln\left(\dfrac{R_e}{R_i}\right)$
 Becomes a triangle if b_2 or b_1 are set equal to zero	$A = \dfrac{b_1 + b_2}{2}(R_e - R_i) \quad R = \dfrac{R_i(2b_1 + b_2) + R_e(2b_2 + b_1)}{3(b_1 + b_2)}$ $A_m = \dfrac{b_1R_e - b_2R_i}{R_e - R_i}\ln\left(\dfrac{R_e}{R_i}\right) - b_1 + b_2$
 Becomes a circle of radius b if $h=b$	$A = \pi bh$ $A_m = \dfrac{2\pi b}{h}\left(R - \sqrt{R^2 - h^2}\right)$
 Becomes a circular ring of external radius R_e and internal radius R_i if $R_e=h_1=b_1$ and $R_i=h_2=b_2$	$A = \pi(b_1h_1 - b_2h_2)$ $A_m = 2\pi\left(\dfrac{b_1R}{h_1} - \dfrac{b_2R}{h_2} - \dfrac{b_1}{h_1}\sqrt{R^2 - h_1^2} + \dfrac{b_2}{h_2}\sqrt{R^2 - h_2^2}\right)$
Segment of circle $k=\sqrt{\lvert b^2-a^2\rvert}$ 	$A = b^2\theta - \dfrac{b^2}{2}\sin 2\theta \quad R = a + \dfrac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$ For $a > b$ $A_m = 2a\theta - 2b\sin\theta - \pi k + 2k\sin^{-1}\left(\dfrac{b + a\cos\theta}{a + b\cos\theta}\right)$ For $b > a$ $A_m = 2a\theta - 2b\sin\theta + 2k\ln\left(\dfrac{b + a\cos\theta + k\sin\theta}{a + b\cos\theta}\right)$
Segment of circle $k=\sqrt{\lvert b^2-a^2\rvert}$ 	$A = b^2\theta - \dfrac{b^2}{2}\sin 2\theta \quad R = a - \dfrac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)}$ $A_m = 2a\theta + 2b\sin\theta - k\pi - 2k\sin^{-1}\left(\dfrac{b - a\cos\theta}{a - b\cos\theta}\right)$

Table 11.1 Continued

Segment of elipse 	$A = \dfrac{\pi b h}{2} \qquad R = a - \dfrac{4h}{3\pi}$ $A_m = 2b + \dfrac{\pi b}{h}(a - \sqrt{a^2 - h^2}) - \dfrac{2b}{h}\sqrt{a^2 - h^2}\,\sin^{-1}\left(\dfrac{h}{a}\right)$

In Table 11.1, we present formulas for computing A_m, A and R for several symmetric about their ζ axis cross sections of curved beams. The values of A_m, A and R for symmetric about their ζ axis composite cross sections, composed of n parts, may be established using the following formulas:

$$A_m = \sum_{i=1}^{n} (A_m)_i \qquad A = \sum_{i=1}^{n} A_i \qquad R = \frac{\sum_{i=1}^{n} A_i R_i}{\sum_{i=1}^{n} A_i} \tag{11.30}$$

In Example 6 of Section 7.8.3 the components of stress acting on the particles of a thin curved beam of rectangular cross section subjected at its ends to equal and opposite bending moments M_2 are established on the basis of the theory of elasticity. In Table 11.2 we tabulate, for various values of h/R, the ratios of the maximum value of the circumferencial component of stress $\tau_{\theta\theta}^{M}$ obtained on the basis of relation (11.29) to that obtained on the basis of the theory of elasticity $\tau_{\theta\theta}^{E}$. Moreover, in Table 11.2 we tabulate the ratios of the maximum circumferencial component of stress $\tau_{\theta\theta}^{S.B.}$ obtained on the basis

Table 11.2 Ratios of the maximum circumferencial stress $\tau_{\theta\theta}^{M}$ or $\tau_{\theta\theta}^{S.B.}$ acting on the cross sections of curved beams of rectangular cross sections obtained on the basis of relation (11.29) or (9.12b), respectively, to that obtained on the basis of the theory of elasticity $\tau_{\theta\theta}^{E}$.

h/R	$\tau_{\theta\theta}^{M}/\tau_{\theta\theta}^{E}$	$\tau_{\theta\theta}^{S.B}/\tau_{\theta\theta}^{E}$
3/2	1.040	0.450
1	0.990	0.526
1/2	0.997	0.654
1/3	0.999	0.888
1/5	0.999	0.933

of formula (9.12b) for straight beams to that obtained on the basis of the theory of elasticity $\tau_{\theta\theta}^{E}$. Referring to Table 11.2, we can make the following observations:

1. The results obtained on the basis of relation (11.29) are accurate for all values of h/R $(3/2 > h/R > 1/5)$ with a maximum error of 4% for $h/R = 3/2$.

2. When $h/R < 1/5$, the error in the circumferencial component of stress $\tau_{\theta\theta}$ obtained by using the formula for straight beams (9.12b) is less than 6.7%.

In what follows, we present three examples.

Example 1 Consider the cantilever planar curved beam of constant rectangular cross section of width b, depth h, and h/R ratio of 0.25, shown in Fig. a. The beam is subjected to a bending moment M_2^* at its unsupported end. Compute the maximum normal component of stress acting on the cross sections of the beam.

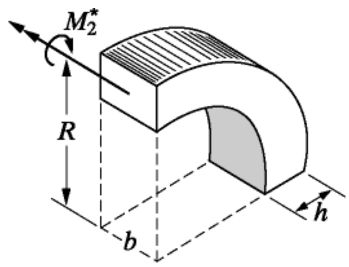


Figure a Geometry and loading of the beam.

Solution Referring to Fig. a and to Table 11.1, we have

$$A = bh$$

$$A_m = b\ln\left(\frac{R_e}{R_i}\right) = b\ln\left(\frac{2+\frac{h}{R}}{2-\frac{h}{R}}\right) = b\ln\left(\frac{2.25}{1.75}\right) = b\ln(1.2857) = 0.251314b \qquad \text{(a)}$$

$$N = 0$$

Substituting relations (a) into (11.29), using relation (11.26) and $h/R = 0.25$, we obtain

$$\tau_{\theta\theta} = \frac{M_2^*\left(\frac{A}{r} - A_m\right)}{A(RA_m - A)} = \frac{M_2^*}{bh}\left[\frac{\frac{bh}{r} - (0.251314)b}{Rb(0.251314) - bh}\right] = \frac{M_2^*}{0.005256bh^2}\left[\frac{1}{4-\frac{\zeta}{h}} - 0.251314\right] \qquad \text{(b)}$$

The neutral axis is located at

$$\tau_{\theta\theta} = 0$$

or

$$\zeta = 0.02091h \qquad \text{(c)}$$

The distribution of the normal component of stress on a cross section of the planar curved beam under consideration and of a straight beam of the same cross section subjected to the same moment is shown in Fig. b; referring to this figure we can make the following observations:

1. The neutral axis of the curved beam does not pass through the centroid of its cross sections.
2. The maximum circumferencial stress acting on the cross sections of a curved beam with h/R = 1/4 is 9.1% bigger than that computed by formula (9.12b) for the normal component of stress acting on the cross sections of a straight beam of the same rectangular cross section when subjected to the same moment.

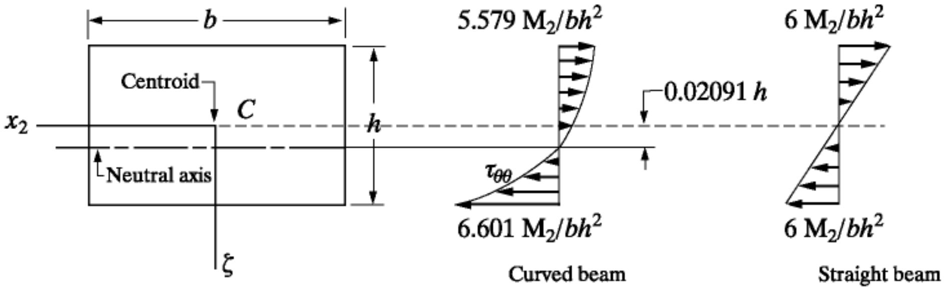
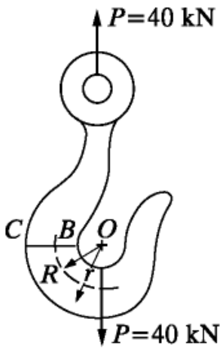


Figure b Distribution of the normal component of stress.

Example 2 The crane hook shown in Fig. a is made from an isotropic, linearly elastic material and is subjected to a force P = 40kN. Compute the circumferencial component of stress $\tau_{\theta\theta}$ acting on its cross section BC whose geometry is shown in Fig. b.



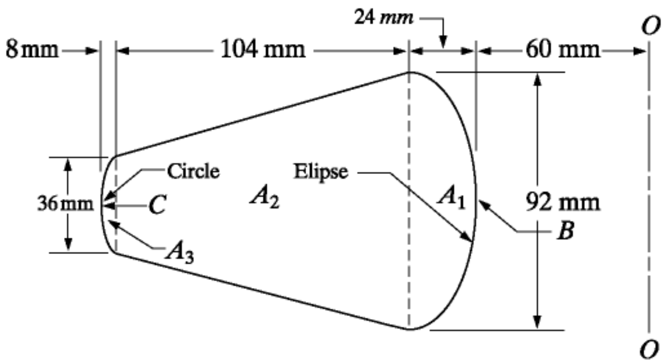


Figure a Crane hook.

Figure b Geometry of the cross section BC of the crane hook.

Solution We divide the cross section of the crane hook into three sub-regions whose area we denote by A_1, A_2, A_3 (see Fig. b).

For the region of area A_1 referring to Fig b and to Table 11.1, we obtain

$$a = 84 \text{ mm} \qquad h = 24 \text{ mm} \qquad b = 46 \text{ mm} \tag{a}$$

Referring to Table 11.1, we have

$$A_1 = \frac{\pi bh}{2} = \frac{\pi(46)(24)}{2} = 1{,}734.16 \text{ mm}^2$$

$$A_{m1} = 2b + \frac{\pi b}{h}\left(a - \sqrt{a^2 - h^2}\right) - \frac{2b}{h}\left(\sqrt{a^2 - h^2}\right)\sin^{-1}\left(\frac{h}{a}\right) = 23.67 \tag{b}$$

$$R_1 = a - \frac{4h}{3\pi} = 73.81$$

For the region of area A_2 referring to Fig b and to Table 11.1, we get

$$R_i = 84 \text{ mm} \qquad R_e = 188 \text{ mm} \qquad b_1 = 92 \text{ mm} \qquad b_2 = 36 \text{ mm} \tag{c}$$

Moreover,

$$A_2 = \frac{b_1 + b_2}{2}(R_e - R_i) = 6{,}656 \text{ mm}^2$$

$$A_{m2} = \frac{b_1 R_e - b_2 R_i}{R_e - R_i} \ln\left(\frac{R_e}{R_i}\right) - b_1 + b_2 = 54.56 \text{ mm} \tag{d}$$

$$R_2 = \frac{R_i(2b_1 + b_2) + R_e(2b_2 + b_1)}{3(b_1 + b_2)} = 128.42 \text{ mm}$$

For the region of area A_3 referring to Fig. b and Table 11.1, we have

$$b^2 = 18^2 + (b-8)^2 \qquad \text{or} \qquad b = 24.25 \text{ mm}$$

$$\sin\theta = \frac{18}{24.25} = 0.742268 \qquad \text{or} \qquad \theta = 47°55.5' = 0.836 \text{ rad}$$

Moreover,

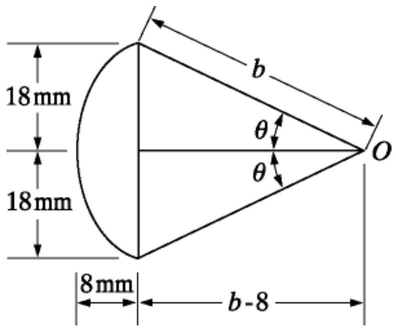


Figure b Region of area A_3.

$$a = 171.75 \text{ mm}$$

$$A_3 = b^2\theta - \frac{b^2}{2}\sin 2\theta = 199.09\,\text{mm}^2$$

$$A_{m3} = 2a\theta - 2b\sin\theta - \pi\sqrt{a^2-b^2} + 2\sqrt{a^2-b^2}\sin^{-1}\left(\frac{b + a\cos\theta}{a + b\cos\theta}\right) = 1.04 \text{ mm}^2 \tag{e}$$

$$R_3 = a + \frac{4b\sin^3\theta}{3(2\theta - \sin 2\theta)} = 191.25\,\text{mm}^2$$

Substituting the values of $A_i\, A_{mi}$ and R_i $(i = 1, 2, 3)$ computed above in relations (11.30), we obtain

$$A = A_1 + A_2 + A_3 = 1{,}734.16 + 6{,}656 + 199.09 = 8{,}587.25 \text{ mm}^2$$

$$A_m = A_{m1} + A_{m2} + A_{m3} = 23.7 + 54.56 + 1.04 = 79.27 \text{ mm}$$

$$R = \frac{\sum_{i=1}^{3} A_i R_i}{\sum_{i=1}^{3} A_i} = \frac{1734.16(73.81) + 6{,}656(128.42) + 199.09(191.25)}{8{,}587.25} = 118.88 \text{ mm} \tag{f}$$

Substituting the values of A, A_m and R from (f) into relation (11.29) and noting that $M_2 = PR$, we find the following expression for the circumferential component of stress:

$$\tau_{\theta\theta}(r) = \frac{N}{A} + \frac{M_2\left(\frac{A}{r} - A_m\right)}{A(RA_m - A)}$$

$$= \frac{40}{8{,}587.25} + \frac{118.88(40)\left(\frac{8{,}587.25}{r} - 79.27\right)}{8{,}587.25[118.88(79.27) - 8{,}587.25]} = \frac{5.6855}{r} - 0.0478\,\frac{\text{kN}}{\text{mm}^2}$$

Thus,

$$(\tau_{\theta\theta})_{\text{max tension}} = \tau_{\theta\theta}(60) = 46.93 \text{ MPa tension}$$

$$(\tau_{\theta\theta})_{\text{max compr.}} = \tau_{\theta\theta}(196) = 18.79 \text{ MPa compression}$$

Example 3 Consider a cantilever curved beam of radius $R = 100$mm made from a steel angle whose cross sectioned properties are given below. The beam is subjected to a bending moment $M_2 = 24$kN·m at its free end. Compute the distribution of the normal component of stress $\tau_{\theta\theta}$ acting on the cross sections of the beam and locate their neutral axis.

$$a_1 = 16.15 \qquad a_2 = 51 - 16.15 = 34.85 \text{ mm}$$
$$h = 51 \text{ mm} \qquad t = 9.5 \text{ mm} \tag{a}$$
$$A = 877 \text{ mm} \qquad a_3 = a_1 - t = 16.15 - 9.5 = 6.65 \text{ mm}$$

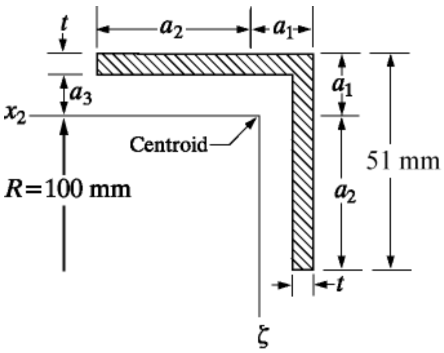


Figure a Cross section of the beam.

Solution We first compute the constants J_{22}, $J_{\zeta\zeta}$, $J_{2\zeta}$ of the cross sections of the beam. Referring to relations (a), we have

$$n = \ln\left(\frac{R + a_3}{R + a_1}\right) \qquad m = \ln\left(\frac{R - a_2}{R + a_3}\right) \tag{b}$$

Substituting the values of the geometric constants (a) into relations (b), we get

$$\begin{aligned} m &= \ln\left(\frac{100 - 34.85}{100 + 6.65}\right) = \ln(0.610877) = -0.4929 \\ n &= \ln\left(\frac{100 + 6.65}{100 + 16.15}\right) = \ln(0.9184) = -0.08533 \end{aligned} \tag{c}$$

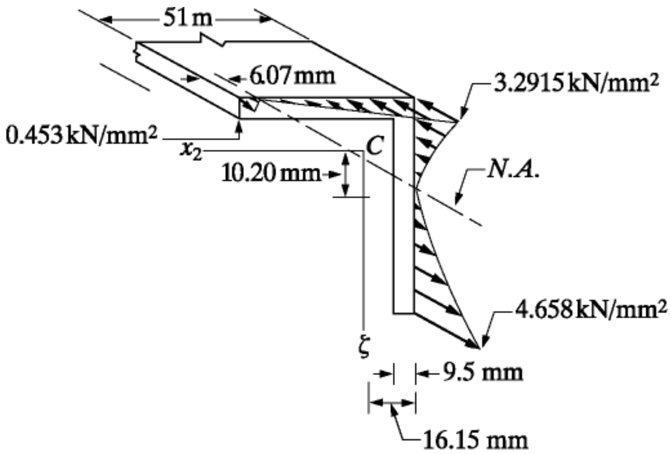


Figure b Results.

Using relation (11.28) and (c), referring to Fig. a and integrating relations (11.18), we have

$$J_{22} = \iint_A \frac{\zeta^2 dA}{1 - \frac{\zeta}{R}} = -R^2[A + R(tm + hn)]$$

$$= -100^2[877 + 100[9.5(-0.4929) + 51(-0.08533)]]$$

$$= 264.38(10^3)\ \text{mm}^4$$

$$J_{\zeta\zeta} = \iint_A \frac{x_2^2 dA}{1 - \frac{\zeta}{R}} = -\frac{R}{3}[(a_1^3 - a_3^3)m + (a_1^3 + a_2^3)n]$$

$$= -\frac{100}{3}[[(16.15)^3 - (6.65)^3](-0.4929) \qquad \text{(d)}$$

$$+ [(16.15)^3 + (34.85)^3](-0.08533)] = 196.747(10^3)\text{mm}^4$$

$$J_{2\zeta} = \iint_A \frac{x_2 \zeta dA}{1 - \frac{\zeta}{R}} = -\frac{R^2}{2}[(a_3^2 - a_1^2)m + (a_2^2 - a_1^2)n]$$

$$= -\frac{100^2}{2}[[(6.65)^2 - (16.15)^2](-0.4929)$$

$$+ [(34.85)^2 - (16.15)^2](-0.08533)] = -126.915(10^3)\ \text{mm}^4$$

From relations (d), we get

$$J_{22}J_{\zeta\zeta} - J_{2\zeta}^2 = (264.38)(10^3)(196.747)(10^3) - (126.914)^2(10^6) = 35{,}908.5(10^6)\ \text{mm}^8 \qquad \text{(e)}$$

Substituting relations (d) and (e) into (11.23), we obtain

$$\tau_{\theta\theta} = -\frac{M_2}{RA} + \frac{M_2 J_{\zeta\zeta}}{J_{22}J_{\zeta\zeta} - J_{2\zeta}^2}\left(\frac{\zeta}{1 - \frac{\zeta}{R}}\right) - \frac{M_2 J_{2\zeta}}{J_{22}J_{\zeta\zeta} - J_{2\zeta}^2}\left(\frac{x_2}{1 - \frac{\zeta}{R}}\right)$$

$$= -\frac{24{,}000}{100(877)} + \frac{24{,}000(196.747)(10^3)(100)}{35{,}908.5(10^6)}\left(\frac{\zeta}{100 - \zeta}\right) \qquad \text{(f)}$$

$$-\frac{24{,}000(-126.915)(10^3)(100)}{35908.5(10^6)}\left(\frac{x_2}{100 - \zeta}\right)$$

$$= -0.27366 + 13.1499\left(\frac{\zeta}{100 - \zeta}\right) + 8.48256\left(\frac{x_2}{100 - \zeta}\right)$$

At the outer surface of the curved beam ($\zeta = -16.15$ mm), relation (f) gives

$$\tau_{\theta\theta} = -2.102079 + 0.073031 x_2 \qquad \text{(g)}$$

From relation (f) we find that the equation of the neutral axis ($\tau_{\theta\theta} = 0$) is

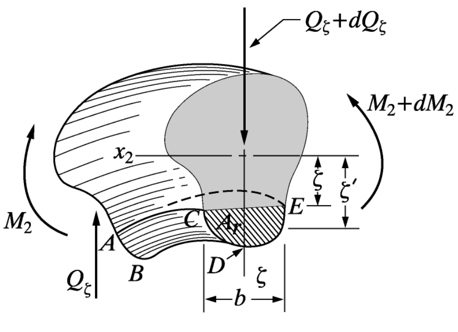
$$13.4236\zeta + 8.48256x_2 = 27.366 \tag{h}$$

The results are shown in Fig. b

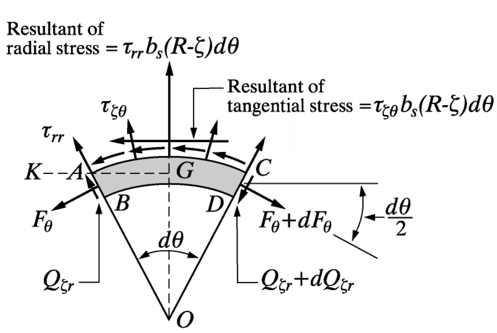
11.4 Computation of the Radial and Shearing Components of Stress in Curved Beams

In Section 11.3 we derive a formula for the circumferential component of stress $\tau_{\theta\theta}$ acting on the cross sections of prismatic planar curved beams. This formula is based on the assumption that the radial component of stress τ_{rr} is negligible. This assumption is satisfactory for curved beams, which do not have thin webs, as, for example, beams of circular, rectangular or trapezoidal cross sections. In Fig. b of Example 6 of Section 7.8.3, a thin curved beam of rectangular cross section having $R_e = 2R_i$ and $h/R = 2/3$ is considered. It is found using the theory of elasticity, that when this beam is subjected to equal and opposite bending moments M_2 at its ends, a radial component of stress τ_{rr} exists, whose maximum value is approximately 13% of the maximum value of the circumferential stress $\tau_{\theta\theta}$. However, it is found that the maximum value of τ_{rr} occurs slightly below the center line of the cross sections of the beam where $\tau_{\theta\theta}$ is small. Consequently, the radial component of stress does not affect the ability of such beams to carry the applied moment, except when they are made from an anisotropic material, like wood, which does not have much strength in the radial direction. Moreover, the maximum value of the radial stress in I- or T-beams having thin webs may exceed the maximum value of the circumferential stress and thus, for such beams, it cannot be disregarded.

In what follows we derive formulas for the radial component of stress τ_{rr} and the shearing component of stress $\tau_{\theta\zeta}$ acting on the particles of planar curved beams of constant cross section subjected to bending moments whose vector is normal to their plane as well as to external forces whose line of action lies in a plane which is parallel to the plane of the beam and contains the shear centers of its cross sections. For this



(a) Segment of a curved beam



(b) Portion *ABCDE* of the segment of Fig. 11.4a

Figure 11.4 Radial component of stress in prismatic planar curved beams.

purpose, we consider a segment *BDFG* of length *ds* of a prismatic planar curved beam cut by two radial planes (see Fig. 11.4a). Moreover, we consider the equilibrium of a portion of this segment *ABCDE* cut from it by a cylindrical surface $r = R - \zeta$ = constant (see Fig. 11.4b). The end surfaces of this segment are subjected to circumferential and shearing stresses. In Fig. 11.4b the resultant forces of the circumstantial stresses are designated as F_θ and $F_\theta + \Delta F_\theta$ while the resultant forces of the radial stresses are designated by $Q_{\zeta r}$ and $Q_{\zeta r} + \Delta Q_{\zeta r}$. Moreover, as shown in Fig. 11.4b, the part *AC* of the lateral surface of the portion under consideration is subjected to radial stresses. Referring to Fig. 11.4b, we have

$$\sum F_{\bar{OG}} = 0 = (2F_\theta + dF_\theta)\sin\left(\frac{d\theta}{2}\right) - \tau_{rr} b_s (R-\zeta) d\theta + (Q_{\zeta r} + dQ_{\zeta r} - Q_{\zeta r})\cos\left(\frac{d\theta}{2}\right) = 0 \tag{11.31a}$$

$$\sum F_{\bar{GK}} = 0 = dF_\theta \cos\left(\frac{d\theta}{2}\right) - (2Q_{\zeta r} + dQ_{\zeta r})\sin\left(\frac{d\theta}{2}\right) - \tau_{\theta\zeta} b_s \left(1 - \frac{\zeta}{R}\right) R\, d\theta = 0 \tag{11.31b}$$

where A_r is the area of the portion $CDE = A_r$ of the cross section of the beam (see Fig. 11.4a) and

$$F_\theta = \iint_{A_r} \tau_{\theta\theta}\, dA \tag{11.32a}$$

$$Q_{\zeta r} = \left\{ \begin{array}{l} \text{resultant force of the shearing component of} \\ \text{stress } \tau_{\theta\zeta} \text{ acting on the cross-sectional area } A_r \end{array} \right\} = \iint_{A_r} \tau_{\theta\zeta}\, dA \tag{11.32b}$$

Substituting relation (11.23) into (11.32a), we get

$$F_\theta = \frac{NA_r}{A} - \frac{M_2 A_r}{AR} + \frac{M_2 J_{\zeta\zeta} + M_\zeta J_{2\zeta}}{J_{\zeta\zeta} J_{22} - J_{\zeta 2}^2} \iint_{A_r} \frac{\zeta'}{1 - \frac{\zeta'}{R}}\, dA - \frac{M_\zeta J_{22} + M_2 J_{2\zeta}}{J_{\zeta\zeta} J_{22} - J_{\zeta 2}^2} \iint_{A_r} \frac{x_2}{1 - \frac{\zeta'}{R}}\, dA \tag{11.33}$$

Noting that $\sin(d\theta/2) \approx d\theta/2$, $\cos d\theta/2 \approx 1$ and $ds = R d\theta$, from relations (11.31), we obtain

$$\tau_{rr} = \frac{R}{b_s(R-\zeta)}\left(\frac{F_\theta}{R} + \frac{dQ_{\zeta r}}{ds}\right) \tag{11.34a}$$

and

$$\tau_{\theta\zeta} = \frac{R}{b_s(R-\zeta)}\left(\frac{dF_\theta}{ds} - \frac{Q_{\zeta r}}{R}\right) \tag{11.34b}$$

Referring to relation (11.34b), we see that the shearing component of stress $\tau_{\theta\zeta}$ depends on the shearing force $Q_{\zeta r}$ which is not known. Thus, we must eliminate $Q_{\zeta r}$ from relation (11.34b), using relation (11.32b). This leads to an integral equation on the dependent variable $\tau_{\theta\zeta}$. In order to avoid having to solve such an equation, we disregard the effect of curvature of a curved beam on the magnitude of $Q_{\zeta r}$. That is, we assume that in relations (11.32b) $\tau_{\theta\zeta}$ may be approximated by the shearing stress τ_{13} acting on the cross section of a straight beam whose cross section is identical to that of the curved beam under consideration. That is, referring to Fig. 11.5, substituting relations (9.67) into (11.32b) and recalling that $q_{\theta\zeta} = \tau_{\theta\zeta} b$ and $\mathrm{d}A = b\,\mathrm{d}\zeta$, we get

$$Q_{\zeta r} = K(\zeta) Q_\zeta \tag{11.35}$$

where

$$K(\zeta) = \iint_{A_r} \frac{\tau_{\theta\zeta}(\zeta')}{Q_\zeta} dA = \int_{\zeta'=\zeta}^{\zeta'=h_b} \frac{q_{\theta\zeta}(\zeta')d\zeta'}{Q_\zeta} = \int_{\zeta'=\zeta}^{\zeta'=h_b} \frac{\bar{\zeta}_r A_r'}{I_2} d\zeta' \tag{11.36}$$

and

A_r = area of portion DCE of the cross section of Fig. 11.5a.
A_r' = cross-hatched area of the cross section of Fig. 11.5a.
$\bar{\zeta}_r$ = distance of the centroid of area A_r' measured from the x_2 axis.
I_2 = moment of inertia of the cross section about the x_2 axis.

For example, referring to Fig. 11.5, for a rectangular cross section, we have

$$\bar{\zeta}_r = \zeta' + \frac{1}{2}\left(\frac{h}{2} - \zeta'\right) = \frac{1}{2}\left(\zeta' + \frac{h}{2}\right) \qquad A_r' = \left(\frac{h}{2} - \zeta'\right) b \qquad I_2 = \frac{bh^3}{12} \tag{11.37}$$

Substituting relations (11.37) into (11.36), we obtain

$$K(\zeta) = \int_\zeta^{\frac{h}{2}} \frac{\left(\frac{h^2}{4} - (\zeta')^2\right)}{2I_2} b\, d\zeta' = \frac{b}{2I_2}\left[\frac{h^2\zeta'}{4} - \frac{(\zeta')^3}{3}\right]_\zeta^{\frac{h}{2}} = \frac{1}{2}\left[1 - 3\left(\frac{\zeta}{h}\right) + 4\left(\frac{\zeta}{h}\right)^3\right] \tag{11.38}$$

Substituting relations (11.33) and (11.35) into (11.34a) and using relation (11.5c), we get

$$\begin{aligned}\tau_{rr} = \frac{1}{b_s(R-\zeta)}\Bigg[& N\left(\frac{A_r}{A} - K\right) - \frac{M_2 A_r}{AR} + \frac{M_2 J_{\zeta\zeta} + M_\zeta J_{2\zeta}}{J_{\zeta\zeta}J_{22} - J_{2\zeta}^2}\iint_{A_r} \frac{\zeta}{1 - \frac{\zeta}{R}} dA \\ & - \frac{M_\zeta J_{22} + M_2 J_{2\zeta}}{J_{\zeta\zeta}J_{22} - J_{2\zeta}^2}\iint_{A_r} \frac{x_2}{1 - \frac{\zeta}{R}} dA - KRp_\zeta\Bigg]\end{aligned} \tag{11.39}$$

If the cross section of a beam is symmetric about the ζ axis, referring to relations (11.18) we see that $J_{2\zeta}$ vanishes. Moreover, if in addition $M_\zeta = 0$, relation (11.39) reduces to

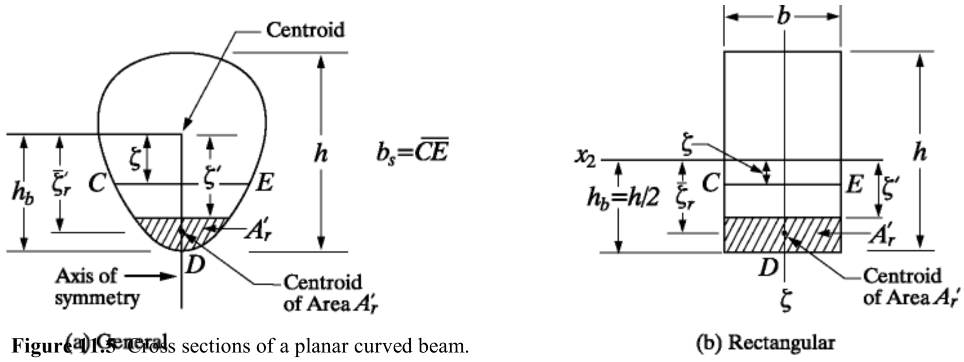


Figure 11.5 Cross sections of a planar curved beam.

$$\tau_{rr} = \frac{1}{b_s(R-\zeta)}\left[N\left(\frac{A_r}{A} - K\right) - KRp_\zeta + M_2\left[-\frac{A_r}{RA} + \frac{1}{J_{22}}\iint_{A_r}\frac{\zeta\, dA}{\left(1-\frac{\zeta}{R}\right)}\right]\right] \tag{11.40}$$

Notice that, as R increases, τ_{rr} decreases. Moreover, notice that

$$\iint_{A_r}\frac{\zeta\, dA}{\left(1-\frac{\zeta}{R}\right)} = -R\iint_{A_r} dA + R\iint_{A_r}\frac{dA}{\left(1-\frac{\zeta}{R}\right)} = R\left(-A_r + RA_{mr}\right) \tag{11.41}$$

where

$$A_{mr} = \iint_{A_r}\frac{dA}{r} \tag{11.42}$$

Substituting relations (11.26) and (11.41) into (11.40) and using relation (11.28), we get

$$\tau_{rr}(s,\zeta) = \frac{1}{b_s r}\left[N\left(\frac{A_r}{A} - K\right) - KRp_\zeta + \frac{M_2}{A}\left(\frac{AA_{mr} - A_rA_m}{RA_m - A}\right)\right] \tag{11.43}$$

where A_{mr}, A_m and r are defined by relations (11.42), (11.27) and (11.26), respectively; A is the total area of the cross section of the beam; A_r is the area of the portion DCE of the cross section of Fig. 11.5a of the beam; R is the radius of curvature of the axis of the beam.

Consider a planar beam subjected to bending moments whose vector is normal to the plane of the beam and to external forces whose line of action lies on a plane which is parallel to the plane of the beam and contains the shear centers of its cross sections. For this beam $M_\zeta = Q_2 = M_\theta = 0$. Taking this into account and substituting relations (11.33) and (11.35) into (11.34b) and using relations (11.7b), (11.6c) and (11.5a), we obtain

$$\tau_{\theta\zeta}(s,\zeta) = \frac{RQ_\zeta}{b_s(R-\zeta)}\left[\frac{1}{J_{\zeta\zeta}J_{22} - J_{\zeta 2}^2}\left[J_{\zeta\zeta}\iint_{A_r}\frac{\zeta}{R-\zeta}\,dA - J_{2\zeta}\iint_{A_r}\frac{x_2}{R-\zeta}\,dA\right] - \frac{K}{R}\right]$$

$$- \frac{RA_r p_\theta}{b_s A(R-\zeta)} \tag{11.44}$$

If the cross section of a beam is symmetric about the ζ axis, $J_{2\zeta}$ vanishes. Thus, using relations (11.41) and (11.42), relation (11.44) reduces to

$$\begin{aligned} \tau_{\theta\zeta} &= \frac{RQ_\zeta}{b_s(R-\zeta)}\left[\frac{1}{J_{22}}\iint_{A_r}\frac{\zeta}{R-\zeta}\,dA - \frac{K}{R}\right] - \frac{RA_r p_\theta}{b_s A(R-\zeta)} \\ &= \frac{RQ_\zeta}{b_s(R-\zeta)}\left[\frac{1}{A}\left(\frac{AA_{mr} - A_r A_m}{RA_m - A}\right) - \frac{K}{R}\right] - \frac{RA_r p_\theta}{b_s A(R-\zeta)} \end{aligned} \tag{11.45}$$

In what follows we present one example.

Example 4 Consider the planar curved beam of constant cross section shown in Figs. a and b, subjected to a force $P = 80$ kN. Determine the circumferential, the radial and the shearing components of stress acting on the particles of line EF of cross section ADH of this beam.

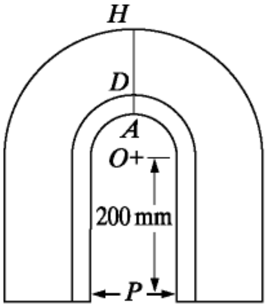


Figure a Geometry and loading of the beam.

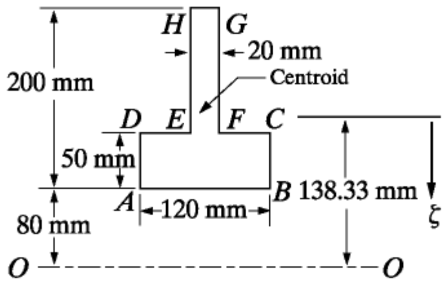


Figure b Cross section ADH of the beam.

Solution Referring to Figs. a and b, we have

$$\begin{aligned} A &= (150)(20) + (50)(120) = 9{,}000 \text{ mm}^2 \\ R &= \frac{[(50)(120)(105) + 20(150)(205)]}{9{,}000} = 138.33 \text{ mm} \end{aligned} \tag{a}$$

Moreover, considering the area of the cross section of the beam as the sum of parts $ABCD$ and $EFGH$ and referring to relations (11.30) and to Table 11.1, we have

$$A_m = \sum_{j=1}^{2}(A_m)_j = \sum_{j=1}^{2} b_j \ln\left(\frac{R_e}{R_i}\right)_j = 120\ln\left(\frac{130}{80}\right) + 20\ln\left(\frac{280}{130}\right) = 73.606 \text{ mm} \tag{b}$$

Furthermore, referring to Fig. a, we have

$$N = 80 \text{ kN} \quad M_2 = (138.33 + 200)\,80 = 27{,}066.4 \text{ kN mm} \qquad Q_\zeta = 0 \qquad p_\theta = 0 \tag{c}$$

Substituting relations (a), (b) and (c) into (11.29), we obtain

$$\tau_{\theta\theta} = \frac{N}{A} + \frac{M_2\left(\dfrac{A}{r} - A_m\right)}{A(RA_m - A)} = \frac{80}{9{,}000} + \frac{27{,}066.4\left(\dfrac{9{,}000}{r} - 73.606\right)}{9{,}000\left[(138.33)(73.606) - 9{,}000\right]}$$

$$= 0.0088889 + \frac{22.90}{r} - 0.18729\left(\frac{\text{kN}}{\text{mm}^2}\right) = -178.40 + \frac{22{,}900}{r}\ (\text{MPa})$$

Thus,

$$(\tau_{\theta\theta})_{r=130} = -2.25 \text{ MPa}$$

$$(\tau_{\theta\theta})_{r=80} = 107.85 \text{ MPa} \tag{d}$$

$$(\tau_{\theta\theta})_{r=280} = -96.61 \text{ MPa}$$

In order to compute the radial stresses at the junction of the flange and web of the beam, we calculate the following quantities:

$$A_r = 50 \times 120 = 6{,}000 \text{ mm}^2$$

$$A_{mr} = A_m \text{ of rectangle } ABCD \text{ (see Table 11.1)} = b \ln\left(\frac{R_e}{R_i}\right) = 120 \ln\left(\frac{130}{80}\right) = 58.26$$

$$r = 130 \text{ mm}$$

$$I_2 = \frac{120(50)^3}{12} + 50(120)(33.33)^2 + \frac{20(150)^3}{12} + 20(150)(75 - 8.33)^2 = 2{,}687.4(10^4) \text{ mm}^4 \tag{e}$$

$$\bar{\zeta}_r = \frac{58.33 - \zeta'}{2} + \zeta' = 29.165 + \frac{\zeta'}{2}$$

$$A'_r = (58.33 - \zeta')120$$

Referring to Fig. b and substituting relations (e) into (11.36), we obtain

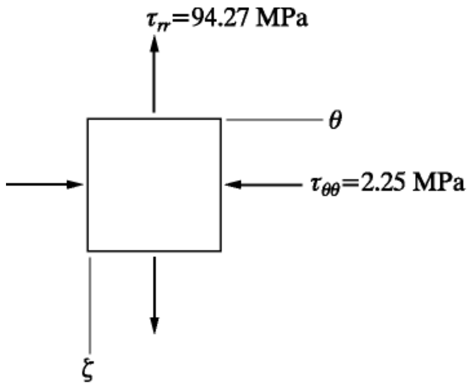


Figure c State of stress of the particles of line *EF*.

$$K(8.33) = \int_{\zeta'=\zeta}^{\zeta'=h_b} \frac{\bar{\zeta}'_r A'_r}{I_2} d\zeta' = \int_{\zeta'=8.33}^{\zeta'=58.33} \frac{(29.165 + \frac{\zeta'}{2})(58.33 - \zeta')(120)}{2{,}687.4(10^4)} d\zeta' = 0.23255 \tag{f}$$

Substituting relations (a), (c) and (e) into (11.43), we have

$$\tau_{rr}(8.33) = \frac{1}{r}\left[\left(\frac{A_r}{A} - K\right)\frac{N}{b} + \frac{M_2}{Ab}\left(\frac{AA_{mr} - A_r A_m}{RA_m - A}\right)\right] \tag{g}$$

$$= \left(\frac{80}{(130)(20)}\left(\frac{6{,}000}{9{,}000} - 0.23255\right) + \frac{27{,}066.4}{9{,}000(20)(130)}\left(\frac{9{,}000(58.26) - 6{,}000(73{,}606)}{(138.33)(73.606) - 9{,}000}\right)\right)$$

$$= 0.094274\left(\frac{\text{kN}}{\text{mm}^2}\right) = 94.27\ \text{MPa}$$

The results are shown in Fig. c. Notice that big values of the radial component of stress occur at points where the circumferential component of stress is small.

Referring to relations (c) and (11.45), we see that on the cross section *ADH* of the beam under consideration, we have

$$\tau_{\theta\zeta} = 0 \tag{h}$$

11.5 Problems

1. A crane hook is subjected to the force P_3 = 120 kN as shown in Fig. 11P1. The crane has the trapezoidal cross section shown in Fig. 11P1 and is made from an isotropic, linearly elastic material. Determine the distribution of the circumferential, and the radial components of stress acting on its cross section BC.

Ans. $\tau_{\theta\theta}(r) = \dfrac{716.07}{r} - 8.914$

$\tau_{rr}(r) = 252.9$ MPa

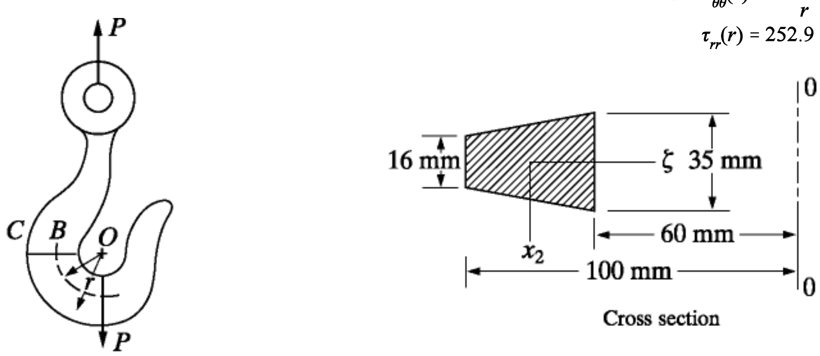


Figure 11P1

2. Determine the maximum allowable force *P* that the crane hook shown in Fig. 11P2 can carry, if the maximum allowable circumferential stress is 150 MPa.

Ans. $P_{max} = 37.55$ kN

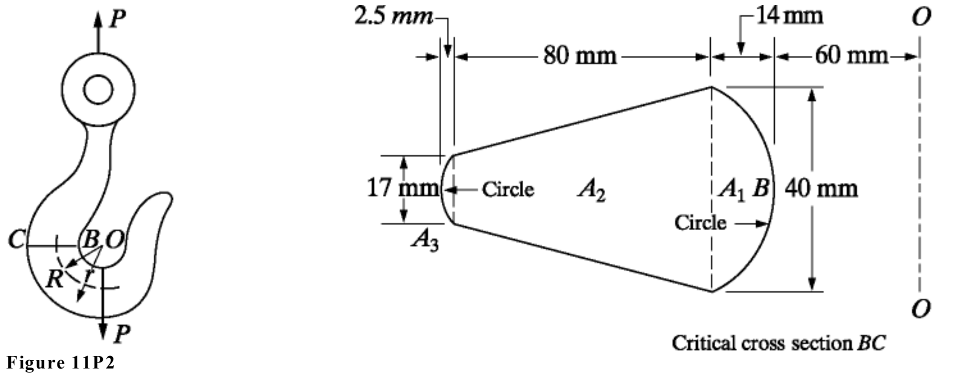


Figure 11P2

3. and **4**. The curved beam of Fig. 11P3 has a T cross section and is subjected to a force $P = 120$ kN. Determine the circumferential and the radial components of stress acting on the particles of the beam located on its cross section BC on the line DE of intersection of the flange with the web. Repeat with the beam of Fig. 11P4.

Ans. 3 $\tau_{\theta\theta}(r) = \left(\dfrac{360.656}{r} - 3.856\right) \dfrac{\text{kN}}{\text{mm}^2}$ $\quad$ Ans. 4 $\tau_{\theta\theta}(r) = \left(\dfrac{5.245}{r} - 0.0273\right) \dfrac{\text{kN}}{\text{mm}^2}$

$\tau_{rr}(r) = 515.36$ MPa $\quad$ $\tau_{rr}(r) = 26$ MPa

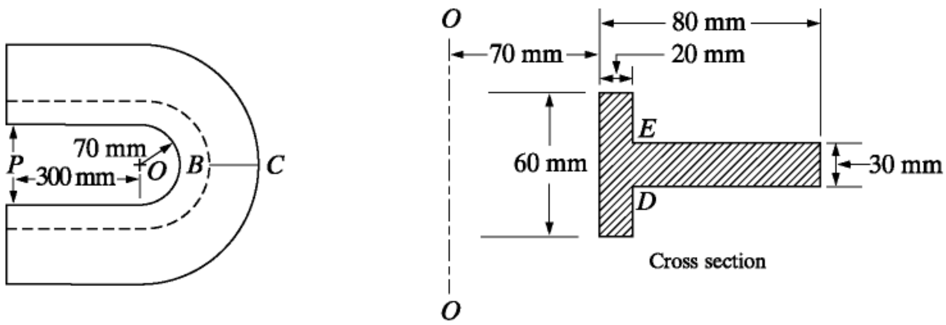


Figure 11P3

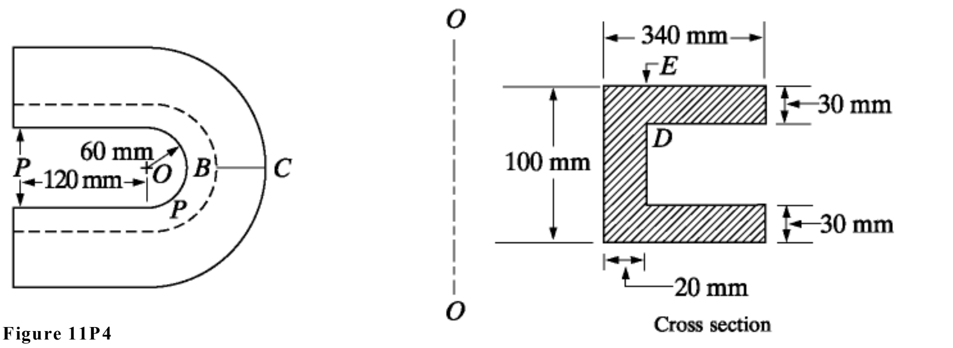


Figure 11P4

5. Consider the cantilever curved beam shown in Fig. 11P5 subjected to a force $P = 240$ kN. The beam is made from an isotropic, linearly elastic material. Determine the values of the maximum circumferential and radial components of stress acting on the particles of this beam.

Ans. 5 $\tau_{\theta\theta}(r) = -\dfrac{135.64}{r} + 0.519$

$\tau_{rr}(r) = -21.36$ MPa

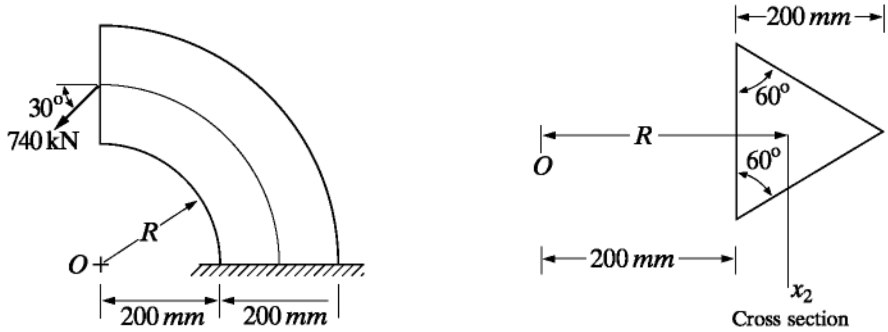


Figure 11P5

6. A gluelam (glued laminated) Douglas fir beam of constant cross section will be used in a roof system. The beam will be simply supported. It will have a span of 8 m and its middle half will be curved with a radius $R = 8$, as shown in Fig. 11P6. The snow load and dead load that the beam should be designed to carry a load of 8 kN/m. The allowable circumferential and radial stresses for Douglas fir as given by the American Institute of Timber Construction are $(\tau_{\theta\theta})_{\text{allowable}}$ = 15.8 MPa and $(\tau_{rr})_{\text{allowable}}$ = 0.119 MPa, respectively. The depth of the beam will be $h = 1.0$ m. Determine the required width of the beam.

Ans. $b \geq 68.8$ mm

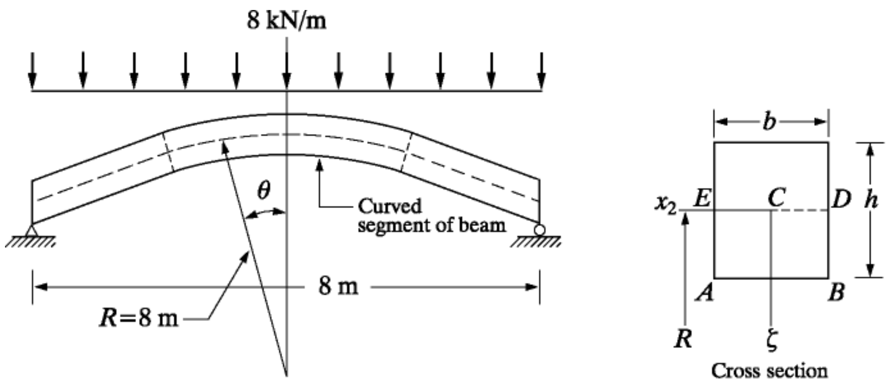


Figure 11P6

Chapter
12

Thin-Walled, Tubular[†] Members

12.1 Introduction

In Section 8.14 we present the theory of mechanics of materials for prismatic members of solid or hollow (thin- or thick-walled) circular cross sections subjected to torsional moments. In Chapter 9 we present the theories of mechanics of materials for prismatic beams subjected to transverse forces and bending moments at their ends and/or along their length. These theories apply to beams of arbitrary simply or multiply connected cross sections. However, as discussed in Section 9.5, the theories of mechanics of materials can be used to compute directly the shearing components of stress acting on the cross sections of beams, only if their geometry is such, that we know a priori a direction along which the shearing component of stress, normal to that direction, may be considered constant. For example, formula (9.66) cannot be used directly to compute the shearing component of stress in a prismatic beam of hollow thin-walled cross sections, if its cross sections do not have an axis of symmetry and the plane of the transverse forces acting on the beam does not contain the axis of symmetry.

Thin-walled, tubular members of circular cross sections are used in many structures and machines to resist torsional and/or bending moments. Moreover, thin-walled, tubular members of non-circular cross sections are often used in light structures such as aircrafts and spacecrafts to resist torsional and/or bending moments. Thin-walled, tubular members may be classified as single-cell when their cross sections have only one hole and multi-cell when their cross sections have more than one hole.

In this chapter we use the theory of mechanics of materials to establish formulas for computing the stress and displacement fields of prismatic thin-walled, tubular members of arbitrary cross section when subjected to one or more of the following external actions:

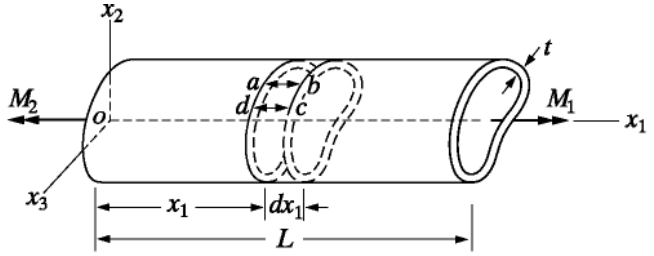
1. Equal and opposite torsional moments at their ends
2. Transverse forces and bending moments

[†] Tubular members are prismatic bodies with one or more holes.

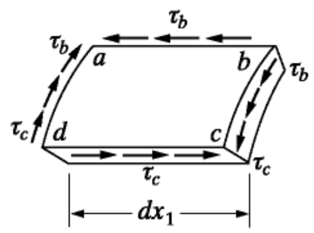
12.2 Computation of the Shearing Stress Acting on the Cross Sections of Thin-Walled, Single-Cell Tubular Members Subjected to Equal and Opposite Torsional Moments at Their Ends.

Consider a prismatic thin-walled, single-cell, tubular member of arbitrary cross section subjected to equal and opposite torsional moments at its ends (see Fig. 12.1). The thickness t of the wall of the member may vary around its cross section. The shearing component of stress acting on a particle located at a point of the boundary of a cross section of this member must be tangent to the boundary at this point (see Fig. 12.1c). Moreover, since the thickness of the wall of the member is small, it is anticipated that the intensity of the shearing stress varies very little across its thickness. Consequently, *we assume that the shearing stress acting on the cross sections of a tubular member does not vary along their thickness.* This assumption simplifies considerably the analysis of thin-walled, tubular members.

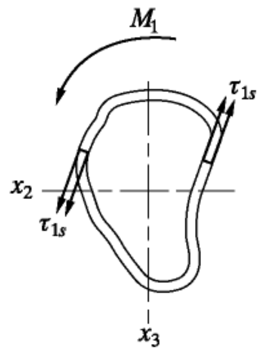
In Fig. 12.1b we show a segment *abcd* of length dx_1 of a thin-walled, single-cell, tubular member cut by two planes perpendicular to its axis and by two planes which are parallel to the axis of the member and normal to its middle surface. The thickness of this segment is constant in the axial direction (*ab* or *cd*), but could vary in the circumferential direction (*bc* or *ad*). We denote by t_b and t_c the thickness of the longitudinal surfaces *ab* and *cd*, respectively. When the thickness of a member varies in the circumferential direction, the intensity of the shearing stress varies in that direction. We denote by τ_b and τ_c the shearing components of stress acting on the surfaces *ab* and *cd*, of the segment



(a) Thin-walled single-cell member



(b) Segment of the member of Fig. 12.1a



(c) Distribution of shearing stress on a cross section of a thin-walled, single-cell, tubular member subjected to torsional moments at its ends.

Figure 12.1 Thin-walled, single-cell, tubular member subjected to equal and opposite torsional moments at its ends.

abcd, respectively. As we have shown in Section 2.13 the components stress are symmetric. Consequently, the shearing stress τ_b acting on the longitudinal surface *ab* must be equal to the shearing stress acting on the end cross sections (*ad* and *bc*) of the segment of Fig. 12.1b at points *a* and *b*. Moreover, the shearing stress τ_c acting on the longitudinal surface *cd* must be equal to the shearing stress acting on the end cross sections of the segment of Fig. 12.1b at points *c* and *d*. We denote by F_b and F_c the resultant shearing forces acting on the longitudinal surfaces *ab* and *cd*, respectively. That is,

$$\begin{aligned} F_b &= \tau_b t_b dx_1 \\ F_c &= \tau_c t_c dx_1 \end{aligned} \tag{12.1}$$

Inasmuch as the element *abcd* is in equilibrium, we have

$$\sum F_1 = 0 = F_c - F_b = (\tau_c t_c - \tau_b t_b) dx_1$$

or

$$\tau_b t_b = \tau_c t_c \tag{12.2}$$

Since the longitudinal cuts *ab* and *cd* are arbitrary, relation (12.2) indicates that the product of the shearing stress τ acting on a particle of the cross section of a single-cell thin-walled, tubular member and the thickness t of the wall of the cross section at the point where this particle is located is constant. This product is known as the *shear flow* and represents the force per unit circumferential length acting on the cross sections of the tubular member. Denoting the shear flow by q, we have

$$q = \tau t = \text{constant} \tag{12.3}$$

In what follows, we proceed to relate the shear flow q acting on a cross section of a tubular member to the torsional moment M_1 acting on this cross section. For this purpose, referring to Fig. 12.2, we consider a portion *AB* of a cross section of the member of thickness t and circumferential length dx_s. The total shearing force acting on this portion of the cross section is equal to

$$dF = q\ dx_s \tag{12.4}$$

The moment of the force dF about an arbitrary point O is equal to

$$dM_1 = rq\ dx_s \tag{12.5}$$

where, referring to Fig. 12.2, r is the distance *OD* from point O to the tangent to the median line of the portion *AB* of the cross section. Thus,

$$M_1 = q \oint r\ dx_s \tag{12.6}$$

The integral is taken counterclockwise around the median line of the cross section of the member. Referring to Fig. 12.2, we have

$$r\ dx_s = 2\ d\Omega \tag{12.7}$$

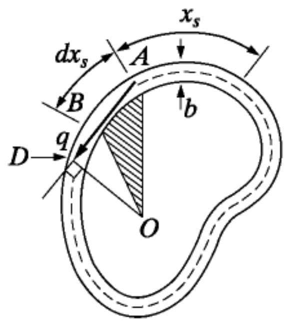


Figure 12.2 Cross section of a single-cell, thin-walled, tubular member.

where $d\Omega$ is the area of the triangle ABO. Integrating relation (12.7) along the median line of the cross section, we get

$$\oint r\, dx_s = 2\oint d\Omega = 2\,\Omega \tag{12.8}$$

where Ω is the area enclosed by the median line of the cross section of the member. Substituting relation (12.8) into (12.6), we have

$$q = \frac{M_1}{2\,\Omega} \tag{12.9}$$

Substituting relation (12.3) into (12.9), we obtain the following formula for the shearing stress τ_{1s} acting on a cross section of a single-cell, thin-walled, tubular member, subjected to equal and opposite torsional moments at its ends

$$\tau_{1s} = \frac{M_1}{2\Omega t} \tag{12.10}$$

Relations (12.9) and (12.10) have been derived in this section for a thin-walled, single-cell, tubular member subjected to equal and opposite torsional moments at its ends. However, it can be used to obtain approximate expressions for the shearing stress τ_{1s} of thin-walled, single-cell, tubular members subjected to torsional moments along their length. In this case M_1 and consequently q and τ_{1s} are functions of x_1.

12.3 Computation of the Angle of Twist per Unit Length of Thin-Walled, Single-Cell, Tubular Members Subjected to Equal and Opposite Torsional Moments at Their Ends

In this section we derive a formula for computing the angle of twist per unit length α of a cross section of a thin-walled, single-cell, tubular member of arbitrary cross section subjected to equal and opposite torsional moments at its ends.

Consider a particle located at point P of a cross section of a prismatic body of arbitrary cross section which may be solid or it may have one or more holes. We denote by O the

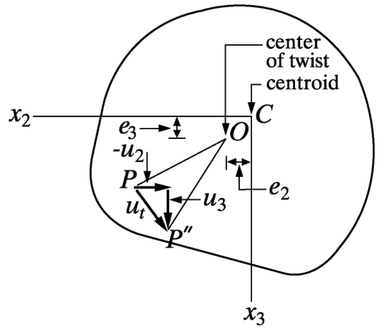


Figure 12.3 Displacement of a particle of a prismatic member subjected to tortional moments at its ends.

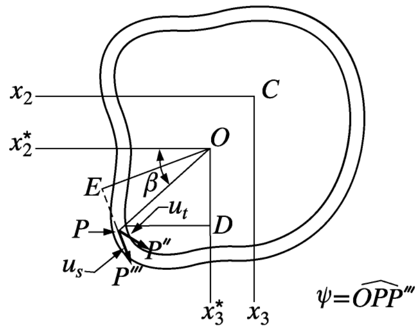


Figure 12.4 Cross section of a single-cell, thin-walled, tubular member subjected to torsional moments at its ends.

center of twist† of the cross section, which as shown in Section 13.11 coincides with the shear center of the cross section. When this body is subjected to equal and opposite torsional moments at its ends, as shown in Section 6.4 the particle under consideration moves to point P' and the before-deformation straight line $\overline{OP}$ becomes a space curve $\overline{OP'}$ whose projection on the cross section of the member is a straight line $\overline{OP''}$ (see Fig. 12.3). The projection of every line of a cross section of the body rotates by the same angle $\theta_1(x_1)$ known as *the angle of twist of the cross section*. The rate of change of the angle of twist along the axis of the member is known as *the angle of twist per unit length of the member* and we denote it by α . That is,

$$\alpha = \frac{d\theta_1}{dx_1} \tag{12.11}$$

In Section 6.4 we show that the component of displacement of point P in the plane of the cross section is normal to the line $\overline{OP}$. That is, the radial component of displacement of any particle of a cross section of a prismatic member subjected to equal and opposite torsional moments at its ends vanishes. Consequently, the displacement vector $\hat{\mathbf{u}}$ of any particle of such a member has only a transverse component of displacement $\hat{u}_t$ and an axial $\hat{u}_1$. Therefore,

$$\hat{\mathbf{u}} = \hat{u}_1\mathbf{i}_1 + \hat{u}_t\mathbf{i}_t \tag{12.12}$$

where $\mathbf{i}_t$ is the unit vector normal to the line OP. Referring to Fig. 12.3, we see that the transverse component of displacement u_t can be expressed as

$$\hat{u}_t\mathbf{i}_t = \hat{u}_2\mathbf{i}_2 + \hat{u}_3\mathbf{i}_3 \tag{12.13}$$

In Section 6.3 we show [see relations (6.42)] that

$$\hat{u}_3 = \alpha x_1(x_2 - e_2) \tag{12.14}$$

†The center of twist of a cross section of a prismatic body subjected tortional moments is the point on the plane of the cross section about which the cross section rotates.

$$\hat{u}_2 = -\alpha x_1(x_3 - e_3) \tag{12.14}$$

where α is angle of twist per unit length of the cross sections of the member, while e_2 and e_3 are the coordinates of the center of twist of the cross sections of the member. Thus, referring relation (12.13) and to Fig. 12.3 and using relations (12.14), we have

$$u_t = \sqrt{u_2^2 + u_3^2} = \alpha x_1\sqrt{(x_3 - e_3)^2 + (x_2 - e_2)^2} = \alpha x_1\overline{OP} \tag{12.15}$$

The transverse component of displacement may be decomposed into tangential $\hat{u}_s$ and normal $\hat{u}_n$ components. In Fig. 12.4 the lines $\overline{PP}^{//}$ and $\overline{PP}^{///}$ represent the transverse $\hat{u}_t$ and the tangential $\hat{u}_s$ components of displacement of point P, respectively, while $\psi = \angle P^{///}PO$ is the angle which the tangential component of displacement makes with the line $\overline{OP}$. Referring to Fig. 12.4, we have

$$\hat{u}_s = \overline{PP}^{///} = \overline{PP}^{//}\cos(\angle P^{//}PP^{///}) = \hat{u}_t\cos(\angle P^{//}PP^{///}) = \hat{u}_t\cos(\psi - 90) = \hat{u}_t\sin\psi \tag{12.16}$$

Substituting relation (12.15) into (12.16), we get

$$\hat{u}_s = \alpha x_1\overline{OP}\sin\psi \tag{12.17}$$

In Fig. 12.4 line OE is normal to the tangent to the median line of the cross section of the member at point P; referring to this figure, we have

$$\angle EPO = 180 - \psi \tag{12.18}$$

and

$$r = \overline{OE} = \overline{OP}\sin(\angle EPO) = \overline{OP}\sin(180 - \psi) = \overline{OP}\sin\psi \tag{12.19}$$

Using relation (12.19), relation (12.17) becomes

$$\hat{u}_s = \alpha x_1 r \tag{12.20}$$

Substituting relation (12.20) into (2.16), we obtain

$$e_{1s} = \frac{1}{2}\left(\frac{\partial\hat{u}_s}{\partial x_1} + \frac{\partial\hat{u}_1}{\partial x_s}\right) = \frac{1}{2}\alpha r + \frac{1}{2}\frac{\partial\hat{u}_1}{\partial x_s} \tag{12.21}$$

Using the stress–strain relations (3.47), relation (12.21) may be rewritten as

$$\frac{\partial\hat{u}_1}{\partial x_s} = \frac{\tau_{1s}}{G} - \alpha r \tag{12.22}$$

We assume that the change of $\hat{u}_1$ over the thickness of the wall of the member is negligible. Consequently, on a cross section $\hat{u}_1$ may be considered as function of x_s only and relation (12.22) can be rewritten as

$$d\hat{u}_1 = \frac{\tau_{1s}}{G} dx_s - \alpha r dx_s \tag{12.23}$$

Integrating relation (12.23) around the median line of the cross section of the member, taking into account that $t\tau_{1s} = q$ and noting that the integral of $d\hat{u}_1$ over a closed curve is zero, we get

$$\oint d\hat{u}_1 = 0 = \frac{1}{G}\oint \frac{q\,dx_s}{t} - \alpha \oint r\,dx_s \tag{12.24}$$

Substituting relation (12.8) into the above and noting that q is constant on a cross section, we obtain

$$\alpha = \frac{q}{2\Omega G}\oint \frac{dx_s}{t} \tag{12.25}$$

Substituting relation (12.9) into the above, we get

$$\alpha = \frac{M_1}{R_C G} \tag{12.26}$$

where the torsional constant R_C of the prismatic thin-walled, single-cell, tubular member under consideration is equal to

$$R_C = \frac{4\Omega^2}{\oint \frac{dx_s}{t}} \tag{12.27}$$

For thin-walled single-cell tubular members of constant wall thickness t, relation (12.27) reduces to

$$R_C = \frac{4\Omega^2 t}{S} \tag{12.28}$$

where S is the length of the median line of the cross section of the member.

Relation (12.26) with (12.27) has been established on the basis of results obtained in Chapter 6 for prismatic bodies subjected to equal and opposite torsional moments at their ends. However, it can be used to approximate the angle of twist per unit length of thin-walled, tubular members subjected to torsional moments along their length.

In order to get an indication of the range of validity of the formula (12.27), we compare the torsional constant obtained from this formula for a thin-walled, single-cell, tubular member of circular cross section (see Fig. 12.5) with that obtained on the basis of the linear theory of elasticity. The value of the torsional constant for such a member obtained on the basis of the theory of elasticity [see relation (x) of the example of Section 6.6] is

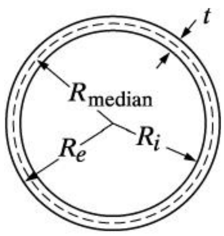


Figure 12.5 Circular cross section of a tubular member.

$$R_C^E = \frac{\pi}{2}(R_e^4 - R_i^4) = \frac{\pi}{2}(R_e^2 - R_i^2)(R_e^2 + R_i^2) \tag{12.29}$$

Notice that

$$R_{\text{median}} = R = \frac{R_e + R_i}{2} \tag{12.30a}$$

$$\Omega = \pi R^2_{\text{median}} = \pi\left(\frac{R_e + R_i}{2}\right)^2 \tag{12.30b}$$

$$S = 2\pi R_{\text{median}} = \pi(R_e + R_i) \tag{12.30c}$$

$$t = R_e - R_i \tag{12.30d}$$

Using relations (12.30b) to (12.30d) the approximate value R_C^A of the torsional constant obtained on the basis of formula (12.28) is

$$R_C^A = \frac{4t\Omega^2}{S} = \frac{\pi}{4}(R_e^2 - R_i^2)(R_e + R_i)^2 \tag{12.31}$$

Using relations (12.29) and (12.31), we get

$$\frac{R_C^A}{R_C^E} = \frac{1}{2}\left(1 + \frac{2k}{1+k^2}\right) \tag{12.32a}$$

where

$$k = \frac{R_i}{R_e} \tag{12.32b}$$

The % error in the approximate value of R_C is

$$\%\ \text{error} = \left(\frac{R_C^A - R_C^E}{R_C^E}\right)100 = \left(\frac{R_C^A}{R_C^E} - 1\right)100 = 50\left(\frac{2k}{1+k^2} - 1\right) \tag{12.33}$$

For a tubular member with a t/R_{median} ratio of 0.1, referring to relations (12.30a), (12.30d) and (12.32b), we have

$$\frac{t}{R_{\text{median}}} = 2\frac{(R_e - R_i)}{R_e + R_i} = 2\left(\frac{1-k}{1+k}\right) = 0.1$$

Consequently,

$$k = \frac{19}{21}$$

Substituting the above value of k into relation (12.33), we get

$$\%\ \text{error} = -0.25\%$$

This result indicates that even for tubular members with not very thin walls, as, for example, $t/R_{\text{median}} = 0.10$, the theory of mechanics of materials for tubular members presented in this section gives very good results.

Example 1 Consider a thin-walled, tubular member whose cross section is shown in Fig. a. The member is fixed at its end ($x_1 = 0$) and is subjected to a torsional moment M_1^L (kN·m) at its end ($x_1 = L$). The member is made from an isotropic, linearly elastic material with shear modulus G (kN/mm^2). Derive formulas for the stress distribution on the cross sections of the member and the angle of twist of its unsupported end.

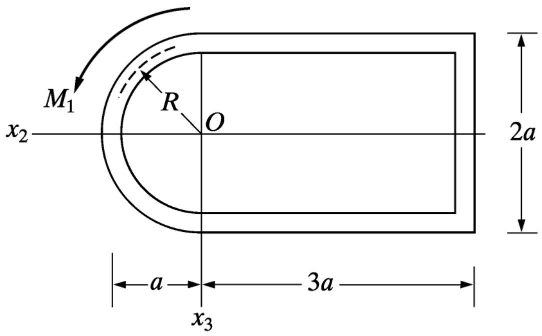


Figure a Geometry of the member.

Solution Referring to Fig. a, we have

$$\Omega = (3a)2a + \frac{\pi a^2}{2} = \frac{a^2}{2}(12 + \pi)\ \text{mm}^2 \tag{a}$$

$$S = 2(3a) + 2a + \pi a = a(8 + \pi)\ \text{mm} \tag{b}$$

Substituting relation (a) into relation (12.10) the shear stress acting on the cross sections of the member is equal to

$$\tau_{1s} = \frac{M_1}{2t\Omega} = \frac{M_1}{ta^2(12 + \pi)} = 0.0660\left(\frac{M_1}{ta^2}\right) \tag{c}$$

Substituting relations (a) and (b) into (12.28), the torsional constant of the member is equal to

$$R_C = \frac{4\Omega^2 t}{S} = \frac{a^3 t(12 + \pi)^2}{(8 + \pi)} \tag{d}$$

Substituting relation (d) into (12.26), we get

$$\alpha = \frac{M_1}{R_C G} = \frac{M_1(8 + \pi)}{Ga^3 t(12 + \pi)^2} = 0.04860\left(\frac{M_1}{Ga^3 t}\right) \tag{e}$$

The total angle of twist of the cross section of the member at its unsupported end is equal to

$$\theta_1(L) = L\alpha = 0.04860\left(\frac{M_1 L}{Ga^3 t}\right) \tag{f}$$

12.4 Prismatic Thin-Walled, Single- cell, Tubular Members with Thin Fins Subjected to Torsional Moments

In Section 6.10 we derive an approximate formula for the shearing stress and the angle of twist per unit length of prismatic members of thin rectangular cross sections. Referring to Fig. 12.6 these formulas are

$$\tau_{12} = 0 \qquad \tau_{13} = \frac{6M_1 x_2}{ht^3} \tag{12.34a}$$

$$\alpha = \frac{M_1}{GR_C} \qquad R_C = \frac{ht^3}{3} \tag{12.34b}$$

Moreover, in Section 12.2 and 12.3 we derive formulas for the shearing stress and the angle of twist per unit length of thin-walled, single-cell, tubular members subjected to torsional moments. In this section we use these formulas to approximate the response to torsional moments of composite members consisting of a thin-walled, single-cell, tube having a number of thin rectangular plates (fins) connected to it (see Fig. 12.7). When such a composite member is *subjected to equal and opposite torsional moments at its ends, the fins and the tube rotate by the same amount*. Consequently, the twist per unit

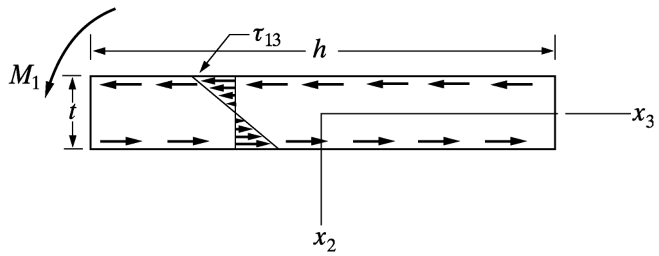


Figure 12.6 Thin rectangular cross section.

length of the tube α^T and of each fin α^{F_i} is equal to that (α) of the composite member. That is,

$$\alpha = \alpha^{F_i} = \alpha^T = \frac{M_1}{GR_C} = \frac{M_1^{F_i}}{GR_C^{F_i}} = \frac{M_1^T}{GR_C^T} \tag{12.35}$$

where $M_i^{F_i}$ and M_1^T are the torsional moments of the distribution of shear stress acting on the cross section of the i^{th} fin and of the tube, respectively. R_C, $R_C^{F_i}$ and $R_C{}^T$ are the torsional constants of the composite member, the i^{th} fin and the tube, respectively. Referring to relations (12.27) and (12.34b), we have

$$R_C^T = \frac{4\Omega^2}{\oint \frac{dx_s}{t}} \tag{12.36a}$$

$$R_C^{F_i} = \frac{b_i t_i^3}{3} \tag{12.36b}$$

where Ω is the area enclosed by the median line of the cross section of the tube; t is the thickness of the tube; b_i and t_i are the width and the thickness of the i^{th} fin. In relation (12.36a) the integration is carried counter-clockwise around the median line of the cross section of the tube.

The total moment acting on the cross section of a composite member with n fins is equal to

$$M_1 = \sum_{i=1}^{n} M_1^{F_i} + M_1^T \tag{12.37}$$

Substituting the expression for the torsional moments $M_i^{F_i}$ and M_1^T from relations (12.35) into (12.37), we obtain

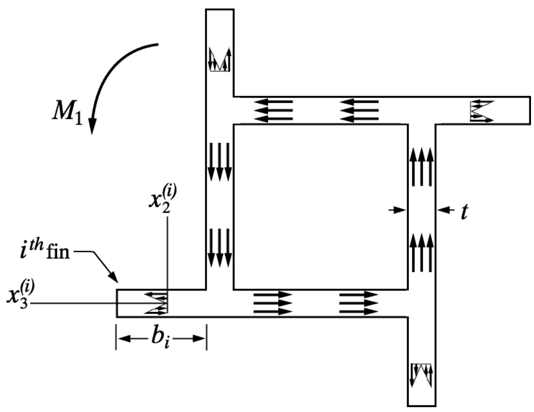


Figure 12.7 Cross section of a thin-walled, single-cell, tubular member with fins.

$$M_1 = \alpha G R_C = \alpha G\left(\sum_{i=1}^{n} R_C^{F_i} + R_C^T\right) \tag{12.38}$$

Thus, using relations (12.36), from relation (12.38), we get

$$R_C = \sum_{i=1} R_C^{F_i} + R_C^T = \sum_{i=1} \frac{b_i t_i^3}{3} + \frac{4\Omega^2}{\oint \frac{dx_s}{t}} \tag{12.39}$$

Referring to relations (12.34a) and (12.10), the shearing stresses $\tau_{1s}^{F_i}$ and $\tau_{1T}^{F_i}$ acting on the cross section of the i^{th} fin and of the tube respectively, are equal to

$$\tau_{1s}^{F_i} = \frac{6M_1^{F_i} x_2^{(i)}}{b_i t_i^3} \qquad \tau_{1s}^T = \frac{M_1^T}{2\Omega t} \tag{12.40}$$

where $x_2^{(i)}$ is the coordinate measured along the principal centroidal axis of the i^{th} fin which is normal to its long edge (see Fig. 12.6). Referring to relation (12.35) and (12.36), the torsional moments M_1^{Fi} and M_1^T may be expressed as

$$M_1^{F_i} = G\alpha R_C^{F_i} = \frac{G\alpha b_i t_i^3}{3} = \frac{M_1 b_i t_i^3}{3R_C} \tag{12.41}$$

$$M_1^T = G\alpha R_C^T = \frac{4G\alpha\Omega^2}{\oint \frac{dx_s}{t}} = \frac{4M_1\Omega^2}{R_C \oint \frac{dx_s}{t}} \tag{12.42}$$

Substituting relations (12.41) and (12.42) into (12.40), we get

$$\tau_{1s}^{F_i} = \frac{2M_1 x_2^{(i)}}{R_C} \tag{12.43}$$

$$\tau_{1s}^T = \frac{2M_1\Omega}{R_C t \oint \frac{dx_s}{t}} \tag{12.44}$$

In what follows, we illustrate the computation of the distribution of the shearing stress on the cross sections of a composite member subjected to equal and opposite torsional moments at its ends.

Example 2 Consider the thin-wall prismatic composite member of length $L = 1$ m whose cross section is shown in Fig. a. The member consists of a tube of circular cross section of thickness $t = 5$ mm and four thin plates (fins) of thickness $t_i = 5$ mm attached to it. The internal radius of the tube is $R_i = 60$ mm. The member is made from an isotropic linearly elastic material of shear modulus $G = 100$ kN/mm^2. The member is fixed at its end $x_1 = 0$

and is subjected to a torsional moment of magnitude $M_1 = 12$ kN·m at its end $x_1 = L$. Compute

(a) The total angle of rotation of the end cross section ($x_1 = L$) of the member
(b) The distribution of the shearing stress on the cross sections of the fins and the tube and show them on a sketch
(c) The percentage of the total moment carried by the tube

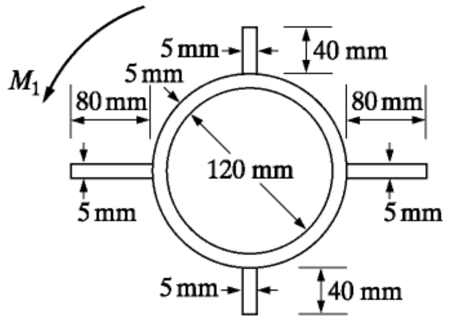


Figure a Geometry of the cross section of the composite member.

Solution Referring to relation (12.39), the torsional constant for the composite cross section of Fig. a is

$$R_C = \frac{1}{3}\sum_{i=1}^{4} b_i t_i^3 + \frac{4\Omega^2}{\oint \frac{dx_s}{t}} = \frac{1}{3}\sum_{i=1}^{4} b_i t_i^3 + \frac{4\Omega^2 t}{S} \tag{a}$$

where, referring to Fig. a, the area Ω enclosed by the median line of the cross section of the tube is equal to

$$\Omega = \pi(60 + 2.5)^2 = 12{,}272 \text{ mm}^2 \tag{b}$$

Moreover, the length of the median line of the cross section of the tube is equal to

$$S = 2\pi(60 + 2.5) = 392.7 \text{ mm} \tag{c}$$

Referring to Fig. a and substituting relations (b) and (c) into (a), we obtain

$$R_C = \frac{2(40)(5)^3}{3} + \frac{2(80)(5)^3}{3} + \frac{4(12{,}272)^2(5)}{392.7} = 7{,}680{,}078 \text{ mm}^4 \tag{d}$$

Substituting relation (d) into (12.35), the total angle of twist of the one end of the member of Fig. a, relative to its other end, is equal to

$$\alpha L = \frac{M_1 L}{G R_C} = \frac{12(1000)(1000)}{100(7{,}680{,}078)} = 0.01562 \text{ rad} = 0.895° \tag{e}$$

Substituting relation (d) into (12.43), the shearing stress acting on the cross sections of

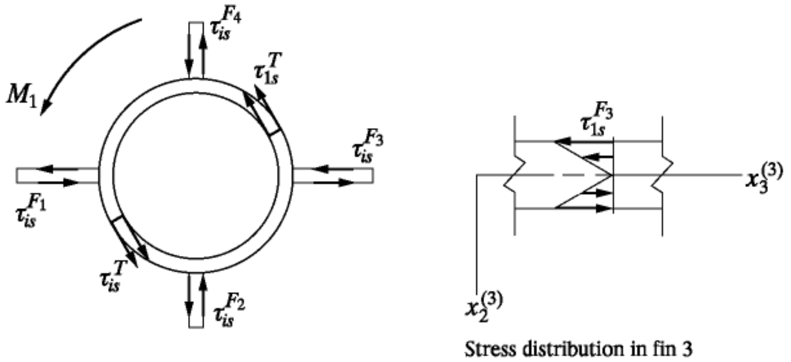


Figure b Distribution of shearing stresses on the cross section of the composite member.

the fins is equal to

$$\tau_{1s}^{F_i} = \frac{2(12{,}000)x_2^{(i)}}{7{,}680{,}078} = 0.00312x_2^{(i)}$$

Thus,

$$(\tau_{1s}^{F_i})_{\max}\Big|_{x_2^{(i)}=2.5} = 0.0078 \;\frac{\text{kN}}{\text{mm}^2}$$

Substituting relations (b), (c) and (d) into (12.44) the shearing stress acting on the walls of the tube is equal to

$$\tau_{1s}^{T} = \frac{2M_1\Omega}{R_cS} = \frac{2(12{,}000)(12{,}272)}{(7{,}680{,}078)392.7} = 0.097706 \;\frac{\text{kN}}{\text{mm}^2}$$

Referring to relations (12.35), the portion M_1^T of the total moment M_1 resisted by the wall of the tube is equal to

$$M_1^T = G\alpha R_C^T$$

Moreover,

$$M_1 = G\alpha R_C$$

Hence,

$$\begin{array}{l}\text{\% of total moment resisted}\\ \text{by the wall of the tube}\end{array} = \frac{M_1^T}{M_1}(100) = \frac{R_C^T}{R_C}(100) = \frac{7{,}670{,}078}{7{,}680{,}078} = 99.87\%$$

Area of the cross section of the tube = $\pi\,(65^2 - 60^2) = 1963.49\ \text{mm}^2$
Area of the cross section of the Fins = $2(5)80 + 2(5)40 = 1200\ \text{mm}^2$

$$\begin{array}{l}\text{\% of total area of the cross section}\\ \text{of the member alocated to the tube}\end{array} = \frac{1943.49(100)}{1200 + 1963.49} = 62\%$$

That is, the tube resists 99.87% of the applied torsional moment while the area of its cross section is 62% of the area of the total cross section of the member.

12.5 Thin-Walled, Multi-Cell, Tubular Members Subjected to Torsional Moments

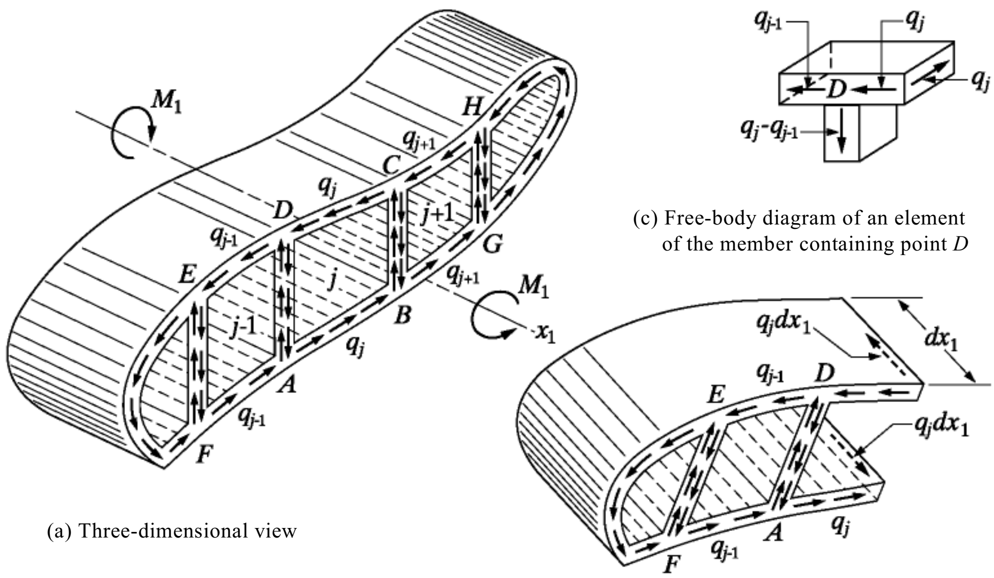
In Sections 12.2 and 12.3 we consider prismatic thin-walled, single-cell, tubular members subjected to torsional moments. For such members we derive formulas for computing:

1. The magnitude of the shearing stress acting on their cross sections
2. The angle of twist per unit length of their cross sections

We derive the formula for the shearing stress by considering the equilibrium of an element of length dx_1 of such a member and taking into account that the distribution of the shearing stress over the thickness of each wall may be considered approximately constant. That is, the problem considered in Section 12.2 and 12.3 is statically determinate.

Thin-walled, multi-cell, tubular members having n cells subjected to torsional moments are statically indeterminate to the $n-1$ degree. Consequently, in addition to the equation of equilibrium we must establish $(n-1)$ equations by requiring that the angles of twist of each one of the cells of a member are equal.

Consider a thin-walled, n-cell, tubular member of arbitrary cross section subjected to equal and opposite torsional moments at its ends of magnitude M_1. Referring to Fig. 12.8a we denote by q_{j-1}, q_j and q_{j+1} the shear flows acting on the walls ED, DC and CH, respectively, and we consider them positive when they act in the directions shown in Fig.



(a) Three-dimensional view

(c) Free-body diagram of an element of the member containing point D

(b) Free-body diagram or a portion of the member of length dx_1

Figure 12.8 Thin-walled, multi-cell, tubular member subjected to torsional moments and free-body diagram of two parts of it.

12.8a. The equilibrium of the forces acting on the portion of the member shown in Fig. 12.8b requires that the shear flow acting on the wall *AB* is equal and opposite to that acting on *DC*. Similarly, the shear flow acting on the wall *FA* or *BG* is equal and opposite to that acting on the wall *ED* or *CH*, respectively. Moreover, the equilibrium of the forces acting on the element of the member shown in Fig. 12.8c requires that the shear flow acting on the wall *DA* is equal to $q_j - q_{j-1}$, respectively. We denote by $M_1^{(j)}$ the moment of the portion q_j of the shear flow acting on all the walls of the j^{th} cell about an axis parallel to the axis x_i. Thus, referring to relation (12.9), we have

$$M_1^{(j)} = 2q_j\Omega_j \tag{12.45}$$

where Ω_j is the area enclosed by the median line of the cross section of the j^{th} cell of the multi-cell member. The moment of the shear flows acting on any cross section of the member about any axis parallel to the x_1 axis must be equal to the torsional moment M_1 acting on this cross section. Thus, using relation (12.45), we have

$$M_1 = \sum_{j=1}^{n} M_1^{(j)} = 2\sum_{j=1}^{n} q_j\Omega_j \tag{12.46}$$

The angle of twist per unit length of the cross section of a multi-cell, tubular member is equal to that of the corresponding cross section of any one of its cells subjected to the actual shear flows acting on its walls. That is, denoting by $q^{(j)}$ the shear flow on the walls of the j^{th} cell of an n-cell, thin-walled, tubular member and referring to relation (12.25), the angle of twist of the j^{th} cell, is equal to

$$\alpha = \alpha_j = \frac{\oint_{s_j} q^{(j)}\frac{dx_s}{t}}{2G\Omega_j} = \frac{1}{2G\Omega_j}\left(q_j\oint_{s_j}\frac{dx_s}{t} - q_{j-1}\int_{DA}\frac{dx_s}{t} - q_{j+1}\int_{BC}\frac{dx_s}{t}\right) \qquad j = 1, 2, .., n \tag{12.47}$$

where s_j is the length of the median line of the wall of the j^{th} cell. If the j^{th} cell of a multi-cell member having n cells is bounded by m_j cells, we have

$$\alpha = \alpha_j = \frac{1}{2G\Omega_j}\left(q_j\oint_{s_j}\frac{dx_s}{t} - \sum_{r=1}^{m_j} q_r\int_{s_{jr}}\frac{dx_s}{t}\right) \qquad j = 1, 2, ..., n \tag{12.48}$$

where s_{jr} is the length of the wall between the j^{th} and the r^{th} cells.

Relations (12.48) are called the *equations of consistent deformation.* The shear flow satisfying these relations is the only one which corresponds to equal angles of twist for each cell component of the multi-cell member. Relation (12.48) may be rewritten as

$$\delta_{jj}q_j + \sum_{r=1}^{m_j} q_r\delta_{jr} - 2\Omega_j\alpha = 0 \qquad j = 1, 2, ..., n \tag{12.49}$$

where

$$\delta_{jj} = \frac{1}{G}\oint_{s_j}\frac{dx_s}{t} \qquad \delta_{jr} = -\frac{1}{G}\int_{s_{jr}}\frac{dx_s}{t} \qquad r = 1, 2, ..., m_j \tag{12.50}$$

For a thin-walled, multi-cell, tubular member of specified geometry, the coefficients δ_{jj} and δ_{jr} ($r = 1, 2, ..., m_j$) can be computed using relations (12.50). Consequently, for any given value of the torsional moment M_1 equations (12.49), together with equation (12.46), can be solved to establish the unknown shear flows q_j ($j = 1, 2, ..., n$) and the unknown angle of twist α.

In what follows we illustrate the computation of the shear flow and the angle of twist of prismatic thin-walled, multi-cell, tubular members by one example.

Example 3 Consider the thin-walled, three-cell, tubular member of length L whose cross section is shown in Fig. a. The thickness of the wall of the member is constant and we denote it by t. The member is made from an isotropic, linearly elastic material of shear modulus $G = 100$ GPa. The member is subjected to equal and opposite torsional moments of magnitude M_1 (kN·m) at its ends. Derive formulas for the distribution of the shearing component of stress on the cross sections and the twist per unit length of this member.

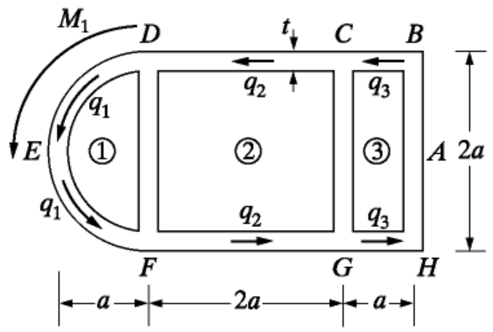


Figure a Cross section of the thin-walled, three-cell, tubular member.

Solution Referring to Fig. a, from relation (12.46), we obtain

$$M_1 = 2\sum_{j=1}^{3} q_j\Omega_j = 2\left[q_1\left(\frac{\pi a^2}{2}\right) + q_2(2a)^2 + q_3(2a^2)\right] = \pi a^2 q_1 + 8a^2 q_2 + 4a^2 q_3 \tag{a}$$

Referring to relation (12.47) and to Fig. a, for cell 1, we have

$$\begin{aligned}\alpha &= \frac{1}{2G\Omega_1}\left[q_1\oint_{DEFD}\frac{dx_s}{t} - q_2\int_{FD}\frac{dx_s}{t}\right] = \frac{1}{2G\left(\frac{\pi a^2}{2}\right)}\left[q_1\left(\frac{\pi a + 2a}{t}\right) - q_2\left(\frac{2a}{t}\right)\right] \\ &= \frac{1}{G\pi a t}\left[(\pi + 2)q_1 - 2q_2\right]\end{aligned} \tag{b}$$

Moreover, referring to relation (12.47) and to Fig. a, for cell 2, we have

$$\alpha = \frac{1}{2G\Omega_2}\left[q_2\oint_{CDFGC}\frac{dx_s}{t} - q_1\int_{DF}\frac{dx_s}{t} - q_3\int_{GC}\frac{dx_s}{t}\right] \tag{c}$$

$$= \frac{1}{2G(4a^2)}\left[q_2\left(\frac{8a}{t}\right) - q_1\left(\frac{2a}{t}\right) - q_3\left(\frac{2a}{t}\right)\right] = \frac{1}{4Gat}(4q_2 - q_1 - q_3) \tag{c}$$

Furthermore, referring to relation (12.47) and to Fig. a, for cell 3, we get

$$\begin{aligned}\alpha &= \frac{1}{2G\Omega_3}\left(q_3\oint_{BCGHB}\frac{dx_s}{t} - q_2\int_{CG}\frac{dx_s}{t}\right) = \frac{1}{2G(2a^2)}\left[q_3\left(\frac{2a+4a}{t}\right) - q_2\left(\frac{2a}{t}\right)\right] \\ &= \frac{1}{2Gat}(3q_3 - q_2)\end{aligned} \tag{d}$$

From relations (b) and (d), we obtain

$$\begin{aligned} q_1 &= \frac{\pi Gat\alpha + 2q_2}{2 + \pi} \\ q_3 &= \frac{2Gat\alpha + q_2}{3} \end{aligned} \tag{e}$$

Substituting relations (e) into (c), we get

$$q_2 = \frac{17\pi + 28}{11\pi + 16}Gat\alpha = 1.61\ Gat\alpha \tag{f}$$

Substituting relation (f) into (e), we obtain

$$\begin{aligned} q_1 &= \left[\frac{11\pi^2 + 50\pi + 56}{(16 + 11\pi)(2 + \pi)}\right]Gat\alpha = 1.237\,Gat\alpha \\ q_3 &= \left[\frac{22\pi^2 + 49\pi + 28}{3(11\pi + 16)}\right]Gat\alpha = 2.631\ Gat\alpha \end{aligned} \tag{g}$$

Substituting relations (g) and (f) into (a), we obtain

$$M_1 = [1.237\pi + 8(1.61) + 4(2.631]Ga^3t\alpha = 27.290\,Ga^3t\alpha \tag{h}$$

Thus,

$$\alpha = 0.03664\left(\frac{M_1}{Ga^3t}\right) \tag{i}$$

Substituting relation (i) into (g) and (f), we get

$$q_1 = 1.237\left(\frac{0.03664\,M_1}{a^2}\right) = 0.0453\left(\frac{M_1}{a^2}\right) \tag{j}$$

$$q_2 = 1.61\left(\frac{0.03664 M_1}{a^2}\right) = 0.0590\left(\frac{M_1}{a^2}\right)$$
$$q_3 = 2.630\left(\frac{0.03664 M_1}{a^2}\right) = 0.0964\left(\frac{M_1}{a^2}\right) \tag{j}$$

The distribution of the shearing component of stress on the walls of the member, referring to Fig. a, is equal to

$$\tau_{1s}^{(DEF)} = \frac{q_1}{t} = 0.0453\left(\frac{M_1}{ta^2}\right)$$
$$\tau_{13}^{(DF)} = \frac{q_2 - q_1}{t} = (0.0590 - 0.0453)\left(\frac{M_1}{ta^2}\right) = 0.0137\left(\frac{M_1}{ta^2}\right)$$
$$\tau_{12}^{(CD)} = -\tau_{12}^{(GF)} = \frac{q_2}{t} = 0.0590\left(\frac{M_1}{ta^2}\right)$$
$$\tau_{13}^{(CG)} = \frac{q_3 - q_2}{t} = (0.0964 - 0.0594)\left(\frac{M_1}{ta^2}\right) = 0.0370\left(\frac{M_1}{ta^2}\right)$$
$$\tau_{12}^{(BC)} = -\tau_{13}^{(BH)} = -\tau_{12}^{(HG)} = \frac{q_3}{t} = 0.0964\left(\frac{M_1}{ta^2}\right)$$

The multi-cell member under consideration is made from the single-cell member of the example of Section 12.3 by welding to it two diaphrams. Comparing the angle of twist and the magnitude of shearing stress acting on the cross sections of the two members, we can make the following observations:

1. The addition of the diaphrams has increased the torsional stiffness of the multi-cell member. Thus, the angle of twist per unit length of the multi-cell member is smaller than that of the single-cell member.
2. The addition of the diaphrams resulted in a non-uniform distribution of the shearing stresses on the cross sections of the member. The maximum shearing stress in the multi-cell member is almost 50% more than that in the single-cell member.

12.6 Thin-Walled, Single-Cell, Tubular Bea ms Subjected to Bending without Twisting

Consider a thin-walled, single-cell, prismatic tubular beam subjected to distributed and concentrated transverse forces and bending moments acting along its length and concentrated transverse forces and bending moments acting on its end surfaces ($x_1 = 0$ and $x_1 = L$) (see Fig. 12.9a). The line of action of the transverse forces lies in a plane which contains the shear centers of the cross sections of the beam. In general, on any cross section of this beam there is a distribution of normal component of stress τ_{11} and a distribution of shearing component of stress τ_{1s} (see Fig 12.9b).

At any particle located at a point of the boundary of a cross section of a thin-walled,

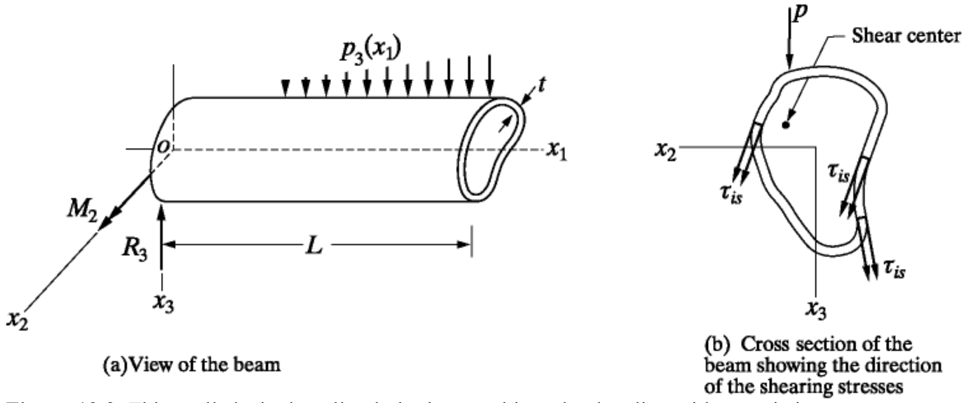


Figure 12.9 Thin-walled, single-cell, tubular beam subjected to bending without twisting.

tubular beam, the shearing component of stress must act tangent to the boundary of the cross section at that point. Inasmuch as the thickness of the tubular beam under consideration is small, the component of shearing stress in the direction normal to the boundary of the beam acting at particles which are not located on the boundary must be very small. Moreover, the tangential component of shearing stress varies little across the thickness of the cross sections of the beam. *Thus, we assume that the direction and magnitude of the shearing component of stress are constant across the thickness of the cross sections of the beam.*

The normal component of stress τ_{11} acting on the cross sections of any thin-walled, tubular beam subjected to bending without twisting can be computed using relation (9.12a or b). However, as dicussed in Section 9.5 the shearing conponent of stress acting on the cross sections of a beam cannot always be computed using relation (9.66). This becomes apparent if we recall that relation (9.66) has been established by considering the equilibrium of segment *ABDHEFG* of length dx_1, shown in Fig. 12.10b. This segment is cut from the beam by two planes perpendicular to its axis and a plane parallel to its axis which contains line *AD* (see Fig. 12.10a). In general, the magnitude F_1 of the resultant of the normal components of stress acting on the end surface *EFG* of the segment under consideration will be different than the magnitude F_2 of the resultant of the normal component of stress acting on the end surface *ADH* of the segment. Thus, a shearing force dF is required on the surface *ABDEF* of the segment *ABDHEFG* in order to keep it in equilibrium. Thus,

$$dF = F_2 - F_1 \tag{12.51}$$

The forces F_2 and F_1 can be computed from the distribution of the normal component of stress acting on the end surfaces *ABDH* and *EFG*, respectively, of the segment of the beam shown in Fig 12.10b. That is, referring to relation (9.63), we have

$$q = \frac{dF}{dx_1} = \frac{Q_3 I_{33} - Q_2 I_{23}}{I_{22}I_{33} - I_{23}^2} \iint_{A_n} x_3 dA + \frac{Q_2 I_{22} - Q_3 I_{23}}{I_{22}I_{33} - I_{23}^2} \iint_{A_n} x_2 dA \tag{12.52}$$

where q is the shear flow, that is, the force per unit length, acting on the surface *ABDEF*. A_n is the cross-hatched area *ABDH* of Fig. 12.10a. If the shearing stress $\tau_{n1} = \tau_{1n}$ acting on the plane *ABDEF* is constant along line *AD*, we can compute it as

$$\tau_{1n} = \frac{dF}{dx_1}\left(\frac{1}{\overline{AD}}\right) = \frac{q}{\overline{AD}} \tag{12.53}$$

On the basis of the foregoing presentation, the determination of the shear flow acting on the cross sections of a solid prismatic beam subjected to bending without twisting is a statically determinate problem. However, as it would become clear later the determination of the shear flow acting on the wall of single-cell, tubular beams subjected to transverse forces, which bend them without twisting them, is in general a statically indeterminated problem of the first degree. Consequently, in addition to the equation which we can obtain from statics we need a relation which will ensure that the shear flow acting on the cross sections of the beam does not twist it. This relation is established by setting the angle of twist per unit length α, equal to zero.

Consider a thin-walled, single-cell, tubular beam whose cross section is shown in Fig. 12.10c. We choose the shear flow q_{AA} acting on line AA as the unknown quantity which we call the *redundant shear flow*. We express the shear flow q_{BB} at any other line BB in terms of q_{AA} by considering the equilibrium of the segment $AABBCCDDEF$ of length dx_1 of the beam under consideration, cut from it by two planes normal to its axis and two planes normal to its cross sections. One of these planes contains line AA, while the other contains line BB. These lines are normal to the median line of the cross section of the beam. In general, the magnitude F_1 and F_2 of the resultant forces of the normal component of stress acting on the cross sections $CCDDE$ and $AABBF$, respectively, are not equal. Consequently, since segment $AABBCCDDEF$ is in equilibrium, two shearing forces dF_{AA} and dF_{BB} exist on the surfaces $AADD$ and $BBCC$, respectively. That is,

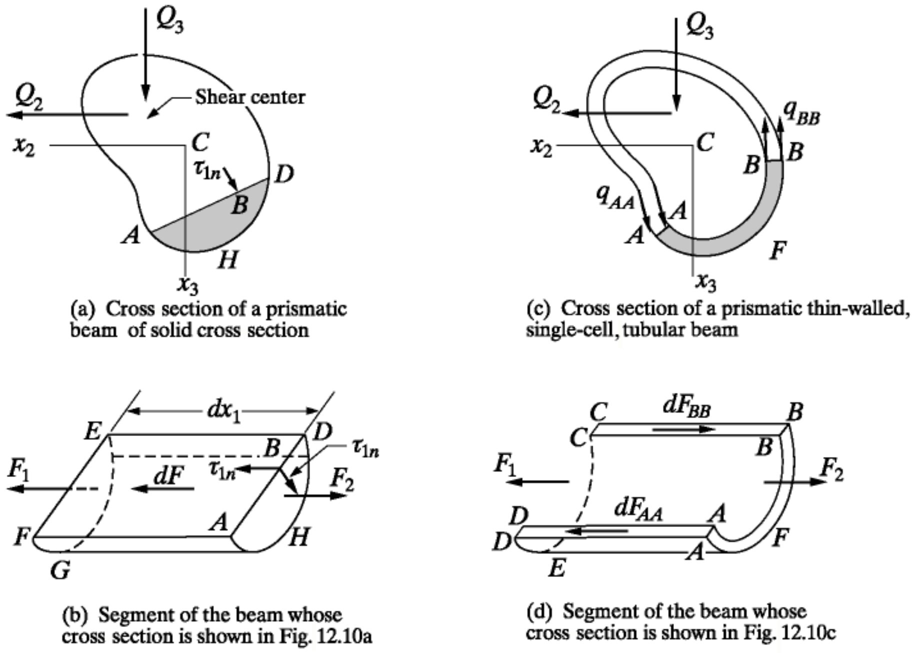


Figure 12.10 Cross sections and segments of prismatic beam subjected to bending without twisting.

$$dF_{AA} - dF_{BB} = F_2 - F_1 \tag{12.54}$$

In the above relation, we assume that the shear flow q is positive as shown in Fig. 12.10c; that is, if it tends to turn counterclockwise, the cross sections whose outward normal is in the direction of the positive x_1 axis. The force $F_2 - F_1$ can be computed from the distribution of the normal component of stress acting on the end surfaces $AABBF$ and $CCDDE$ of the segment of Fig. 12.10d using relation (9.63). That is,

$$\begin{aligned} q_{AA} - q_{BB} &= \frac{dF_{AA}}{dx_1} - \frac{dF_{BB}}{dx_1} = \frac{F_2 - F_1}{dx_1} \\ &= \frac{Q_3 I_{33} - Q_2 I_{23}}{I_{22}I_{33} - I_{23}^2} \iint_{A_n} x_3 dA + \frac{Q_2 I_{22} - Q_3 I_{23}}{I_{22}I_{33} - I_{23}^2} \iint_{A_n} x_2 dA \end{aligned} \tag{12.55}$$

where

q_{AA} = shear flow on transverse surface $AADD$.
q_{BB} = shear flow on transverse surface $BBCC$.
A_n = area of the cross-hatched surface $AABBF$ of Fig. 12.10c.

Referring to relation (9.63), we see that the right side of relation (12.55) represents the shear flow at line BB of an auxiliary prismatic beam, whose cross section is shown in Fig. 12.11, subjected to the same loads and supported the same way as the actual beam. The auxiliary beam is obtained from the actual beam by making a slit across line AA along its entire length. Thus, in relation (12.55), we express the shear flow $q_{BB} = q$ at any line BB of a cross section of the beam (see Fig. 12.10c) as the difference of the redundant shear flow q_{AA} at some reference line AA and the shear flow q_{aux} of the auxiliary beam at line BB (see Fig. 12.11). That is,

$$q = q_{BB} = q_{AA} - q_{\text{aux}} \tag{12.56}$$

where

$$q_{\text{aux}} = \frac{F_2 - F_1}{dx_1} = \frac{Q_3 I_{33} - Q_2 I_{23}}{I_{22}I_{33} - I_{23}^2} \iint_{A_n} x_3 dA + \frac{Q_2 I_{22} - Q_3 I_{23}}{I_{22}I_{33} - I_{23}^2} \iint_{A_n} x_2 dA \tag{12.57}$$

A_n is the area of the shaded surface of Fig. 12.11.

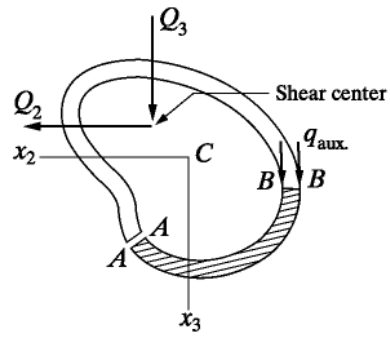


Figure 12.11 Cross section of the auxiliary thin-walled, tubular beam.

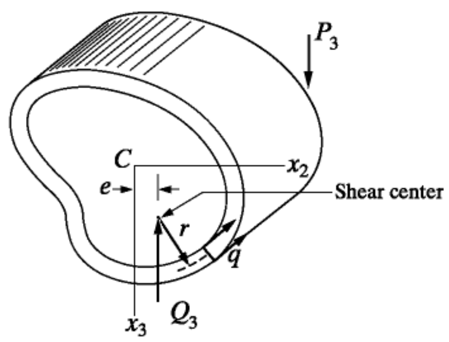


Figure 12.12 Cross section of a thin-walled, single-cell, tubular beam subjected to bending without twisting.

It is apparent that the equation of equilibrium (12.55) is not sufficient to compute the shear flows q_{AA} and q_{BB}. Therefore, the problem is statically indeterminate. Consequently, we have to take into account that the cross sections of the beam under consideration do not twist. That is, referring to relation (12.25), we have

$$\alpha = \frac{1}{2\Omega G} \oint q\frac{dx_s}{t} = 0 \tag{12.58}$$

Substituting relation (12.56) into (12.58), we obtain

$$q_{AA} = \frac{\oint q_{\text{aux}} \frac{dx_s}{t}}{\oint \frac{dx_s}{t}} \tag{12.59}$$

Substituting relation (12.59) into (12.56), we obtain that the shear flow acting on the cross sections of a thin-walled, single-cell, tubular beam is equal to

$$q = -q_{\text{aux}} + \frac{\oint q_{\text{aux}} \frac{dx_s}{t}}{\oint \frac{dx_s}{t}} \tag{12.60}$$

Substituting relation (12.57) into (12.60), we get

$$\begin{aligned} q &= -\frac{I_{22}\iint_{A_n} x_2 dA - I_{23}\iint_{A_n} x_3 dA - K_2}{I_{22}I_{33} - I_{23}^2}Q_2 - \frac{I_{33}\iint_{A_n} x_3 dA - I_{23}\iint_{A_n} x_2 dA - K_3}{I_{22}I_{33} - I_{23}^2}Q_3 \\ &= -\frac{I_{22}A_n\bar{x}_{2n} - I_{23}A_n\bar{x}_{3n} - K_2}{I_{22}I_{33} - I_{23}^2}Q_2 - \frac{I_{33}A_n\bar{x}_{2n} - I_{23}A_n\bar{x}_{3n} - K_3}{I_{22}I_{33} - I_{23}^2}Q_3 \end{aligned} \tag{12.61}$$

where

$$K_2 = \frac{\oint \left(I_{22}\iint_{A_n} x_2 dA - I_{23}\iint_{A_n} x_3 dA\right)\frac{dx_S}{t}}{\oint \frac{dx_S}{t}} = \frac{\oint (I_{22}A_n\bar{x}_{2n} - I_{23}A_n\bar{x}_{3n})\frac{dx_s}{t}}{\oint \frac{dx_s}{t}} \tag{12.62}$$

and

$$K_3 = \frac{\oint \left(I_{33}\iint_{A_n} x_3 dA - I_{23}\iint_{A_n} x_2 dA\right)\frac{dx_s}{t}}{\oint \frac{dx_s}{t}} = \frac{\oint (I_{33}A_n\bar{x}_{3n} - I_{23}A_n\bar{x}_{2n})\frac{dx_s}{t}}{\oint \frac{dx_s}{t}} \tag{12.63}$$

The shear flow q is positive when it acts counterclockwise on the cross section whose normal is in the direction of the positive x_1 axis.

On the basis of the foregoing presentation, we adhere to the following steps in order to establish the shear flow at a line BB normal to the middle line of a cross section of a single-cell, tubular beam subjected to bending without twisting:

STEP 1 We choose a reference line AA normal to the middle line of the cross section of the beam.

STEP 2 We establish the products $A_n\tilde{x}_{3n}$ and $A_n\bar{x}_{2n}$ for line BB where A_n is the area of the portion of the cross section from the reference line AA to line BB while $\bar{x}_{2n}$ and $\bar{x}_{3n}$ are the coordinates of the centroid of this area.

STEP 3 We substitute the expressions for $A_n\bar{x}_{2n}$ and $A_n\bar{x}_{3n}$ established in step 2 in relations (12.62) and (12.63) to compute the values of K_2 and K_3. The line integrals in these relations are evaluated counterclockwise.

STEP 4 The expressions for the products $A_n\bar{x}_{2n}$ and $A_n\bar{x}_{3n}$ established in step 2 and the values of K_2 and K_3 established in step 3 are substituted into relation (12.61) to obtain the shear flow at line BB of the cross section of the beam.

After the distribution of the shear flow in a thin-walled, single-cell, beam subjected to bending without twisting is computed, the shear center of its cross sections may be established. In order to accomplish this, we assume that the tubular member is a cantilever beam subjected at its unsupported end to a force $P_3 = Q_3$ which is parallel to the x_3 axis and its line of action passes through the shear center of its unsupported end. Referring to Fig. 12.12 from the equilibrium of the end segment of the beam we have

$$Q_3 e = \oint rq \, dx_s \tag{12.64}$$

where q is the shear flow on the negative cross section.

Example 4 Consider the cantilever beam whose cross section is shown in Fig. a. The beam is subjected at its unsupported end to a transverse force P_3 whose line of action

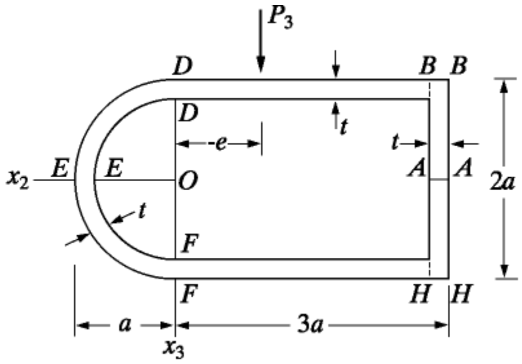


Figure a Cross section of the beam.

passes through the shear center of its cross section. The beam is made from an isotropic, linearly elastic material of shear modulus G (GPa). Compute the shear flow on the cross sections of the beam and locate their shear center. Assume that the thickness of the beam is negligible compared to its other dimensions.

Solution Since the x_2 axis is an axis of symmetry of the cross sections of the beam, it is a principal axis ($I_{23} = 0$) for any point on it. Referring to Fig. a, the moment of inertia of the cross sections of the beam about the x_2 axis is equal to

$$I_2 = I_2^{(HB)} + 2I_2^{(BD)} + I_2^{(DEF)}$$
$$= \frac{t(2a)^3}{12} + 2(3at)a^2 + \frac{1}{2}\left[\frac{\pi}{4}[a^4 - (a-t)^4]\right] \approx ta^3\left(\frac{20}{3} + \frac{\pi}{2}\right) = 8.237ta^3 \tag{a}$$

The shear flow on the cross sections of the beam is given by relation (12.61), which referring to relation (12. 63), for the beam under consideration ($Q_2 = 0, I_{23} = 0$), reduces to

$$q = -\left[\frac{\iint_{A_n} x_3 dA}{I_2} - \frac{K_3}{I_2 I_3}\right] Q_3 = -\left[A_n \bar{x}_{3n} - \frac{\oint A_n \bar{x}_{3n} dx_s}{\oint dx_s}\right]\frac{Q_3}{I_2} \tag{b}$$

STEP 1 We choose line AA, shown in Fig. b, as the reference line.

STEP 2 We establish expressions for the product $A_n \bar{x}_{3n}$ and we compute the integrals $\oint A_n \bar{x}_{3n} dx_s$ and $\oint dx_s$. Referring to Fig. b, for the portion of the cross section of the beam from line AA to line BB, we have

$$A_n = x_3 t \qquad \bar{x}_{3n} = -\frac{x_3}{2} \qquad A_n \bar{x}_{3n} = -\frac{x_3^2 t}{2} \tag{c}$$

Referring to Fig. c, for the portion of the cross section of the beam from line BB to line

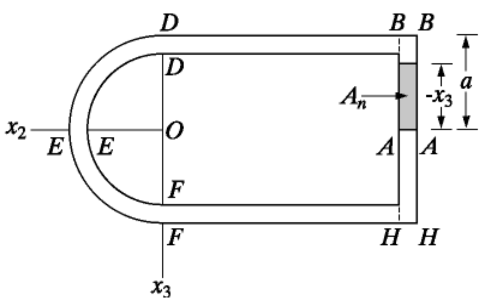


Figure b Cross section of the beam showing the area A_n used in the computation of the shearing stress, acting on the portion of the cross section from line AA to line BB.

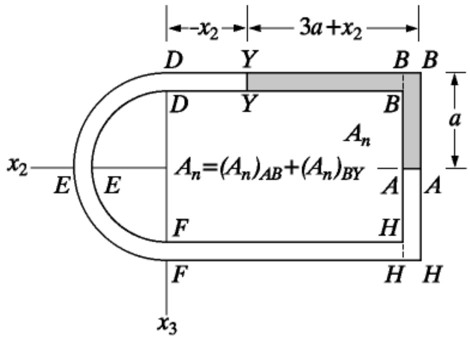


Figure c Cross section of the beam showing the area A_n used in the computation of the shearing stress acting on the portion of the cross section from line BB to line DD.

DD, we have

$$A_n \bar{x}_{3n} = (A_n \bar{x}_{3n})_{AB} + (A_n \bar{x}_{3n})_{BY} = -\frac{ta^2}{2} - ta(3a + x_2) = -\frac{at}{2}(7a + 2x_2) \tag{d}$$

Referring to Fig. d, for the portion of the cross section of the beam from line *DD* to line *FF,* we have

$$A_n \bar{x}_{3n} = (A_n \bar{x}_{3n})_{AB} + (A_n \bar{x}_{3n})_{BD} + \iint_{DZ} x_3 dA \tag{e}$$

where

$$(A_n \bar{x}_{3n})_{AB} = -at\left(\frac{a}{2}\right)$$

$$(A_n \bar{x}_{3n})_{BD} = -3at(a) \tag{f}$$

$$\iint_{DZ} x_3 dA = -\int_0^\theta a\cos\phi(tad\phi) = -a^2 t \sin\theta$$

Substituting relations (f) into (e), we get

$$A_n \bar{x}_{3n} = -\frac{a^2 t}{2}(7 + 2\sin\theta) \tag{g}$$

Referring to Fig. e and to relation (g) for the portion of the cross section of the beam kfrom line *FF* to line *HH,* we have

$$\begin{aligned} A_n \bar{x}_{3n} &= (A_n \bar{x}_{3n})_{AB} + (A_n \bar{x}_{3n})_{BD} + (A_n \bar{x}_{3n})_{DF} + (A_n \bar{x}_{3n})_{FX} \\ &= -\frac{at^2}{2} - 3at(a) + 0 - tax_2 = -\frac{at}{2}(7a + 2x_2) \end{aligned} \tag{h}$$

For the portion of the cross section of the beam from line *HH* to line *AA,* we have

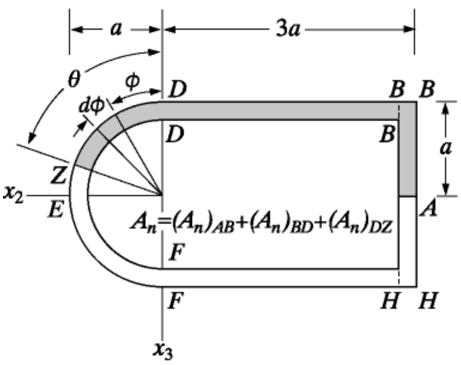


Figure d Cross section of the beam showing the area A_n used in the computations of the shearing stress, acting on the portion of the cross section from line *DD* to line *FF*.

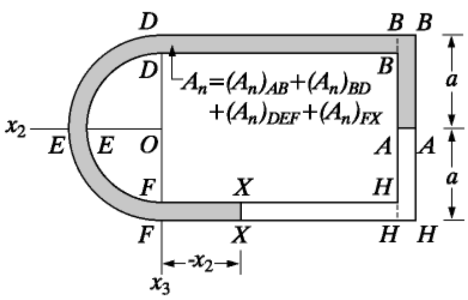


Figure e Cross section of the beam showing the area A_n used in the computation of the shearing stress, acting on the portion of the cross section from line *FF* to line *HH.*

$$A_n\bar{x}_{3n} = (A_n\bar{x}_{3n})_{AB} + (A_n\bar{x}_{3n})_{BD} + (A_n\bar{x}_{3n})_{DF} + (A_n\bar{x}_{3n})_{FH} + (A_n\bar{x}_{3n})_{H(a-x_3)}$$

$$= -\frac{a^2t}{2} - 3a^2t + 0 + 3a^2t + (a - x_3)t\left[x_3 + \frac{a - x_3}{2}\right] = -\frac{x_3^2t}{2} \tag{i}$$

STEP 3 Using relations (c), (d), (g), (h) and (i), we obtain

$$\oint A_n\bar{x}_{3n}dx_s = -\int_0^{-a}\frac{x_3^2t}{2}(-dx_3) - \int_{-3a}^{0}\frac{at}{2}(7a + 2x_2)\,dx_2 - \frac{a^3t}{2}\int_0^{\pi}(7 + 2\sin\theta)\,d\theta$$

$$- \int_0^{-3a}\frac{at}{2}(7a + 2x_2)(-dx_2) - \int_a^{0}\frac{x_3^2t}{2}(-dx_3)$$

$$= \frac{a^3t}{6} + 6a^3t - \frac{a^3t}{2}\left[7\pi - 2\cos\theta\Big|_0^{\pi}\right] - 6a^3t + \frac{a^3t}{6} = -25.329a^3t \tag{j}$$

In obtaining the above relation, we took into account, referring to Fig. a that in the portion AB of the cross section $x_s = -x_3$, in the portion BD of the cross section $x_s = 3a + x_2$ where $0 \le x_s \le 3a$ and $-3a \le x_2 \le 0$. In the portion DEF of the cross section $x_s = a\theta$ where $0 \le \theta \le \pi$, in the portion FH of the cross section $x_s = -x_2$ where $0 \le x_s \le 3a$ and $0 \le x_2 \le -3a$. In the portion HA of the cross section $x_s = a - x_3$ where $0 \le x_s \le a$ and $a \le x_3 \le 0$. Moreover, referring to Fig. a, we have

$$\oint dx_s = 2a + 3a + \pi a + 3a = (\pi + 8)a = 11.1416a \tag{k}$$

STEP 4 Substituting relations (a), (c), (j) and (k) into (b), we get

$$q_{AB} = -\left(-\frac{x_3^2t}{2} + \frac{25.329a^3t}{11.1416a}\right)\frac{P_3}{8.237ta^3} = -\left(-0.06070\frac{x_3^2}{a^2} + 0.2760\right)\frac{P_3}{a} \tag{l}$$

Substituting relations (a), (d), (j) and (k) into (b), we have

$$q_{BD} = -\left[-\frac{at}{2}(7a + 2x_2) + 2.2733a^2t\right]\frac{P_3}{8.237ta^3} = \left(0.12140\frac{x_2}{a} + 0.14893\right)\frac{P_3}{a} \tag{m}$$

Substituting relations (a), (g), (j) and (k) into (b), we obtain

$$q_{DF} = -\left[-\frac{a^2t}{2}(7 + 2\sin\theta) + 2.2733a^2t\right]\frac{P_3}{8.237ta^3} = (0.12140\sin\theta + 0.14893)\frac{P_3}{a} \tag{n}$$

Substituting relation (a), (h), (j) and (k) into (b), we have

$$q_{FH} = -\left[-\frac{a^2t}{2}\left(7 + \frac{2x_2}{a}\right) + 2.2733a^2t\right]\frac{P_3}{8.237ta^3} = \left(0.12140\frac{x_2}{a} + 0.14893\right)\frac{P_3}{a} \tag{o}$$

Substituting relations (a), (i), (j) and (k) into (b), we get

$$q_{HA} = -\left(-\frac{x_3^2t}{2} + 2.2733a^2t\right)\frac{P_3}{8.237ta^3} = \left(0.06070\frac{x_3^2}{a^2} - 0.276\right)\frac{P_3}{a} \tag{p}$$

The results are shown in Fig. f.

In order to locate the shear center of the cross sections of the beam we use relation (12.64). Thus, referring to Fig. f, we have

$$\begin{aligned} P_3 e = -\Bigg[& \left(-0.2153\frac{P_3}{a}\right)(2a)(3a) - \frac{2}{3}(0.2760 - 0.2153)(2a)(3a)\frac{P_3}{a} \\ & - \left(0.2153\frac{P_3}{a}\right)\left(\frac{1.22677a}{2}\right)a + \left(0.14893\frac{P_3}{a}\right)\left(\frac{0.177323a}{2}\right)a + \\ & + \int_0^{\pi}\left[(0.1214\sin\theta + 0.14893)\frac{P_3}{a}(a^2)\right]d\theta \\ & + \left(0.14893\frac{P_3}{a}\right)\left(\frac{1.77323a}{2}\right)a - \left(0.2153\frac{P_3}{a}\right)\left(\frac{1.22677a}{2}\right)a \Bigg] \end{aligned}$$

or

$$e = -0.811a \tag{q}$$

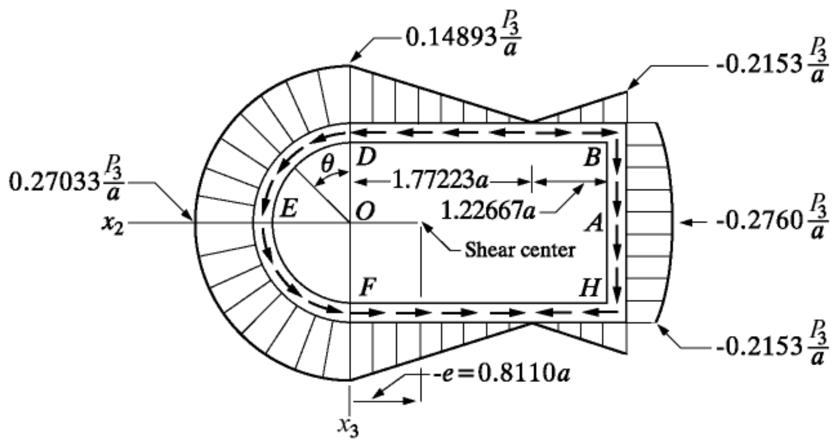
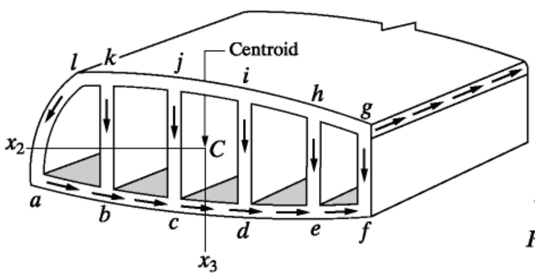


Figure f Results. Showing the shear flow on the positive cross section.

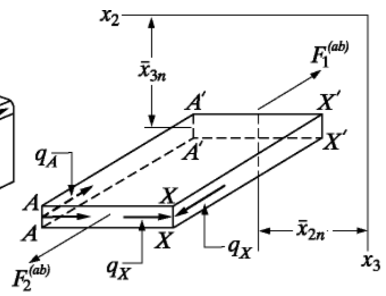
12.7 Thin-Walled, Multi-Cell, Tubular Beams Subjected to Bending without Twisting

Consider an n-cell, thin-walled, tubular beam subjected to bending without twisting due to the application on it of transverse forces whose line of action lies in the plane which contains the shear centers of the cross sections of the beam. An n-cell, tubular beam is a statically indeterminate structure to the n^{th} degree. That is, there are n more unknown shear flows than the number of available equations of statics and, consequently, we need n equations of compatibility in addition to the equations of statics. Thus, in order to establish the shear flow acting on the cross sections of an n-cell beam we adhere to the following steps:

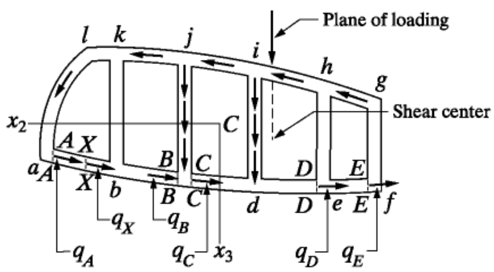
STEP 1 We choose a line normal to the median line of the wall of each cell of the beam in a way that if we make slits extending over the whole length of the beam by planes normal to its cross sections, each of which contains one of the chosen lines, the resulting auxiliary beam has an open cross section (see Fig. 12.13d). We choose the shear flows corresponding to the chosen lines as the unknown quantities, which we call the *redundant shear flows*. They are considered positive when they tend to turn the cell on whose walls they act counterclockwise. We express the shear flow at any line normal to the median line of the cross sections of the walls of the beam in terms of one or more redundant shear flows by considering the equilibrium of parts of the wall of the beam. For example, referring to Fig. 12.13b, we choose lines AA, BB, CC, DD and EE of the five-cell tubular beam shown



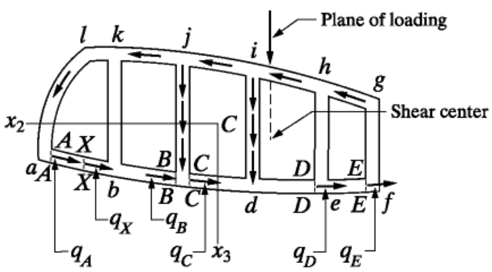
(a) View of a five-cell, thin-walled, tubular beam



(c) Free-body diagram of segment $AAXXX'X'A'A'$ of the beam



(b) Cross section of the five-cell, thin-walled, tubular beam of Fig. 12.13a showing the chosen redundant shear flows.



(d) Cross section of the auxiliary beam of the beam of Fig. 12.13a

Figure 12.13 Thin-walled, multi-cell beam.

in Fig. 12.13a and the shear flows q_A, q_B, q_C, q_D and q_E acting on these lines as the redundant shear flows (see Fig. 12.13b). We can establish the shear flow q_X at any line normal to the median line of part *ab* of this five-cell beam by considering the equilibrium of seqment *AAXXA'A'X'X* of length dx_1 shown in Fig. 12.13c. This segment is cut from the beam by two planes normal to its axis at x_1 and $x_1 + dx_1$ and by two planes normal to its cross sections, the one containing line *AA* and the other line *XX*. We denote by F_2 and F_1 the resultants of the normal components of stress acting on the faces *AAXX* and *A'A'X'X'* of the segment under consideration, respectively. Referring to Fig. 12.13c, we have

$$q_X = q_A - \frac{F_2 - F_1}{dx_1} \tag{12.65}$$

STEP 2 We denote by $(q_X)_{aux}$ the shear flow at line *XX* of an auxiliary beam obtained from the actual beam by making slits along its entire length by planes normal to the cross sections of the beam each of which contains one of the lines *AA, BB, DD, EE* or *FF* (see Fig. 12.13d). The auxiliary beam has thin-walled open cross sections and the shear flows acting on its walls can be established using relation (9.63). That is, referring to relation (12.57), we have

$$(q_X)_{\text{aux.}} = \frac{F_2 - F_1}{dx_1} = -\left(\frac{Q_3 I_{23} - Q_2 I_{22}}{I_{22} I_{33} - I_{23}^2}\right)\bar{x}_{2n} A_n + \left(\frac{Q_3 I_{33} - Q_2 I_{23}}{I_{22} I_{33} - I_{23}^{\,2}}\right)\bar{x}_{3n} A_n \tag{12.66}$$

where

A_n = area of the part *AAXX* of the cross section of the beam.

$\bar{x}_{2n}, \bar{x}_{3n}$ = distance of the centroid of the part *AAXX* of the cross section of the beam from its x_3 or x_2 centroidal axis, respectively (see Fig. 12.13c).

I_{22}, I_{33}, I_{23} = moments and product of inertia of the cross section of the beam about the its x_2 and x_3 centroidal axes.

Using relation (12.66) relation (12.65) can be rewritten as

$$q_x = q_A - (q_X)_{\text{aux.}} \tag{12.67}$$

We compute the shear flows acting on the walls of the auxiliary beam using relation (12.66) and we substitute them into relation (12.67) to obtain the shear flows at any line normal to the middle line of the cross sections of the beam in terms of the redundant shear flows.

STEP 3 We write one equation of consistent deformation for each cell of the beam. We accomplish this by noting that inasmuch as the thin-walled, multi-cell beam under consideration does not twist, the angle of twist per unit length of each cell must vanish. That is, referring to relation (12.25), we have

$$\alpha_j = \frac{1}{2G\Omega}\oint \frac{q^{(j)}}{t}\, dx_s = 0 \qquad j = 1, 2, ..., n \tag{12.68}$$

where $q^{(j)}$ is the shear flows acting on the cross sections of the walls of the j^{th} cell of the beam established in step 2. We substitute $q^{(j)}$ in relations (12.68) to obtain n linear algebraic equations involving the n unknown redundant shear flows and the known shear flows acting on the cross sections of the auxiliary beam. We solve these equations to obtain the redundant shear flows. Relations (12.68) are referred to as the equations of consistant deformation of the beam.

STEP 4 We substitute the values of the redundant shear flows obtained in step 3 and the shear flows acting on the cross sections of the auxiliary beam established in step 2 in relation (12.67) to obtain the shear flows acting on the cross sections of the walls of the thin-walled, multi-cell, tubular beam under consideration.

In what follows we present an example.

Example 5 Consider the thin-walled, three-cell, tubular beam whose cross section is shown in Fig. a. The beam is fixed at its one end and is subjected to a concentrated transverse force of magnitude P_3 at its other end. The walls of the beam have a constant thickness t. The line of action of this force passes through the shear center of the unsupported end cross section of the beam. The beam is made from an isotropic linearly elastic material whose shear modulus is G. Compute the distribution of shearing stress on the cross sections of the beam and locate their shear center.

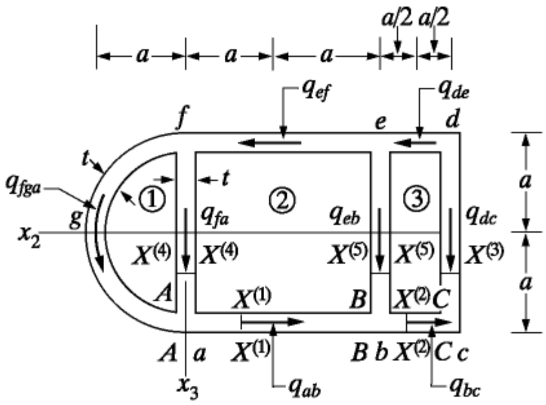


Figure a Geometry of the cross sections of the tubular beam and positive shear flows.

Solution We denote by q_{ab}, q_{bc}, q_{de}, q_{ef}, q_{fga}, q_{fa}, q_{eb} and q_{dc} the shear flows acting on the parts *ab, bc, de, ef, fga, fa, eb* and *dc,* respectively, of the cross sections of the beam. We consider them positive as shown in Fig. a. Inasmuch as, the cross sections of the beam are symmetric with respect to the x_2 axis, the x_2 and x_3 axes are principal. Moreover,

$$q_{ab} = q_{ef} \qquad\qquad q_{bc} = q_{de} \tag{a}$$

STEP 1 Referring to Fig. a we choose the shear flows at lines AA, BB and CC of the walls of cells 1, 2 and 3, respectively, as the redundant shear flows and we denote them by q_A, q_B and q_C, respectively. We establish the shear flows q_{ab}, q_{bc}, q_{fga}, q_{fa}, q_{eb} and q_{dc} in terms of the redundant shear flows by considering the equilibrium of the seqments of length dx_1 of the beam shown in Fig. b. That is

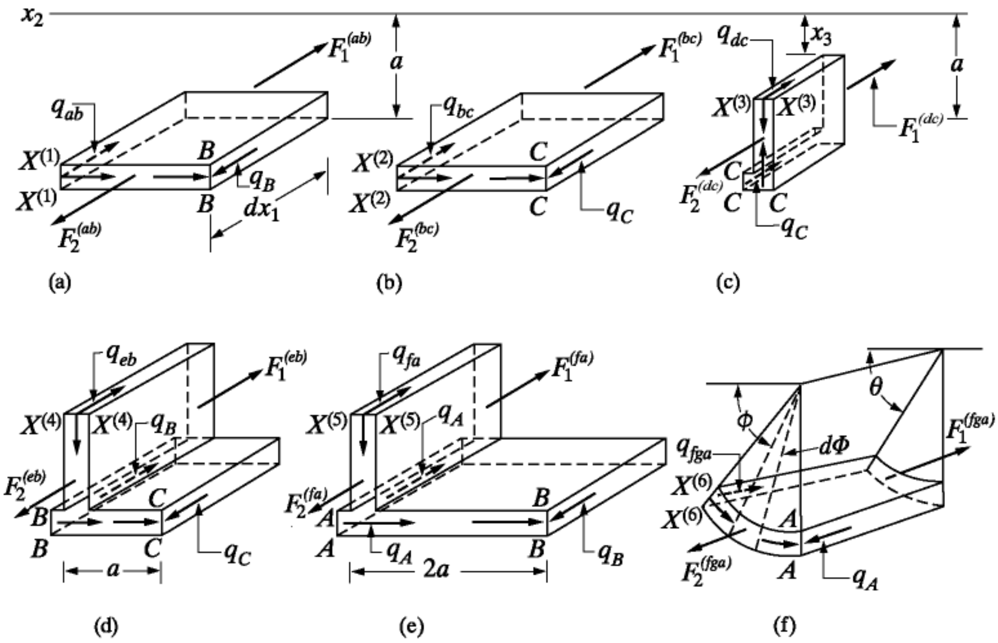


Figure b Free-body diagrams of segments of infinitesimal length of a beam.

Referring to Fig. ba

$$q_{ab} = \frac{F_2^{(ab)} - F_1^{(ab)}}{dx_1} + q_B = q_{ab}^{\text{aux}} + q_B \tag{b}$$

Referring to Fig. bb

$$q_{bc} = \frac{F_2^{(bc)} - F_1^{(bc)}}{dx_1} + q_C = q_{bc}^{\text{aux}} + q_C \tag{c}$$

Referring to Fig. bc

$$q_{dc} = \frac{F_2^{(dc)} - F_1^{(dc)}}{dx_1} - q_C = q_{dc}^{\text{aux}} - q_C \tag{d}$$

Referring to Fig. bd

$$q_{eb} = \frac{F_2^{(eb)} - F_1^{(eb)}}{dx_1} + q_C - q_B = q_{eb}^{\text{aux}} + q_C - q_B \tag{e}$$

Referring to Fig. be

$$q_{fa} = \frac{F_2^{(fa)} - F_1^{(fa)}}{dx_1} - q_A + q_B = q_{fa}^{\text{aux}} - q_A + q_B \tag{f}$$

Referring to Fig. bf

$$q_{fga} = \frac{F_2^{(fga)} - F_1^{(fga)}}{dx_1} + q_A = q_{fga}^{\text{aux}} + q_A \tag{g}$$

STEP 2 We form an auxiliary beam by making slits across lines AA, BB and CC extending over the entire length of the beam. A cross section of the auxiliary beam is shown in Fig. c. We compute the shear flow acting on parts *ab, bc, fga, fa, eb* and *dc,* of the cross sections of the auxiliary beam. That is, referring to Fig. ba and denoting by $A_n^{(ab)}$ area $X^{(1)}$ $X^{(1)}$ BB, we have

$$q_{ab}^{aux} = \frac{P_3 A_n^{(ab)} \bar{x}_{2n}^{(ab)}}{I_2} = \frac{P_3(2a + x_2)ta}{I_2} \tag{h}$$

Referring to Fig. bb and denoting by $A_n^{(bc)}$ area $X^{(2)}$ $X^{(2)}$ CC, we get

$$q_{bc}^{aux} = \frac{P_3 A_n^{(bc)} \bar{x}_{2n}^{(bc)}}{I_2} = \frac{P_3(3a + x_2)ta}{I_2} \tag{i}$$

Referring to Fig. bc and denoting by $A_n^{(dc)}$ area CC $X^{(3)}$ $X^{(3)}$, we obtain

$$q_{dc}^{aux} = \frac{P_3 A_n^{(dc)} \bar{x}_{3n}^{(dc)}}{I_2} = \frac{P_3(a - x_3)t\left(\dfrac{a + x_3}{2}\right)}{I_2} \tag{j}$$

Referring Fig. bd and denoting by $A_n^{(eb)}$ area $X^{(4)}$ $X^{(4)}$ $BBCC$, we have

$$q_{eb}^{aux} = \frac{P_3 A_n^{(eb)} \bar{x}_{3n}^{(eb)}}{I_2} = \frac{P_3\left[a^2 t + (a - x_3)t\left(\dfrac{a + x_3}{2}\right)\right]}{I_2} \tag{k}$$

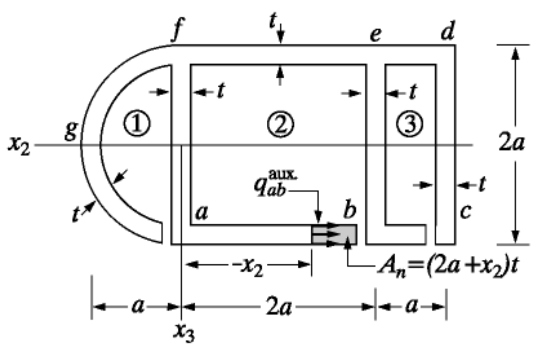


Figure c Cross section of the auxiliary beam.

Referring to Fig. be and denoting by $A_n^{(fa)}$ area $X^{(5)}\, X^{(5)}\, AABB$, we get

$$q_{fa}^{\text{aux}} = \frac{P_3 A_n^{(fa)} \bar{x}_{3n}^{(fa)}}{I_2} = \frac{P_3\left[4a^2t + (a^2 - x_3^2)t\right]}{2I_2} \tag{l}$$

Referring to Fig. bf and denoting by $A_n^{(fga)}$ area $X^{(6)}\, X^{(6)}\, AA$, we obtain

$$q_{fga}^{\text{aux}} = \frac{P_3 A_n^{(fga)} \bar{x}_{3n}}{I_2} = \frac{P_3 t a^2}{I_2}\int_{\phi=\theta}^{\phi=\pi/2} \sin\phi \, d\phi = \frac{P_3 t a^2}{I_2}\cos\theta \tag{m}$$

STEP 3 Substituting relations (f) and (g) into relation (12.68) and using relations (l) and (m), we obtain

$$2G\Omega t\alpha_{fgaf} = -\int_{x_s=0}^{x_s=2a} q_{fa}\,dx_s + \int_{x_s=0}^{x_s=a\pi} q_{fga}\,dx_s = \int_{x_3=a}^{x_3=-a} (q_{fa}^{\text{aux}} - q_A + q_B)\,dx_3 + \int_{\theta=-\pi/2}^{\theta=\pi/2} (q_{fga}^{\text{aux}} + q_A)\,d\theta$$

$$= \int_{x_3=a}^{x_3=-a}\left[P_3\left(\frac{4a^2t + (a^2 - x_3^2)t}{2I_2}\right) - q_A + q_B\right]dx_3 + \int_{\theta=-\pi/2}^{\theta=\pi/2}\left(\frac{P_3 t a^2}{I_2}\cos\theta + q_A\right) a\,d\theta = 0 \tag{n}$$

In obtaining the above relation we have taken into account that in the portion af of the cross section of the beam $x_s = a - x_3$ where $0 \le x_s \le 2a$ and $a \le x_3 \le -a$, while in the portion fga of the cross section $x_s = a\theta + a\pi/2$ where $0 \le x_s \le a\pi$ and $-\pi/2 \le \theta \le \pi/2$. Moreover, positive q_{fa} is acting clockwise while in the integrand of relation (12.68) positive $q^{(j)}$ is acting counterclockwise; thus, $q^{(j)} = -q_{fa}$. Substituting relations (b), (e) and (f) into (12.68) and using relations (a), (h), (k) and (l), we have

$$2G\Omega t\alpha_{abefa} = \int_{x_s=0}^{x_s=2a} q_{ab}\,dx_s - \int_{x_s=0}^{x_s=2a} q_{eb}\,dx_s + \int_{x_s=0}^{x_s=2a} q_{ef}\,dx_s + \int_{x_s=0}^{x_s=2a} q_{fa}\,dx_s$$

$$= 2\int_{x_2=0}^{x_2=-2a} (q_{ab}^{\text{aux}} + q_B)(-dx_2) + \int_{x_3=a}^{x_3=-a} (q_{eb}^{\text{aux}} + q_C - q_B)\,dx_3 + \int_{x_3=-a}^{x_3=a} (q_{fa}^{\text{aux}} - q_A + q_B)\,dx_3$$

$$= -2\int_{x_2=0}^{x_2=-2a}\left[P_3 a t\left(\frac{2a + x_2}{I_2}\right) + q_B\right]dx_2 + \int_{x_3=a}^{x_3=-a}\left[P_3\left(\frac{2a^2t + (a^2 - x_3^2)t}{2I_2}\right) - q_C - q_B\right]dx_3$$

$$+ \int_{x_3=-a}^{x_3=a}\left[P_3\left(\frac{4a^2t + (a^2 - x_3^2)t}{2I_2}\right) - q_A + q_B\right]dx_3 = 0 \tag{o}$$

In obtaining the above relation we have taken into account that in the portion *ab* of the cross section $x_s = -x_2$, in the portion *be* of the cross section of the beam $x_s = a - x_3$ where $0 \le x_s \le 2a$ and $a \le x_3 \le -a$, while in the integrand of relation (12.68) $q^{(j)} = -q_{eb}$. Moreover, in the portion of *fa* of the cross section of the beam $x_s = a + x_3$ where $0 \le x_s \le 2a$ and $-a \le x_3 \le a$ while in the integrand of relation (12.68) $q^{(j)} = q_{fa}$. Substituting relations (c), (d) and (e) into (12.67) and using relations (a), (i), (j) and (k), we get

$$2G\Omega t\alpha_{bcdeb} = 2\int_{x_2=-2a}^{x_2=-3a}(q_{bc}^{\text{aux}} + q_C)(-dx_2) + \int_{x_3=a}^{x_3=-a}(q_{dc}^{\text{aux}} - q_C)dx_3 + \int_{x_3=-a}^{x_3=a}(q_{eb}^{\text{aux}} + q_C - q_B)dx_3$$

$$= -2\int_{x_2=-2a}^{x_2=-3a}\left[P_3at\left(\frac{3a+x_2}{I_2}\right) + q_C\right]dx_3 + \int_{x_3=a}^{x_3=-a}\left[P_3t\left(\frac{a^2-x_3^2}{2I_2}\right) - q_C\right]dx_3 + \int_{x_3=-a}^{x_3=a}\left[P_3t\left(\frac{3a^2-x_3^2}{2I_2}\right) + q_C - q_B\right]dx_3 = 0 \tag{p}$$

In obtaining the above relation we have taken into account that in the portion *bc* of the cross section of the beam $x_s = -x_2 - 2a$ where $0 \le x_s \le a$ and $-2a \le x_2 \le -3a$, in the portion *cd* of the cross section $x_s = a - x_3$ where $0 \le x_s \le 2a$ and $a \le x_3 \le -a$, while in the integrand of relation (12.68) $q^{(j)} = -q_{dc}$. Moreover, in the portion *eb* of the cross section of the beam $x_s = a + x_3$ where $0 \le x_s \le 2a$ and $-a \le x_3 \le a$ while in the integrand of relation (12.68) $q^{(j)} = q_{eb}$. Integrating relations (n) to (p), we have

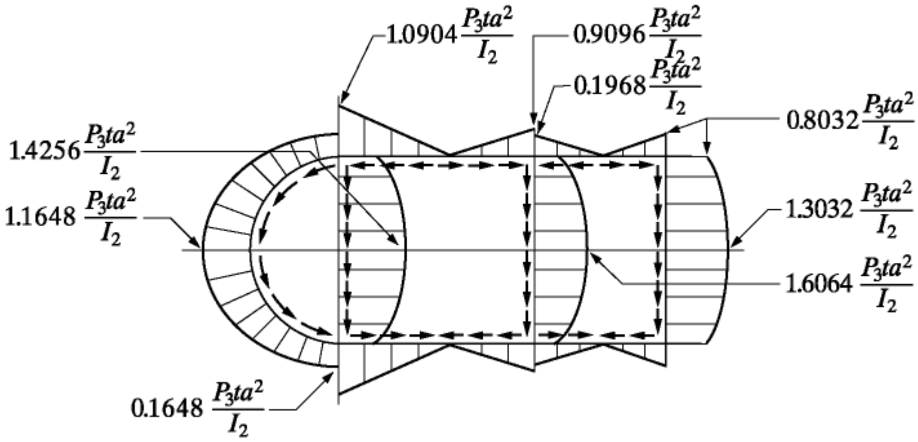


Figure d Results.

$$
\begin{aligned}
(\pi + 2)q_A - 2q_B &= \frac{8P_3ta^2}{3I_2} \\
4q_B - q_C - q_A &= -\frac{3P_3a^2t}{I_2} \\
2q_B - 6q_C &= \frac{P_3ta^2}{I_2}
\end{aligned}
\tag{q}
$$

Solving relations (q), simultaneously, we obtain

$$
q_A = 0.16483\frac{P_3ta^2}{I_2} \qquad q_B = -0.90959\frac{P_3ta^2}{I_2} \qquad q_C = -0.80397\frac{P_3ta^2}{I_2} \tag{r}
$$

Substituting relations (r) into relations (b) to (g), and using relations (h) to (m), we get expressions for the shear flows $q_{ab,b}$ q_{bc}, q_{dc}, q_{eb}, q_{fa} and q_{fga}. They are plotted in Fig. d. It can be shown that the sum of the resultant forces of the computed shear flows is equal to the external force P_3. This is a partial check of the calculations.

12.8 Single-Cell, Tubular Beams with Longitudinal Stringers Subjected to Bending Without Twisting

Single-cell, thin-walled, tubular beams are also known as *monocoque*. These beams are not well suited to resist transverse forces and bending moments since their walls offer little resistance to local buckling. For this reason these beams are often reinforced with longitudinal stringers (see Fig. 12.14). Such stiffened tubular beams are called *semimonocoque*.

When analyzing semimonocoque beams, it is assumed that the stringers carry all the normal components of stress while the walls (skins) of the beams transmit the shearing components of stress from one stringer to another. Recalling the derivation of the formula for computing the shearing components of stress acting on the cross sections of beams we can deduce that the assumption that the normal component to stress acting on the skins of a semimonocoque beam is negligible implies that *the shear flow acting on the skins of such a beam is constant between stringers*. The normal component of stress acting on the cross sections of the stringers of a semimonocoque beam is determined using relation

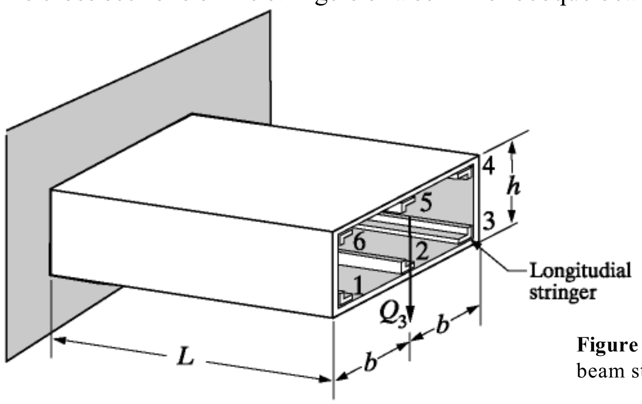


Figure 12.14 Thin-walled, single-cell tubular beam stiffened with longitudinal stringers.

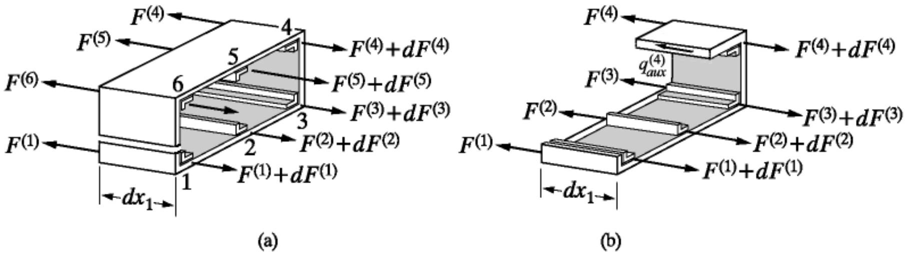


Figure 12.15 Segments of the auxiliary beam of the thin-walled, single-cell, semimonocoque beam.

(9.12a), where I_{22}, I_{33} and I_{23} are the moments and the product of inertia of the cross sections of the stringers about the x_2 and x_3 centroidal axes of the cross section of the beam. The shearing component of stress is computed following the procedure adhered to in Section 12.6 for computating the shear flow in monocoque beams. For example, for the beam of Fig. 12.14, we form an auxiliary beam having an open cross section obtained from the actual beam by making a slit at the left side of stringer 1 running along the entire length of the beam. The auxiliary beam is subjected to the given loading of the actual beam. A segment of this auxiliary beam of the length dx_1 is shown in Fig. 12.15a; in this figure we denote by $F^{(i)}$ and $F^{(i)} + dF^{(i)}$ $(i = 1, 2, 3, ..., 6)$ the resultants of the normal component of stress τ_{11} acting on the end surfaces of the i^{th} stringer at x_1 and at x_1+dx_1, respectively. We compute the shear flow $q_{\text{aux}}^{(k)}$ acting on the skin of the auxiliary beam which extends between the k^{th} and the $(k + 1)^{th}$ stringer. In Fig. 12.15b we show a portion of the segment of the auxiliary beam shown in Fig. 12.15a. From the equilibrium of this portion we have

$$q_{\text{aux}}^{(k)} = \sum_{i=1}^{k} \frac{dF^{(i)}}{dx_1} \tag{12.69}$$

Referring to relations (9.12a) the forces $F^{(i)}$ are equal to

$$F^{(i)} = \tau_{11}^{(i)} A_i = A_i \left[-\left(\frac{M_2 I_{23} + M_3 I_{22}}{I_{22} I_{33} - I_{23}^2} \right) x_2^{(i)} + \left(\frac{M_2 I_{33} + M_3 I_{23}}{I_{22} I_{33} - I_{23}^2} \right) x_3^{(i)} \right] \tag{12.70}$$

where $\tau_{11}^{(i)}$ is the average value of the normal component of stress acting on the cross section of the i^{th} stringer which could be approximated by the value of the normal component of stress at the centroid of the i^{th} stringer. Substituting relation (12.70) into (12.69) and using relations (8.20) and (8.21) we obtain

$$q_{\text{aux}}^{(k)} = \left(\sum_{i+1}^{k} A_i \bar{x}_2^{(i)} \right) \left(\frac{Q_2 I_{22} - Q_3 I_{23}}{I_{22} I_{33} - I_{23}^2} \right) + \left(\sum_{i=1}^{k} A_i \bar{x}_3^{(i)} \right) \left(\frac{Q_3 I_{33} - Q_2 I_{23}}{I_{22} I_{33} - I_{23}^2} \right) \tag{12.71}$$

This relation may be rewritten as

$$q_{\text{aux}}^{(k)} = \left(\frac{I_{22} \sum_{i=1}^{k} A_i \bar{x}_2^{(i)} - I_{23} \sum_{i=1}^{k} A_i \bar{x}_3^{(i)}}{I_{22} I_{33} - I_{23}^2} \right) Q_2 + \left(\frac{I_{33} \sum_{i=1}^{k} A_i \bar{x}_3^{(i)} - I_{23} \sum_{i=1}^{k} A_i \bar{x}_2^{(i)}}{I_{22} I_{33} - I_{23}^2} \right) Q_3 \tag{12.72}$$

In the above relations I_{22}, I_{33} and I_{23} are equal to

$$I_{22} = \sum_{i=1}^{n} A_i[x_3^{(i)}]^2 \qquad I_{33} = \sum_{i=1}^{n} A_i[x_2^{(i)}]^2 \qquad I_{23} = \sum_{i=1}^{n} A_i x_2^{(i)} x_3^{(i)} \tag{12.73}$$

Once the shear flow $q_{\text{aux}}^{(k)}$ (k = 1, 2, ..., 6) acting on the skins of the auxiliary beam is established the shear flow $q^{(k)}$ acting on the skins of the actual beam can be computed using relation (12.60). That is,

$$q^{(k)} = -q_{\text{aux}}^{(k)} + \frac{\oint q_{\text{aux}} \frac{dx_s}{t}}{\oint \frac{dx_s}{t}} \tag{12.74}$$

In the above relation the line integrals are taken counterclockwise.

In what follows we present an example.

Example 6 Compute the shear flow acting on the skin of the thin-walled, single-cell, tubular reinforced with stringers cantilever beam of Fig a. subjected to a concentrated force P_3 at its unsupported end. The stringers have the same cross section of area A.

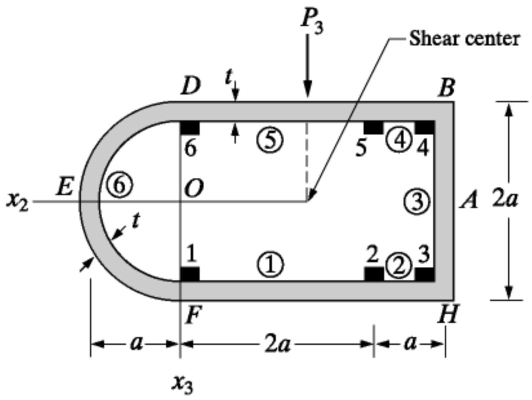


Figure a Geometry and loading of the cross section of the beam.

Solution We form the auxiliary beam of Fig. b by making a slit adjacent to stringer 1 of the actual beam extending over its entire length. Noting that the axes x_2 and x_3 are principal, for this auxiliary beam relation (12.72) reduces to

$$q_{\text{aux}}^{(k)} = \frac{\left(\sum_{i=1}^{k} A_i \bar{x}_3^{(i)}\right) P_3}{I_2} = \frac{A P_3 \sum_{i=1}^{k} \bar{x}_3^{(i)}}{I_2} \tag{a}$$

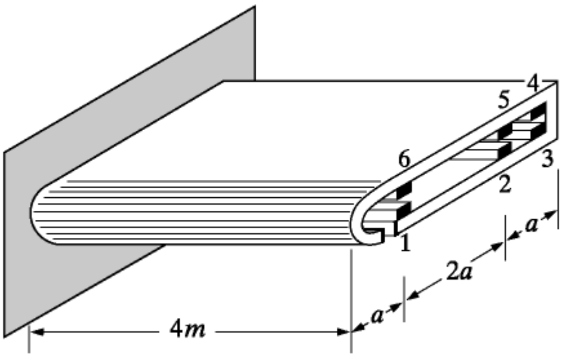


Figure b Auxiliary beam.

where

$$I_2 = \sum_{i=1}^{6} A_i[\bar{x}_3^{(i)}]^2 = A\sum_{i=1}^{6}[\bar{x}_3^{(i)}]^2 = 6Aa^2 \tag{b}$$

Substituting relation (b) into (a), we obtain

$$q_{\text{aux}}^{(k)} = \frac{P_3\sum_{i=1}^{k}\bar{x}_3^{(i)}}{\sum_{i=1}^{6}[\bar{x}_3^{(i)}]^2} = \frac{P_3\sum_{i=1}^{k}\bar{x}_3^{(i)}}{6a^2} \tag{c}$$

Thus,

$$\begin{aligned} q_{\text{aux}}^{(1)} &= \frac{P_3}{6a} & q_{\text{aux}}^{(2)} &= \frac{P_3}{3a} & q_{\text{aux}}^{(3)} &= \frac{P_3}{2a} \\ q_{\text{aux}}^{(4)} &= \frac{P}{3a} & q_{\text{aux}}^{(5)} &= \frac{P_3}{6a} & q_{\text{aux}}^{(6)} &= 0 \end{aligned} \tag{d}$$

Substituting relation (c) into (12.74) and integrating, we obtain

$$q^{(k)} = -q_{\text{aux}}^{(k)} + \frac{\frac{P_3}{6at}(2a + 2a + 6a + 2a + 2a)}{\frac{1}{t}(2a + a + 2a + a + 2a + \pi a)} = -q_{\text{aux}}^{(k)} + \frac{14P_3}{6(8+\pi)a} \tag{e}$$

Substituting relations (d) into (e), we obtain

$$q^{(1)} = q^{(5)} = \frac{P_3}{a}\left(-\frac{1}{6} + \frac{14}{6(8+\pi)}\right) = \frac{P_3(6-\pi)}{6a(8+\pi)} = 0.042759\frac{P_3}{a}$$

$$q^{(2)} = q^{(4)} = \frac{P_3}{a}\left(-\frac{1}{3} + \frac{7}{3(8+\pi)}\right) = -\frac{P_3(1+\pi)}{3(8+\pi)} = -0.12391\frac{P_3}{a}$$

$$q^{(3)} = \left(-\frac{1}{2} + \frac{7}{3(8 + \pi)} \right) = -\frac{P_3(10 + 3\pi)}{6(8 + \pi)} = -0.29057 \frac{P_3}{a}$$

$$q^{(6)} = \frac{7P_3}{3(8 + \pi)a} = 0.20943 \frac{P_3}{a}$$

The results are shown in Fig. c.

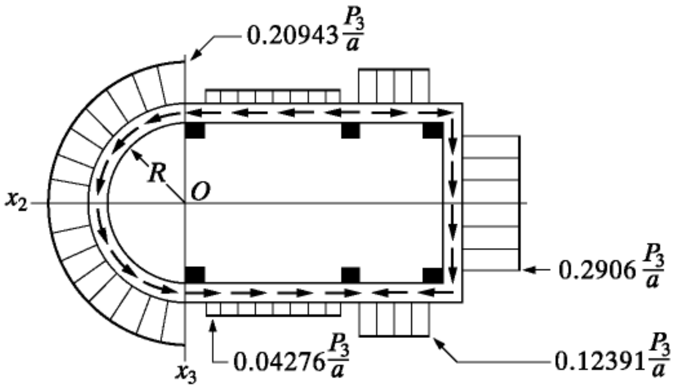


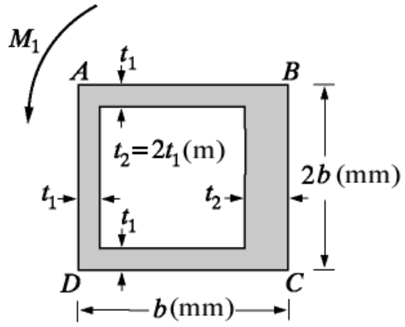
Figure c Shear flow on the cross sections of the beam of Fig. a.

12.9 Problems

1. to **4.** Consider the thin-walled, single-cell cantilever member whose cross section is shown in Fig 12P1. The member is made from an isotropic, linearly elastic material (G = 80 GPa) and is subjected to a torsional moment M_1(kN·m) at its unsupported end. Determine the magnitude of the shear stress acting on the cross sections of the member as well as the twist per unit length. Repeat with the members whose cross sections are shown in Figs. 12P2 to 12P4.

Ans. 1 $\alpha = \frac{5(10^3)M_1}{16Gb^3t_1}\left(\frac{\text{rad}}{\text{mm}}\right)$ $\quad \tau_{1s}^{BADC} = \frac{10^3 M_1}{4b^2t_1}\left(\frac{\text{kN}}{\text{mm}^2}\right)$ Ans. 2 $\alpha = 1.435(10^{-8})\frac{M_1}{G}\left(\frac{\text{rad}}{\text{m}}\right)$ $\quad \tau_{1s} = 341.84M_1\left(\frac{\text{kN}}{\text{m}^2}\right)$

Ans. 3 $\alpha = 0.625(10^{-11})\frac{M_1}{a^3t}\left(\frac{\text{rad}}{\text{m}}\right)$ $\quad \tau_{1s} = 0.2887\frac{M_1}{ta^2}\left(\frac{\text{kN}}{\text{m}^2}\right)$ Ans. 4 $\alpha = 2.0833(10^{-8})M_1\left(\frac{\text{rad}}{\text{m}}\right)$ $\quad \tau_{1s} = 416.67M_1\left(\frac{\text{kN}}{\text{m}^2}\right)$



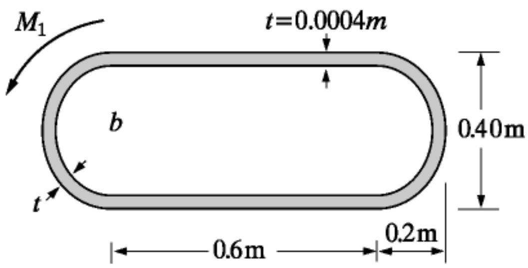


Figure 12P1 **Figure 12P2**

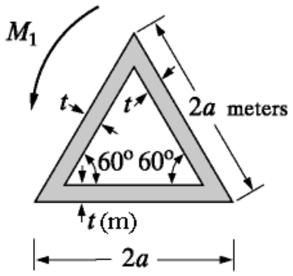


Figure 12P3

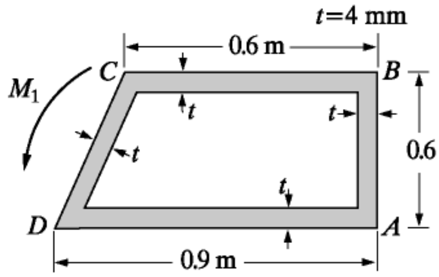


Figure 12P4

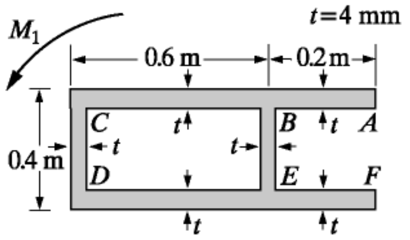


Figure 12P5

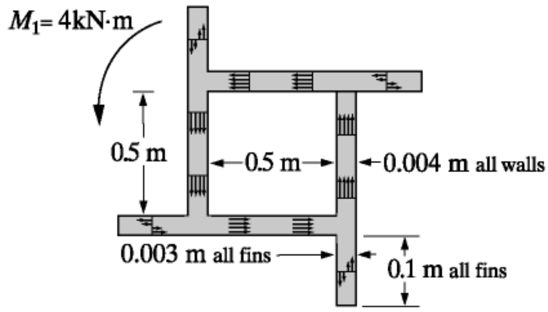


Figure 12P6

5. and **6.** Consider the thin-walled, single-cell, tubular cantilever member with fins whose cross section is shown in Fig. 12P5. The member is made from an isotropic, linearly elastic material G(GPA) and it is subjected to a torsional moment M_1 (kN·m) at its unsupported end. Determine the distribution of shearing stress on the cross sections of the member as well as the twist per unit length. Repeat with the member whose cross section is shown in Fig 12P6.

Ans. 5 $\alpha = 2170.1\dfrac{M_1}{10^6 G}\left(\dfrac{\text{rad}}{\text{m}}\right)$ $\tau_{1s}^{F_i} = 4340 M_1 x_2^{(i)}\left(\dfrac{\text{kN}}{\text{m}^2}\right)$ $\tau_{1s}^{T} = 520.8 M_1\left(\dfrac{\text{kN}}{\text{m}^2}\right)$

Ans. 6 $\alpha = 1952.7\dfrac{M_1}{10^6 G}\left(\dfrac{\text{rad}}{\text{m}}\right)$ $\tau_{1s}^{F_i} = 3095.5 M_1 x_2^{(i)}\left(\dfrac{\text{kN}}{\text{m}}\right)^2$ $\tau_{1s}^{T} = 492.09 M_1\left(\dfrac{\text{kN}}{\text{m}^2}\right)$

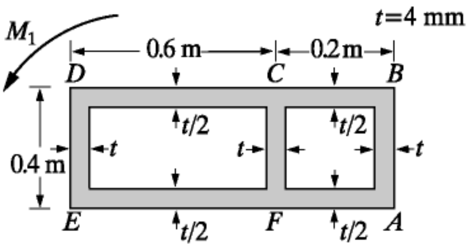


Figure 12P7

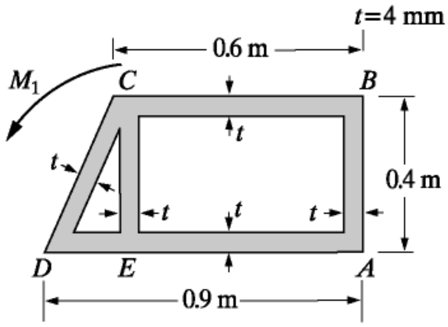


Figure 12P8

7. to **10**. Consider the thin-walled, multi-cell, cantilever member whose cross section is shown in Fig. 12P7. The member is made from an isotropic, linearly elastic material (G = 80 GPa) and is subjected to a torsional moment M_1 (kN·m) at its unsupported end.

Determine the distribution of shearing stress on the cross sections of the member as well as the twist per unit length. Repeat with the members whose cross sections are shown in Figs. 12P8 to 12P10.

Ans. 7 $\alpha = 0.3027(10^{-4})M_1 \dfrac{\text{rad}}{\text{m}}$ $\quad \tau_{1s}^{AB} = 0.34375 M_1$ MPa $\quad \tau_{1s}^{DE} = 0.40625 M_1$ MPa

Ans. 8 $\alpha = 7.9676(10^{-8})\dfrac{M_1}{t}\dfrac{\text{rad}}{\text{m}}$ $\quad \tau_{1s}^{AB} = 1.7760\dfrac{M_1}{t}$ MPa $\quad \tau_{1s}^{CD} = 1.2294\dfrac{M_1}{t}$ MPa $\quad \tau_{1s}^{CE} = 0.5466\dfrac{M_1}{t}$

Ans. 9 $\alpha = 0.1103(10^{-6})a^3 M_1 \dfrac{\text{rad}}{\text{m}}$ $\quad \tau_{1s}^{ABC} = 288.75 M_1 = \tau_{1s}^{DEF}$ MPa $\quad \tau_{1s}^{CD} = 369.5 M_1 = \tau_{1s}^{FA}$ MPa

Ans. 10 $\alpha = 1.0986(10^{-3})M_1 \dfrac{\text{rad}}{\text{m}}$ $\quad \tau_{1s}^{AB} = 7812.5 M_1 = \tau_{1s}^{DH} = \tau_{1s}^{BE}$ MPa

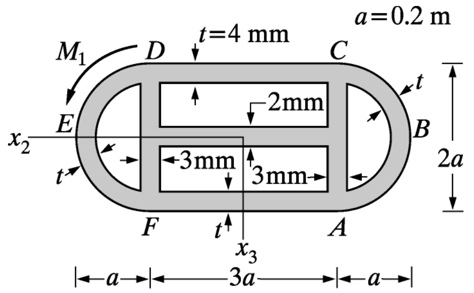


Figure 12P9

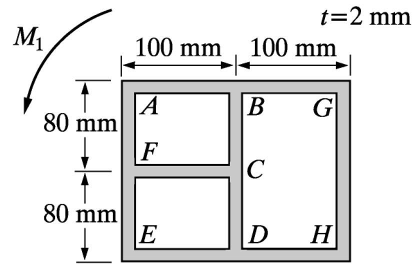


Figure 12P10

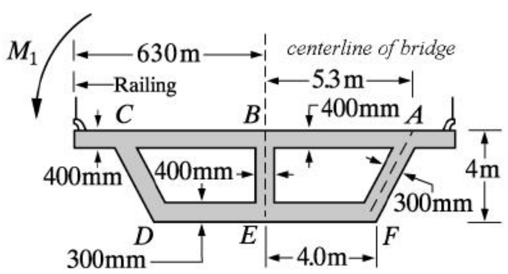


Figure 12P11

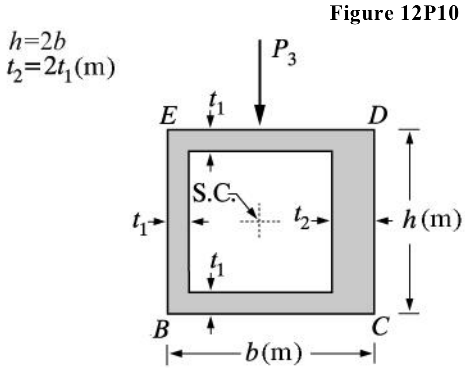


Figure 12P12

11. The cross section of a reinforced concrete hollow girder bridge is as shown in Fig. 12P11. A cross section of the girder is subjected to a torsional moment M_1. Consider the girder as a thin-walled, tubular member and determine the shear flow acting on this cross section of the girder. Disregard the effect of the cantilevering part of the cross section of the girder.

Ans. $\alpha = 0.1467\dfrac{M_1}{G}$ $\quad \tau_{1s}^{AB} = 0.3360 M_1$ $\quad \tau_{1s}^{CD} = 0.4480 M_1$

12. to 14. Consider the thin-walled, single-cell, tubular cantilever beam whose cross section is shown in Fig. 12P12. The beam is subjected to a transverse force P_3 acting on its unsupported end as shown in Fig. 12P12. Determine the shear flow acting on the cross section of the beam. Locate the shear center of the cross sections of the beam. Repeat with the beam whose cross section is shown in Figs. 12P13 and 12P14.

Ans. 12 $e = \dfrac{41b}{60}$ $\quad$ Ans. 13 $e_2 = 0.262\,\text{m}$ $\quad e_3 = 0.1816\,\text{m}$ $\quad$ Ans. 14 $e = 0.309'\text{m}$

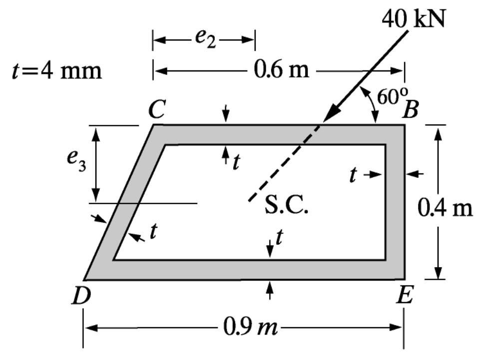


Figure 12P13

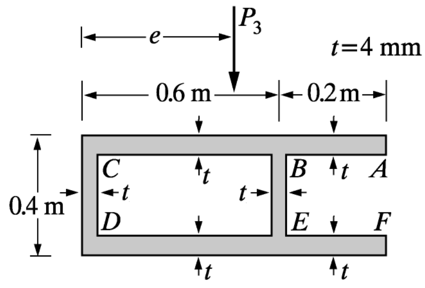


Figure 12P14

15. The cross section of a hollow reinforced concrete girder of a bridge is shown in Fig 12P11. Consider the girder as a thin-walled, tubular beam and determine the shear flow acting on one of its cross sections which is subjected to a shearing force Q_3 acting through its shear center.

16. to 19. Consider the 2 m long thin-walled, multi-cell, cantilever beam whose cross section is shown in Fig. 12P16. The beam is made from an isotropic, linearly elastic material (G = 80 GPa) and is subjected to a transverse force P_3 at its unsupported end which passes through the shear center of the cross section at the unsupported end. Determine the distribution of shearing stress on the cross sections of the beam. Locate the shear center of the cross section of the beam. Repeat with the beams whose cross sections are shown in Figs. 12P17 to 12P19. In problem 12.17 use: $I_{22} = 0.3042(10^{-3})$ m^4, I_{33} = $0.9189(10^{-3})$ m^4, $I_{23} = 0.0971(10^{-3})$ m^4.

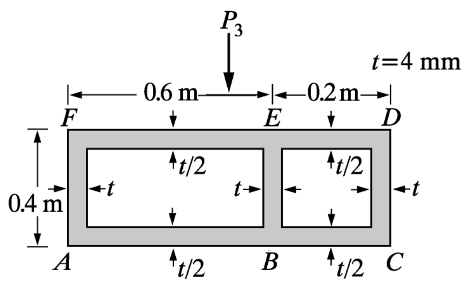


Figure 12P16

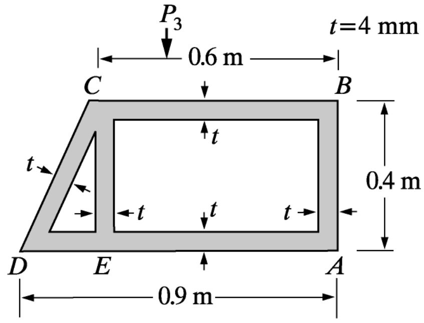


Figure 12P17

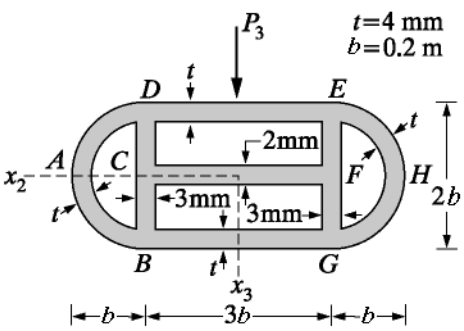


Figure 12P18

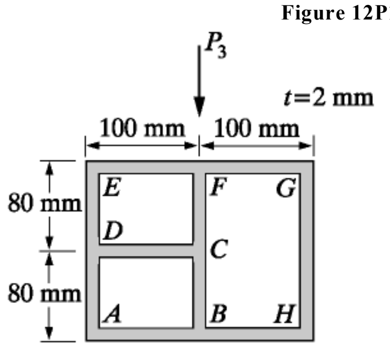


Figure 12P19

Ans. 16 $e = 0.156$ m Ans. 17 $e = 0.1914$ m Ans. 18 $q_{AB} = P_3(0.1346 + 0.1081\cos\theta)$
Ans. 19 shear center coincides with point C

20. and **21**. Consider 2 m long thin-walled, single-cell, cantilever beam reinforced with longitudinal stringers as shown in Fig. 12P20, the beam is subjected at its free end and to a concentrated transverse force P_3 whose line of action passes through the shear center of the cross section at the unsupported end of the beam. The thickness of the walls of the beam is as shown in Fig. 12P20 while the area of the cross section of each stringer is 800 mm^2. Assume that the stringers carry all the normal components of stress, while the skin is carrying the shearing components of stress. Determine
(a) The distribution of the normal component of stress acting on the cross section of the stringers at the fixed end of the beam
(b) The shear flow acting on the cross section of the walls of the beam
Repeat with the beam of Fig. 12P21.

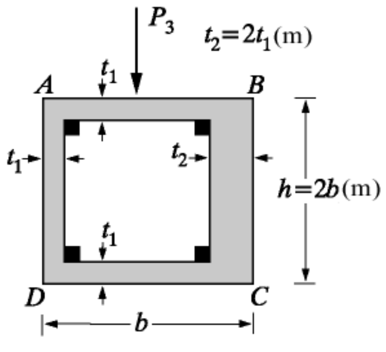


Figure 12P20

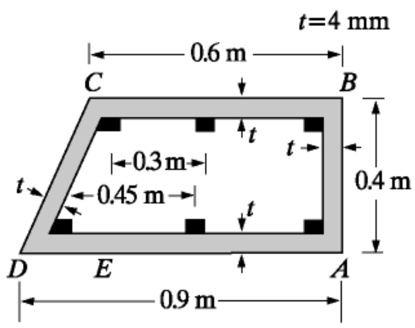


Figure 12P21

Chapter 13

Integral Theorems of Structural Mechanics

13.1 A Statically Admissible Stress Field and an Admissible Displacement Field of a Body

Consider a body initially in a stress–free, strain–free state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, the body is subjected to external loads as a result of which it deforms and reaches a second state of mechanical equilibrium at some known temperature distribution $T(x_1, x_2, x_3)$. We define the following quantities for this body:

1. A *statically admissible stress field* $\tilde{\tau}_{ij}(x_1, x_2, x_3)$ $(i, j = 1, 2, 3)$ in the body in the deformed state of mechanical equilibrium is one which satisfies the requirements for equilibrium of its particles. This implies that the statically admissible components of stress have the following attributes:

(a) They have first derivatives at every point inside the volume of the body.

(b) They are symmetric $(\tilde{\tau}_{ij} = \tilde{\tau}_{ji})$. This ensures that the sum of the moments of all the forces acting on each particle of the body vanishes.

(c) They satisfy the equations of equilibrium (2.69) at every point inside the volume of the body. This ensures that the sum of all the forces acting on each particle inside the volume of the body vanishes.

(d) When substituted in the traction–stress relations (2.73), they yield the specified components of traction at the points of the surface of the body where components of traction are specified. This ensures that the sum of all the forces acting on each particle of the surface of the body vanishes.

Notice that an infinite number of statically admissible distributions of components of stress $\tilde{\tau}_{ij}(x_1, x_2, x_3)$ $(i, j = 1, 2, 3)$ exist in a statically indeterminate body. Generally, for a given material (given stress–strain relations) the components of strain obtained from a statically admissible stress field may not yield the specified components of displacement at the points of the surface of the body where components of displacement are specified.

2. We denote by $\tilde{\tilde{u}}_1(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ the components of a vector field which have derivatives of any order required at every point inside the volume of the body. In the literature they are usually called *admissible components of the "virtual" displacement field*. We shall adopt this label. However it should be emphasized that they are not necessarily components of displacement of the body.

3. We define as the components of a *geometrically admissible "virtual" displacement field* any set of functions $\tilde{\tilde{u}}_1(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ which have the following properties:

(a) They are admissible.
(b) They yield the specified components of displacement at the points of the surface of the body where components of displacement are specified.

4. We define *an admissible or a geometrically admissible "virtual" strain field* $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i,j = 1, 2, 3)$ as one which is related to an admissible or geometrically admissible "virtual" displacement field, respectively, by relations (2.16).

13.2 Derivation of the Principle of Virtual Work for Deformable Bodies

Consider a deformable body initially in a reference stress–free, strain–free state of mechanical and thermal equilibrium at a uniform temperature T_o. In this state the body is not subjected to external loads and heat does not flow in or out of it. Subsequently, the body is subjected to a specified distribution of specific body forces $B_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ throughout its volume, to a specified temperature distribution $T(x_1, x_2, x_3)$, to specified components of traction $\overset{n}{T}{}_i^s(x_1^s, x_2^s, x_3^s)$ $(i = 1, 2, 3)$ at the points of the portion S_t of its surface and to specified components of displacement $\hat{u}_i^s(x_1^s, x_2^s, x_3^s)$ $(i = 1, 2, 3)$ at the points of the remaining portion of its surface. As a result of these loads the body deforms and reaches a second state of mechanical but not necessarily thermal equilibrium.

Consider an infinitesimal portion (particle) of dimensions dx_1, dx_2 and dx_3 and volume dV of the body under consideration in the second state of mechanical equilibrium and denote by $\tilde{\tau}_{ij}$ a distribution of stress in the body which is statically admissible to the given external forces (specific body forces and surface tractions). The forces acting on the faces of this particle which are normal to the x_1 axis and the components of the specific body force **B** acting on it, are shown in Fig. 13.1. Moreover, consider an admissible vector field $\tilde{\tilde{\mathbf{u}}}(x_1, x_2, x_3)$ and corresponding "virtual" strain field $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i, j = 1, 2, 3)$. Referring to Fig. 13.1, we denote by $\tilde{\tilde{u}}_1$, $\tilde{\tilde{u}}_2$ and $\tilde{\tilde{u}}_3$ the average value, of the components of the vector field $\tilde{\tilde{\mathbf{u}}}(x_1, x_2, x_3)$ on the face *OAFE* of the particle under consideration. The average value of the components of the vector field $\tilde{\tilde{\mathbf{u}}}(x_1, x_2, x_3)$ on the face *BGDC* of this particle may be expressed as

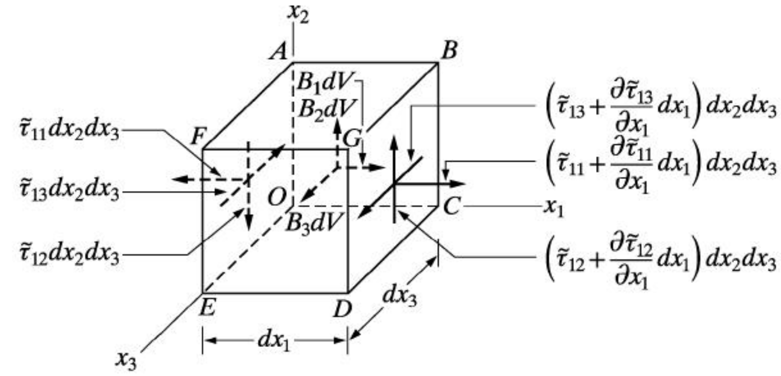


Figure 13.1 Forces acting on the faces of a particle of a body which are normal to the x_1 axis.

$$\tilde{\tilde{\hat{u}}}_i\Big|_{x_1+dx_1} = \tilde{\tilde{\hat{u}}}_i\Big|_{x_1} + \frac{\partial\tilde{\tilde{\hat{u}}}_i}{\partial x_1}\bigg|_{x_1} dx_1 \qquad i = 1,2,3 \tag{13.1}$$

The sum of the products of each one of the statically admissible components of stress acting on the two faces of the particle under consideration, which are normal to the x_1 axis, with the corresponding component of the vector field $\tilde{\tilde{\hat{\mathbf{u}}}}(x_1,x_2,x_3)$ is equal to

$$\begin{aligned} dW^{(1)} = {} & -\tilde{\tau}_{11}dx_2dx_3\tilde{\tilde{\hat{u}}}_1 - \tilde{\tau}_{12}dx_2dx_3\tilde{\tilde{\hat{u}}}_2 - \tilde{\tau}_{13}dx_2dx_3\tilde{\tilde{\hat{u}}}_3 + \left(\tilde{\tau}_{11} + \frac{\partial\tilde{\tau}_{11}}{\partial x_1}dx_1\right)dx_2dx_3\left(\tilde{\tilde{\hat{u}}}_1 + \frac{\partial\tilde{\tilde{\hat{u}}}_1}{\partial x_1}dx_1\right) \\ & + \left(\tilde{\tau}_{12} + \frac{\partial\tilde{\tau}_{12}}{\partial x_1}dx_1\right)dx_2dx_3\left(\tilde{\tilde{\hat{u}}}_2 + \frac{\partial\tilde{\tilde{\hat{u}}}_2}{\partial x_1}dx_1\right) + \left(\tilde{\tau}_{13} + \frac{\partial\tilde{\tau}_{13}}{\partial x_1}dx_1\right)dx_2dx_3\left(\tilde{\tilde{\hat{u}}}_3 + \frac{\partial\tilde{\tilde{\hat{u}}}_3}{\partial x_1}dx_1\right) \end{aligned} \tag{13.2a}$$

Simplifying and disregarding infinitesimals of higher order, the above relation reduces to

$$dW^{(1)} = \left(\tilde{\tau}_{11}\frac{\partial\tilde{\tilde{\hat{u}}}_1}{\partial x_1} + \tilde{\tau}_{12}\frac{\partial\tilde{\tilde{\hat{u}}}_2}{\partial x_1} + \tilde{\tau}_{13}\frac{\partial\tilde{\tilde{\hat{u}}}_3}{\partial x_1} + \frac{\partial\tilde{\tau}_{11}}{\partial x_1}\tilde{\tilde{\hat{u}}}_1 + \frac{\partial\tilde{\tau}_{12}}{\partial x_1}\tilde{\tilde{\hat{u}}}_2 + \frac{\partial\tilde{\tau}_{13}}{\partial x_1}\tilde{\tilde{\hat{u}}}_3\right)dV \tag{13.2b}$$

This result can be extended to establish the sum of the products of each one of the statically admissible components of stress acting on all the faces of the particle under consideration with the corresponding component of the vector field $\tilde{\tilde{\hat{\mathbf{u}}}}(x_1,x_2,x_3)$. Moreover, the sum of the products of each one of the components of the body force acting on the particle under consideration with the corresponding component of the vector field $\tilde{\tilde{\hat{\mathbf{u}}}}(x_1,x_2,x_3)$ can be computed. That is,

$$dW = \left(\tilde{\tau}_{11}\frac{\partial\tilde{\tilde{\hat{u}}}_1}{\partial x_1} + \tilde{\tau}_{12}\frac{\partial\tilde{\tilde{\hat{u}}}_2}{\partial x_1} + \tilde{\tau}_{13}\frac{\partial\tilde{\tilde{\hat{u}}}_3}{\partial x_1} + \tilde{\tau}_{21}\frac{\partial\tilde{\tilde{\hat{u}}}_1}{\partial x_2} + \tilde{\tau}_{22}\frac{\partial\tilde{\tilde{\hat{u}}}_2}{\partial x_2} + \tilde{\tau}_{23}\frac{\partial\tilde{\tilde{\hat{u}}}_3}{\partial x_2} + \tilde{\tau}_{31}\frac{\partial\tilde{\tilde{\hat{u}}}_1}{\partial x_3} + \tilde{\tau}_{32}\frac{\partial\tilde{\tilde{\hat{u}}}_2}{\partial x_3} + \tilde{\tau}_{33}\frac{\partial\tilde{\tilde{\hat{u}}}_3}{\partial x_3}\right)dV$$

$$+\left(\frac{\partial\tilde{\tau}_{11}}{\partial x_1}\tilde{\tilde{u}}_1+\frac{\partial\tilde{\tau}_{12}}{\partial x_1}\tilde{\tilde{u}}_2+\frac{\partial\tilde{\tau}_{13}}{\partial x_1}\tilde{\tilde{u}}_3+\frac{\partial\tilde{\tau}_{21}}{\partial x_2}\tilde{\tilde{u}}_1+\frac{\partial\tilde{\tau}_{22}}{\partial x_2}\tilde{\tilde{u}}_2+\frac{\partial\tilde{\tau}_{23}}{\partial x_2}\tilde{\tilde{u}}_3+\frac{\partial\tilde{\tau}_{31}}{\partial x_3}\tilde{\tilde{u}}_1+\frac{\partial\tilde{\tau}_{32}}{\partial x_3}\tilde{\tilde{u}}_2+\frac{\partial\tilde{\tau}_{33}}{\partial x_3}\tilde{\tilde{u}}_3\right.$$
$$\left.+\tilde{B}_1\tilde{\tilde{u}}_1+\tilde{B}_2\tilde{\tilde{u}}_2+\tilde{B}_3\tilde{\tilde{u}}_3\right)dV \tag{13.3}$$

Inasmuch as the stress field $\tilde{\tau}_{ij}(i,j = 1,2,3)$ is statically admissible, it satisfies the equations of equilibrium (2.69). Referring to these equations we see that the sum of the terms in the second parenthesis on the right side of relation (13.3) is equal to zero. Moreover, using the strain–displacement relations (2.16) relation (13.3) is simplified to the following:

$$dW = (\tilde{\tau}_{11}\tilde{\tilde{e}}_{11} + 2\tilde{\tau}_{12}\tilde{\tilde{e}}_{12} + 2\tilde{\tau}_{13}\tilde{\tilde{e}}_{13} + 2\tilde{\tau}_{23}\tilde{\tilde{e}}_{23} + \tilde{\tau}_{22}\tilde{\tilde{e}}_{22} + \tilde{\tau}_{33}\tilde{\tilde{e}}_{33})dV \tag{13.4a}$$

where $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i,j = 1,2,3)$ are a set of functions of the space coordinates obtained from the admissible vector field $\tilde{\tilde{\mathbf{u}}}(x_1,x_2,x_3)$ on the basis of relations (2.16). Relation (13.4a) may be rewritten as

$$dW = \sum_{p=1}^{3}\sum_{q=1}^{3}\tilde{\tau}_{pq}\tilde{\tilde{e}}_{pq}dV \tag{13.4b}$$

Integrating relation (13.4b), we obtain

$$W = \iiint_V dW = \iiint_V\left(\sum_{p=1}^{3}\sum_{q=1}^{3}\tilde{\tau}_{pq}\tilde{\tilde{e}}_{pq}\right)dV \tag{13.4c}$$

Notice that, two adjacent particles of a body have a common boundary and the components of stress acting on this boundary of the one particle are equal and opposite to those acting on the common boundary of the other particle. Thus, the sum of the products of each one of these equal and opposite components of stress with the corresponding component of the "virtual" strain field $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i,j = 1,2,3)$, vanishes. Consequently, W is equal to

$$W = W_{\text{ext forces}} + W_{\text{reactions}} \tag{13.5}$$

where

$W_{\text{ext forces}}$ = the sum of the products of each one of the known external forces (body forces and surface tractions) acting on the body with the corresponding component of the vector field $\tilde{\tilde{\mathbf{u}}}(x_1,x_2,x_3)$.

$W_{\text{reactions}}$ = the sum of the products of each one of the unknown statically admissible reactions of the supports of the body with the corresponding component of the vector field $\tilde{\tilde{\mathbf{u}}}(x_1,x_2,x_3)$.

The reactions of a body are the unknown tractions at the points of its surface where components of displacement have been specified. If the supports of the body do not move and moreover the vector field $\tilde{\tilde{\mathbf{u}}}(x_1, x_2, x_3)$ is geometrically admissible, then $W_{\text{reactions}}$ vanishes.

Substituting relation(13.5) into (13.4c), we get

$$W_{\text{ext forces}} + W_{\text{reactions}} = \iiint_V \left(\sum_{p=1}^{3} \sum_{q=1}^{3} \tilde{\tau}_{pq} \tilde{\tilde{e}}_{pq} \right) dV \tag{13.6a}$$

Relation (13.6a) may be rewritten as

$$W_{\text{ext forces}} + W_{\text{reactions}} = \iiint_V [\tilde{\tilde{e}}]^T [\tilde{\tau}] dV \tag{13.6b}$$

where

$$[\tilde{\tilde{e}}] = \begin{Bmatrix} \tilde{\tilde{e}}_1 \\ \tilde{\tilde{e}}_2 \\ \tilde{\tilde{e}}_3 \\ \tilde{\tilde{e}}_4 \\ \tilde{\tilde{e}}_5 \\ \tilde{\tilde{e}}_6 \end{Bmatrix} = \begin{Bmatrix} \tilde{\tilde{e}}_{11} \\ \tilde{\tilde{e}}_{22} \\ \tilde{\tilde{e}}_{33} \\ \tilde{\tilde{\gamma}}_{12} \\ \tilde{\tilde{\gamma}}_{13} \\ \tilde{\tilde{\gamma}}_{23} \end{Bmatrix} \qquad [\tilde{\tau}] = \begin{Bmatrix} \tilde{\tau}_1 \\ \tilde{\tau}_2 \\ \tilde{\tau}_3 \\ \tilde{\tau}_4 \\ \tilde{\tau}_5 \\ \tilde{\tau}_6 \end{Bmatrix} = \begin{Bmatrix} \tilde{\tau}_{11} \\ \tilde{\tau}_{22} \\ \tilde{\tau}_{33} \\ \tilde{\tau}_{12} \\ \tilde{\tau}_{13} \\ \tilde{\tau}_{23} \end{Bmatrix} \tag{13.7}$$

and

$$\tilde{\tilde{\gamma}}_{ij} = 2\tilde{\tilde{e}}_{ij} \qquad i,j = 1,2,3 \tag{13.8}$$

The sum of the products of each one of the components of the given external forces (specific body forces $\mathbf{B}$ and surface tractions $\overset{n}{\mathbf{T}}{}^{s}$) acting on the body with the corresponding component of the vector field $\tilde{\tilde{\mathbf{u}}}(x_1, x_2, x_3)$ is equal to

$$W_{\text{ext forces}} = \iint_{S_t} \overset{\mathbf{n}}{\mathbf{T}}{}^{\mathbf{s}} \cdot \tilde{\tilde{\mathbf{u}}}\, dS + \iiint_V \mathbf{B} \cdot \tilde{\tilde{\mathbf{u}}}\, dS = \iint_{S_t} \sum_{i=1}^{3} \overset{n}{T}{}_i^{s} \tilde{\tilde{u}}_i dS + \iiint_V \sum_{i=1}^{3} B_i \tilde{\tilde{u}}_i dV \tag{13.9}$$

The sum of the products of each one of the components of the statically admissible reactions with the corresponding component of the vector field $\tilde{\tilde{\mathbf{u}}}(x_1, x_2, x_3)$ is equal to

$$W_{\text{reactions}} = \iint_{S-S_t} \sum_{i=1}^{3} \overset{n}{\tilde{T}}_i\, \tilde{\tilde{u}}_i^{s} dS \tag{13.10}$$

where $\overset{n}{\tilde{T}}_i$ $(i = 1,2,3)$ are the unknown statically admissible components of traction (reactions) exerted on the body by its supports (points of its surface where components of displacement are specified).

Relation (13.6a) or (13.6b) is known as *the principle of virtual work for a deformable body*. In obtaining this relation, we have not employed the stress–strain relations for the material from which the body is made. Consequently, this relation is valid for bodies made from any material (elastic or inelastic).

We have shown that *a stress field* $\tilde{\tau}_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3)$ *which is statically admissible to the given external forces acting on a body satisfies the principle of virtual work (13.6) for any admissible vector field (admissible "virtual" displacement field)* $\tilde{\hat{u}}_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ *and corresponding "virtual" strain field* $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i, j = 1, 2, 3)$. Moreover, it can be shown that *a set of functions* $\tilde{\tau}_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3)$*, which together with the given external forces acting on a body satisfy the principle of virtual work (13.6) for every admissible vector field* $\tilde{\tilde{\hat{u}}}_i(x_1, x_2, x_3)(i = 1,2,3)$ *and corresponding "virtual" strain field* $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i = 1, 2, 3)$*, represents a set of components of stress which are statically admissible to the given ex ternal forces.* This implies that *a geometrically admissible set of functions* $\hat{u}_i(x_1, x_2, x_3)(i = 1, 2, 3)$ *is the actual displacement field of a body if when substituted in relations (2.16) gives a strain field* $e_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3)$*, which on the basis of the stress–strain relations for the material from which the body is made gives a stress field* $\tau_{ij}(x_1, x_2, x_3)$*, which together with the given set of external forces acting on the body satisfies the principle of virtual work (13.6) for every admissible vector field* $\tilde{\tilde{\hat{u}}}_i(x_1, x_2, x_3)(i = 1, 2, 3)$ *and corresponding admissible "virtual" strain field* $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i = 1, 2, 3)$.

On the basis of the foregoing discussion the boundary value problem for computing the components of displacement, strain and stress of the particles of a deformable body can be formulated as follows:

Find the geometrically admissible vector field $\hat{u}_i(x_1, x_2, x_3)$ *which when substituted into the strain–displacement relations (2.16) gives a strain field* $e_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3)$ *which when substituted into the stress–strain relations for the material from which the body is made gives a stress field* $\tau_{ij}(x_1, x_2, x_3)(i, j = 1, 2, 3)$ *which together with the given set of external forces acting on the body satisfies the principle of virtual work (13.6) for every admissible vector field* $\tilde{\tilde{\hat{u}}}_i(x_1, x_2, x_3)(i = 1, 2, 3)$ *and corresponding strain field* $\tilde{\tilde{e}}_{ij}(x_1, x_2, x_3)$ $(i = 1, 2, 3)$. This formulation of the boundary value under consideration is called *weak* and is equivalent to its strong formulation described in Section 5.2. It is used in conjunction with the finite element method to obtain approximations to the components of displacement of the particles of a body.

Notice, that if we choose as the "virtual" displacement field, the actual displacement field of a deformable body subjected to given loads, and as the statically admissible stress field the actual stress field of the body, $W_{\text{ext forces}}$ and $W_{\text{reactions}}$ represent the work of the given loads and of the unknown reactions due to the deformation of the body. If the supports of the body do not move, $W_{\text{reactions}}$ vanishes. Moreover, the right side of relation (13.6) represents the sum of the work of the components of stress acting on all the particles of the body. That is, in this case, the principle of virtual work reduces to "the sum of the work performed by the external forces acting on a body and by its reactions

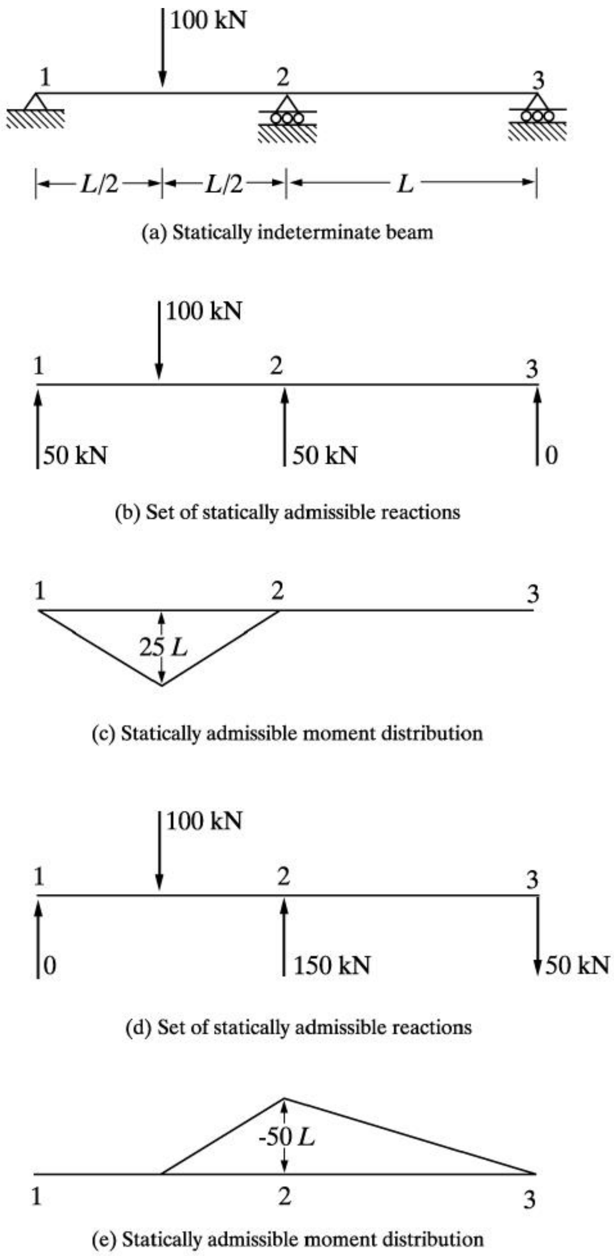


Figure 13.2 Statically admissible sets of reactions and corresponding moment distributions.

due to its deformation is equal to the sum of the work of the components of stress acting on all the particles of the body due to their deformation."

13.3 Statically Admissible Reactions and Internal Actions of Framed Structures

In statically determinate framed structures only the actual internal actions are in equilibrium with the given external actions, while in statically indeterminate framed structures, we can find an infinite number of distributions of internal actions which satisfy the equations of equilibrium for every portion of the structure. These distributions of internal actions are referred to as *statically admissible*. In addition to being statically admissible, the actual distribution of internal actions must yield components of displacement which are continuous functions of the space coordinates and satisfy the specified conditions at the supports of the structure (displacement boundary conditions).

When a framed structure is externally statically indeterminate, an infinite number of sets of reactions can be found which are in equilibrium with the given external actions acting on the structure. These sets of reactions are referred to as *statically admissible*. The one set of statically admissible reactions which yields components of displacement compatible with the constraints of the structure is the actual set of reactions. For example, two statically admissible sets of reactions for the beam shown in Fig. 13.2a are shown in Figs. 13.2b and d. The corresponding statically admissible distributions of moment are shown in Figs. 13.2c and e.

13.4 The Principle of Virtual Work for Framed Structures

13.4.1 Prismatic Members Subjected to Axial Centroidal Forces and to Uniform Change of Their Temperature

Consider a member of a framed structure of length L and cross-sectional area A, made from an isotropic, linearly elastic material. The member is originally in a stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. The member reaches a second state of mechanical equilibrium at the temperature $T(x_1)$ such that $\delta T_2 = \delta T_3 = 0$ (see Section 8.8) due to the application on it of the following forces:

1. Distributed axial centroidal forces $p_1(x_1)$ given per unit length of the member

2. Concentrated axial centroidal forces $P_1^{(n)}$ ($n = 1, 2, ..., n_1$) acting along the length of the member

This loading induces on the cross sections of the member only normal components of stress $\tau_{11}^{(A)}$ which does not vary in the directions normal to the axis of the member. Thus, a statically admissible state of stress in the member under consideration has the following form:

$$[\tilde{\tau}^{(A)}] = \begin{bmatrix} \tilde{\tau}_{11}^{(A)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13.11}$$

If we denote by $\tilde{N}(x_1)$ the resultant force of the statically admissible distribution of stress acting on the cross sections of the member, we have

$$\tilde{\tau}_{11}^{(A)} = \frac{\tilde{N}}{A} \tag{13.12}$$

Consider an admissible "virtual" displacement field $\tilde{\tilde{u}}_1(x_1)$ and the corresponding "virtual" strain field $\tilde{\tilde{e}}_{11}(x_1)$ defined as

$$\tilde{\tilde{e}}_{11} = \frac{d\tilde{\tilde{u}}_1}{dx_1} \tag{13.13}$$

Moreover, for members made from an isotropic linearly elastic material, we define the following quantity†:

$$\tilde{\tilde{N}} = EA\left(\frac{d\tilde{\tilde{u}}_1}{dx_1} - \tilde{\tilde{H}}_1\right) \tag{13.14a}$$

where as we shall see later $\tilde{\tilde{H}}_1$ is chosen either zero or equal to

$$\tilde{\tilde{H}}_1 = \alpha\Delta T_c = \alpha(T_c - T_o) \tag{13.14b}$$

T_c is the uniform temperature of the member in the deformed state. We introduced $\tilde{\tilde{H}}_1$ in order to be able to include the effect of the temperate when we apply the principle of virtual work to compute a component of displacement of a point of a member due to its deformation using the unit load method (see Section 13.5). In this method we choose the function $\tilde{\tilde{u}}_1$ to be the actual translation of the cross sections of the member of a structure and we want the function $\tilde{\tilde{N}}$ to be the actual axial internal force acting on the cross sections of the member. Substituting relations (13.11), (13.12) and (13.13) into (13.6) and using relation (13.14a), we get

$$\begin{aligned} W_{ext\,forces} + W_{reactions} &= \iiint_V \tau_{11}^{(A)} e_{11}^{(A)} dV = \int_0^L \left(\frac{\tilde{N}}{A}\frac{d\tilde{\tilde{u}}_1}{dx_1}\iint_A dA\right) dx_1 \\ &= \int_0^L \tilde{N}\frac{d\tilde{\tilde{u}}_1}{dx_1} dx_1 = \int_0^L \frac{\tilde{N}\tilde{\tilde{N}}}{EA} dx_1 + \int_0^L \tilde{N}\tilde{\tilde{H}}_1 dx_1 \end{aligned} \tag{13.15}$$

13.4.2 Prismatic Members Subjected to Torsional Moments

As we have seen in Chapter 6 when a prismatic member is free to warp and it is subjected to equal and opposite torsional moments at its ends, the state of stress of its particles is specified by the shearing components of stress τ_{12} and τ_{13}. These components of stress are obtained from the stress function ψ (x_2, x_3) on the basis of relation (6.34). The stress function is a property of the geometry of the cross section of the member which has been established for only a few cross sections as, for example, solid elliptical and circular (see example of Section 6.5); hollow elliptical or circular (see

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† $\tilde{\tilde{N}}(x_1)$ is a quantity obtained from the admissible "virtual" displacement field $\tilde{\tilde{u}}_1$ on the basis of relation (13.14). However, referring to relation (8.58a) we see that, if $\tilde{\tilde{u}}_1(x_1)$ is the actual displacement field of the member and if $\tilde{\tilde{H}}_1$ is given by relation (13.14b), then $\tilde{\tilde{N}}(x_1)$ is the actual distribution of the internal force acting on its cross sections.

example of Section 6.6); triangular, thin, rectangular (see Section 6.10) and simply connected composite consisting of a number of thin rectangular components. Thus, a statically admissible stress field in the member under consideration has the following form:

$$[\tilde{\tau}^T] = \begin{bmatrix} 0 & \tilde{\tau}_{12}^{(T)} & \tilde{\tau}_{13}^{(T)} \\ \tilde{\tau}_{12}^{(T)} & 0 & 0 \\ \tilde{\tau}_{13}^{(T)} & 0 & 0 \end{bmatrix} \tag{13.16a}$$

where

$$\tilde{\tau}_{12}^{(T)}(x_2, x_3) = \frac{d\tilde{\theta}_1}{dx_1} G \frac{\partial \psi}{\partial x_3} \qquad \tilde{\tau}_{13}^{(T)}(x_2, x_3) = \frac{d\tilde{\theta}_1}{dx_1} G \frac{\partial \psi}{\partial x_2} \tag{13.16b}$$

where $\tilde{\theta}(x_1)$ is defined by the above relation. However, if $\tilde{\tau}_{12}$ and $\tilde{\tau}_{13}$ are the actual components of stress acting on the particles of a member subjected to equal and opposite torsional moments at its ends, $\tilde{\theta}_1$ is its angle of twist [see relations (6.34)].

As we have seen in Chapter 6 when a prismatic member is free to warp and it is subjected to equal and opposite torsional moments at its ends, any material line of a cross section which is straight before deformation in general deforms into a space curve whose projection on the x_2x_3 plane is a straight line obtained by rotating the before-deformation line by an angle θ_1 about an axis normal to the cross section through its center of twist. All lines of a cross section rotate by the same angle θ_1. This angle is called the angle of twist of the cross section and its relation to the torsional moment may be expressed as

$$\alpha = \frac{d\theta_1}{dx_1} = \frac{M_1}{GR_C} \tag{13.17}$$

where α is the constant angle of twist per unit length and M_1 is the constant torsional moment acting on the cross sections of the member. R_C is the torsional constant of the member defined by relation (6.33). It depends on the geometry of the cross sections of the member. For a member of circular cross section R_C is equal to its polar moment of inertia.

We assume that relations (13.16a), (13.16b) and (13.17) are also valid for prismatic members subjected to torsional moments along their length. In this case, the twist per unit length α and the internal moment M_1 are functions of x_1.

Consider a prismatic member of a framed structure of length L made from an isotropic, linearly elastic material and supported in such a way that its cross sections can be assumed free to warp. Originally, the member is in a stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. We choose as the x_1 axis, the axis of the member. Subsequently, the member is subjected to the following loading and reaches a second state of mechanical equilibrium:

1. Distributed axial component of moment $m_1(x_1)$ given per unit of length of the member

2. Concentrated axial component of moment $M_1^{(m)}$ (m = 1, 2, ...,, m_1) acting along the length of the member and on its end faces.

We choose the following "virtual"strain field

$$\tilde{\tilde{e}}_{12} = \frac{1}{2}\frac{d\tilde{\tilde{\theta}}_1}{dx_1}\frac{\partial\psi}{\partial x_3} \qquad \tilde{\tilde{e}}_{13} = -\frac{1}{2}\frac{d\tilde{\tilde{\theta}}_1}{dx_1}\frac{\partial\psi}{\partial x_2} \tag{13.18}$$

where $\tilde{\tilde{\theta}}_1$ is any admissible function of x_1. It is called the admissible "virtual" twist.

Taking into account relation (13.16a) and using relation (13.16b), (13.17) and (13.18) into (13.6b), we get

$$W^{(T)}_{ext\,forces} + W^{(T)}_{reactions} = \iiint_V \left(2\tilde{\tau}^{(T)}_{12}\tilde{\tilde{e}}^{(T)}_{12} + 2\tilde{\tau}^{(T)}_{13}\tilde{\tilde{e}}^{(T)}_{13}\right)dV \tag{13.19a}$$

$$= \iiint_V \frac{d\tilde{\theta}_1}{dx_1}\frac{d\tilde{\tilde{\theta}}_1}{dx_1}\left[\left(\frac{\partial\psi}{\partial x_2}\right)^2 + \left(\frac{\partial\psi}{\partial x_3}\right)^2\right]dV = \int_0^L \left(\frac{\tilde{M}_1\tilde{\tilde{M}}_1}{GR_C^2}D\right)dx_1 = \int_0^L \frac{\tilde{M}_1}{R_C}D\frac{d\tilde{\tilde{\theta}}_1}{dx_1}dx_1$$

where

$$D = \iint_A \left[\left(\frac{\partial\psi}{\partial x_2}\right)^2 + \left(\frac{\partial\psi}{\partial x_3}\right)^2\right]dA \tag{13.19b}$$

$$\tilde{\tilde{M}}_1 = GR_C\tilde{\tilde{a}} = GR_C\frac{d\tilde{\tilde{\theta}}_1}{dx_1} \tag{13.19c}$$

Referring to relations (13.17), we see that if $\tilde{\tilde{\theta}}_1(x_1)$ is the actual twist of the cross sections of the member, then $\tilde{\tilde{a}}$ is their actual twist per unit length and $\tilde{\tilde{M}}_1$ is the actual torsional moment acting on the cross sections of the member.

It can be shown that relation (13.19a) is valid for prismatic members whose cross sections have any given geometry provided that the appropriate expressions are used for their torsional constant R_C and the function $\psi(x_2, x_3)$.

For a member of solid elliptical cross section referring to Fig. 13.3a and to relations (e) and (h) of the example of Section 6.5, we have

$$\psi = -a^2\frac{b^2}{a^2+b^2}\left(\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1\right) \qquad R_C = \frac{\pi a^3 b^3}{a^2+b^2}$$

Thus,

$$\left(\frac{\partial\psi}{\partial x_2}\right)^2 + \left(\frac{\partial\psi}{\partial x_3}\right)^2 = \frac{4a^4b^4}{a^2+b^2}\left(\frac{x_2^2}{a^4} + \frac{x_3^2}{b^4}\right)$$

Substituting the above relation into (13.19b) and integrating the resulting relation, we obtain

$$D = \frac{4a^4b^4}{(a^2+b^2)^2}\left(\frac{1}{a^4}\iint_A x_2^2 dA + \frac{1}{b^4}\iint_A x_3^2 dA\right)$$
$$= \frac{4a^4b^4}{(a^2+b^2)^2}\left[\frac{I_3}{a^4} + \frac{I_2}{b^4}\right] = \frac{\pi a^3 b^3}{a^2+b^2} = R_C \tag{13.20}$$

For a member of circular cross section $D = R_C$ is equal to the polar moment of inertia of its cross section.

Similarly, it can be shown, that for a member of hollow elliptical cross section (see Fig. 13.3b), we have

$$D = R_C = \frac{\pi a^3 b^3}{a^2+b^2}(1-m^4) \tag{13.21a}$$

where

$$m = \frac{a_1}{a} = \frac{b_1}{b} \tag{13.21b}$$

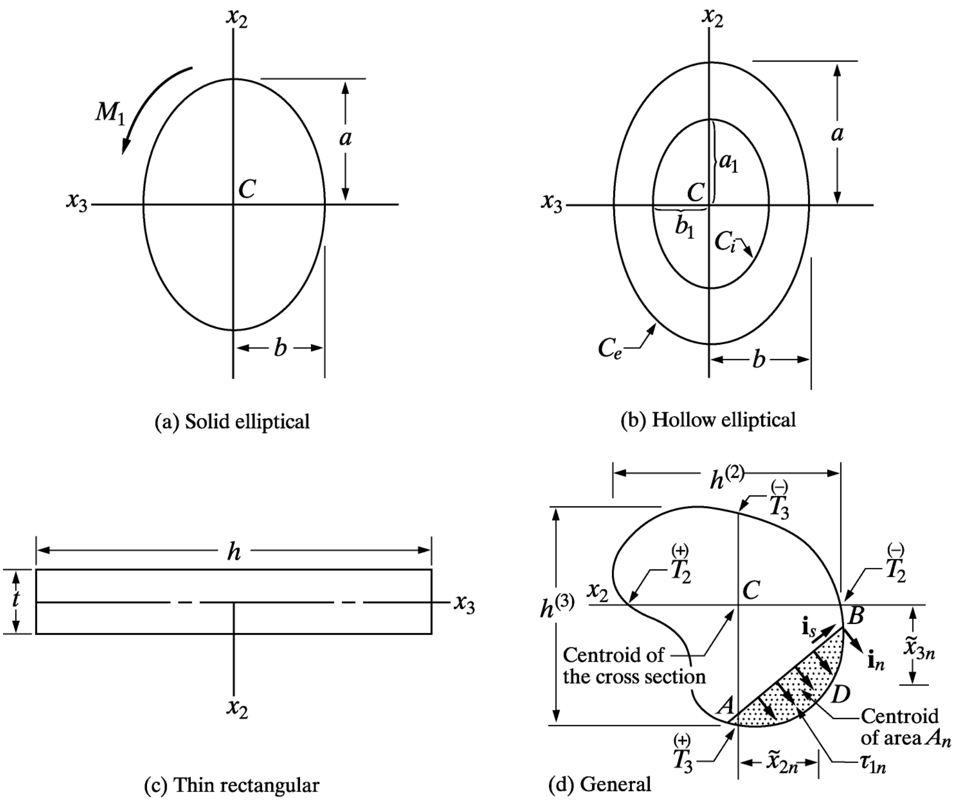


Figure 13.3 Cross sections of members.

For members of thin rectangular cross section referring to relations (6.30), (6.88) and (13.17) and to Fig. 13.3c, we have

$$\psi = \frac{\phi}{aG} = \frac{M_1}{aGR_C}\left(x_2 - \frac{t^2}{4}\right) = \left(x_2^2 - \frac{t^2}{4}\right)$$

Substituting the above relation into (13.19b) and integrating, we get

$$D = \frac{ht^3}{3} = R_C \tag{13.22}$$

For prismatic members of a simply connected composite cross section consisting of n thin rectangular components, it can be shown that

$$D = \sum_{i=1}^{n} \frac{h_i t_i^3}{3} = R_C \tag{13.23}$$

In order to simplify our presentation throughout this chapter, we limit our attention to structures whose members, which are subjected to torsional moments, have cross sections with $D = R_C$.

13.4.3 Members Subjected to Bending without Twisting

Consider a member of a framed structure of length L made from an isotropic, linearly elastic material. The member is originally in a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, it reaches a second state of mechanical equilibrium at a temperature $T(x_1, x_2, x_3)$ due to the application on it of the following actions, which bend it about the x_2 axis without twisting it:

1. Distributed external transverse forces $p_3(x_1)$ (including the weight of the member) given per unit of length of the member. The lines of action of these forces lie in a plane parallel to the x_3x_1 plane which contains the shear center of the cross sections of the member.
2. Distributed external moments $m_2(x_1)$ given per unit of length of the member.
3. Concentrated forces $P_3^{(n)}$ ($n = 1, 2, ..., n_3$) acting along the length of the member. The line of action of these forces lies in a plane parallel to the x_1x_3 plane, which contains the shear center of the cross sections of the member.
4. Concentrated moments $M_2^{(m)}$ ($m = 1, 2, ..., m_2$) acting along the length of the member.

As discussed in Sections 9.1 and 9.5 this loading induces on the cross sections of the member normal τ_{11} and shearing τ_{1n} and possibly τ_{1s} components of stress; where $\mathbf{i}_n$ and $\mathbf{i}_s$ are mutually perpendicular unit vectors in the plane of the cross section (see Fig. 13.3d). *In this section we limit our attention to principal centroidal axes x_2 and x_3.* Thus, a statically admissible state of stress of a particle of a member subjected to the loading described above has the following form:

$$[\hat{\tau}^{B2}] = \begin{bmatrix} \tau_{11}^{(B2)} & \tau_{1n}^{(B2)} & \tau_{1s}^{(B2)} \\ \tau_{1n}^{(B2)} & 0 & 0 \\ \tau_{1s}^{(B2)} & 0 & 0 \end{bmatrix} \tag{13.24}$$

Referring to relations (9.12b) and (9.67), we have

$$\tilde{\tau}_{11}^{(B2)} = \frac{\tilde{M}_2 x_3}{I_2} \tag{13.25}$$

$$\tilde{\tau}_{1n}^{(B2)} = \frac{\tilde{Q}_3 \bar{x}_{3n} A_n}{I_2 b_s} \tag{13.26}$$

A_n = area of the portion *ABDA* of the cross section shown in Fig. 13.3d.
$\bar{x}_{3n}$ = x_3 coordinate of the centroid of the portion of cross section of Area A_n (see Fig. 13.3d).

Moreover, we choose a "virtual" strain field of the following form:

$$\tilde{\tilde{e}}_{11}^{(B2)} = \frac{\tilde{\tilde{M}}_2 x_3}{EI_2} + \tilde{\tilde{H}}_2 x_3 = x_3 \frac{d\tilde{\tilde{\theta}}_2}{dx_1} \tag{13.27}$$

$$\tilde{\tilde{e}}_{1n}^{(B2)} = \frac{\tilde{\tilde{\tau}}_{1n}}{2G} = \frac{\tilde{\tilde{Q}}_3 \bar{x}_{3n} A_n}{2GI_2 b_s} \tag{13.28a}$$

where $\tilde{\tilde{H}}_2$ is taken as either zero or referring to Fig. 13.3d as

$$\tilde{\tilde{H}}_2 = \frac{\alpha(T_3^{(+)} - T_3^{(-)})}{h^{(3)}} = \frac{\delta T_3}{h^{(3)}} \tag{13.28b}$$

where α is the coefficient of thermal expansion. We include $\tilde{\tilde{H}}_2$ for the same reason we have included $\tilde{\tilde{H}}_1$(see Section 13.4.1). Referring to relation (9.9a), we see that if $\tilde{\tilde{e}}_{11}^{(B2)}(x_1, x_3)$ is the actual component of strain of the particles of the body and if $\tilde{\tilde{H}}_2$ is given by relation (13.28b), then $\tilde{\tilde{M}}_2(x_1)$ is the actual moment acting on the cross sections of the member.

Taking into account relation (13.24), relation (13.6a) reduces to

$$W_{ext\,forces} + W_{reactions} = \iiint_V \left[\tilde{\tau}_{11}^{(B2)} \tilde{\tilde{e}}_{11}^{(B2)} + 2\,\tilde{\tau}_{1n}^{(B2)} \tilde{\tilde{e}}_{1n}^{(B2)} + 2\,\tilde{\tau}_{1s}^{(B2)} \tilde{\tilde{e}}_{1s}^{(B2)} \right] dV \tag{13.29}$$

The second and third terms of the integrand of the above relation represent the effect of shear deformation of the particles of the member. For members whose lengths are large as compared to their other dimensions these terms are small compared to the first term and can be disregarded. For such members relation (13.29) reduces to

$$W_{ext\,forces} + W_{reactions} = \iiint_V \tilde{\tau}_{11}^{(B2)} \tilde{\tilde{e}}_{11}^{(B2)} dV \tag{13.30}$$

Substituting relations (13.25) and (13.27) into (13.30), we get

$$W_{extforces} + W_{reactions} = \iiint_V \left[\frac{\tilde{M}_2 x_3^2}{I_2} \left(\frac{\tilde{\tilde{M}}_2}{EI_2} + \tilde{\tilde{H}}_2 \right) \right] dV = \int_0^L \left[\frac{\tilde{M}_2}{I_2} \left(\frac{\tilde{\tilde{M}}_2}{EI_2} + \tilde{\tilde{H}}_2 \right) \iint_{A_n} x_3^2 \, dA \right] dx_1$$

$$= \int_0^L \tilde{M}_2 \left(\frac{\tilde{\tilde{M}}_2}{EI_2} + \tilde{\tilde{H}}_2 \right) dx_1 = \int_0^L \tilde{M}_2 \frac{d\tilde{\tilde{\theta}}_2}{dx_1} dx_1 \qquad (13.31)$$

Similarly for a member subjected to bending about its x_3 axis without twisting, it can be shown that

$$W^{(B3)}_{extforces} + W^{(B3)}_{reactions} = \iiint_V \left[\tilde{\tau}^{(B3)}_{11} \tilde{\tilde{e}}^{(B3)}_{11} \right] dV = \int_0^L \tilde{M}_3 \left(\frac{\tilde{\tilde{M}}_3}{EI_3} - \tilde{\tilde{H}}_3 \right) dx_1 = \int_0^L \tilde{M}_3 \frac{d\tilde{\tilde{\theta}}_3}{dx_1} dx_1 \qquad (13.32a)$$

where $\tilde{\tilde{H}}_3$ equal either to zero or referring to Fig. 13.3d it is taken as

$$\tilde{\tilde{H}}_3 = \frac{\alpha (T_2^{(+)} - T_2^{(-)})}{h^{(2)}} = \frac{\delta T_2}{h^{(2)}} \qquad (13.32b)$$

Consider a framed structure consisting of NE members originally in a stress-free, strain-free state of mechanical and thermal equilibrium at a uniform temperature T_o. Subsequently, the structure reaches a second state of mechanical equilibrium at the temperature $T(x_1, x_2, x_3)$ due to the application on it of the following loading:

1. Distributed external forces (including the weight) given per unit length of the member on which they act. The line of action of the forces acting on each member of this structure is normal to its axis and lies in a plane which contains the shear center of its cross sections.
2. Distributed external moments given per unit length of the member on which they act.
3. Concentrated external forces. Some of these forces act on the members of the structure, while the remaining act on its joints. The line of action of the forces acting on each member of the structure is normal to its axis and lies in a plane, which contains the shear center of its cross sections.
4. Concentrated external moments. Some of these moments are applied to the members of the structure, while the remaining are applied to its joints.

For this loading we choose a set of statically admissible reactions of the supports of the structure which we denote by $\tilde{R}^{(s)}$ $(s = 1, 2, ..., S)$. Moreover, we denote the corresponding statically admissible internal axial force, shearing forces, torsional moment and bending moments acting on the cross sections of a member of the structure by $\tilde{N}(x_1), \tilde{Q}_2(x_1), \tilde{Q}_3(x_1), \tilde{M}_1(x_1), \tilde{M}_2(x_1)$ and $\tilde{M}_3(x_1)$, respectively.

We denote the additional "virtual" translation and rotation vectors of the cross sections of the member of the structure under consideration by $\{\tilde{\tilde{u}}\}^T = [\tilde{\tilde{u}}_1(x_1) \; \tilde{\tilde{u}}_2(x_1) \; \tilde{\tilde{u}}_3(x_1)]$ and $\{\tilde{\tilde{\theta}}(x_1)\}^T = [\tilde{\tilde{\theta}}_1(x_1) \; \tilde{\tilde{\theta}}_2(x_1) \; \tilde{\tilde{\theta}}_3(x_1)]$, respectively. Moreover, we denote by $\{\tilde{\tilde{\Delta}}^s\}$ the matrix of the components of the "virtual" displacement field evaluated at the supports of the structure.

This matrix is conjugate to the matrix $\{\tilde{R}\}$. That is, $\tilde{\tilde{\Delta}}_i^S$ represents the "virtual" component of displacement of the same support and in the same direction as the statically admissible reaction $\tilde{R}_i$. Thus, the sum of the products of each one of the components of the statically admissible reactions of the structure with the corresponding component of the "virtual" displacement field evaluated at the supports of the structure is equal to

$$W_{reactions} = \{\tilde{\tilde{\Delta}}^S\}^T \{\tilde{R}\} \tag{13.33}$$

Substituting relations (13.33), (13.17), (13.23), (13.31), and (13.32) into the principle of virtual work (13.9), we obtain

$$\begin{aligned}
&\text{The product of the given external actions with the admissible virtual displacement field} + \{\tilde{\tilde{\Delta}}^S\}^T\{\tilde{R}\} \\
&= \sum_{e=1}^{NE}\left[W^{(A)} + W^{(T)} + W^{(B2)} + W^{(B3)}\right]^{(e)} = \sum_{e=1}^{NE}\left[\int_0^{L_e}\left[\tilde{N}\frac{d\tilde{\tilde{u}}_1}{dx_1} + \tilde{M}_1\frac{d\tilde{\tilde{\theta}}_1}{dx_1} + \tilde{M}_2\frac{d\tilde{\tilde{\theta}}_2}{dx_1} + \tilde{M}_3\frac{d\tilde{\tilde{\theta}}_3}{dx_1}\right]dx_1\right]^{(e)} \\
&= \sum_{e=1}^{NE}\left[\int_0^{L_e}\left(\frac{\tilde{N}\tilde{\tilde{N}}}{EA} + \frac{\tilde{M}_1\tilde{\tilde{M}}_1}{R_C G} + \frac{\tilde{M}_2\tilde{\tilde{M}}_2}{EI_2} + \frac{\tilde{M}_3\tilde{\tilde{M}}_3}{EI_3}\right)dx_1 + \int_0^{L_e}\left(\tilde{N}\tilde{\tilde{H}}_1 + \tilde{M}_2\tilde{\tilde{H}}_2 - \tilde{M}_3\tilde{\tilde{H}}_3\right)dx_1\right]^{(e)}
\end{aligned} \tag{13.34}$$

The superscript (e) indicates that the quantities of the terms inside the bracket pertain to member e. The components of rotation $\tilde{\tilde{\theta}}_2$ and $\tilde{\tilde{\theta}}_3$ are obtained from the components of translation $\tilde{\tilde{u}}_3$ and $\tilde{\tilde{u}}_2$ on the basis of relations (9.27a) and (9.27b), respectively. $\tilde{\tilde{H}}_1$, $\tilde{\tilde{H}}_2$ and $\tilde{\tilde{H}}_3$ are taken as equal to zero or are defined by relations (13.14b), (13.28b) and (13.32b), respectively.

Relation (13.34) represents the *principle of virtual work for framed structures*. In general it is only required that the "virtual" displacement field be admissible. If, however, it is chosen to be geometrically admissible, the last term on the left-hand side of relation (13.34) represents the sum of the products of the statically admissible reactions of the supports of the structure with their specified components of displacement.

On the basis of our discussion in Section 13.2 the following statements are valid:

1. The actual components of displacements of the members of a framed structure, when substituted into relations (8.58), (8.69) and (9.29), give components of internal actions $N^{(e)}(x_1)$, $M_1^{(e)}(x_1)$, $M_2^{(e)}(x_1)$ and $M_3^{(e)}(x_1)(e = 1, 2, ..., NE)$ which satisfy the principle of virtual work (13.34) for any admissible distribution of the components of the "virtual" displacement field $\tilde{\tilde{u}}_1^{(e)}(x_1)$, $\tilde{\tilde{u}}_2^{(e)}(x_1)$, $\tilde{\tilde{u}}_3^{(e)}(x_1)$ and $\tilde{\tilde{\theta}}_1^{(e)}(x_1)(e = 1, 2, ..., NE)$.

2. A set of continuous functions $u_1^{(e)}(x_1)$ and $\theta_1^{(e)}(x_1)(e = 1, 2, ..., NE)$ and a set of functions $u_2^{(e)}(x_1)$ and $u_3^{(e)}(x_1)(e = 1, 2, ..., NE)$ which have continuous first derivatives and give the specified components of displacements of the supports of a framed structure and moreover, give on the basis of relations (8.58), (8.69) and (9.29) components of internal action for its members which satisfy the principle of virtual work (13.34) for every admissible "virtual" displacement field $\tilde{u}_i^{(e)}(x_1)(i = 1, 2, 3)$ and $\tilde{\tilde{\theta}}_1^{(e)}$ $(e = 1, 2, ..., NE)$ are the actual components of displacement of the members of the structure.

3. Any set of internal actions $N^{(e)}(x_1)$, $M_1^{(e)}(x_1)$, $M_2^{(e)}(x_1)$ and $M_3^{(e)}(x_1)(e = 1, 2, ..., NE)$ acting on the members of a structure which are statically admissible to the given external actions acting on the structure satisfy the principle of virtual work (13.34) for any admissible distribution of the components of "virtual" displacement $\tilde{u}_1^{(e)}(x_1)$, $\tilde{u}_2^{(e)}(x_1)$, $\tilde{u}_3^{(e)}(x_1)$ and $\tilde{\theta}_1^{(e)}(x_1)(e = 1, 2, ..., NE)$ of the members of the structure.

4. A set of functions $N^{(e)}(x_1)$, $M_1^{(e)}(x_1)$, $M_2^{(e)}(x_1)$ and $M_3^{(e)}(x_1)(e = 1, 2, ..., NE)$ which together with the given external actions acting on a framed structure satisfy relation (13.34) for every admissible "virtual" displacement field $\tilde{u}_1^{(e)}(x_1)$, $\tilde{u}_2^{(e)}(x_1)$, $\tilde{u}_3^{(e)}(x_1)$ and $\tilde{\theta}_1^{(e)}(x_1)(e = 1, 2, ..., NE)$ is statically admissible to the given external actions acting on the structure. If it happens that $N^{(e)}(x_1)$, $M_1^{(e)}(x_1)$, $M_2^{(e)}(x_1)$ and $M_3^{(e)}(x_1)$ are obtained from a geometrically admissible "virtual" displacement field $u_1^{(e)}(x_1)$, $u_2^{(e)}(x_1)$, $u_3^{(e)}(x_1)$ and $\theta_1^{(e)}(x_1)(e = 1, 2, ..., NE)$ on the basis of relations (8.58), (8.69) and (9.29), then $u_1^{(e)}(x_1)$, $u_2^{(e)}(x_1)$, $u_3^{(e)}(x_1)$, $\theta_1^{(e)}(x_1)$, $N^{(e)}(x_1)$, $M_1^{(e)}(x_1)$, $M_2^{(e)}(x_1)$,and $M_3^{(e)}(x_1)(e = 1, 2, ..., NE)$ are the actual components of displacement and of internal actions of the members of the structure.

13.5 The Unit Load Method

In this section we describe† the unit load method, also known as the method of "virtual" work, or the dummy load method and we apply it to several examples. This method has been used extensively in classical structural analysis in establishing the following:

1. The component of translation u_m in the direction of the unit vector $\mathbf{i}_m$, of any point A of a framed structure

2. The component of rotation θ_m about an axis specified by the unit vector $\mathbf{i}_m$ at any point A of a framed structure

The component of displacement u_m and the component of rotation θ_m at a point of a structure may be due to external actions, to a change of temperature or to specified movement of the supports of the structure. The structure may be statically determinate or statically indeterminate.

In order to establish the component of displacement u_m, or the component of rotation θ_m at any point A of a structure, we introduce in the principle of virtual work the following quantities:

1. As "virtual" displacement and strain fields we choose the actual (real) displacement field of the structure subjected to the given loading. With this choice $\{\tilde{\Delta}^S\}$ is the matrix of the specified components of displacements of the supports of the structure.

† For a more detailed description and many examples see Armenàkas, A. E., *Classical Structural Analysis: A Modern Approach*, McGraw-Hill, New York, 1988, Chapter 5.

2. As the loading and corresponding statically admissible distribution of internal actions $\tilde{N}(x_1)$, $\tilde{Q}_2(x_1)$, $\tilde{Q}_3(x_1)$, $\tilde{M}_1(x_1)$, $\tilde{M}_2(x_1)$, $\tilde{M}_3(x_1)$ in the members of the structure, we choose an auxiliary loading and a corresponding statically admissible distribution of internal actions in the members of the structure. If we want to establish the component of translation u_m in the direction of the unit vector $\mathbf{i}_m$ at point A of the structure, the "auxiliary" loading consists of a unit force acting at point A in the direction of the unit vector $\mathbf{i}_m$. If we want to establish the component of rotation θ_m in the direction of the unit vector $\mathbf{i}_m$ at point A of the structure, the "virtual" loading consists of a unit moment acting at point A in the direction of the unit vector $\mathbf{i}_m$.

On the basis of the foregoing discussion, we adhere to the following steps in order to compute a component of translation or of rotation of a point of a framed structure made from an isotropic, linearly elastic material:

STEP 1 We establish the actual components of internal actions $N(x_1)$, $Q_2(x_1)$, $Q_3(x_1)$, $M_1(x_1)$, $M_2(x_1)$, $M_3(x_1)$ in the members of the structure as functions of their axial coordinate. For statically determined structures this can be done by considering the equilibrium of appropriate segments of each member. For statically indeterminate structures this can be done by analyzing the structure using one of the classical methods (for example, the force method presented in Chapter 14).

STEP 2 We subject the structure to the "auxiliary" loading described previously and establish a set of statically admissible reactions $\{\tilde{R}\}$ of the structure and corresponding internal actions $\tilde{N}(x_1)$, $\tilde{Q}_2(x_1)$, $\tilde{Q}_3(x_1)$, $\tilde{M}_1(x_1)$, $\tilde{M}_2(x_1)$, $\tilde{M}_3(x_1)$ of its members.

STEP 3 We substitute the internal actions and the reactions $\{\tilde{R}\}$ established in steps 1 and 2 into the principle of virtual work (13.34) which can be rewritten as

$$1 \times d + \{\Delta^S\}^T\{\tilde{R}\} = \sum_{e=1}^{ME}\left[\int_0^{L_e}\left(\frac{N\tilde{N}}{EA} + \frac{M_2\tilde{M}_2}{EI_2} + \frac{M_3\tilde{M}_3}{EI_3} + \frac{M_1\tilde{M}_1}{KG}\right)dx_1 + \int_0^{L_e}\left(H_1\tilde{N} + H_2\tilde{M}_2 - H_3\tilde{M}_3\right)dx_1\right]^{(e)} \tag{13.35}$$

where d is either u_n or θ_m; the superscript (e) indicates that the quantities of the terms inside the brackets pertain to member e; $\{\Delta^S\}$ is the matrix of the specified components of displacements of the supports of the structure; $\{\tilde{R}\}$ is the matrix of statically admissible reactions of the supports of the structure subjected to the "auxiliary" loading; H_1, H_2, H_3 are specified by relations (13.14b), (13.28b) and (13.32b) respectively, where in ΔT_c, δT_2, δT_3 are the given changes of temperature of the members of the structure. This procedure gives the desired component of translation or rotation.

In what follows we illustrate the unit load method with two examples.

Example 1 Using the unit load method compute the total translation of joint 2 of the truss loaded as shown in Fig. a. The members of the truss have the same constant cross section ($A = 4 \times 10^3$ mm^2) and are made of steel ($E = 210$ GPa).

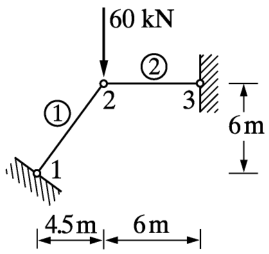


Figure a Geometry and loading of the truss.

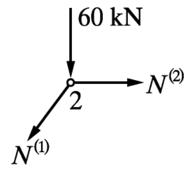


Figure b Free-body diagram of joint 2 of the truss of Fig. a.

Solution

STEP 1 The truss of Fig. a is statically determinate. Consequently, we can readily compute the internal forces in its members resulting from the application of the given external force. Referring to Fig. b, from the equilibrium of joint 2 of the truss, we have

$$\sum F_V = 0 \qquad 60 + \frac{4N^{(1)}}{5} = 0 \qquad \text{or} \qquad N^{(1)} = -75\,\text{kN} \tag{a}$$

$$\sum F_h = 0 \qquad \frac{3N^{(1)}}{5} - N^{(2)} = 0 \qquad \text{or} \qquad N^{(2)} = -\frac{3(75)}{5} = -45\ \text{kN} \tag{b}$$

STEP 2 In order to find the total translation of joint 2 of the truss we establish its vertical and horizontal components. This may be accomplished using the unit load method in conjunction with the auxiliary loading of Figs. c and d. However, the horizontal component of translation of joint 2 may be established by noting that it is equal to the shortening of member 2. That is,

$$u_h^{(2)} = \frac{N^{(2)}L_2}{AE} = \frac{45(6{,}000)}{4(10^3)210} = 0.321\ \text{mm} \tag{c}$$

Referring to Fig. e from the equilibrium of joint 2 of the truss of Fig. c, we have

$$\sum F_V = 0 \qquad 1 + \frac{4\tilde{N}^{(1)}}{5} = 0 \qquad \text{or} \qquad \tilde{N}^{(1)} = -\frac{5}{4} \tag{d}$$

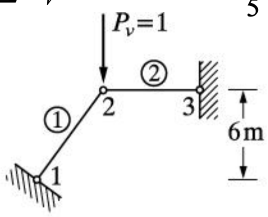


Figure c Auxiliary loading for computing the vertical component of translation of joint 2.

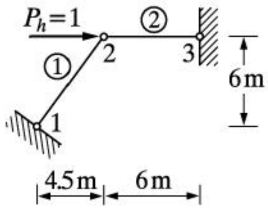


Figure d Auxiliary loading for computing the horizontal component of translation of joint 2.

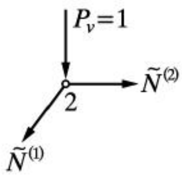


Figure e Free- body diagram of joint 2 of the truss of Fig. c.

$$\sum F_h = 0 \qquad \tilde{N}^{(2)} = \frac{3\tilde{N}^{(1)}}{5} = -\frac{3}{4} \tag{e}$$

STEP 3 We substitute results (a), (b), (d) and (e) into relation (13.35) to obtain the vertical component of translation of joint 2 of the truss. That is,

$$u_V^{(2)} = \sum_{e=1}^{2}\left[\int_0^{L_t}\frac{\tilde{N}N}{EA}dx_1\right]^{(e)} = \frac{1}{EA}\left[N^{(1)}\tilde{N}^{(1)}L_1 + N^{(2)}\tilde{N}^{(2)}L_2\right]$$
$$= \frac{(-75)(-5/4)7{,}500 + (-45)(-3/4)6{,}000}{(210)4(10^3)} = 1.078 \text{ mm} \tag{f}$$

The total translation of joint 2 of the truss is equal to

$$u^{(2)} = [(1.078)^2 + (0.321)^2]^{1/2} = 1.125 \text{ mm}$$

Example 2 The beam shown in Fig. a has a constant cross section [$I_2 = 360\ (10^6)$ mm^4, depth of cross section h = 300 mm] and it is made from an isotropic, linearly elastic material (E = 200 GPa, $\alpha = 10^{-5}$/°C). Compute the deflection and the rotation of point 3 of the beam due to following loading cases:

1. The forces shown in Fig. a
2. A difference in temperature between the top and bottom fibers of the beam (T_t = 40°C, T_b = 10°C)
3. A 5 mm downward settlement of support 2

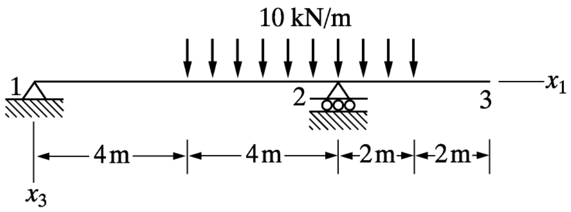


Figure a Geometry and loading of the beam.

Solution

Part a Computation of the deflection and the rotation of point 3 of the beam due to the forces shown in Fig. a:

STEP 1 Referring to Fig. b, we obtain the following distribution of moments due to the forces shown in Fig. a.

$$M_2 = 7.5x_1 \text{ kN}\cdot\text{m} \qquad 0 \le x_1 \le 4 \text{ m}$$

$$M_2 = 7.5x_1 - \frac{10(x_1-4)^2}{2} \text{ kN}\cdot\text{m} \qquad 4 \text{ m} \le x_1 \le 8 \text{ m} \tag{a}$$

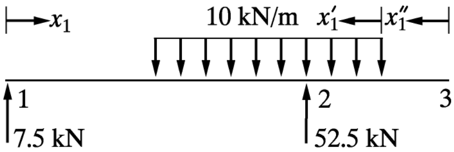


Figure b Free-body diagram of the beam loaded with the actual loading.

$$M_2 = -\frac{10(x_1')^2}{2} \text{ kN}\cdot\text{m} \qquad 0 \leq x_1' \leq 2$$
$$M_2 = 0 \qquad 0 \leq x_1'' \leq 2 \tag{a}$$

Notice that in order to reduce the required algebra, the moment in the beam segment 2, 3 was expressed in terms of the coordinates x_1', x_1'', shown in Fig. b.

Computation of the deflection of point 3

STEP 2 In order to compute the deflection of point 3 of the beam, the auxiliary loading shown in Fig. c is employed. The distribution of moments corresponding to this loading is

$$\tilde{M}_2 = -\frac{x_1}{2} \qquad 0 \leq x_1 \leq 8$$
$$\tilde{M}_2 = -(x_1' + 2) \qquad 0 \leq x_1' \leq 2 \tag{b}$$
$$\tilde{M}_2 = -x_1'' \qquad 0 \leq x_1'' \leq 2$$

STEP 3 We apply the unit load method to compute the deflection of the beam. Substituting relations (a) and (b) into (13.35), we get

$$u_V^{(3)} = \frac{10^9}{EI_2}\left\{-\int_0^4 7.5x_1\left(\frac{x_1}{2}\right)dx_1 - \int_4^8\left[7.5x_1 - \frac{10(x_1-4)^2}{2}\right]\frac{x_1}{2}dx_1 + \int_0^2 \frac{10(x_1')^2}{2}(x_1'+2)\,dx_1'\right\} \tag{c}$$

In the above relation, if E is in megapascals and I_2 is in mm^4, the deflection $u_v^{(3)}$ is in millimeters. Integrating relation (c), we get

$$u_V^{(3)} = -\frac{710 \text{ x } 10^9}{3EI_2} = -\frac{710 \text{ x } 10^9}{3(200)(3.6 \text{ x } 10^8)} = -3.29 \text{ mm} \tag{d}$$

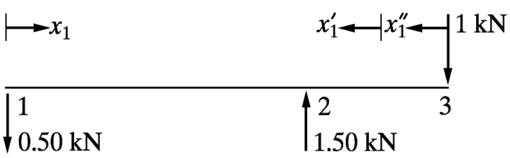


Figure c Free body-diagram of the beam loaded with the auxiliary loading for computing the deflection of point 3.

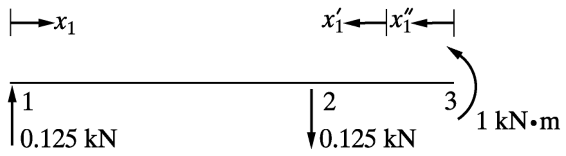


Figure d Free-body diagram of the beam loaded with the virtual loading for computing the rotation at point 3.

The minus sign indicates that point 3 moves in the direction opposite to the unit load shown in Fig. c. That is, it moves upward.

Computation of the rotation of point 3

STEP 2 In order to compute the rotation of the elastic curve of the beam at point 3, the auxiliary loading shown in Fig. d is employed. The internal moments corresponding to this loading are

$$\begin{aligned} \tilde{M}_2 &= \frac{x_1}{8} \qquad 0 \leq x_1 \leq 8 \\ \tilde{M}_2 &= 1 \qquad 0 \leq x_1' \leq 2 \end{aligned} \tag{e}$$

STEP 3 Substituting relations (a) and (e) into relation (13.35), we have

$$\begin{aligned} \theta_2^{(3)} &= \frac{10^6}{EI_2}\left\{\int_0^4 7.5x_1 \frac{x_1}{8} dx_1 + \int_4^8 \left[7.5x_1 - \frac{10(x_1-4)^2}{2}\right]\frac{x_1}{8} dx_1 + \int_0^2 \frac{-10(x_1')^2}{2} dx_1'\right\} \\ &= \frac{53.33 \text{ x } 10^6}{EI_2} = \frac{53.33 \text{ x } 10^6}{(200)(3.6 \text{ x } 10^8)} = 7.41 \text{ x } 10^{-4} \text{ rad} \end{aligned} \tag{f}$$

The plus sign indicates that the rotation $\theta^{(3)}$ of the beam is in the direction of the applied unit moment shown in Fig. d. The deformed configuration of the beam is shown in Fig. e.

Part b Computation of the deflection and the rotation of point 3 of the beam due to the difference in temperature between its top and bottom surfaces

STEPS 1, 2, 3 The deflection and the rotation of point 3 of the beam due to the given change of temperature may be established using relation (13.35) in conjunction with the auxiliary loading of Figs. c and d, and, consequently, the distribution of moments given by relations (b) and (e), respectively. Thus,

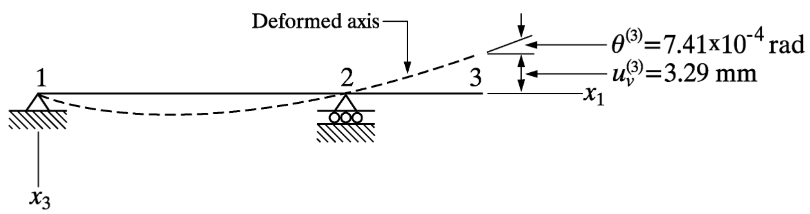


Figure e Deformed configuration of the beam subjected to the given external forces.

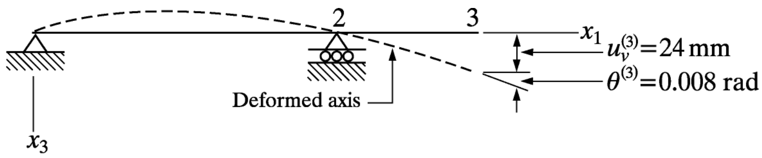


Figure f Deformation of the beam due to the change of temperature.

$$u_V^{(3)} = \int_0^L \alpha \tilde{M}_2 \frac{\delta T_3}{h} dx_1 = -\int_0^8 \frac{\alpha}{h}\frac{x_1}{2}(10-40)dx_1 - \int_0^4 \frac{\alpha}{h}(10-40)x_1'' dx_1'' \tag{g}$$

$$= \frac{720\alpha}{h} = \frac{720(10^{-5})}{0.30} = 24 \text{ mm}\downarrow$$

$$\theta_2^{(3)} = \int_0^L \alpha \tilde{M}_2 \frac{\delta T_3}{h} dx_1 = \int_0^8 \frac{\alpha}{h}\frac{x_1}{8}(10-40)dx - \int_0^4 \frac{\alpha}{h} 30 dx_1 \tag{h}$$

$$= -\frac{240\alpha}{h} = -\frac{240(10^{-5})}{0.30} = 0.008 \text{ rad clockwise}$$

The minus sign indicates that the rotation $\theta_2^{(3)}$ of the beam is as shown in Fig. f. That is, in the direction opposite to the applied unit moment in Fig. d.

Part c Computation of the deflection and the rotation of point 3 of the beam due to the settlement of its support 2

STEPS 1, 2, 3 We use relation (13.35) to compute the deflection and the rotation of point 3 of the beam due to the 5 mm settlement of support 2. Thus, we have

$$u_V^{(3)} + 5\tilde{R}^{(s)} = 0 \qquad\qquad \theta_2^{(3)} + 5\left[\tilde{R}^{(s)}\right]' = 0 \tag{i}$$

where $\tilde{R}^{(s)}$ and $[\tilde{R}^{(s)}]'$ are the reactions at support 2 when the beam is subjected to the auxiliary loading shown in Figs. c and d, respectively. The reactions $\tilde{R}^{(s)}$ and $[\tilde{R}^{(s)}]'$ are positive in the direction of the settlement $\Delta^{(s)}$, i.e., when they act downward. Thus,

$$\tilde{R}^{(s)} = -1.5 \qquad\qquad \left[\tilde{R}^{(s)}\right]' = 0.125 \tag{j}$$

consequently,

$$u_3^{(3)} = 7.5 \text{ mm} \tag{k}$$

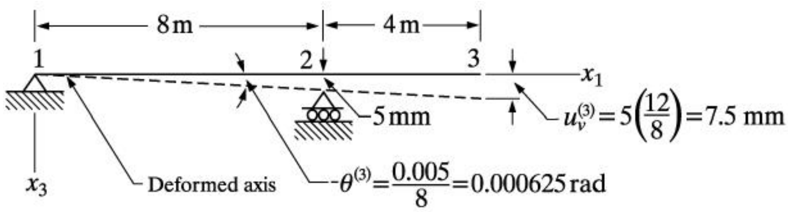


Figure g Deformation of the beam due to the settlement of support 2.

and

$$\theta_2^{(3)} = -\frac{5(0.125)}{1000} = -0.000625 \text{ rad}$$

13.5.1 Application of the Unit Load Method to Curved Beams

The principle of virtual work (13.35) is valid for straight and curved (with small h/R ratio) beams made from an isotropic, linearly elastic material. However, for curved beams referring to Fig. 11.1 the following substitutions must be made in relation (13.35) or (13.35); x_1 becomes x_s, x_3 becomes ζ; subscripts: 1 becomes t and 3 becomes ζ. The coordinate s is measured along the curved axis of the beam; ζ is measured along the normal to the axis of the beam. L is the total length of the axis of the beam.

In this section the unit load method is employed in establishing the components of displacement of the unsupported end of a curved cantilever beam whose axis lies in one plane and is an arc of a circle. The beam is subjected to a concentrated force acting in its plane.

Example 3 Consider the cantilever curved beam of constant cross sections and radius R shown in Fig. a. Assuming that the thickness-to-radius ratio of the beam is very small, compute the components of translation of point 2.

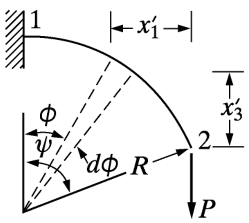


Figure a Geometry and loading of the curved beam.

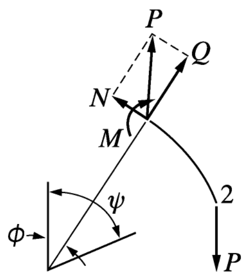


Figure b Free-body diagram of a segment of the curved beam subjected to the given loading.

Solution

STEP 1 We compute the internal moment and axial force in the curved beam subjected to the given loading. Referring to Fig. a, the coordinates (x_1', x_3') of a point of the beam specified by the angle ϕ are

$$x_1' = R(\sin\psi - \sin\phi) \qquad x_3' = R(\cos\phi - \cos\psi) \tag{a}$$

From the equilibrium of the segment of the curved beam shown in Fig. b we have

$$M_2(x_1') = -Px_1' = -PR(\sin\psi - \sin\phi) \qquad N(x_1') = P\sin\phi \tag{b}$$

STEP 2 In order to find the total translation of point 2 of the beam we establish its horizontal and vertical components. This is accomplished by using the principle of virtual work (13.35) in conjunction with the auxiliary loadings of Figs. c and e, respectively. From the equilibrium of the segment of the beam shown in Fig. d, we get

$$\tilde{M}_2 = x_3' = R(\cos\phi - \cos\psi) \qquad \tilde{N} = \cos\phi \tag{c}$$

Moreover, from the equilibrium of the segment of the beam shown in Fig. f we have

$$\tilde{M}_2 = x_1' = -R(\sin\psi - \sin\phi) \qquad \tilde{N} = \sin\phi \tag{d}$$

STEP 3 For the beam of Fig. a the principle of virtual work (13.35) reduces to

$$u_n^{(2)} = \int_0^S \left(\frac{N\tilde{N}}{EA} + \frac{M_2\tilde{M}_2}{EI_2} \right) dx_s = \int_{\phi=0}^{\phi=\psi} \left(\frac{N\tilde{N}}{EA} + \frac{M_2\tilde{M}_2}{EI_2} \right) R d\phi \tag{e}$$

where S is the length of the axis of the beam. Substituting relations (b) and (c) into (e), we obtain

$$u_h^{(2)} = -\int_0^{\psi} \frac{PR^2(\sin\psi - \sin\phi)(\cos\phi - \cos\psi) R d\phi}{EI_2} + \int_0^{\psi} \frac{P\sin\phi\cos\phi R d\phi}{AE}$$

$$= -\frac{PR^3}{EI_2}\left[\sin 2\psi + \frac{\cos 2\psi}{4} - \frac{1}{4} - \frac{\psi \sin 2\psi}{2} - \cos 2\psi + \cos\psi\right] + \frac{PR}{AE}\left[-\frac{\cos 2\psi}{4} + \frac{1}{4}\right] \tag{f}$$

Substituting relation (b) and (d) into (e), we get

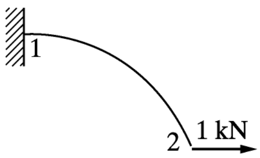


Figure c Auxiliary loading for computing the horizontal component of translation of point 2.

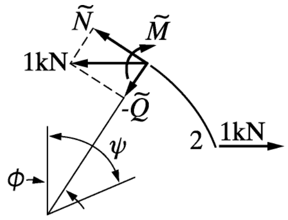


Figure d Free-body diagram of a segment of the curved beam loaded as shown in Fig. c.

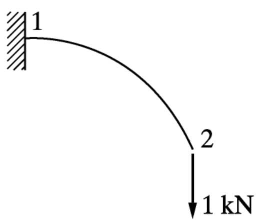


Figure e Auxiliary loading for computing the vertical component of translation of point 2.

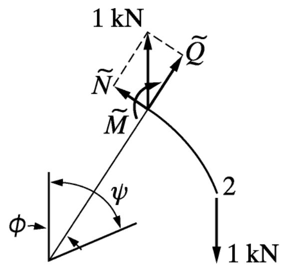


Figure f Free-body diagram of a segment of the curved beam loaded as shown in Fig. e.

$$u_V^{(2)} = \int_0^{\psi} \frac{PR^2(\sin\psi - \sin\phi)(\sin\psi - \sin\phi)R\,d\phi}{EI} + \int_0^{\psi} \frac{P\sin 2\phi\, R\,d\phi}{AE}$$
$$= \frac{PR^3}{EI_2}\left[\psi\sin 2\psi + \frac{3}{2}\cos\psi\sin\psi + \frac{\psi}{2} - 2\sin\psi\right] + \frac{PR}{2AE}[\psi - \sin\psi\cos\psi] \tag{g}$$

The first term on the right side of relations (f) and (g) represents the effect of the bending moment while the second term represents the effect of the axial force. For a beam of rectangular cross section, we have

$$I_2 = \frac{bh^3}{12} \qquad A = bh \qquad \frac{I_2}{A} = \frac{h^2}{12} \tag{h}$$

Thus, substituting relations (h) into (f) and (g), we can readily see that for this example the effect of the axial force is of the order of h^2/R^2 as compared to unity. As stated previously, we are limiting our attention to curved beams with small h/R ratios; consequently, h^2/R^2 may be disregarded as compared to unity. Hence, in this example the effect of the axial force is negligible.

For $\psi = 90^\circ$, disregarding terms of the order of h^2/R^2, relations (f) and (g) yield

$$u_h^{(2)} = -\frac{PR^3}{2EI_2} \qquad u_V^{(2)} = \frac{PR^3}{2EI_2}\left(\frac{3\pi}{2} - 4\right) \tag{i}$$

The negative sign for $u_h^{(2)}$ indicates that its direction is opposite to that of the unit force of Fig. c.

13.6 The Principle of Virtual Work for Framed Structures, Including the Effect of Shear Deformation

In Section 13.3 we derive the principle of virtual work for framed structures disregarding the effect of shear deformation of the members of the structure. For most structures of practical interest, this effect is small and it is neglected. In this section we derive the principle of virtual work for framed structures which includes the effect of shear deformation of their members. We establish this effect first on the basis of the Timoshenko theory of beams and then on the basis of a more accurate theory.

In the theories of beams (classical and Timoshenko) the shearing components of stress acting on a cross section are established from the shearing components of force acting on this cross section on the basis of relation (9.67); that is

$$\tau_{1n} = \frac{Q_2\bar{x}_{2n}A_n}{I_3 b_s} + \frac{Q_3\bar{x}_{3n}A_n}{I_2 b_s} \tag{13.36}$$

I_2 and I_3 are the moments of inertia of the cross section of the member under consideration about its principal centroidal axes x_2 and x_3, respectively; τ_{1n} is the shearing component of stress at a point of a cross section in the direction of the unit vector $\mathbf{i}_n$.

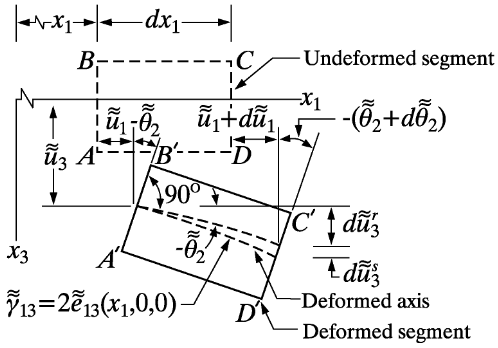


Figure 13.4 Deformation of an element of a member of a planar structure subjected to bending without twisting.

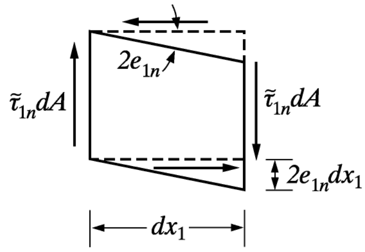


Figure 13.5 shearing deformation of an infinitesimal element and statically admissible shearing components of stress.

Referring to Fig. 13.3d, A_n is the area of the shaded portion of the cross section while $\bar{x}_{jn}(j = 2 \text{ or } 3)$ is the distance of the centroid of area A_n from the x_i axis ($i = 3$ or 2, $i \neq j$). Moreover, b_s is the length of the line AB (see Fig. 13.3d).

As discussed in Section 9.5, in general the formula for τ_{1n} gives its average value along line AB (see Fig. 13.3d). However, if the variation of τ_{1n} along the direction normal to the unit vector $\mathbf{i}_n$ (line AB) is negligible, the average value of τ_{1n} is equal to its actual value. This occurs in members having one of the cross sections shown in Fig. 9.11 in the directions indicated in that figure. Moreover, in members having cross sections like the ones shown in Fig. 9.11b, c, d and e, the shearing component of stress normal to τ_{1n} is negligible while for members whose cross sections are like the ones shown in Fig. 9.11a and f, the components of stress normal to τ_{1n} are antisymmetric and, consequently, do not contribute to the deflection of the member. Thus, we consider only the effect of τ_{1n}.

Consider a structure subjected to bending without twisting by given external forces (actual loading). The undeformed and the deformed configurations of an element of length dx_1 of this structure are shown in Fig. 13.4. The geometry of the latter is based on the assumption that plane sections normal to the axis of the element prior to deformation can be considered plane subsequent to deformation but not normal to its deformed axis. As discussed in Section 9.1, in this case the increments du_3 and du_2 of the transverse components of translation consist of two parts. One part du_3^r and du_2^r is due to the rotation of the element as a rigid body about the x_2 and x_3 axis, respectively; the other part du_3^s and du_2^s is due to the shear deformation of the element. Referring to Fig. 13.4 and to relations (9.20c), (9.22b), (9.45) and (9.48), we have

$$du_3^s = 2e_{13}(x_1, 0, 0)dx_1 = \frac{Q_3}{\lambda_3 GA}dx_1 \tag{13.37a}$$

$$du_2^s = 2e_{12}(x_1, 0, 0)dx_1 = \frac{Q_2}{\lambda_2 GA}dx_1 \tag{13.37b}$$

Consider the same structure subjected only to the auxiliary loading for computing a component of displacement of one of its points (see Section 13.5), and denote by $\tilde{Q}_2$ and $\tilde{Q}_3$ statically admissible shearing forces acting on the faces of an element of length dx_1 of a member of the structure. The sum of the products of the shearing forces $\tilde{Q}_2$ and $\tilde{Q}_3$ with the corresponding increments of the components of translation du_3^s and du_2^s is equal to

$$dW^s = \tilde{Q}_2 du_2^s + \tilde{Q}_3 du_3^s \tag{13.38}$$

Substituting relations (13.37) into (13.38), we have

$$dW^s = \frac{Q_3\tilde{Q}_3 dx_1}{\lambda_3 GA} + \frac{Q_2\tilde{Q}_2 dx_1}{\lambda_2 GA} \tag{13.39}$$

Integrating relation (13.39), we obtain

$$W^s = \sum_{k=1}^{NM}\left[\int_0^L \frac{Q_3\tilde{Q}_3 dx_1}{\lambda_3 GA} + \int_0^L \frac{Q_2\tilde{Q}_2 dx_1}{\lambda_2 GA}\right]^{(k)} \tag{13.40}$$

Adding W^s to the right-hand side of relation (13.34) we obtain the principle of virtual work which includes the effect of shear deformation of the members of a structure on the basis of the Timonshenko theory of beams.

In what follows we present a more accurate theory than the Timonshenko theory of beams for establishing W^s. In this theory, the component of strain e_{1n} of the particles of the members of the actual structure is computed from the corresponding components of stress τ_{1n}, as

$$2e_{1n} = \frac{\tau_{1n}}{G} \tag{13.41}$$

Using relations (13.36) and (13.41) from relation (13.29), we have

$$\begin{aligned} W^s &= \int_0^L \iint_A \tilde{\tau}_{1n} 2e_{1n}\Big|_{\text{due to}\,Q_2} dx_1\, dA + \int_0^L \iint_A \tilde{\tau}_{1n} 2e_{1n}\Big|_{\text{due to}\,Q_3} dx_1\, dA \\ &= K_2 \int_0^L \frac{Q_2\tilde{Q}_2}{GA} dx_1 + K_3 \int_0^L \frac{Q_3\tilde{Q}_3}{GA} dx_1 \end{aligned} \tag{13.42}$$

where

$$K_2 = \iint_A \frac{(\bar{x}_{2n}A_n)^2}{I_3^2 b_s^2} dA \qquad K_3 = \iint_A \frac{(\bar{x}_{3n}A_n)^2}{I_2^2 b_s^2} dA \tag{13.43}$$

The factors K_2 and K_3 depend on the geometry of the cross sections of the member and are referred to as *form factors.*

Adding the effect of shear deformation (13.42) to the principle of virtual work (13.35), we get

$$d + \sum_{s+1}^{S} \tilde{R}^{(s)}\Delta^{(s)} = \sum_{e=1}^{NM}\left[\int_0^L\left(\frac{N\tilde{N}}{EA} + \frac{M_1\tilde{M}_1}{KG} + \frac{M_2\tilde{M}_2}{EI_2} + \frac{M_3\tilde{M}_3}{EI_3}\right)dx_1 + K_2\int_0^L \frac{Q_2\tilde{Q}_2}{GA}dx_1\right.$$

$$\left. + K_3\int_0^L \frac{Q_3\tilde{Q}_3}{GA}dx_1 + \int_0^L\left(\alpha\tilde{N}\Delta T_c + \alpha\tilde{M}_2\frac{\delta T_3}{h^{(3)}} - \alpha\tilde{M}_3\frac{\delta T_2}{h^{(2)}}\right)dx_1\right]^{(e)} \tag{13.44}$$

This is the principle of virtual work for framed structures which includes the effect of shear deformation of the members of the structure.

Notice that W^s given by relation (13.42) has the same form as that obtained on the basis of the Timoshenko theory of beams (13.40). However, the factors K_2 and K_3 are obtained from relations (13.43) while the factors λ_2 and λ_3 represent the ratio of the average to the maximum values of the shearing components of stress τ_{12} and τ_{13}, respectively [see relations (9.49)]. As shown in the example at the end of this section, for a rectangular cross section $K_2 = K_3 = 1.2$, while as shown in Section 9.4, $\lambda_2 = \lambda_{13} = 2/3$ ($1/\lambda_2 = 1/\lambda_3 = 1.5$). This indicates that the effect of shear deformation on the components of displacement of a particle of a structure whose members have rectangular cross sections, obtained on the basis of the Timoshenko theory of beams, is 1.25 times larger than that obtained on the basis of the principle of virtual work (13.44).

In the sequel, we apply the method of virtual work in computing the deflection of a beam, including the effect of shear deformation. It is shown that the effect of shear deformation is negligible when the ratio of the depth to the length of the beam is small.

Example 4 Consider a cantilever beam of rectangular cross section subjected to a concentrated force as shown in Fig. a. Compute the deflection of the beam at point 1. Include the effect of shear deformation and establish the range of the parameters characterizing the geometry of the beam for which the effect of shear deformation is negligible.

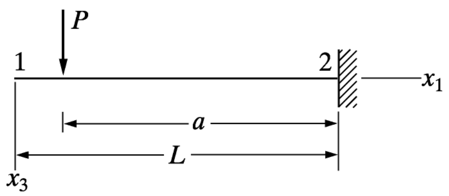


Figure a Geometry and loading of the beam.

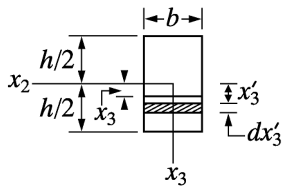


Figure b Cross section of beam.

Solution The effect of shear deformation is represented by the second and third terms on the right-hand side of relation (13.44). To compute this effect, we must calculate the value of the form factor K_3 of the beam [see relation (13.43)]. Thus, referring to Fig. b, we have

$$\bar{x}_{3n}A_n = \int_{x_3}^{h/2} x_3' b\, dx_3' = \frac{b}{2}\left(\frac{h^2}{4} - x_3^2\right) \tag{a}$$

and

$$(\bar{x}_{3n}A_n)^2 = \frac{b^2}{4}\left(\frac{h^4}{16} - \frac{h^2x_3^2}{2} + x_3^4\right) \tag{b}$$

Substituting relation (b) into (13.43), we obtain

$$K_3 = \int_{-h/2}^{h/2} \frac{(A_n\bar{x}_{3n})^2 A}{I_2^2 b} dx_3 = \frac{b^2h^6}{120 I_2^2} = 1.2 \tag{c}$$

Referring to Figs. a and c, we get

$$\left.\begin{array}{ll} Q = 0 & M = 0 \\ \tilde{Q} = -1 & \tilde{M} = x_1 \end{array}\right\} 0 \le x_1 \le (L - a)$$

$$\left.\begin{array}{ll} Q = -P & M = P(x_1 - L + a) \\ \tilde{Q} = -1 & \tilde{M} = x_1 \end{array}\right\} (L - a) \le x_1 \le L \tag{d}$$

Substituting relations (c) and (d) into (13.44), we have

$$\bar{u}_3^{(1)} = \int_{L-a}^{L} \frac{P(x_1 - L + a)x_1 dx_1}{EI_2} + 1.2\int_{L-a}^{L} \frac{P}{GA} dx_1 = \frac{Pa}{Ebh}\left[\frac{2a}{h}\left(\frac{3L}{h} - \frac{a}{h}\right) + 2.4(1 + \nu)\right] \tag{e}$$

The last term in the bracket represents the effect of shear deformation. It is apparent that the effect of shear deformation depends on the a/h and L/h ratios. For the case $\nu = 0.3$ and $a = L$, relation (e) reduces to

$$\bar{u}_3^{(1)} = \frac{PL}{Ebh}\left[4\left(\frac{L}{h}\right)^2 + 3.12\right] \tag{f}$$

For $L/h = 2$, the effect of shear deformation is approximately equal to 20%, while for $L/h = 6$, it is less than 2%, and may be disregarded. Thus, it is only for very short and deep beams that the effect of shear deformation is not negligible.

Figure c Beam subjected to the auxiliary loading for the computation of the deflection of point 1.

13.7 The Strong Form of One-Dimensional, Linear Boundary Value Problems

The strong or classical form of one-dimensional, linear boundary value problems involves the determination of a function $X(x_1)$ called the *state variable* which has the following attributes:

1. It satisfies an ordinary differential equation at every point of a line $0 \le x_1 \le L$ called the *domain of the problem* which can be written in the following form:

$$L(X) + J = 0 \tag{13.45}$$

where L is a linear differential operator; J is a scalar function of x_1 called the *source function*. It represents the distribution of the sources located inside the domain of the problem which together with the sources located on the boundary of the domain produce the state variable $X(x_1)$.

As an example consider a boundary value problem involving a state variable $X(x_1)$ satisfying the following differential equation of order $2m$ where m is an integer:

$$a_{2m}(x_1)\frac{d^{2m}X(x_1)}{dx_1^{\,2m}} + a_{2m-1}(x_1)\frac{d^{2m-1}X(x_1)}{dx_1^{\,2m-1}} + ,\ldots, + a_1(x_1)\frac{dX(x_1)}{dx_1} + a_o(x_1)X(x_1) + J(x_1) = 0 \tag{13.46}$$

The differential operator L for this equation is

$$L = a_{2m}(x_1)\frac{d^{2m}}{dx_1^{\,2m}} + a_{2m-1}(x_1)\frac{d^{2m-1}}{dx_1^{\,2m-1}} + ,\ldots, + a_1(x_1)\frac{d}{dx_1} + a_o(x_1) \tag{13.47}$$

where $a_0(x_1)$, $a_1(x_1)$,. . . ,$a_{2m-1}(x_1)$, and $a_{2m}(x_1)$ are known functions which usually represent material properties.

2. It satisfies appropriate equations, called *boundary conditions*, relating the values of the state variable $X(x_1)$ and/or some of its derivatives of order less than that of the differential operator L at the points of the boundary of the domain. The boundary of the problems considered in this section consists of the two end points $x_1 = 0$ and $x_1 = L$ of their domain $0 \le x_1 \le L$. A boundary value problem of order $2m$ requires m boundary conditions to be specified at each one of its two end points. Boundary conditions involving a differential equation of order up to $m - 1$ are called *essential*, while boundary conditions involving a differential equation from order m up to $2m - 1$ are called *natural*. Thus, for the correct formulation of second order one-dimensional, linear boundary value problems, one boundary condition must be specified at each end of the domain. If we suppose that an essential boundary condition is specified at $x_1 = 0$ and a natural at $x_1 = L$, the boundary conditions for a second order, one-dimensional, linear boundary value problem have the following form:

$$X + \hat{G} = 0 \qquad \text{at } x_1 = 0 \tag{13.48a}$$

$$B(X) + G = 0 \qquad \text{at } x_1 = L \tag{13.48b}$$

where B is a linear differential operator involving the first derivative while G and $\hat{G}$ are given functions of x_1.

In addition to the state variable in a boundary value problem we are interested in establishing one or more quantities known as *fluxes*, which for the problems that we are considering are equal to a linear combination of derivatives of the state variable.

For example, consider the second order, one-dimensional, linear boundary value problem for computing the axial component of translation, $u_1(x_1)$ of the member of Fig. 8.23a subjected to axial centroidal forces. For this problem $u_1(x_1)$ is the *state variable* while the axial component of the internal force $N(x_1)$ is the *flux*. The domain of this problem is a line (the axis of the member $0 \le x_1 \le L$). Its boundary consists of the two end points $x_1 = 0$ and $x_1 = L$. That is, why such boundary value problems are called *two point*. Moreover, comparing relations (13.45) with (8.65) the differential operator L and the function $J(x_1)$ for this boundary value problem are equal to

$$X(x_1) = u_1(x_1) \qquad\qquad L = \frac{d}{dx_1}\left(EA\frac{d}{dx_1}\right) \tag{13.49a}$$

and

$$J = J_1(x_1) = p_1(x_1)\Delta(x_1 - c_1) + \sum_{n=1}^{n_1} P_1^{(n)}\,\delta(x_1 - a_{1n}) - \frac{d}{dx_1}(EAH_1) \tag{13.49b}$$

where n_1 is the total number of concentrated axial centroidal forces $P_1^{(n)}$ acting along the length of the member. H_1 is defined by relation (8.57). $J_1(x_1)$ is called the *source function.* It includes all the loads that produce the state variable. $u_1(x_1)$ Furthermore comparing relations(13.48) with (8.58) the differential operator B and the functions G and $\hat{G}$ for the one-member structure of Fig. 8.23a are equal to

$$N(L) = \left[EA\left(\frac{du_1}{dx_1} - H_1\right)\right]_{x_1 = L} = P_1^L \rightarrow \begin{cases} B = \left[EA\dfrac{d}{dx_1}\right]_{x_1 = L} \\ G = -P_1^L - (EAH_1)_{x_1 = L} \end{cases} \tag{13.50}$$

$$\hat{G} = 0$$

For the correct formulation of fourth order, one-dimensional, linear boundary value problems two boundary conditions must be specified at each end of the domain. If we suppose that two essential boundary conditions are specified at $x_1 = 0$ and two natural at $x_1 = L$, the boundary conditions have the following form:

$$X + \hat{G}_1 = 0 \qquad \text{at } x_1 = 0 \tag{13.51a}$$

$$\frac{dX}{dx_1} + \hat{G}_2 = 0 \qquad \text{at } x_1 = 0 \tag{13.51b}$$

$$B_1(X) + G_1 = 0 \qquad \text{at } x_1 = L \tag{13.51c}$$

$$B_2(X) + G_2 = 0 \qquad \text{at } x_1 = L \tag{13.51d}$$

where $\hat{G}_1, \hat{G}_2$, G_1, G_2 are given functions of x_1, while B_1, B_2 are linear differential operators involving the second and third derivatives, respectively.

Boundary value problems of the type considered in this section exist in many fields of science and engineering and our presentation applies to all of them.

As an example, consider the forth order, one-dimensional, linear boundary value problem for computing the deflection $u_2(x_1)$ of a beam subjected to bending about its x_3 principal centroidal axis. For this problem $u_2(x_1)$ is the *state variable* while the shearing force $Q_2(x_1)$ and the bending moment $M_3(x_1)$ are the fluxes. Moreover, comparing relation (13.45) with (9.34a), the differential operator L and the function $J(x_1)$ for the beam of Fig. 13.6 is equal to

$$X(x_1) = u_2(x_1)$$

$$L = \frac{d^2}{dx_1^2}\left[EI_3\left(\frac{d^2}{dx_1^2}\right)\right] \tag{13.52a}$$

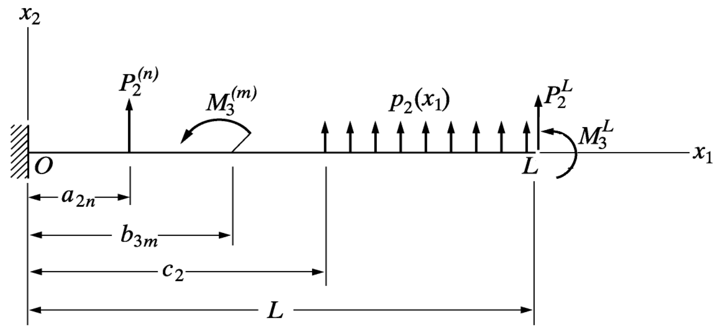


Figure 13.6 Beam subjected to bending about its x_3 axis.

$$J = J_2(x_1) = -p_2\Delta(x_1 - c_2) - \sum_{n=1}^{n_2} P_2^{(n)}\delta(x_1 - a_{2n}) - \sum_{m=1}^{m_3} M_3^{(m)}\delta^I(x_1 - b_{3m}) + \frac{d^2}{dx_1^2}(EI_3H_3) \tag{13.52b}$$

where n_2 is the total number of concentrated transverse forces $P_2^{(n)}$ ($n = 1, 2, ..., n_2$) acting on the member and m_3 is the total number of concentrated bending moments $M_3^{(m)}$ ($m = 1, 2, ..., m_3$) acting on the member.

Furthermore, referring to Fig. 13.6 the differential operators B_1 and B_2 and the functions G_1, G_2, $\hat{G}_1$ and $\hat{G}_2$ of relations (13.51) for the one member structure of Fig. 13.6 subjected to bending about its x_3 axis, are equal to

$$u_2(0) = 0 \quad \rightarrow \quad \hat{G}_1 = 0 \tag{13.53a}$$

$$\left.\frac{du_2}{dx_1}\right|_{x_1=0} = 0 \quad \rightarrow \quad \hat{G}_2 = 0 \tag{13.53b}$$

$$M_3(L) = M_3^L \quad \rightarrow \quad \begin{cases} B_1 = -\left[EI_2\dfrac{d^2}{dx_1^2}\right]_{x_1=L} \\ G_1 = -M_2^L - EI_2H_2 \end{cases} \tag{13.53c}$$

$$Q_3(L) = P_3^L \quad \rightarrow \quad \begin{cases} B_2 = \left[\dfrac{d}{dx_1}\left(EI_2\dfrac{d^2}{dx_1^2}\right)\right]_{x_1=L} \\ G_2 = -P_3^L - \dfrac{d}{dx_3}EI_2H_2 \end{cases} \tag{13.53d}$$

13.8 Approximation of the Solution of One-Dimensional, Linear Boundary Value Problems Using Trial Functions

Only simple boundary value problems can be solved exactly with the available mathematical methods. The rest are solved approximately. In order to use a digital computer to obtain approximate solutions of boundary value problems they must be *discretized*. That is, they must be recast in an algebraic form involving a finite number of unknown coefficients. The methods used to discretize boundary value problems can be classified into two groups. Those applied directly to their strong or classical form [such as finite differences (see Sections 6.12 and 9.10)] and those applied directly to one of *their integral forms*; for example, their weighted residual form (see Sections 13.9 to 13.11) or their modified weighted residual (weak) form (see Sections 13.12 and 13.13). The most important by far of the second group of methods is the finite element method (see Chapter 15) which in the last three decades has been used extensively in writing programs for solving complex engineering problems by a computer.

In order to discretize one-dimensional, linear boundary value problems, using one of their integral forms, we construct approximate solutions $\tilde{X}(x_1)$ of the following form:

$$\tilde{X}(x_1) = \phi_0(x_1) + \sum_{s=1}^{S} c_s \phi_s(x_1) \tag{13.54}$$

where

$c_s (s = 1, 2, ..., S)$ = undeterminate coefficients, also known as *degrees of freedom*

$\phi_0 (x_1)$ = continuous function of x_1 chosen to satisfy at least the essential boundary conditions of the problem.

$\phi_s (x_1)(s = 1, 2,..., S)$ = linearly independent functions of x_1 known as *interpolation or trial functions*. They are chosen to satisfy the homogeneous part of the boundary conditions of the problem which were satisfied by the function $\phi_0 (x_1)$. Thus the function $\phi_0 (x_1)$ and $\phi_s (x_1)(s = 1, 2, ..., S)$ are chosen so that the approximation to the state variable (13.54) satisfies at least the essential boundary conditions of the problem.

The choice of the function $\phi_0(x_1)$ and $\phi_s(x_1)(s = 1, 2, ..., S)$ affects the accuracy of the approximate solution (13.54). In order to ensure that as s increases the approximate solution (13.54) converges to the actual solution of the boundary value problem, the trial functions must be a sequence of functions from a *complete in energy infinite sequence of functions* starting from the lowest order up to the order S without missing an intermediate term. A sequence of trial functions $\phi_s (s = 1, 2, ..., S)$ is complete in energy if a set of parameters $c_s (s = 1, 2, ..., S)$ can be found so that the function $\tilde{X}(x_1)$ given by relation (13.54) approaches closely the exact solution $X(x_1)$ as s increases. The closeness is measured by the energy error. For instance, the set of functions $\sin(s\pi x_1/L)$ or $\cos(s\pi x_1/L)$ $(s = 1, 2, ..., S)$ is a complete set, because, as known from the theory of Fourier series, any function of x_1 which is continuous in an interval $a \le x_1 \le b$ can be expanded in a convergent series of these functions in this interval. Moreover, the functions $(x_1 - a)^s$ $(s = 1, 2, ..., S)$ are also a set of complete functions, because any function of x_1 having continuous derivatives of any order in the interval $a \le x_1 \le b$ can be expanded into a Taylor series in that interval. That is,

$$\tilde{X}(x_1) = \sum_{s=1}^{\infty} \frac{X^s(a)}{s!}(x_1 - a)^s = \sum_{s=1}^{\infty} c_s (x_1 - a)^s \tag{13.55}$$

where $X^s(a)$ represents the s^{th} derivative of the function $X(x_1)$ evaluated at $x_1 = a$. Notice that the Fourier series can represent continuous and discontinuous functions, while the Taylor series can represent only functions which have derivatives of all orders.

Due to the inclusion of only a finite number of trial functions a pointwise error $E(x_1)$ is introduced in the approximate solution (13.54) called the *discretization error*. It is defined as

$$E(x_1) = [X(x_1) - \tilde{X}_n(x_1)] \tag{13.56}$$

where $\tilde{X}_n(x_1)$ represents the approximate solution corresponding to n trial functions while $X(x_1)$ is the actual solution. We say that the approximate solution (13.54) converges pointwise if it satisfies the following relation:

$$E(x_1) = [X(x_1) - \tilde{X}_n(x_1)] \rightarrow 0 \qquad \text{as} \quad n \rightarrow \infty \tag{13.57}$$

It is clear that if a sequence of approximate solutions converges pointwise, then two different approximate solutions $\tilde{X}_m(x_1)$ and $\tilde{X}_n(x_1)$ will approach each other as m and n increase to infinity. That is,

$$[\tilde{X}_m(x_1) - \tilde{X}_n(x_1)] \rightarrow 0 \qquad \text{as} \quad m, n \rightarrow \infty \tag{13.58}$$

Notice that although relation (13.57) implies (13.58), the reverse is not true. That is, relation (13.58) does not imply (13.57). A sequence of approximate solutions may converge according to relation (13.58) but the limit to which it converges may not be the exact solution of the boundary value problem. However, if in constructing approximate solutions, of the type (13.54), to a boundary value problem we choose a set of trial functions from a complete set of functions, there is a high likehood that if the sequence of shape functions converges according to (13.58), it will also converge according to (13.57). In such an eventuality we can examine the accuracy of a solution by finding a sequence of approximate solutions and examining their difference.

A measurement of the closeness of approximate solutions to the exact solution is the *energy error*. For example, the energy error of an approximate solution of a boundary value problem involving a state variable of one space coordinate is defined as

$$\text{energy error} = \left[\int_0^L E(x_1)\, L\left[E\,(x_1)\right] dx_1 \right]^{1/2} \tag{13.59}$$

where $E(x_1)$ is the pointwise error defined by relation (13.56) and L is the differential operator of the governing differential equation of the problem under consideration.

When we say that a sequence of functions *is complete in energy,* we mean that when it is used to form approximate solutions of a boundary value problem, the energy error defined by relations (13.59) can be made as small as desired by increasing the number of terms in the series of relation (13.54). Convergence of an approximate solution in energy does not ensure its pointwise converge. *However, when the approximate solutions of most properly posed physical problems converge in energy, they also converge pointwise.*

13.9 The Classical Weighted Residual Form for Second Order, One-Dimensional, Linear Boundary Value Problems

When the approximate solution (13.54) is substituted into the differential equation

(13.45) there may be a residual R_d. That is,

$$R_d = L(\tilde{X}) + J \tag{13.60}$$

Moreover, when the approximate solution (13.54) is substituted into the boundary conditions (13.48) there may be residuals. That is,

$$\hat{R}_b = \tilde{X} + \hat{G} \tag{13.61a}$$

$$R_b = B(\tilde{X}) + G \tag{13.61b}$$

The functions $\phi_0(x_1)$ and $\phi_s(x_1)$ ($s = 1, 2, ..., S$) are chosen in a way that the approximate solution (13.54) satisfies as many of the boundary conditions of the problem as possible. The residuals $\hat{R}_b$ and/or R_b corresponding to these boundary conditions vanish. *We assume that the functions* $\phi_0(x_1)$ and $\phi_s(x_1)$ *are chosen in a way that the approximate solution (13.54) satisfies at least the essential boundary conditions of the problem.* Thus, we are interested in establishing a set of values of the parameters c_s(s = 1, 2, ..., S) which reduce the value of the residual R_d and of the non-vanishing residual R_b if there are any.

A way to reduce the residual R_d, uniformly throughout the domain of the problem and the residual R_b on its boundary (say at $x_1 = L$) is to choose the coefficients c_s(s = 1, 2, ..., S) of the approximate solution (13.54) so that the following S integral equations are satisfied:

$$\int_0^L W_r R_d \, dx_1 + \epsilon \left(W_{br} R_b\right)^{x_1=L} = 0 \qquad r = 1, 2, ..., S \tag{13.62}$$

where

$$\epsilon = \begin{cases} 0 & \text{when the approximate solution (13.54) satisfies the natural boundary condition (13.48b)} \\ 1 & \text{when the approximate solution (13.54) does not satisfy the natural boundary condition (13.48b)} \end{cases} \tag{13.63}$$

W_r and W_{br}(r = 1, 2, ..., S) are called the *weighting functions*. They are sequences of linearly independent functions of x_1. Each sequence could be chosen independently. Substituting relations (13.60) and (13.61b) into (13.62), we get

$$\int_0^L W_r [L(\tilde{X}) + J] dx_1 + \epsilon \left[W_{br} \left[(B(\tilde{X}) + G\right]\right]^{x_1=L} = 0 \qquad r = 1, 2, ..., S \tag{13.64}$$

Equation (13.64) is called the *weighted residual equation* for the boundary value problem (13.45) with (13.48). It is apparent that the exact solution of this boundary value problem satisfies the weighted residual equation (13.64) for any weighing function W_r. Moreover, a function $X(x_1)$ which satisfies the essential boundary conditions of the problem and the weighted residual equation (13.64) for every set of weighting functions W_r and W_{br} is the solution of the boundary value problem (13.45) and (13.48). In order to prove this statement let us assume that there exists a function $\tilde{X}(x_1)$ which satisfies the essential boundary conditions of the problem and its weighted residual equation (13.64) for every set of weighting functions W_r and W_{br} but it does not satisfy equation (13.45) at some points of the domain of the problem. That is, $L(\tilde{X}) + J \neq 0$ at some points of the domain. For such a function it is apparent that we can find a set of weighting functions W_r and W_{br} which do not satisfy the weighted residual equation (13.64). Consequently, our assumption leads to a contradiction. Thus, any function $\tilde{X}(x_1)$ which satisfies the essential boundary conditions of the problem and its weighted residual equation (13.64) for every set of weighting functions W_r and W_{br} must also satisfy relations (13.45) and (13.48). That is, it must be the solution of the boundary value problem.

On the basis of the foregoing discussion the weighted residual form of the boundary value problem under consideration involves the determination of the function $X(x_1)$ which satisfies the essential boundary conditions of the problem and its weighted residual equations (13.64) for every set of weighting functions W_r and W_{br}. The weighted residual form of a boundary value problem is equivalent to its strong form.

Referring to relation (13.49) and (13.50), the weighted residual equation (13.64) for the one member structure of Fig. 8.23a, is

$$\int_0^L W_r \left[\frac{d}{dx_1} \left[EA \left(\frac{d\tilde{u}_1}{dx_1} - H_1 \right) \right] + p_1(x_1)\Delta(x_1 - c_1) + \sum_{n=1}^{n_1} P_1^{(n)} \delta(x_1 - a_{1n}) \right] dx_1$$
$$+ \epsilon \left[W_{br} \left[EA \left(\frac{d\tilde{u}}{dx_1} - H_1 \right) - P_1^L \right] \right]_{x_1 = L} = 0 \qquad (13.65)$$
$$r = 1, 2, 3, \ldots, S$$

13.10 The Classical Weighted Residual Form for Fourth Order, One-Dimensional, Linear Boundary Value Problems

When the approximate solution (13.54) is substituted into the differential equation (13.45) of a fourth order, one-dimensional, linear boundary value problem, there could be a residual R_d. Moreover, when the approximate solution (13.54) is substituted into the boundary conditions (13.51) of such a problem there could be residuals $\hat{R}_{b1}, \hat{R}_{b2}, R_{b1}, R_{b2}$. That is, referring to relations (13.45) and (13.51), we have

$$R_d(c_1, c_2, \ldots, c_s) = L(\tilde{X}) + J \qquad (13.66a)$$

$$\hat{R}_{b1}(c_1, c_2, ..., c_S) = \tilde{X} + \hat{G}_1 \qquad (13.66b)$$

$$\hat{R}_{b2}(c_1, c_2, ..., c_s) = \frac{d\tilde{X}}{dx_1} + \hat{G}_2 \qquad (13.66c)$$

$$R_{b1}(c_1, c_2, \ldots, c_S) = B_1(X) + G_1 \qquad (13.66d)$$

$$R_{b2}(c_1, c_2, \ldots, c_S) = B_2(X) + G_2 \qquad (13.66e)$$

The function $\phi_0(x_1)$ and $\phi_s(x_1)$ ($s = 1, 2, ..., S$) of the approximate solution (13.54) are chosen so as to satisfy as many of the boundary conditions of the problem as possible. The residuals $\hat{R}_{b1}, \hat{R}_{b2}, R_{b1}, R_{b2}$ which correspond to these boundary conditions vanish. *We assume that the functions* $\phi_0(x_1)$ and $\phi_s(x_1)$ *are chosen in a way that the approximate solution* (13.54) *satisfies at least the essential boundary conditions of the problem.* Thus, we are interested in establishing a set of values of the coefficients $c_s (s = 1, 2, ..., S)$ which reduce the value of the residual R_d uniformly throughout the domain of the problem as well as the value of those of the residuals R_{b1} or R_{b2} which do not vanish. One way to accomplish this is to choose the coefficients $c_s (s = 1, 2, ..., S)$ so that the following S integral equations are satisfied:

$$\int_0^L W_r R_d dx_1 + \epsilon_1 (W_{br1} R_{b1})_{x_1 = L} + \epsilon_2 (W_{br2} R_{b2})_{x_1 = L} = 0 \qquad r = 1, 2, ..., S \tag{13.67}$$

where

$$\epsilon_i \,(i = 1,2) \begin{cases} 0 & \text{if the approximate solution (13.54) is chosen to satisfy the corresponding natural boundary condition.} \\ 1 & \text{if the approximate solution (13.54) is not chosen to satisfy the corresponding natural boundary condition.} \end{cases} \tag{13.68}$$

$W_r(x_1)$ and $W_{bri}(x_1)$ ($i = 1$ or 2) = functions of the axial coordinate called weighting ($r = 1, 2, ..., S$) functions.

The functions W_r are such that they do not make infinite the integrals in equations (13.67). Equation (13.67) is the *weighted residual equation* for one-dimensional, fourth order, linear, boundary value problems. It is apparent that the exact solution of such a boundary value problem satisfies the weighted residual equation (13.67) for any set of weighting functions W_r and W_{bri} ($i = 1, 2$). Moreover, it can be shown that a function $X(x_1)$ which satisfies the essential boundary conditions of the problem and the weighted residual equation (13.67) for every set of weighting functions $W_r(x_1)$ and $W_{bri}(x_1)$ ($i = 1, 2$) is the solution of the boundary value problem (13.45) and (13.51).

On the basis of the foregoing discussion the weighted residual form of the one-dimensional, fourth order, linear boundary value problem (13.45) and (13.51) involves the determination of the function $X(x_1)$ which satisfies the essential boundary conditions of the problem and the weighed residual equation (13.67) for every set of weighting functions W_r and W_{bri} ($i = 1, 2$). The weighted residual form of a boundary value problem is equivalent to its strong form.

Substituting relations (13.52) and (13.53) into (13.66) and the resulting relations into (13.67) and using relation (9.32b), the weighted residual equation for the beam of Fig. 13.6, is

$$\int_0^L W_r \frac{d^2 \tilde{M}_3}{dx_1^2} dx_1 + \int_0^L W_r J_2 dx_1 + \epsilon_1 \Big[W_{rb1} (\tilde{Q}_2 - P_2^L) \Big]_{x_1 = L} \qquad r = 1, 2, \ldots, S$$
$$+ \epsilon_2 \Big[W_{br2} (\tilde{M}_3 - M_3^L) \Big]_{x_1 = L} = 0 \tag{13.69}$$

13.11 Discretization of Boundary Value Problems Using the Classical Weighted Residual Methods

There are several methods available in the literature for constructing approximate solutions of boundary value problems using their weighted residual form. They are known as *classical weighted residual methods*. In these methods the state variable is approximated by relation (13.54). Each of the functions ϕ_0 and ϕ_s ($s = 1, 2, ..., S$) is defined over the entire domain of the problem by a single function which has derivatives of any order. Moreover the functions ϕ_0 and ϕ_s ($s = 1, 2, ..., S$) are usually chosen so that the approximate solution (13.54) satisfies all the boundary conditions of the problem. Thus, for these methods the weighted residual equations (13.62) and (13.67) reduce to

$$\int_0^L W_r R_d dx_1 = \int_0^L W_r[L(\tilde{X}) + J]dx_1 = 0 \qquad r = 1, 2, ..., S \tag{13.70}$$

A boundary value problem is discretized and the coefficients c_s($s = 1, 2, ..., S$) are evaluated by substituting the approximate solution (13.54) into relation (13.70). Thus,

$$\int_0^L W_r[L(\phi_0) + \sum_{s=1}^{S} c_s L(\phi_s) + J]dx_1 = 0 \qquad r = 1, 2, 3, \ldots, S \tag{13.71}$$

Relations (13.71) are a set of S linear algebraic equations which can be written as

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdot & \cdot & \cdot & S_{1S} \\ S_{21} & S_{22} & S_{23} & \cdot & \cdot & \cdot & S_{2S} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S_{S1} & S_{S2} & S_{S3} & \cdot & \cdot & \cdot & S_{SS} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_S \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ F_S \end{Bmatrix} \tag{13.72a}$$

or

$$[S]\ \{c\} = \{F\} \tag{13.72b}$$

where

$$S_{rs} = -\int_0^L W_r L(\phi_s) dx_1 \qquad r, s = 1, 2, \ldots, S \tag{13.73}$$

$$F_r = \int_0^L W_r[L(\phi_0) + J]dx_1 \qquad r = 1, 2, \ldots, S \tag{13.74}$$

The matrix $[S]$ is called the *stiffness matrix* while its terms S_{rs} are called the *stiffness* coefficients. Moreover, the matrix $\{F\}$ is called the *load vector*. Referring to relation (13.73) we see that the stiffness matrix $[S]$ may or may not be symmetric.

When the terms $S_{rs}(r, s = 1, 2, ..., S)$ and $F_r(r = 1, 2, ..., S)$ of the matrices $[S]$ and $\{F\}$ are evaluated, the system of algebraic equations (13.72) can be solved to obtain the coefficients c_s($s = 1, 2, ..., S$). These coefficients can then be substituted into relation (13.54) to give an approximate solution of the boundary value problem under consideration.

The weighted residual methods defer only in the choice of the weighting functions W_r($r = 1, 2, ..., S$). They have been applied to obtain approximate solutions of boundary value problems involving a state variable of usually one and occasionally two space coordinates. The most extensively used weighted residual method is the Gallerkin. In this method the weighting function W_r is taken as

$$W_r = \boldsymbol{\phi}_r \tag{13.75}$$

Substituting relation (13.75) into (13.73), it can be seen that the stiffness matrix in the Gallerkin method usually is not symmetric ($S_{rs} \neq S_{sr}$) and, consequently, it could be difficult to invert.

13.12 The Modified Weighted Residual (Weak) Form of One-Dimensional, Linear Boundary Value Problems

The modified residual (weak) form of the boundary value problems described in Section 13.7 is obtained by adhering to the following steps:

STEP 1 We form the weighted residual equation (13.62) or (13.67).

STEP 2 We reduce the order of the derivative of $\tilde{X}$ present in the integrand of the first integral in relation (13.62) or (13.67). Since X is a function of one space coordinate, this is accomplished by integration by parts.

STEP 3 We choose the weighting function W_{br} for a second order boundary value problem or W_{br1} and W_{br2} for a fourth order, boundary value problem, in a way that some of the boundary terms in the integral equation obtained in step 2 are eliminated. The resulting equation is *the weak form of the boundary value problem*.

In the *classical weighted residual methods*, we use a function $\boldsymbol{\phi}_0$ and shape functions $\boldsymbol{\phi}_s$ ($s = 1, 2, ..., S$) each of which is defined over the entire domain of the problem by a single expression which has derivatives of any order. Thus, in these methods it is not important to lower the order of differentiation of the state variable in the integrand of the first integral of relation (13.62) or (13.67), except when such a reduction results in a symmetric stiffness matrix $[S]$ as, for example, in the Gallerkin method. For this reason in the classical weigthed residual methods the weighted residual form (13.62) or (13.67) is usually employed.

In Chapter 15, we present the finite element method. As shown in that chapter this is a weighted residual method in which the domain of the problem is subdivided into a finite number of subdomains called *elements*. Moreover, the trial functions $\boldsymbol{\phi}_s$ are defined by different expressions over some elements. Thus, the derivatives of these functions above a certain order do not exist at some interelement boundaries. In order to be able to choose simple trial functions in the finite element method, it is desirable that the derivatives of the state variable and of the weighting functions be of as small order as possible. For this reason in the finite element method we discretize a boundary value problem, using its modified weighted residual (weak) form.

As a first example, we establish the modified weighted residual (weak) form of the boundary value problem for computing the component of translation $u_1(x_1)$ and the internal force $N(x_1)$ of the one member structure of Fig. 8.23a by adhering to the following steps:

STEP 1 We form its weighted residual equation [see relation (13.65)].

STEP 2 We reduce the order of the derivative of u_1 present in the first term of the integrand of the integral in relation (13.65). This is accomplished by integration by parts. Thus, relation (13.65) becomes

$$-\int_0^L EA\frac{dW_r}{dx_1}\left(\frac{d\tilde{u}_1}{dx_1} - H_1\right) dx_1 + \int_0^L W_r\left[p_1(x_1)\Delta(x_1 - c_1) + \sum_{n=1}^{n_1} P_1^{(n)}\delta(x_1 - a_{1n})\right] dx_1$$
$$+\left[W_r EA\left(\frac{d\tilde{u}_1}{dx_1} - H_1\right)\right]_{x_1=0}^{x_1=L} + \epsilon\left[W_{br}\left[EA\left(\frac{d\tilde{u}_1}{dx_1} - H_1\right) - P_1^L\right]\right]^{x_1=L} = 0 \qquad r = 1, 2, \ldots, S \tag{13.76}$$

STEP 3 We choose the weighting functions W_{br} as follows:

$$W_{br} = -W_r \tag{13.77}$$

If the approximate solution $\tilde{u}_1(x_1)$ does not satisfy the natural boundary condition of the one member structure of Fig. 8.23a then $\epsilon = 1$. Taking this into account and using relations (13.77) and (8.58), the last two terms of relation (13.76) reduce to

$$\left[W_r EA\left(\frac{d\tilde{u}_1}{dx_1} - H_1\right)\right]_{x_1=0}^{x_1=L} + \epsilon\left[W_{br}\left[EA\left(\frac{d\tilde{u}_1}{dx_1} - H_1\right) - P_1^L\right]\right]^{x_1=L} = -W_r(0)\tilde{N}(0) + W_r(L)P_1^L \tag{13.78}$$

If the approximate solution $\tilde{u}_1(x_1)$ satisfies the natural boundary condition of the one member structures of Fig. 8.23a, then $\epsilon = 0$ and $[EA(d\tilde{u}/dx_1 - H_1)]_{x_1=L} = P_1^L$. Consequently, relation (13.78) is still valid. Using relations (13.77) and (13.78), relations (13.76) can be rewritten as

$$-\int_0^L \frac{dW_r}{dx_1}\tilde{N}\, dx_1 + \int_0^L W_r[p_1(x_1)\Delta(x_1 - c_1) + \sum_{n=1}^{n_1} P_1^{(n)}\delta(x_1 - a_{1n})]\, dx_1$$
$$-W_r(0)\tilde{N}(0) + W_r(L)P_1^L = 0 \qquad r = 1, 2, \ldots, S \tag{13.79}$$

Where $\tilde{N}(x_1)$ is an approximation to the internal axial centroidal force. Referring to relation (8.58), it is equal to

$$\tilde{N}(x_1) = EA\left(\frac{d\tilde{u}_1}{dx_1} - H_1\right) \tag{13.80}$$

Relation (13.79) is the *modified weighted residual equation* for the one-dimensional, second order, linear boundary value problem for computing the component of translation $u_1(x_1)$ of the one-member structure of Fig. 8.23a. *The modified weighted residual (weak) form of the boundary value problem for the one-member structure of Fig. 8.23a involves the determination of the function* $\tilde{u}_1(x_1)$ *which satisfies the essential boundary* *conditions of the member and the modified weighted residual equation (13.79) for every weighting function W_r.*

Notice that the weighted residual form for the boundary value problem of computing the component of translation $u_1(x_1)$ of a structure subjected to the axial centroidal forces having only essential boundary conditions, is obtained from relation (13.79) by replacing $W_r(L)P_1^L$ by $W_r(L)\tilde{N}(L)$.

As a second example, we establish the modified weighted residual (weak) form of the boundary value problem for computing the deflection $u_2(x_1)$ of the beam of Fig. 13.6 by adhering to the following steps:

STEP 1 We form its weighted residual equation (13.69).

STEP 2 We reduce the fourth order of the derivative of $u_2(x_1)$ present in the integral of the first integral in relation (13.69) to second order. This is accomplished by integrating by parts twice this integral. Thus, referring to relations (13.69), we get

$$\int_0^L \tilde{M}_3 \frac{d^2 W_r}{d^2 x_1} dx_1 + \int_0^L W_r J_2 dx_1 + \left(W_r \frac{d\tilde{M}_3}{dx_1} \right)_{x_1=0}^{x_1=L} - \left(\frac{dW_r}{dx_1} \tilde{M}_3 \right)_{x_1=0}^{x_1=L}$$
$$+ \epsilon_1 \left[W_{br1} (\tilde{Q}_2 - P_2^L) \right]_{x_1=L} + \epsilon_2 \left[W_{br2} (\tilde{M}_3 - M_3^L) \right]_{x_1=L} = 0$$
$$r = 1, 2, \ldots, S \tag{13.81}$$

STEP 3 We chose the weighting functions W_{br1} and W_{br2} as follows:

$$W_{br1} = -W_r \qquad W_{br2} = \frac{dW_r}{dx_1} \tag{13.82}$$

If the approximate solutions $\tilde{u}_2(x_1)$ [see relation (13.54)] do not satisfy the natural boundary conditions of the beam of Fig 12.6, then $\epsilon_1 = \epsilon_2 = 1$. Taking this into account and using relations (13.82) and (8.21), the last four terms of relation (13.81) reduce to

$$-\epsilon_1 \left[W_r \left(\tilde{Q}_2 - P_2^L \right) \right]_{x_1=L} + \epsilon_2 \left[\frac{dW_r}{dx_1} \left(\tilde{M}_3 - M_3^L \right) \right]_{x_1=L} + \left(W_r \frac{d\tilde{M}_3}{dx_1} \right)_{x_1=0}^{x_1=L} - \left(\frac{dW_r}{dx_1} \tilde{M}_3 \right)_{x_1=0}^{x_1=L}$$
$$= -\left(W_r \frac{d\tilde{M}_3}{dx_1} \right)_{x_1=0} + \left(\frac{dW_r}{dx_1} \tilde{M}_3 \right)_{x_1=0} + \left(W_r P_2^L \right)_{x_1=L} - \left(\frac{dW_r}{dx_1} M_3^L \right)_{x_1=L} \tag{13.83}$$

If the approximate solution (13.54) satisfies the natural boundary conditions of the beam of Fig 12.6 then $\epsilon_1 = \epsilon_2 = 0$ and $\left(\tilde{M}_3 - M_3^L\right)_{x_1=L} = \left(\tilde{Q}_2 - P_2^L\right)_{x_3=L} = 0$. Consequently, relation (13.83) is still valid. Using relations (13.82) and (13.83), relation (13.81) can be rewritten as

$$\int_0^L \tilde{M}_3 \frac{d^2W_r}{d^2x_1}dx_1 + \int_0^L W_r\, J_2 dx_1 + \left(\frac{dW_r}{dx_1}\tilde{M}_3\right)_{x_1=0} - \left(W_r \frac{d\tilde{M}_3}{dx_1}\right)_{x_1=0}$$

$$- W_r(L)P_2^L - \left(\frac{dW_r}{dx_1}M_3^L\right)_{x_1=L} = 0 \tag{13.84}$$

$$r = 1, 2, \dots, S$$

The integral $\int_0^L J_2 W_r dx_1$ can be computed, using functions of discontinuity. For example, if

$$J_2(x_1) = -p_2\Delta(x_1 - c_1) - \sum_{n=1}^{n_3} P_2^{(n)}\delta(x_1 - a_{2n}) - \sum_{m=1}^{m_2} M_3^{(m)}\delta^I(x_1 - b_{3m}) \tag{13.85}$$

referring to relations (G.3), (G.12) and (G.17), we have

$$\int_0^L J_2(x_1)W_r dx_1 = -\int_{c_1}^L p_2(x_1)W_r(x_1)dx_1 - \sum_{n=1}^{n_2} P_2^{(n)}W_r(a_{3n}) + \sum_{m=1}^{m_3} M_3^{(m)}\frac{dW_r}{dx_1}\bigg|_{x_1=b_{3m}} \tag{13.86}$$

Relation (13.84) is the *modified weighted residual equation* for the one-dimensional, fourth order, linear boundary value for computing the deflection $u_2(x_1)$ of the beam of Fig. 13.6.

On the basis of the foregoing presentation we can deduce that a function $u_2(x_1)$, which satisfies the essential boundary conditions for a beam and on the basis of relation (9.32a) gives a function $M_3(x_1)$, which satisfies the modified weighted residual equation (13.84) for every function $W_r(x_1)$, is the actual deflection $u_3(x_1)$ of the beam. *Hence, the modified weighted residual (weak) form, of the boundary value problem under consideration involves the determination of the function* $u_2(x_1)$, *which when substituted into relation (9.32a) gives a bending moment* $M_3(x_1)$, *which satisfies the modified weighted residual equation (13.84) for every weighting function* W_r. The modified weighted residual (weak) form of a boundary value problem is equivalent to its weighted residual form and consequently, to its strong form.

Notice that if we replace $W_r(x_1)$ by $\tilde{\tilde{u}}_1(x_1)$ and take into account that $\tilde{N}(0) = -\tilde{R}_1^0$, relation (13.78) assumes the form of the principle of virtual work (13.15) for the one member structures of Fig. 8.23a. Moreover, if we replace W_r by $\tilde{\tilde{u}}_2(x_1)$ and take into account relations (9.27a) and (8.22) and noting that $\tilde{Q}_2(0) = -\tilde{R}_2^0$, relations (13.84) assume the form of the principle of virtual work (13.31) for the one member structure of Fig. 13.6. However, in the principle of virtual work $\tilde{N}(x_1)$ and $\tilde{M}_3(x_1)$ are statically admissible internal actions, while in relations (13.84) are approximate expressions for the internal actions obtained from a displacement field, which satisfies the essential boundary conditions of the problem.

In what follows we present an example.

Example 5 Using the modified Gallerkin method with $S = 2$ and $S = 4$, establish an approximate expression for the axial component of translation of the tapered (n = 0.5) one- member structure of length L = 4 m subjected to the loading shown in Fig. a. The structure has a constant thickness t and it is made from an isotropic, linearly elastic material with modulus of elasticity E.

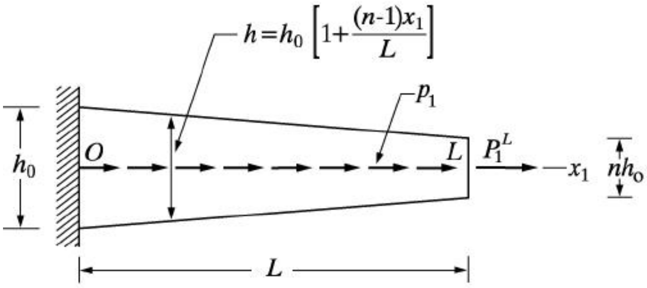


Figure a Geometry and loading of the one- member structure.

Formulation

We assume a solution of the form (13.54) and we substitute it into relation (13.79) with the Gallerkin assumption (13.75) to get for the one-member structure of Fig. a:

$$-\int_0^L EA\frac{d\phi_r}{dx_1}\left(\frac{d\phi_0}{dx_1}+\sum_{s=1}^{S} c_s\frac{d\phi_s}{dx_1}-H_1\right)dx_1+\int_0^L \phi_r\left[p_1(x_1)\Delta(x_1-c)\right.$$

$$\left.+\sum_{n=1}^{n_1} P_1^{(n)}\delta(x_1-a_{1n})\right]dx_1+(\phi_r P_1^L)_{x_1=L}=0 \tag{a}$$

$$r\ ,\ s\ =\ 1,2,\ldots,S$$

Relation (a) represents a set of S linear algebraic equations, which can be written as

$$[S]\ \{c\}\ =\ \{F\} \tag{b}$$

where

$$\{c\}^T\ =\ [c_1, c_2, \ldots, c_S] \tag{c}$$

$$S_{rs}\ =\ \int_0^L EA\frac{d\phi_r}{dx_1}\frac{d\phi_s}{dx_1}dx_1 \qquad\qquad r,\ s\ =\ 1,2,\ldots,S \tag{d}$$

and

$$F_r\ =\ -\int_0^L EA\frac{d\phi_r}{dx_1}\left(\frac{d\phi_0}{dx_1}\right)dx_1\ +\ \int_0^L p_1\phi_r dx_1+\left(\phi_r P_1^L\right)_{x_1=L} \tag{e}$$

$$r\ =\ 1,2,\ldots,S$$

Solution

Case 1 Assumed solution satisfies all the boundary conditions

STEP 1 We choose a function $\phi_0$which satisfies all the boundary conditions. Moreover, we choose trial functions $\phi_s$which satisfy the homogenous part of the boundary conditions (c). That is,

$$\phi_0 = \frac{P_1^L x_1}{EnA_o} \qquad \phi_s = \sin\left(\frac{s\pi x_1}{2L}\right) \qquad (s = 1, 3, 5, \ldots) \tag{f}$$

Substituting relations (f) into (13.54), we obtain

$$\tilde{u}_1(x_1) = \frac{P_1^L x_1}{EnA_o} + \sum_{s=1}^{S} c_s \sin\left(\frac{s\pi x_1}{2L}\right) \tag{g}$$

STEP 2 We compute the matrices $[S]$ and $\{F\}$. Substituting relations (f) into (d), we get

$$S_{rs} = \int_0^L \left(\frac{s\pi}{2L}\right)\left(\frac{r\pi}{2L}\right) EA_o\left[1 + \frac{(n-1)x_1}{L}\right]\cos\left(\frac{s\pi x_1}{2L}\right)\cos\left(\frac{r\pi x_1}{2L}\right) dx_1 \tag{h}$$

If $r = s$, noting that $(s + r)$ is even, relation (h) gives

$$S_{rr} = \left(\frac{r^2\pi^2}{4L^2}\right) EA_o\left[\frac{L}{2} + (n-1)\left\{\frac{L}{4} + \frac{L}{2r^2\pi^2}(\cos r\pi - 1)\right\}\right] \tag{i}$$

$$(r = 1, 3, 5, 7, \ldots)$$

If $r \neq s$, relation (h) gives

$$S_{rs} = \left(\frac{sr}{2L}\right) EA_o\left[\frac{(n-1)}{(s-r)^2}\left(\cos\left[\frac{(s-r)\pi}{2}\right] - 1\right) + \frac{(n-1)}{(s+r)^2}\left[\cos\left[\frac{(s+r)\pi}{2}\right] - 1\right]\right] \tag{j}$$

$$(r\,,\ s = 1, 3, 5, 7, \ldots)$$

Using $L = 4$ m, $n = 0.5$ from relations (i) and (j), we get

$$\begin{aligned}
&S_{11} = 0.262569\ EA_o && S_{31} = 0.093750\ EA_o\\
&S_{13} = 0.093750\ EA_o && S_{33} = 2.1131207\ EA_o\\
&S_{15} = 0.017361\ EA_o && S_{35} = 0.468750\ EA_o\\
&S_{17} = 0.0243056\ EA_o && S_{37} = 0.026250\ EA_o\\
& && \\
&S_{51} = 0.017361\ EA_o && S_{71} = 0.0243055\ EA_o\\
&S_{53} = 0.468750\ EA_o && S_{73} = 0.026250\ EA_o\\
&S_{55} = 5.814222\ EA_o && S_{75} = 1.093750\ EA_o\\
&S_{57} = 1.093750\ EA_o && S_{77} = 11.365880\ EA_o
\end{aligned} \tag{k}$$

Substituting relations (g) into (f), we obtain

$$F_r = -\int_0^L EA_o\left[1 + \frac{(n-1)x_1}{2L}\right]\left(\frac{P_1^L}{EnA_o}\right)\left(\frac{r\pi}{2L}\right)\cos\left(\frac{r\pi x_1}{2L}\right) dx_1$$

$$+ \int_0^L p_1 \sin\left(\frac{r\pi x_1}{2L}\right) dx_1 + P_1^L \sin\left(\frac{r\pi}{2}\right)$$

$$
\begin{aligned}
&= -P_1^L\left[\sin\left(\frac{r\pi}{2}\right) - \frac{2(n-1)}{nr\pi}\right] + \frac{2L}{r\pi}p_1 + P_1^L \sin\left(\frac{r\pi}{2}\right) \\
&= \frac{2P_1^L(n-1)}{rn\pi} + \frac{2Lp_1}{r\pi} \qquad (r = 1, 3, 5, 7, \ldots)
\end{aligned}
\tag{l}
$$

Using $L = 4$ m and $n = 0.5$ from relation (l), we obtain

$$
\begin{aligned}
F_1 &= 0.63662077\, p_1L - 0.63662077\, P_1^L \\
F_3 &= 0.21220659\, p_1L - 0.21220659\, P_1^L \\
F_5 &= 0.12732395\, p_1L - 0.12732395\, P_1^L \\
F_7 &= 0.09045680\, p_1L - 0.09094568\, P_1^L
\end{aligned}
\tag{m}
$$

STEP 3 We compute the coefficients c_s and use them to obtain approximate solutions for the boundary value problem under consideration. Substituting relations (k) and (l) into (b), for $S = 2$, we get

$$
\begin{Bmatrix} c_1 \\ c_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{13} \\ S_{31} & S_{22} \end{bmatrix}^{-1} \begin{bmatrix} F_1 \\ F_3 \end{bmatrix} = \frac{1}{EA_o} \begin{Bmatrix} 2.427568\,(pL - P_1^L) \\ -0.007277\,(pL - P_1^L) \end{Bmatrix}
\tag{n}
$$

Moreover, for $S = 4$ we obtain

$$
\begin{Bmatrix} c_1 \\ c_3 \\ c_5 \\ c_7 \end{Bmatrix} = [S]^{-1}\,[F] = \frac{1}{EA_o} \begin{Bmatrix} 2.427257\,(p_1L - P_1^L) \\ -0.010666\,(p_1L - P_1^L) \\ 0.015262\,(p_1L - P_1^L) \\ 0.001324\,(p_1L - P_1^L) \end{Bmatrix}
\tag{o}
$$

Table a Comparison of the results $L = 4$ m, $n = 0.5$.

No. of Trial Functions	$EA_o u_1(L)$
Two	$2.434845\, p_1L + 5.565155\, P_1^L$
Four	$2.454509\, p_1L + 5.548139\, P_1^L$
Exact solution	$2.454822\, p_1L + 5.545177\, P_1^L$

We substitute the parameters c_s ($s = 1, 3$) or c_s ($s = 1, 3, 5, 7$) given by relation (n) or (o), respectively, into relation (g) to obtain approximate expressions $\tilde{u}_1(x_1)$ for the axial component of translation of the structure of Fig. a. The values of $\tilde{u}_1(L)$ are tabulated in Table a together with its value obtained on the basis of the exact analysis, given (see Problem 11.4 at the end of Chapter 11) as

$$u_1(L) = \frac{L}{EA_o(1 - n)}\left[p_1 L + \left(\frac{p_1 n L}{1 - n} - P_1^L\right)\ln(n)\right] \tag{p}$$

It can be seen that as s increases, the values of the component of translation $u_1(L)$ approaches monotonically the exact solution.

Case 2 Assumed solution satisfies only the essential boundary condition

STEP 1 We choose a function ϕ_0 which satisfies the essential boundary condition (c) but not the natural (d). Moreover, we choose trial functions ϕ_s, which satisfy the homogeneous part of the essential boundary conditions (c). That is,

$$\phi_o = 0 \qquad \phi_s = x_1^s \qquad s = 1, 2, 3 \ldots \tag{q}$$

Substituting relation (l) into (13.54), we get

$$\tilde{u}_1(x_1) = \sum_{s=1}^{s} c_s x_1^s \tag{r}$$

STEP 2 We compute the terms of the matrices $[S]$ and $\{F\}$. Substituting relations (q) into (d), we obtain

$$S_{rs} = \int_0^L EA_o\left[1 + \frac{(n-1)x_1}{L}\right] rsx_1^{r+s-2}dx_1 = EA_o rsL^{r+s-1}\left[\frac{1}{r+s-1} + \frac{n-1}{r+s}\right] \tag{s}$$

From relations (r) using $L = 4$ and $n = 0.5$, we get

$$\begin{aligned}
&S_{11} = 3\,EA_o && S_{21} = 10.667\,EA_o \\
&S_{12} = 10.667\,EA_o && S_{22} = 53.333\,EA_o \\
&S_{13} = 40\,EA_o && S_{23} = 230.4\,EA_o \\
&S_{14} = 153.6\,EA_o && S_{24} = 955.733\,EA_o \\
&S_{31} = 40\,EA_o && S_{41} = 153.6\,EA_o \\
&S_{32} = 230.40\,EA_o && S_{42} = 955.733\,EA_o \\
&S_{33} = 1075.20\,EA_o && S_{43} = 4681.143\,EA_o \\
&S_{34} = 4681.143\,EA_o && S_{44} = 21065.143\,EA_o
\end{aligned} \tag{t}$$

Moreover, substituting relations (q) into (c), we obtain

$$F_r = \int_0^L p_1 x_1^r dx_1 + P_1^L L^r = \frac{L^{r+1}p_1}{r+1} + P_1^L L^r \tag{u}$$

From relation (u), using $L = 4$ m and $n = 0.5$, we have

$$\begin{aligned}
&F_1 = 2p_1 L + 4P_1^L && F_3 = 16p_1 L + 64P_1^L \\
&F_2 = 5.333p_1 L + 16P_1^L && F_4 = 51.20p_1 L + 256P_1^L
\end{aligned} \tag{v}$$

STEP 3 We compute the coefficients c_s and use them to obtain approximate solutions. Substituting relations (t) and (v) into (b) for $S = 2$, we get

$$\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \frac{1}{EA_o} \begin{bmatrix} 3.000 & 10.667 \\ 10.667 & 53.333 \end{bmatrix}^{-1} \begin{Bmatrix} 2p_1L + 4P_1^L \\ 5.333p_1L + 16P_1^L \end{Bmatrix} = \frac{1}{EA_o} \begin{bmatrix} 1.077p_1L + 0.92308P_1^L \\ -0.11543p_1L + 0.11538P_1^L \end{bmatrix} \tag{w}$$

Moreover, for $S = 4$, we obtain

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = [S]^{-1}\{F\} = \frac{1}{EA_o} \begin{bmatrix} 1.00828p_1L + 0.99409P_1^L \\ -0.07693p_1L + 0.073807P_1^L \\ 0.00266p_1L + 0.0014458P_1^L \\ -0.00202p_1L - 0.0018771P_1^L \end{bmatrix} \tag{x}$$

We substitute the values of the coefficients $c_s (s = 1, 2)$ or c_s $(s = 1, 2, 3, 4)$ given by relation (w) or (x), respectively, into relation (r) to obtain approximate expressions $\tilde{u}_1(x_1)$ for the axial component of translation of the structure of Fig. a. The values of $\tilde{u}_1(L)$ are tabulated in Table a together with those obtained on the basis of the exact analysis (p). It can be seen that as s increases the approximate values of the component of translation $u_1(L)$ approach monotonically its exact value.

We substitute the values of the coefficients c_s $(s = 1, 2)$ or c_s $(s = 1, 2, 3, 4)$ given by relation (w) or (x), respectively, in relation (p) and the resulting relation into (8.58) to obtain approximate expressions for the internal axial force in the structure of Fig. a. Thus, at the support ($x_1 = 0$), we have

$$\tilde{N}(0) = EA_0c_1 \tag{y}$$

The values of the axial force $\tilde{N}(0)$ obtained on the basis of relation (y) are tabulated in Table a together with the corresponding exact value obtained by considering the equilibrium of the structure. It can be seen that as S increases, the approximate values of the component of translation $u_1(L)$ and the axial force $N(0)$ approach monotonically the exact values. Since $N(x_1)$ is related to the first derivative of $u_1(x_1)$, as expected, for the same number of trial functions, the error in the approximate values of $N(x_1)$ is larger that

Table a Comparison of the results ($n = 0.5$, $L = 4$ m)

No. of Trial Functions	$EA_0u_1(L)$	% Error	$N(0)$	% Error
Two	$2.46148p_1L + 5.55385P_1^L$	1.45	$1.077p_1L + 0.92308P_1^L$	7.7
Four	$2.45522p_1L + 5.54728P_1^L$	0.09	$1.00828p_1L + 0.99409P_1^L$	0.828
Exact solution	$2.4548p_1L + 5.5452P_1^L$		$p_1L + P_1^L$	

in the approximate values of $u_1(x_1)$.

13.13 Total Strain Energy of Framed Structures

As shown in Section 3.11.1 [see relation (3.80)], the total strain energy of an elastic body subjected to external actions in an environment of constant temperature is equal to the work performed by the external actions and the reactions as they are applied to the body in order to bring it from its stress-free, strain-free state to its deformed state. That is,

$$(U_s)_T = \iiint_V U_s dV = W_{\text{ext actions}} + W_{\text{reactions}} \tag{13.87}$$

Consider a framed structure whose supports do not move (workless supports) consisting of NM members (straight or curved with small h/R ratio) made from elastic materials, subjected in an environment of constant temperature to external actions. As these external actions are applied, the structure deforms. We denote the displacement vector at the point of application of the force $\mathbf{P}^{(n)}$ by $\mathbf{u}^{(n)}$ and the rotation vector at the point of application of the moment $\mathbf{M}^{(m)}$ by $\boldsymbol{\theta}^{(m)}$. Thus, assuming that there are N concentrated forces and M concentrated moments acting on the structure, the work performed by the external actions as they are applied to the structure is equal to

$$(U_s)_T = \frac{1}{2}\left[\sum_{n=1}^{N} \mathbf{P}^{(n)} \cdot \mathbf{u}^{(n)} + \sum_{m=1}^{M} \mathbf{M}^{(m)} \cdot \boldsymbol{\theta}^{(m)} + \sum_{e=1}^{NM}\left[\int_0^{L_e} (\mathbf{p} \cdot \mathbf{u} + \mathbf{m} \cdot \boldsymbol{\theta}) dx_1\right]^{(e)}\right] \tag{13.88}$$

If we choose the actual displacement field of the members of the structure as the geometrically admissible displacement field and the actual internal actions in the members of the structure as the statically admissible internal actions, the principle of virtual work (13.34) gives

$$\sum_{n=1}^{N} \mathbf{P}^{(n)} \cdot \mathbf{u}^{(n)} + \sum_{m=1}^{M} \mathbf{M}^{(m)} \cdot \boldsymbol{\theta}^{(m)} + \sum_{e=1}^{NM}\left[\int_0^{L} (\mathbf{p} \cdot \mathbf{u} + \mathbf{m} \cdot \boldsymbol{\theta}) dx_1\right]^{(e)}$$

$$= \sum_{e=1}^{NM}\left[\int_0^{L_e}\left(N\frac{du_1}{dx_1} + M_1\frac{d\theta_1}{dx_1} + M_2\frac{d\theta_2}{dx_1} + M_3\frac{d\theta_3}{dx_1}\right) dx_1\right]^{(e)}$$

Comparing relation (13.88) with the above, we obtain

$$(U_s)_T = \iiint_V U_s dV = \frac{1}{2}\sum_{e=1}^{NM}\left[\int_0^{L_e}\left(N\frac{du_1}{dx_1} + M_1\frac{d\theta_1}{dx_1} + M_2\frac{d\theta_2}{dx_1} + M_3\frac{d\theta_3}{dx_1}\right) dx_1\right]^{(e)} \tag{13.89}$$

If the members of the structure are made from isotropic, linearly elastic materials taking into account that $H_1 = H_2 = H_3 = 0$, and using relations (13.14a), (6.32) and (9.8) into (13.89), we obtain the following expression for the total strain energy of a structure consisting of NM members, made from isotropic, linearly elastic materials:

$$(U_s)_T = \iiint_V U_s dV = \frac{1}{2}\sum_{e=1}^{NM}\left[\int_0^{L_e}\left(\frac{N^2}{EA} + \frac{M_1^2}{GR_C} + \frac{M_2^2}{EI_2} + \frac{M_3^2}{EI_3}\right)dx_1'\right]^{(e)} \tag{13.90}$$

13.14 Castigliano's Second Theorem

In this section, we present Castigliano's second theorem and we prove it for framed structures made from isotropic, linearly elastic materials, using the principle of virtual work. Moreover, we employ this theorem in computing the components of translation and rotation of points of framed structures.

Theorem Consider a body made from a linearly elastic material, subjected to external actions (*N* concentrated forces, *M* concentrated moments, and distributed forces and moments) in an environment of constant temperature and assume that its supports do not move (workless supports). Moreover, assume that at point *A* of the body, a concentrated force $P_n^{(A)}$ acts in the direction specified by the unit vector $\mathbf{i}_n$, while at point *B* of the body a concentrated moment $M_m^{(B)}$ whose vector acts in the direction specified by the unit vector $\mathbf{i}_m$. The total strain energy $(U_s)_T$ of the body (see Section 13.13) may be considered a function of the external forces and moments and satisfies the following relations:

$$u_n^{(A)} = \frac{\partial (U_s)_T}{\partial P_n^{(A)}} \tag{13.91}$$

$$\theta_m^{(B)} = \frac{\partial (U_s)_T}{\partial M_m^{(B)}} \tag{13.92}$$

where $u_n^{(A)}$ is the component of translation of point *A* in the direction specified by the unit vector $\mathbf{i}_n$; $\theta_m^{(B)}$ is the component of rotation at point *B* in the direction specified by the unit vector $\mathbf{i}_m$. Relations (13.91) and (13.92) were established by Castigliano in 1873 and are referred to as *Castigliano's second theorem*.

13.14.1 Proof of Castigliano's Second Theorem

As mentioned previously Castigliano's second theorem is valid for bodies made from linearly elastic materials. In this subsection, however, we prove it only for framed structures whose members are made from isotropic, linearly elastic materials, using the principle of virtual work.

Consider a framed structure consisting of *NM* members made from isotropic, linearly elastic materials subjected to external actions in an environment of constant temperature and assume that its supports do not move (workless supports). We denote the corresponding internal actions in the members of the structure by N, M_1, M_2, M_3 and the resulting components of displacement by u_1, u_2, u_3, θ_1, θ_2, θ_3. Referring to relation (13.89), we see that the total strain energy of this structure is equal to

$$(U_s)_T = \frac{1}{2}\sum_{e=1}^{NM}\left[\int_0^L \left(N\frac{du_1}{dx_1} + M_1\frac{d\theta_1}{dx_1} + M_2\frac{d\theta_2}{dx_1} + M_3\frac{d\theta_3}{dx_1}\right) dx_1\right]^{(e)} \tag{13.93}$$

If we choose the actual components of displacement (u_1, u_2, u_3, $\theta_1, \theta_2, \theta_3$) of the members of this structure as the geometrically admissible displacement field and the actual internal actions (N, M_1, M_2, M_3) in the members of this structure as the statically admissible distribution of internal actions, the principle of virtual work (13.34) gives

$$\sum_{n=1}^{n_1} \mathbf{P}^{(n)} \cdot \mathbf{u}^{(n)} + \sum_{m=1}^{M} \mathbf{M}^{(m)} \cdot \boldsymbol{\theta}^{(m)} + \sum_{e=1}^{NM}\left[\int_0^L (\mathbf{p}\cdot\mathbf{u} + \mathbf{m}\cdot\boldsymbol{\theta})dx_1\right]^{(e)}$$

$$= \sum_{e=1}^{NM}\left[\int_0^L \left(N\frac{du_1}{dx_1} + M_1\frac{d\theta_1}{dx_1} + M_2\frac{d\theta_2}{dx_1} + M_3\frac{d\theta_3}{dx_1}\right) dx_1\right]^{(e)} \tag{13.94}$$

Consider the same framed structure subjected to a slightly different loading consisting of the external actions of the previous loading except that the concentrated force $P_n^{(A)}$ has increased to $P_n^{(A)} + \hat{d}P_n^{(A)}$. As a result of this loading, the components of internal actions are $N + \hat{d}N$, $M_1 + \hat{d}M_1$, $M_2 + \hat{d}M_2$, $M_3 + \hat{d}M_3$. Moreover, the components of displacement of the points of the structure are $u_1 + \hat{d}u_1$, $u_2 + \hat{d}u_2$, $u_3 + \hat{d}u_3$, $\theta_1 + \hat{d}\theta_1$, $\theta_2 + \hat{d}\theta_2$, $\theta_3 + \hat{d}\theta_3$. The total strain energy of the structure subjected to this second loading is equal to

$$(U_s)_T + \hat{d}(U_s)_T = \frac{1}{2}\sum_{e=1}^{NM}\left\{\int_0^{L_e}\left[(N + \hat{d}N)\left[\frac{du_1}{dx_1} + \hat{d}\left(\frac{du_1}{dx_1}\right)\right] + (M_1 + \hat{d}M_1)\left[\frac{d\theta_1}{dx_1} + \hat{d}\left(\frac{d\theta_1}{dx_1}\right)\right]\right.\right.$$

$$\left.\left. + (M_2 + \hat{d}M_2)\left[\frac{d\theta_2}{dx_1} + \hat{d}\left(\frac{d\theta_2}{dx_1}\right)\right] + (M_3 + \hat{d}M_3)\left[\frac{d\theta_3}{dx_1} + \hat{d}\left(\frac{d\theta_3}{dx_1}\right)\right]\right]dx_1\right\}^{(e)}$$

(13.95)

If we choose the components of displacements (u_1, u_2, u_3, $\theta_1, \theta_2, \theta_3$) of the structure subjected to the first loading as the geometrically admissible displacement field and the internal actions $N + \hat{d}N$, $M_1 + \hat{d}M_1$, $M_2 + \hat{d}M_2$, $M_3 + \hat{d}M_3$ of the structure subjected to the second loading as the statically admissible internal actions, the principle of virtual work (13.34) gives

$$\sum_{n=1}^{N=1} \mathbf{P}^{(n)} \cdot \mathbf{u}^{(n)} + \hat{d}P_n^{(A)}u_n^{(A)} + \sum_{m=1}^{M} \mathbf{M}^{(m)} \cdot \boldsymbol{\theta}^{(m)} + \sum_{e=1}^{NM}\left[\int_0^{L_e} (\mathbf{p}\cdot\mathbf{u} + \mathbf{m}\cdot\boldsymbol{\theta})dx_1\right]^{(e)}$$

$$= \sum_{e=1}^{NM}\left\{\int_0^{L_e}\left[(N + \hat{d}N)\frac{du_1}{dx_1} + (M_1 + \hat{d}M_1)\frac{d\theta}{dx_1} + (M_2 + \hat{d}M_2)\frac{d\theta_2}{dx_1} + (M_3 + \hat{d}M_3)\frac{d\theta_3}{dx_1}\right]dx_1\right\}^{(e)}$$

(13.96)

Substracting relations (13.93) from (13.95) and disregarding infinitesimal of higher order, we get

$$\hat{d}(U_s)_T = \frac{1}{2}\sum_{e=1}^{NM}\left\{\int_0^{L_e}\left[\hat{d}N\frac{du_1}{dx_1} + N\hat{d}\left(\frac{du_1}{dx_1}\right) + \hat{d}M_1\frac{d\theta_1}{dx_1} + M_1\hat{d}\left(\frac{d\theta_1}{dx_1}\right)\right.\right.$$
$$\left.\left. + \hat{d}M_2\frac{d\theta_2}{dx_1} + M_2\hat{d}\left(\frac{d\theta_2}{dx_1}\right) + \hat{d}M_3\frac{d\theta_3}{dx_1} + M_3\hat{d}\left(\frac{d\theta_3}{dx_1}\right)\right]dx_1\right\}^{(e)} \tag{13.97}$$

Noting that $N = EA\, du_1/dx_1$, $M_1 = GR_C\, d\,\theta_1/dx_1$, $M_2 = EI_2\, d\,\theta_2/dx_1$ and $M_3 = EI_3\, d\,\theta_3/dx_1$, relation (13.97) may be rewritten as

$$\hat{d}(U_s)_T = \sum_{e=1}^{NM}\left[\int_0^{L_e}\left(\hat{d}N\frac{du_1}{dx_1} + \hat{d}M_1\frac{d\theta_1}{dx_1} + \hat{d}M_2\frac{d\theta_2}{dx_1} + \hat{d}M_3\frac{d\theta_3}{dx_1}\right)dx_1\right]^{(e)} \tag{13.98}$$

Moreover, substracting relation (13.94) from (13.96), we have

$$dP_n^{(A)}u_n^{(A)} = \sum_{e=1}^{NM}\left[\int_0^{L_e}\left(\hat{d}N\frac{du_1}{dx_1} + \hat{d}M_1\frac{d\theta_1}{dx_1} + \hat{d}M_2\frac{d\theta_2}{dx_1} + \hat{d}M_3\frac{d\theta_3}{dx_1}\right)dx_1\right]^{(e)} \tag{13.99}$$

Comparing relations (13.98) and (13.99), we get

$$\hat{d}P_n^{(A)}u_n^{(A)} = \hat{d}(U_s)_T = \frac{\partial(U_s)_T}{\partial P_n^{(A)}}\hat{d}P_n^{(A)} \tag{13.100}$$

Thus,

$$u_n^{(A)} = \frac{\partial(U_s)_T}{\partial P^{(A)}} \tag{13.101}$$

Similarly we can prove the validity of relation (13.92).

13.14.2 Application of Castigliano's Second Theorem in Computing Components of Displacements of Points of Framed Structures

Using relation (13.90), for a framed structure made from isotropic, linearly elastic materials, we may rewrite relations (13.91) and (13.92) as

$$u_n^{(A)} = \frac{\partial(U_s)_T}{\partial P_n^{(A)}} = \sum_{e=1}^{NM}\left[\int_0^{L_e}\left(\frac{N}{EA}\frac{\partial N}{\partial P_n^{(A)}} + \frac{M_1}{GR_C}\frac{\partial M_1}{\partial P_n^{(A)}} + \frac{M_2}{EI_2}\frac{\partial M_2}{\partial P_{(A)}^{n}} + \frac{M_3}{EI_3}\frac{\partial M_3}{\partial P_n^{(A)}}\right)dx_1\right]^{(e)} \tag{13.102}$$

$$\theta_m^{(B)}\frac{\partial(U_s)_T}{\partial M_m^{(B)}} = \sum_{e=1}^{NM}\left[\int_0^{L_e}\left(\frac{N}{EA}\frac{\partial N}{\partial M_m^{(B)}} + \frac{M_1}{GR_C}\frac{\partial M_1}{\partial M_m^{(B)}} + \frac{M_2}{EI_2}\frac{\partial M_2}{\partial M_m^{(B)}} + \frac{M_3}{EI_3}\frac{\partial M_3}{\partial M_m^{(B)}}\right)dx_1\right]^{(e)} \tag{13.103}$$

Relations (13.102) or (13.103) can be used to compute a component of translation or a component of rotation at a point *A* or *B* of a framed structure in the direction of the unit vector $\mathbf{i}_n$ or $\mathbf{i}_m$, respectively, by adhering to the following steps:

STEP 1 The structure is considered subjected to an auxiliary loading consisting of
1. The given actions, except the components in the direction of the unit vectors $\mathbf{i}_n$, $\mathbf{i}_m$ of the force (moment) acting at point *A(B)* if the given actions include a force (moment) having such a component
2. An unknown force $P_n^{(A)}$ [moment $M_m^{(B)}$] acting at point *A(B)* in the direction of the unit vector $\mathbf{i}_n$, [$\mathbf{i}_m$]

STEP 2 The internal actions N, M_1, M_2, M_3 in the members of the structure subjected to the auxiliary loading described in step 1 are established as functions of the force $P_n^{(A)}$ [moment $M_m^{(B)}$].

STEP 3 The internal actions are differentiated with respect to $P_n^{(A)}$ [$M_m^{(B)}$] and the results are substituted in relation (13.102) [(13.103)]. This gives the translation $u_n^{(A)}$ [rotation $\theta_m^{(B)}$] of the structure subjected to the auxiliary loading described in step 1.

STEP 4 The translation $u_n^{(A)}$ [rotation $\theta_m^{(B)}$] of the structure subjected to the given actions is obtained by setting $P_n^{(A)}$ [$M_m^{(B)}$] equal to the component in the direction of the unit vector $\mathbf{i}_n$, [$\mathbf{i}_m$] of the given force (moment) acting at point *A(B)*. If there is not a force (moment) acting at point *A(B)* in the direction of the unit vector $\mathbf{i}_n$, [$\mathbf{i}_m$], the force $P_n^{(A)}$ [moment $M_m^{(B)}$] is set equal to zero.

In what follows, we illustrate with two examples the use of Castigliano's second theorem in computing components of displacement of points of framed structures made of isotropic, linearly elastic materials.

Example 6 Consider the truss shown in Fig. a. The cross-sectional area of its top and bottom chords is equal to $2.4(10^3)$ mm^2, while the cross-sectional area of its diagonal and vertical members is equal to $1.6(10^3)$ mm^2. The members of the truss are made from an isotropic, linearly elastic material (E = 200 GPa). Compute the vertical component of translation (deflection) of joint 7.

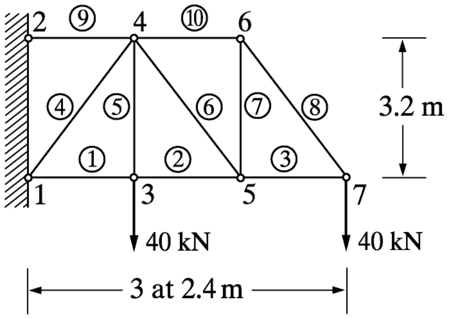


Figure a Geometry and loading of the truss.

Solution In order to compute the vertical component of translation of joint 7, the vertical force of 40 kN acting at joint 7 is replaced by a vertical force *P*. The reactions of the truss are computed by referring to Fig. b and considering its equilibrium. Moreover, the

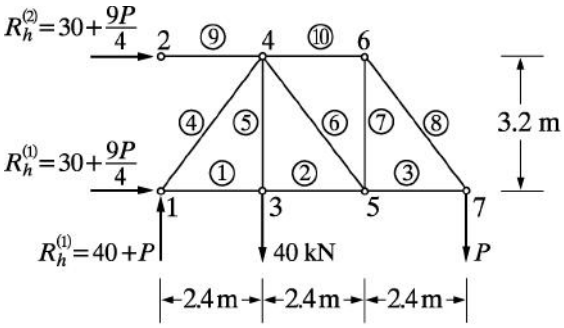


Figure b Free-body diagram of the truss subjected to the auxiliary loading for computing the deflection of joint 7.

internal forces in the members of the truss are computed by considering the equilibrium of its joints.

Equilibrium of joint 1 (see Fig. c)

$$\sum \bar{F}_V = 0 \qquad N^{(4)} = -\frac{5}{4}(40 + P)$$

$$\sum \bar{F}_h = 0 \qquad N^{(1)} + \frac{3}{5}N^{(4)} + 30 + \frac{9}{4}P = 0$$

$$N^{(1)} = -\frac{3}{2}P$$

Equilibrium of joint 3 (see Fig. d)

$$\sum F_V = 0 \qquad N^{(5)} = 40 \text{ kN}$$

$$\sum F_h = 0 \qquad N^{(2)} = -\frac{3}{2}P$$

Equilibrium of joint 4 (see Fig. e)

$$\sum F_V = 0 \qquad \frac{4}{5}N^{(6)} - (40 + P) + 40 = 0 \qquad \text{or} \qquad N^{(6)} = \frac{5}{4}P$$

$$\sum F_h = 0 \qquad \frac{3}{4}P + \frac{3}{4}P = \frac{3}{4}(40 + P) - N^{(9)} = 0 \qquad \text{or} \qquad N^{(9)} = 30 + \frac{9}{4}P$$

Equilibrium of joint 7 (see Fig. f)

$$F_V = 0 \qquad N^{(8)} = \frac{5}{4}P$$

$$F_h = 0 \qquad N^{(3)} = -\frac{3}{4}P$$

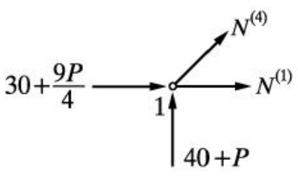


Figure c Free-body diagram of joint 1.

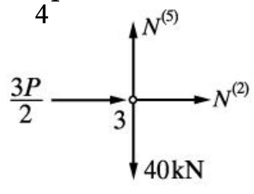


Figure d Free-body diagram of joint 3.

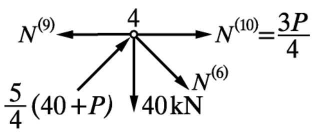


Figure e Free-body diagram of joint 4.

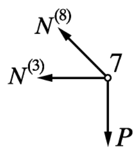


Figure f Free-body diagram of joint 7.

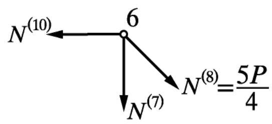


Figure g Free-body diagram of joint 6.

Equilibrium of joint 6 (see Fig. g)

$$\sum F_V = 0 \qquad N^{(7)} = -P$$

$$\sum F_h = 0 \qquad N^{(10)} = \frac{3}{4}P$$

Taking into account that the internal force N is constant in each member of the truss, we find that Castigliano's second theorem (13.102) reduces to

$$u_V^{(7)} = \frac{\partial (U_s)_T}{\partial P} = \sum_{e=1}^{NM} \left(\frac{NL}{EA}\frac{\partial N}{\partial P} \right)^{(e)} \tag{a}$$

The deflection of joint 7 of the truss, subjected to the given loading, may be obtained by substituting in relation (a) the expressions for the internal forces established above, carrying out the differentiation, and setting $P = 40$ kN. This is done in Table a; referring to this table, the vertical component of translation of joint 7 of the truss is

$$u_V^{(7)} = 6 \text{ mm} \downarrow$$

Table a Computation of the deflecti.on $u_V^{(7)}$.

Member	$N\big\vert_{P=40\,kN}$	A, 10^3mm^2	L, mm	$\frac{NL}{EA}\big\vert_{P=40\,kN}$	$\frac{\partial N}{\partial P}\big\vert_{P=40\,kN}$	$\frac{NL}{EA}\frac{\partial N}{\partial P}\big\vert_{P=40\,kN}$
1	-60	2.4	2400	-0.30	-1.5	0.450
2	-60	2.4	2400	-0.30	-1.5	0.450
3	-60	2.4	2400	-0.15	-0.75	0.1125
4	-100	1.6	4000	-1.25	-1.25	1.5625
5	40	1.6	3200	0.40	0	0.000
6	50	1.6	4000	0.625	1.25	0.78125
7	-40	1.6	3200	-0.40	-1.00	0.400
8	50	1.6	4000	0.625	1.25	0.78125
9	120	2.4	2400	0.60	2.25	0.350
10	30	2.4	2400	0.15	0.75	0.1125
Total						6.000

Example 7 Consider a cantilever beam made from an isotropic, linearly elastic material subjected to a uniform load over part of its span, as shown in Fig. a. Compute the deflection and the rotation of the unsupported end of this beam, using Castigliano's second theorem.

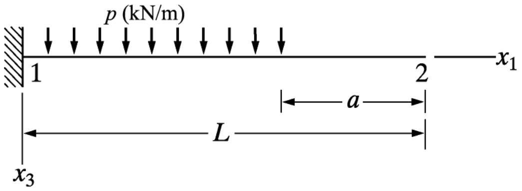


Figure a Geometry and loading of the beam.

Solution

1. *Computation of the deflection of the unsupported end of the beam*

In this case, we consider the beam subjected to the auxiliary loading shown in Fig. b, consisting of the actual load of the beam and a concentrated force P acting at point 2. The bending moment in the beam is given as

$$M_2 = -Px_1' \qquad 0 \leq x_1' \leq a$$

$$M_2 = -Px_1' - \frac{p(x_1' - a)^2}{2} \qquad a \leq x_1' \leq L \tag{a}$$

The deflection $u_v^{(2)}$ of point 2 of the beam of Fig. b is obtained from relation (13.102) which, in this case, reduces to

$$u_V^{(2)} = \frac{\partial (U_s)_T}{\partial P} = \int_0^L \frac{M_2}{EI_2}\frac{\partial M_2}{\partial P} dx_1' \tag{b}$$

Substituting relations (a) into (b), we get

$$u_V^{(2)} = \int_0^a \frac{P(x_1')^2}{EI_2} dx_1' + \int_a^L \frac{\left[Px_1' + p\dfrac{(x_1' - a)^2}{2}\right] x_1' dx_1'}{EI_2} \tag{c}$$

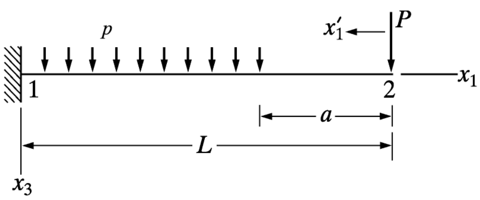


Figure b Beam subjected to the auxiliary loading for the computation of the displacement of point 2.

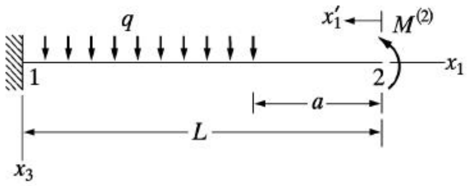


Figure c Beam subjected to the auxiliary loading for the computation of the rotation at point 2.

The deflection of point 2 of the beam of Fig. a may be obtained from relation (c) by setting $P = 0$ and integrating. Thus,

$$u_V^{(2)} = \int_a^L \frac{p(x_1^{/} - a)^2 x_1^{/} dx_1}{2EI_2} = \frac{p}{24EI_2}(3L^4 - 8aL^3 + 8a^2L^2 - a^4) \tag{d}$$

In case $a = 0$, the above relation yields

$$u_V^{(2)} = \frac{pL^4}{8EI_2} \tag{e}$$

2. *Computation of the rotation of the unsupported end of the beam*

In this case, we consider the beam subjected to the auxiliary loading of Fig. c, consisting of the actual load of the beam and a concentrated moment acting at point 2. The bending moment in this beam is given as

$$\begin{aligned} M_2 &= M_2^{(2)} && 0 \leq x_1^{/} \leq a \\ M_2 &= M_2^{(2)} - \frac{p(x_1^{/} - a)^2}{2} && a \leq x_1^{/} \leq L \end{aligned} \tag{f}$$

Substituting relations (f) into (13.103), the rotation of point 2 of the auxiliary beam of Fig. c is obtained as

$$\theta_2^{(2)} = \frac{\partial (U_s)_T}{\partial M_2^{(3)}} = \int_0^a \frac{M_2^{(2)} dx_1^{/}}{EI_2} + \int_a^L \frac{M_2^{(2)} - p(x_1^{/} - a)^2/2}{EI_2} dx_1^{/} \tag{g}$$

The rotation of point 2 of the beam of Fig. a may be obtained from relation (g) by setting $M_2^{(2)} = 0$ and integrating. Thus,

$$\theta_2^{(2)} = -\int_0^L \frac{p(x_1^{/} - a)^2}{2EI_2} dx_1^{/} = -\frac{p}{6EI_2}(L - a)^3 \tag{h}$$

The minus sign indicates that the rotation $\theta^{(2)}$ is opposite to the moment $M^{(2)}$ shown in Fig. c; that is, $\theta^{(2)}$ is clockwise.

13.15 Betti-Maxwell Reciprocal Theorem

Consider a body made from a linearly elastic material subjected to a set of external actions A in an environment of constant temperature. Consider the same body subjected to another set of external actions B in the same environment of constant temperature. We denote the displacement field of the body resulting from the set of external actions A and B by $\mathbf{u}^A$ and $\mathbf{u}^B$, respectively. Betti's reciprocal theorem states that *the work performed by the system of external actions A on the displacement field $\mathbf{u}^B$ is equal to the work performed by the system of external action B on the displacement field $\mathbf{u}^A$*.

For a framed structure, Betti's reciprocal theorem can be stated as follows:

Theorem Consider a framed structure consisting of NM members made from a linearly elastic material and subjected to loading in an environment of constant temperature. The loading consists of N concentrated forces $\mathbf{P}^{(n)}(n = 1, 2, ..., N)$, M concentrated moments $\mathbf{M}^{(m)}(m = 1, 2, ..., M)$, a distributed force $\mathbf{p}$, and a distributed moment $\mathbf{m}$. We denote the resulting distributions of the translation and the rotation vectors by $\mathbf{u}$ and θ, respectively.

Consider the same structure subjected to another loading in the same environment of constant temperature. This loading consists of $\hat{N}$ concentrated forces $\hat{P}^{(n)}$, $\hat{M}$ concentrated moments $\hat{M}^{(m)}$, a distributed force $\hat{p}$ and a distributed moment $\hat{m}$. We denote the resulting distribution of the translation and rotation vectors by $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\theta}}$, respectively.

According to the Betti-Maxwell reciprocal theorem the work performed by the loading $\mathbf{P}^{(n)}$, $\mathbf{M}^{(m)}$, $\mathbf{p}$ and $\mathbf{m}$ on the displacements $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\theta}}$ is equal to the work performed by the loading $\hat{\mathbf{P}}^{(n)}$, $\hat{\mathbf{M}}^{(n)}$, $\hat{\mathbf{p}}$ and $\hat{\mathbf{m}}$ on the displacements $\mathbf{u}$ and θ. That is,

$$\begin{aligned}
&\sum_{n=1}^{N} \mathbf{P}^{(n)} \cdot \hat{\mathbf{u}}^{(n)} + \sum_{m=1}^{M} \mathbf{M}^{(m)} \cdot \hat{\boldsymbol{\theta}}^{(m)} + \sum_{e=1}^{NM} \left[\int_0^L (\mathbf{p} \cdot \hat{\mathbf{u}} + \mathbf{m} \cdot \hat{\boldsymbol{\theta}}) dx_1 \right]^{(e)} \\
&= \sum_{n=1}^{N} \hat{\mathbf{P}}^{(n)} \cdot \mathbf{u}^{(n)} + \sum_{m=1}^{M} \hat{\mathbf{M}}^{(m)} \cdot \boldsymbol{\theta}^{(m)} + \sum_{e=1}^{NM} \left[\int_0^L (\hat{\mathbf{p}} \cdot \mathbf{u} + \hat{\mathbf{m}} \cdot \boldsymbol{\theta}) dx_1 \right]^{(e)}
\end{aligned} \tag{13.104}$$

Proof As previously mentioned, Betti's reciprocal theorem is valid for framed structures made from linearly elastic materials. However, in this text, we limit our attention to framed structures made from isotropic, linearly elastic materials. For this reason, we prove Betti's reciprocal theorem only for framed structures made from such materials.

To prove this theorem, we employ the principle of virtual work for the structure under consideration subjected to the external actions $\mathbf{P}^{(n)}$ $(n = 1, 2, ..., N)$, $\mathbf{M}^{(m)}$ $(m = 1, 2, ..., M)$, $\mathbf{p}$ and $\mathbf{m}$. We choose the resulting distribution of internal actions N, M_1, M_2, M_3 in the members of the structure as the statically admissible distribution of internal actions. Moreover, we choose the displacement fields $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\theta}}$ of the structure subjected to the external actions $\hat{\mathbf{P}}^{(n)}$, $\hat{\mathbf{M}}^{(m)}$, $\hat{\mathbf{p}}$ and $\hat{\mathbf{m}}$ as the geometrically admissible displacement fields. Thus, the principle of virtual work [see relation (13.34)] yields

$$\begin{aligned}
&\sum_{n=1}^{N} \mathbf{P}^{(n)} \cdot \hat{\mathbf{u}}^{(n)} + \sum_{m=1}^{M} \mathbf{M}^{(m)} \cdot \hat{\boldsymbol{\theta}}^{(m)} + \sum_{e=1}^{NM} \left[\int_0^{L_e} (\mathbf{p} \cdot \hat{\mathbf{u}} + \mathbf{m} \cdot \hat{\boldsymbol{\theta}}) dx_1 \right]^{(e)} \\
&= \sum_{e=1}^{NM} \left[\int_0^{L_e} \left(N \frac{d\hat{u}_1}{dx_1} + M_1 \frac{d\hat{\theta}_1}{dx_1} + M_2 \frac{d\hat{\theta}_2}{dx_1} + M_3 \frac{d\hat{\theta}_3}{dx_1} \right) dx_1 \right]^{(e)}
\end{aligned} \tag{13.105}$$

Moreover, we employ the principle to virtual work for the structure under consideration subjected to the external actions $\hat{\mathbf{P}}^{(n)}(n = 1, 2, ..., N)$, $\hat{\mathbf{M}}^{(m)}(m = 1, 2, ..., M)$, $\hat{\mathbf{p}}$ and $\hat{\mathbf{m}}$. We choose the resulting distribution of internal actions $\hat{N}$, $\hat{M}_1$, $\hat{M}_2$, $\hat{M}_3$ as the statically admissible distribution of internal actions. Moreover, we choose the displacement field $\mathbf{u}$ and θ of the structure subjected to the external actions $\mathbf{P}^{(n)}$, $\mathbf{M}^{(m)}$, $\mathbf{p}$ and $\mathbf{m}$ as the

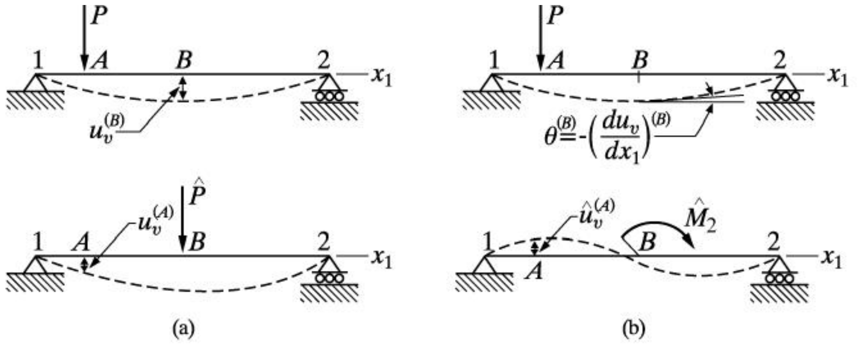


Figure 13.7 Application of Betti-Maxwell's reciprocal theorem.

geometrically admissible displacement fields. Thus, the principle of virtual work [see relation (13.34)] yields

$$\sum_{n=1}^{N} \hat{\mathbf{P}}^{(n)} \cdot \mathbf{u}^{(n)} + \sum_{m=1}^{M} \hat{\mathbf{M}}^{(m)} \cdot \boldsymbol{\theta}^{(m)} + \sum_{e=1}^{NM} \left[\int_0^{L_e} (\hat{\mathbf{q}} \cdot \mathbf{u} + \hat{\mathbf{m}} \cdot \boldsymbol{\theta}) dx_1 \right]^{(e)}$$
$$= \sum_{e=1}^{NM} \left[\int_0^{L_e} \left(\hat{N}\frac{du_1}{dx_1} + \hat{M}_1\frac{d\theta_1}{dx_1} + \hat{M}_2\frac{d\theta_2}{dx_1} + \hat{M}_3\frac{d\theta_3}{dx_1} \right) dx_1 \right]^{(e)} \tag{13.106}$$

Using the relations between the components of internal actions and the corresponding components of displacements for structures made from isotropic linearly elastic materials (i.e., $N/EA = du_1/dx_1$, $M_1/GR_C = d\theta_1/dx_1$, etc.) [see relations (8.9), (6.32) and (9.8)], we can convert the integral on the right-hand side of relation (13.106) to that on the right hand side of relation (13.105). Thus, the left-hand sides of relations (13.105) and (13.106) must be equal. Hence, relation (13.104) is valid, and concomitantly Betti's reciprocal theorem has been proven.

We illustrate Betti's theorem by applying it to the beam loaded as shown in Fig. 13.7a. That is,

$$P\hat{u}_V^{(A)} = \hat{P}u_V^{(B)} \tag{13.107}$$

This relation is referred to as Maxwell's theorem. If $P = \hat{P}$, relation (13.107) reduces to

$$\hat{u}_V^{(A)} = u_V^{(B)} \tag{13.108}$$

Betti's theorem may be applied to the beam loaded as shown in Fig. 13.7b. That is,

$$P\hat{u}_V^{(A)} = \hat{M}_2\boldsymbol{\theta}_2^{(B)} = -\hat{M}_2\left(\frac{du_V}{dx_1}\right)^{(B)} \tag{13.109}$$

13.16 Proof That the Center of Twist of a Cross Section Coincides with Its Shear Center

Consider a cantilever beam subjected to a transverse concentrated force P_3 at its unsupported end ($x_1 = L$) whose line of action passes through the shear center of its end cross section. Under this loading the angle of twist of the cross sections of the beam is equal to zero ($\theta_1^* = 0$). Consider the same beam subjected to a concentrated torsional moment M_1 at its unsupported end. We denote by u_3^M the component of translation, in the direction of the x_3 axis of the shear center of the cross section at the unsupported end of the beam. On the basis of Betti's theorem (see Section 13.15), we have

$$0 = M_1\theta_1^* = P_3 u_3^M$$

Thus,

$$u_3^M = 0$$

Consequently, the shear center of the cross sections of a beam does not translate when the beam is subjected to a torsional moment and, therefore, it coincides with their center of twist.

13.17 The Variational Form of the Boundary Value Problem for Computing the Components of Displacement of a Deformable Body — Theorem of Stationary Total Potential Energy

Consider a variation $\delta\hat{u}_i (i = 1, 2, 3)$ of the solution $\hat{u}_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ of the boundary value problem for computing the components of displacement of a deformable body defined by

$$\delta\hat{u}_i = \hat{u}_i^*(x_1, x_2, x_3) - \hat{u}_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3) \tag{13.110}$$

where the family of the sets of functions $\hat{u}_i^*(x_1, x_2, x_3)(i = 1, 2, 3)$ satisfies the essential (displacement) boundary conditions of the problem on the portion $S - S_t$ of the surface S of the body where such conditions are specified. Consequently

$$\delta\hat{u}_i = 0 \quad (i = 1, 2, 3) \quad \text{on the portion } S - S_t \text{ of } S \tag{13.111}$$

The variation of the components of strain $\delta e_{ij}(i, j = 1, 2, 3)$ is defined as

$$\delta e_{ij} = e_{ij}^*(x_1, x_2, x_3) - e_{i,j}(x_1, x_2, x_3) \quad (i,j = 1, 2, 3) \tag{13.112}$$

where $e_{ij}^*(x_1, x_2, x_3)$ or $e_{ij}(x_1, x_2, x_3)$ are obtained from $\hat{u}_i^*(x_1, x_2, x_3)$ or $\hat{u}_i(x_1, x_2, x_3)$, respectively, on the basis of relations (2.16).

Choosing $\tilde{\tilde{u}}_i = \delta\hat{u}_i$ the principle of virtual work (13.6a) with (13.9) becomes

$$\iiint_V \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ij}\delta e_{ij}\right) dV = \iiint_V \sum_{i=1}^{3} B_i \delta\hat{u}_i dV + \iint_{S_t} \sum_{i=1}^{3} \overset{n}{T}{}_i^s \delta\hat{u}_i dS = \delta W \tag{13.113}$$

where the right side of relation (13.113) represents the work done by the components of body force and surface traction acting on the body due to the variation $\delta\hat{u}_i(i=1,2,3)$ of its components of displacement $\hat{u}_i(x_1, x_2, x_3)(i = 1, 2, 3)$. In obtaining relation (13.113) use has been made of the fact that $\delta\hat{u}_i(i = 1, 2, 3)$ vanishes on the points on the portion

$S - S_t$ of the surface of the body where the components of displacement are specified.

In what follows we consider a body in an environment of constant temperature T_0 made from an elastic material. For such a body, as discussed in Section 2.11, there exists a positive definite function of the nine components of strain, the strain energy density, $U_s(x_1, x_2, x_3\, e_{ij})$, which satisfies the following relation:

$$\tau_{ij} = \frac{\partial U_s}{\partial e_{ij}} \tag{13.114}$$

Substituting relation (13.114) into (13.113), we obtain

$$\iiint_V \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial U_s}{\partial e_{ij}} \delta e_{ij} \right) dV = \iiint_V \sum_{i=1}^{3} B_i \delta \hat{u}_i dV + \iint_{S_i} \sum_{i=1}^{3} \overset{n}{T}{}_i^s \delta \hat{u}_i dS \tag{13.115}$$

The variation of the strain–energy density $U_s\,(x_1, x_2, x_3, e_{ij})$ is defined as

$$\delta U_s = U_s^*(x_1, x_2, x_2, e_{ij} + \delta e_{ij}) - U_s(x_1, x_2, x_3, e_{ij}) \tag{13.116}$$

Expanding $U_s\,(x_1, x_2, x_3, e_{ij} + \delta e_{ij})$ into a Taylor series about e_{ij}, relation (13.116) can be written as

$$\delta U_s = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial U_s}{\partial e_{ij}} \delta e_{ij} + 0(\delta e_{ij}) \tag{13.117}$$

where $0(\delta e_{ij})$ refers to the terms containing second and higher powers of the variations of e_{ij}. For sufficiently small variations δe_{ij} relation (13.117) may be approximated as

$$\delta U_s \approx \delta^{(1)} U_s = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial U_s}{\partial e_{ij}} \delta e_{ij} \tag{13.118}$$

$\delta^{(1)} U_s$ is called the first variation of U_s. Substituting relation (13.118) into (13.115), we get

$$\delta^{(1)} \iiint_V U_s dV - \delta \iiint_V \sum_{i=1}^{3} B_i \hat{u}_i dV - \delta \iint_{S_t} \sum_{i=1}^{3} \overset{n}{T}{}_i^s \hat{u}_i dS = 0 \tag{13.119}$$

In obtaining relation (13.119) it was possible to take the operator δ outside the second and third integral signs because the distributions of the components of body force $B_i\,(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ and surface traction $\overset{n}{T}{}_i^s (i = 1, 2, 3)$ remain constant during the variation of the components of displacement $\delta \hat{u}_i\ (i = 1, 2, 3)$. Relation (13.119) can be rewritten as

$$\delta^{(1)} \Pi_s = 0 \tag{13.120}$$

where

$$\Pi_s = \iiint_V U_s dV - \iiint_V \left(\sum_{i=1}^{3} B_i \hat{u}_i \right) dV - \iint_{S_t} \left(\sum_{i=1}^{3} \overset{n}{T}{}_i^s \hat{u}_i \right) dS \tag{13.121}$$

The functional $\Pi_s(x_1, x_2, x_3, \hat{\mathbf{u}})$ of the function $\hat{u}_i(x_1, x_2, x_3)$ is called the *total potential* *energy of the body.* The first term, on the right side of the relation (13.121), represents the energy stored in the body during an isothermal or adiabatic deformation process. The last two terms represent the work that the external forces (body forces and surface tractions) will perform in going from their position in the deformed configuration to their position in the undeformed configuration of the body.

We have shown that the actual displacement field $\hat{\mathbf{u}}(x_1, x_2, x_3)$ of an elastic body subjected to external forces in an environment of constant temperature satisfies relation (13.120). That is, renders the first variation of the total potential energy Π_s $(x_1, x_2, x_3, \hat{\mathbf{u}})$ defined by relations (13.120) equal to zero. It can be shown that a vector field $\hat{\mathbf{u}}(x_1, x_2, x_3)$ which renders a functional Π_s $(x_1, x_2, x_3, \hat{\mathbf{u}})$ stationary must render its first variation equal to zero (see Section E-4 of Appendix E). Moreover, it can be shown that *a vector field* $\hat{\mathbf{u}}(x_1, x_2, x_3)$ *which renders the first variation of a functional* $\Pi_s(x_1, x_2, x_3, \hat{\mathbf{u}})$ *equal to zero, makes this functional assume stationary values*. Furthermore, it can be shown that *the vector field* $\hat{\mathbf{u}}(x_1, x_2, x_3)$, *which renders a functional* $\Pi_s(x_1, x_2, x_3, \hat{\mathbf{u}})$ *stationary, must satisfy the Euler–Lagrange equations for this functional (see Section E-4 of Appendix E)*. This implies that we have shown that the actual displacement field $\hat{\mathbf{u}}(x_1, x_2, x_3)$ of an elastic body subjected to external forces in an environment of constant temperature renders stationary its total potential energy.

In what follows, we show that if a geometrically admissible displacement field renders *stationary, the total potential energy of an elastic body, subjected to external forces in an environment of constant temperature, is the actual displacement field of this body*. For this purpose we consider a geometrically admissible displacement field $\hat{\hat{u}}_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ and we denote by $\hat{e}_{ij}(x_1, x_2, x_3)$ $(i,j = 1, 2, 3)$ the geometrically admissible strain field obtained from $\hat{\hat{u}}_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ on the basis of relations (2.16). Moreover, we denote by $\hat{U}_s$ the strain energy density corresponding to the components of strain $\hat{e}_{ij}$, on the basis of relation (3.55). Furthermore, we denote by $\hat{\tau}_{ij}$ the components of stress obtained from the strain energy density on the basis of relation (13.114). Using the strain–displacement relation (2.16) the first variation of the strain energy density (13.118) can be written as

$$\delta^{(1)}\hat{U}_s = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\delta\hat{e}_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\left(\frac{\partial\delta\hat{\hat{u}}_i}{\partial x_j}+\frac{\partial\delta\hat{\hat{u}}_j}{\partial x_i}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\left[\frac{\partial\delta\hat{\hat{u}}_i}{\partial x_j}+\frac{1}{2}\left(\frac{\partial\delta\hat{\hat{u}}_j}{\partial x_i}-\frac{\partial\delta\hat{\hat{u}}_i}{\partial x_j}\right)\right] \tag{13.122}$$

It can be shown by expansion that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\left[\frac{1}{2}\left(\frac{\partial\delta\hat{\hat{u}}_j}{\partial x_i}-\frac{\partial\delta\hat{\hat{u}}_i}{\partial x_j}\right)\right] = 0 \tag{13.123}$$

Thus, relation (13.122) reduces to

$$\delta^{(1)}U_s = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\left(\frac{\partial \delta\hat{\hat{u}}_i}{\partial x_j}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i\right) - \frac{\partial \hat{U}_s}{\partial x_j \partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i\right] \tag{13.124}$$

Substituting relation (13.124) into (13.120), we have

$$\begin{aligned}\delta^{(1)}\Pi_s = &\iiint_V \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\right)\delta\hat{\hat{u}}_i\right] dV \\ &- \iiint_V \left(\sum_{i=1}^{3} B_i \delta\hat{\hat{u}}_i\right) dV - \iint_{S_t}\left(\sum_{i=1}^{3} \overset{n}{T}{}_i^{s}\,\delta\hat{\hat{u}}_i\right) dS = 0\end{aligned} \tag{13.125}$$

The surface integral is taken over the portion S_t of the surface of the body where the components of traction are specified. Notice that

$$\sum_{i=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i = \sum_{i=1}^{3}\hat{\tau}_{ij}\,\delta\hat{\hat{u}}_i = \hat{\boldsymbol{\tau}}\cdot\delta\hat{\hat{\mathbf{u}}} = \text{vector} = \mathbf{F}$$

Thus, we can apply the divergence theorem of Gauss† to the first term of the first volume integral of relation (13.125) to obtain

$$\begin{aligned}\iiint_V \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i\right) dV &= \iiint_V \sum_{j=1}^{3}\frac{\partial F_j}{\partial x_j} dV = \iint_S \sum_{j=1}^{3} F_j \lambda_{nj} dS \\ &= \iint_S \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial \hat{U}}{\partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i\lambda_{nj} dS = \iint_{S_t}\sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial \hat{U}}{\partial \hat{e}_{ij}}\delta\hat{\hat{u}}_i\lambda_{nj} dS\end{aligned} \tag{13.127}$$

where λ_{nj} (j = 1, 2, 3) are the components (direction cosines) of the unit vector i_n which is outward normal to the surface element dS. In obtaining relation (13.127), we took into account that at the particles of the portion $S - S_t$ of the surface of the body where the components of displacement are specified $\delta\hat{\hat{u}}_i$ is zero because u_i (x_1, x_2, x_3) is a geometrically admissible displacement field. Substituting relation (13.127) into (13.125), we obtain

† Gauss's theorem states: Consider a vector field $F_i(x_1, x_2, x_3)$ (i = 1, 2, 3) which has continuous derivatives in a region of volume V bounded by a continuous outer surface S_o. The body may have N cavities of surface S_i (i = 1, 2, ..., N) each . Then

$$\iiint_V \sum_{i=1}^{3}\frac{\partial F_i}{\partial x_i} dV = \iint_S \sum_{i=1}^{3} F_i \lambda_{ni} dS \tag{13.126}$$

where λ_{ni}(i = 1, 2, 3) are the direction cosines of the unit vector normal to the surface element dS and $S = S_o + \sum_{i=1}^{N} S_i$.

$$\delta^{(1)}\Pi_s = -\iiint_V \left[\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\right) + \hat{B}_i\right]\delta\hat{u}_i\right]dV + \iint_{S-S_t}\left[\sum_{i=1}^{3}\left[\sum_{j=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\lambda_{nj} - \overset{n}{T}{}_i^s\right]\delta\hat{u}_i\right]dS = 0$$

(13.128)

$\overset{n}{T}{}_i^s$ (i = 1, 2, 3) are the specified components of traction acting on the surface element dS. Inasmuch as $\delta\hat{\hat{u}}_i$ is an arbitary variation of the geometrically admissible displacement field $\hat{\hat{u}}_i$ we may choose it to vanish on the portion S_t of the surface of the body while inside the volume of the body it could assume any value. With this choice, relation (13.128) reduces to

$$\delta^{(1)}\Pi = -\iiint_V \left[\sum_{i=1}^{3}\left[\left[\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\right) + B_i\right]\delta\hat{\hat{u}}_i\right]\right]dV = 0 \qquad (13.129)$$

According to the fundamental lemma of the calculus of variations (see footnote in Appendix E) the term inside the brackets must vanish at every point inside the volume V. Consequently,

$$\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\right) + \hat{B}_i = 0 \qquad (i = 1, 2, 3) \qquad (13.130)$$

Substituting relation (13.130) into (13.128), we obtain

$$\iint_{S_t}\left[\sum_{i=1}^{3}\left[\left[\sum_{j=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\lambda_{nj} - \overset{n}{T}{}_i^s\right]\delta\hat{\hat{u}}_i\right]\right]dS = 0 \qquad (13.131)$$

According to the fundamental lemma of the calculus of variations the term inside brackets must vanish at every point of the portion S_t of the surface of the body. Consequently,

$$\overset{n}{T}{}_i^s = \sum_{j=1}^{3}\frac{\partial \hat{U}_s}{\partial \hat{e}_{ij}}\lambda_{nj} \qquad (i = 1, 2, 3)$$ on the particles of the portion S_t of the surface of the body

(13.132)

Using relation (13.114), relations (13.130) and (13.132) can be rewritten as

$$\sum_{j=1}^{3}\frac{\partial \hat{\tau}_{ij}}{\partial x_j} + B_i = 0 \qquad (i = 1, 2, 3)$$ on every particle inside the volume of the body

(13.133)

and

$$\overset{n}{T}{}_i^s = \sum_{j=1}^{3}\hat{\tau}_{ij}\lambda_{nj} \qquad (i = 1, 2, 3)$$ on the particles of the portion S_t of the surface of the body (13.134)

Equations (13.133) are the Euler–Lagrange equations (see Section E-4 of Appendix E) for the total potential energy Π_s of the elastic body under consideration. That is, the Euler-Lagrange equations for the total potential energy Π_s of an elastic body are the equations of equilibrium (2.69) for the particles of the body.

We have shown above that a geometrically admissible displacement field which renders stationary the total potential energy (13.121) of the body under consideration yields components of stress which satisfy the equations of equilibrium (2.69) at every particle inside the volume of the body and, moreover, give the specified components of traction $\overset{n}{T}{}_i^s$ [see relation (2.73)] at the points of the portion S_t of the surface of the body where the components of traction are specified. Consequently, this geometrically admissible displacement field is the actual one. *Hence, when an elastic body is subjected to external forces in an environment of constant temperature the geometrically admissible displacement field which renders stationary its total potential energy* Π_s *is its actual displacement field. This statement is known as the theorem of stationary total potential energy.*

13.17.1 Theorem of Minimum Total Potential Energy

Consider a deformable body of volume V and surface S made from a stable elastic material. The body is in equilibrium in an environment of constant temperature under the influence of

(a) Specified body forces $B_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ given in units of force per unit volume

(b) Specified components of surface traction $\overset{n}{T}{}_i^s (i = 1, 2, 3)$ given in units of force per unit area at the portion S_t of its surface

(c) Specified components of displacement $\hat{u}_i (i = 1, 2, 3)$ at the portion $S - S_t$ of its surface

The theorem of minimum potential energy states that *the actual displacement field of a body made from a stable elastic material renders its total potential energy an absolute minimum.*

Proof

The total potential energy of the body in an equilibrium state specified by the displacement field $\hat{\boldsymbol{u}}(x_1, x_2, x_3)$ is equal to

$$\Pi_s = \iiint_V U_s(e_{ij})dV - \iiint_V \sum_{i=1}^{3} \left(B_i \hat{u}_i\right) dV - \iint_{S_t} \left(\sum_{i=1}^{3} \overset{n}{T}{}_i^s \hat{u}_i \right) dS \tag{13.135}$$

The total potential energy of the body under consideration corresponding to another geometrically admissible displacement field $\hat{\mathbf{u}} + \boldsymbol{\delta}\hat{\mathbf{u}}$ is equal to

$$\Pi_s + \delta\Pi_s = \iiint_V U_s(e_{ij} + \delta e_{ij})\,dV - \iiint_V \left[\sum_{i=1}^{3} B_i(\hat{u}_i + \delta\hat{u}_i)\right] dV - \iint_{S_t}\left[\sum_{i=1}^{3} \overset{n}{T}{}_i^s(\hat{u}_i + \delta\hat{u}_i)\right] dS \tag{13.136}$$

Thus,

$$\delta\Pi_s = \iiint_V \left[U_s(e_{ij} + \delta e_{ij}) - U_s(e_{ij})\right] dV - \iiint_V \left(\sum_{i=1}^{3} B_i\delta\hat{u}_i\right) dV - \iint_{S_t}\left(\sum_{i=1}^{3} \overset{n}{T}{}_i^s\delta\hat{u}_i\right) dS \tag{13.137}$$

Expanding the strain energy density $U_s(e_{ij} + \delta e_{ij})$ as a Taylor Series about e_{ij}, we have

$$U_s(e_{ij} + \delta e_{ij}) = U_s(e_{ij}) + \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial U_s}{\partial e_{ij}}\delta e_{ij} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 U_s}{\partial e_{ij}\partial e_{mn}}\delta e_{ij}\delta e_{mn} + 0(\delta^3) \tag{13.138}$$

where $0(\delta^3)$ represents the terms containing higher powers than the second of the variations of e_{ij}. Substituting relation (13.138) into (13.137), we get

$$\delta\Pi_s = \iiint_V \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial U_s}{\partial e_{ij}}\delta e_{ij}\right) dV - \iiint_V\left(\sum_{i=1}^{3} B_i\delta u_i\right) dV - \iint_{S_t}\left(\sum_{i=1}^{3}\overset{n}{T}{}_i^s\delta u_i\right) dS$$
$$+ \iiint_V\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 U_s}{\partial e_{ij}\partial e_{mn}}\delta e_{ij}\delta e_{mn}\right) dV + 0(\delta^3) \tag{13.139}$$

Inasmuch as we are considering variations $\delta\hat{u}_i$ of the actual components of displacement $\hat{u}_i$ at the equilibrium state of the body, relation (13.115) is valid and may be substituted in relation (13.139) to yield

$$\delta\Pi_s = \iiint_V\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 U_s}{\partial e_{ij}\partial e_{mn}}\delta e_{ij}\delta e_{mn}\right) dV + 0(\delta^3) \tag{13.140}$$

For sufficiently small variations $\delta\hat{u}_i$ the terms included in $0(\delta^3)$ are negligible compared to the other term on the right side of relation (13.140). In this case the variation of the potential energy is approximated by the volume integral on the right side of relation (13.140) which is known as the second variation of Π_s and is denoted as $\delta^{(2)}\Pi_s$. Thus, using relation (13.114), we have

$$\delta\Pi_s \approx \delta^{(2)}\Pi_s = \iiint_V\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 U_s}{\partial e_{ij}\partial e_{mn}}\delta e_{ij}\delta e_{mn}\right) dV$$
$$= \iiint_V\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial \tau_{mn}}{\partial e_{ij}}\delta e_{ij}\delta e_{mn}\right) dV = \iiint_V\left(\sum_{m=1}^{3}\sum_{n=1}^{3}\delta\tau_{mn}\delta e_{mn}\right) dV \tag{13.141}$$

Referring to relation (3.38b) for a stable elastic material the following inequality holds:

$$\sum_{m=1}^{3}\sum_{n=1}^{3}\delta\tau_{mn}\delta e_{mn} > 0 \tag{13.142}$$

Hence, from relation (13.141), we get

$$\delta\Pi_S \approx \delta^{(2)}\Pi_S > 0 \tag{13.143}$$

Relation (13.143) indicates that for any geometrically admissible variation of its actual displacement field, the variation of the total potential energy of a body in equilibrium made from a stable elastic material is positive. Thus, of all geometrically admissible displacement fields of a body in equilibrium made from a stable elastic material, the actual one renders its total potential energy (13.121) a minimum.

13.17.2 The Ritz Method

The theorem of stationary total potential energy has been used extensively in conjuction with the Ritz method to obtain approximate expressions for the displacement fields of elastic bodies. In this method the components of displacement are approximated as follows:

$$\begin{aligned}
\tilde{u}_1 &= \phi_{1o}(x_1, x_2, x_3) + \sum_{s=1}^{S} c_{1s}\phi_{1s}(x_1, x_2, x_3) \\
\tilde{u}_2 &= \phi_{2o}(x_1, x_2, x_3) + \sum_{s=1}^{S} c_{2s}\phi_{2s}(x_1, x_2, x_3) \\
\tilde{u}_3 &= \phi_{3o}(x_1, x_2, x_3) + \sum_{s=1}^{S} c_{3s}\phi_{3s}(x_1, x_2, x_3)
\end{aligned} \tag{13.144}$$

Usually the functions $\phi_{jo}(j = 1, 2, 3)$ are chosen to satisfy at least the essential boundary conditions of the problem while the functions $\phi_{js}(j = 1, 2, 3)$ $(i = 1, 2..., S)$ are chosen to vanish on the points where essential (displacement) boundary conditions are specified. In the Ritz method the coefficients c_{js} $(j = 1, 2, 3)$ $(s = 1, 2..., S)$ are chosen as to render the total potential energy of the body stationary. For this purpose relations (13.144) are substituted in (13.121) to obtain an approximate expression of the total potential energy Π_s. Then the values of the coefficients $c_{js}(j = 1, 2, 3)$ $(s = 1, 2..., S)$ which render the total potential Π_s of the body stationary are established by imposing the following requirements:

$$\frac{\partial \Pi_s}{\partial c_{1s}} = 0 \qquad \frac{\partial \Pi_s}{\partial c_{2s}} = 0 \qquad \frac{\partial \Pi_s}{\partial c_{3s}} = 0 \qquad s = 1, 2 ..., S \tag{13.145}$$

This yields $3S$ equations for the $3S$ unknown coefficients, c_{js} $(j = 1, 2, 3)$ $(s = 1, 2..., S)$. If the functions ϕ_{jo} and ϕ_{js} $(j = 1, 2, 3)$ $(s = 1, 2, ..., S)$ are chosen so that $\tilde{u}_i$ $(i = 1, 2, 3)$ satisfy all the essential boundary conditions of the body [for the proper choice of the function ϕ_{js} $(j = 1, 2 ,3)$ $(s = 1, 2, ..., S)$ (see Section 13.7)], relations (13.144) converge to the exact displacement field as $S \to \infty$.

In what follows we present an example.

Example 8 Consider the beam of constant width $b = 0.3$ m subjected to the loading shown in Fig. a . The beam is made from an isotropic, linearly elastic material. Establish an approximate expression for the transverse component of translation (deflection) of the beam using the Ritz method with $S = 3$.

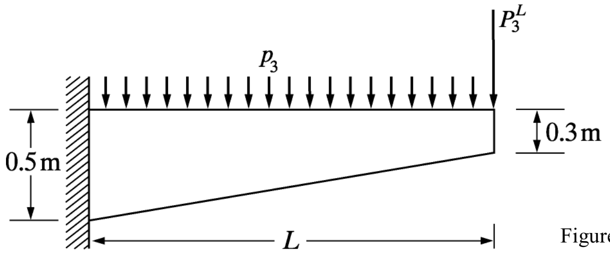


Figure a Geometry and loading of the beam.

Solution Referring to relation (13.90), for the beam under consideration, we have

$$\iiint_V U_s dV = \frac{1}{2}\int_0^L \frac{M_2^2}{EI_2} dx_1 \tag{a}$$

The work of the external forces in going from their position in the deformed configuration of the beam to their position in the undeformed configuration of the beam, is equal to

$$W_{ext\ forces} = -\int_0^L p_3 u_3 dx_1 - P_3^L u_3(L) \tag{b}$$

Substituting relations (a) and (b) into (13.121) and using relation (9.32a) we obtain

$$\begin{aligned} \Pi_s &= \frac{1}{2}\int_0^L \frac{M_2^2}{EI_2} dx_1 - \int_0^L p_3 u_3 dx_1 - P_3^L u_3(L) \\ &= \frac{1}{2}\int_0^L EI_2 \left(\frac{d^2 u_3}{dx_1^2}\right)^2 dx_1 - \int_0^L p_3 u_3 dx_1 - P_3^L u_3(L) \end{aligned} \tag{c}$$

The essential boundary conditions of the beam are

$$u_3(0) = 0 \qquad\qquad \left.\frac{du_3}{dx_1}\right|_{x_1=0} = 0 \tag{d}$$

We assume an approximate solution of the following form:

$$\tilde{u}_s(x_1) = \sum_{s=1}^{3} c_s \phi_s \tag{e}$$

and we choose

$$\phi_1 = x_1^2 \qquad \phi_2 = x_1^3 \qquad \phi_3 = x_1^4 \tag{f}$$

It is apparent that the approximate solution (e) with (f) satisfies the essential boundary conditions (d). Substituting relation (e) into (c), we get

$$\Pi_s = \int_0^L \frac{EI_2}{2}\left(\sum_{s=1}^{3} c_s \frac{d^2\phi_s}{dx_1^2}\right)^2 dx_1 - \int_0^L p_3 \sum_{s=1}^{3} c_s \phi_s dx_1 - P_3^L \sum_{s=1}^{3} c_s \phi_s(L)$$

$$I_2(x_1) = \frac{0.3}{12}\left(0.5 - \frac{0.2x_1}{L}\right)^3 = 0.025\left(0.125 - \frac{0.15x_1}{L} + \frac{0.06x_1^2}{L^2} - \frac{0.008x_1^3}{L^3}\right) \tag{g}$$

The values of c_s ($s = 1, 2, 3$) which render the functional $\Pi_s(c_1, c_2, c_3)$ stationary satisfy the following relations:

$$\frac{\partial \Pi_s}{\partial c_i} = 0 = \int_0^L EI_2\left(\sum_{s=1}^{3} c_s \frac{d^2\phi_s}{dx_1^2}\right)\left(\frac{d^2\phi_i}{dx_1^2}\right) dx_1 - \int_0^L p_3 \phi_i dx_1 - P_3^L \phi_i(L) = \sum_{j=1}^{3} b_{ij} c_j - f_i \tag{h}$$

$$i = 1, 2, 3$$

where

$$b_{ij} = \int_0^L EI_2 \phi_i'' \phi_j'' dx_1 \tag{i}$$

$$f_i = \int_0^L p_3 \phi_i dx_1 + P_3^L \phi_i(L) \tag{j}$$

Relations (h) may be rewritten has

$$[\, b \,]\{ c \} = \{ f \} \tag{k}$$

Referring to Fig. a, we have

$$h(x_1) = 0.5 - \frac{0.2x_1}{L} \tag{l}$$

Moreover, referring to relation (f), we get

$$\phi_1'' = 2 \qquad \phi_2'' = 6x_1 \qquad \phi_3'' = 12x_1^2 \tag{m}$$

Substituting relations (m) and (l) into (i) and integrating, we obtain

$$b_{11} = E\int_0^L I_2 \phi_1'' \phi_1'' dx_1 = 0.1E\left[0.125x_1 - \frac{0.15x_1^2}{2L} + \frac{0.06x_1^3}{3L^2} - \frac{0.008x_1^4}{4L^3}\right]_0^L = 0.0068EL$$

$$b_{12} = b_{21} = E\int_0^L I_2 \phi_1'' \phi_2'' dx_1 = 0.3E\left[\frac{0.125x_1^2}{2} - \frac{0.15x_1^3}{3L} + \frac{0.06x_1^4}{4L^2} - \frac{0.008x_1^5}{5L^3}\right]_0^L = 0.00777EL^2$$

$$b_{13} = b_{31} = E\int_0^L I_2 \phi_1'' \phi_3'' dx_1 = 0.6E\left[\frac{0.125x_1^3}{3} - \frac{0.15x_1^4}{4L} + \frac{0.06x_1^5}{5L^2} - \frac{0.008x_1^6}{6L^3}\right]_0^L = 0.008904EL^3$$

$$b_{22} = E\int_0^L I_2 \phi_2'' \phi_2'' dx_1 = 0.9E\left[\frac{0.125x_1^3}{3} - \frac{0.15x_1^4}{4L} + \frac{0.06x_1^5}{5L^2} - \frac{0.008x_1^6}{6L^3}\right]_0^L = 0.013356EL^3$$

$$b_{23} = E\int_0^L I_2 \phi_2'' \phi_3'' dx_1 = 1.8E\left[\frac{0.125x_1^4}{4} - \frac{0.15x_1^5}{5L} + \frac{0.06x_1^6}{6L^2} - \frac{0.008x_1^7}{7L^3}\right]_0^L = 0.0182EL^4$$

$$b_{33} = E\int_0^L I_2 \phi_3'' \phi_3'' dx_1 = 3.6E\left[\frac{0.125x_1^5}{5} - \frac{0.15x_1^6}{6L} + \frac{0.06x_1^7}{7L^2} - \frac{0.008x_1^8}{8L^3}\right]_0^L = 0.02725EL^5$$

(n)

Substituting relation (f) into (j), we get

$$f_3 = p_3\int_0^L \phi_3 dx_1 + P_3^L L^4 = \frac{p_3 L^5}{5} + P_3^L L^4$$

$$f_1 = p_3\int_0^L \phi_1 dx_1 + P_3^L L^2 = \frac{p_3 L^3}{3} + P_3^L L^2 \qquad \text{(o)}$$

$$f_2 = p_3\int_0^L \phi_2 dx_1 + P_3^L L^3 = \frac{p_3 L^4}{4} + P_3^L L^3$$

Substituting relations (n) and (o) into (k), we have

$$EL\begin{bmatrix} 0.00680 & 0.00777L & 0.008904L^2 \\ 0.00777L & 0.013356L^2 & 0.01820L^3 \\ 0.008904L^2 & 0.01820L^3 & 0.02725L^4 \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = L^2\begin{Bmatrix} p_3\dfrac{L}{3} + P_3^L \\ p_3\dfrac{L^2}{4} + P_3^L L \\ p_3\dfrac{L^3}{5} + P_3^L L^2 \end{Bmatrix} \qquad \text{(p)}$$

Solving relation (p), we get

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 81.54\ p_3 L^2 + 151.4\ P_3^L L \\ -27.04\ p_3 L + 45.8\ P_3^L \\ -1.24\ p_3 L + 43.32\ P_3^L / L \end{Bmatrix} \tag{q}$$

Substituting the values of the coefficients (q) into relation (e) and using (f), we have

$$u_1(L) = c_1 L^2 + c_2 L^3 + c_3 L^4 = 53.26\ p_3 L^4 + 240.52\ P_3^L L^3 \tag{r}$$

13.18 Comments on the Modified Gallerkin Form and the Theorem of Stationary Total Potential Energy

From our presentation in this chapter, we can make the following statements:

1. The modified Gallerkin method can be employed to discretize any boundary value problem provided that a function $\phi_o(x_1)$ can be found which satisfies at least the essential boundary conditions of the problem and trial functions $\phi_s(x_1)(s = 1, 2, ..., S)$ can be found which satisfy the homogeneous part of the boundary conditions satisfied by the function $\phi_o(x_1)$.

2. The theorem of stationary total potential energy can be employed, using the Raleigh–Ritz method, to discretize the boundary value problems for computing the displacement and stress fields of bodies made only from elastic materials provided that a function $\phi_o(x_1)$ can be found which satisfies at least the essential boundary conditions of the problem and trial functions $\phi_s(x_1)(s = 1, 2, ..., S)$ can be found which satisfy the homogeneous part of the boundary conditions satisfied by the function $\phi_o(x_1)$. This method, and the modified Galerkin method, always give the same results. Thus, the Rayleigh–Ritz method does not seem to have any conceptual or computational advantages over the modified Galerkin method. Moreover, it requires knowledge of some elements of calculus of variation. However, the theorem of stationary total potential energy can be employed to establish natural boundary conditions of some problems whose natural boundary conditions are not easily established from physical consideration. Moreover, as we show in Section 18.2, the principle of minimum total potential energy can be employed to find out whether an equilibrium state of a body is stable or not.

13.19 Problems

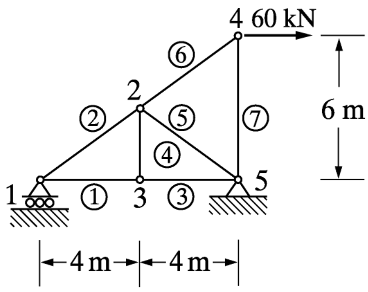
1. and **2.** Using the unit load method, compute the vertical component of translation of joint 3 of the truss subjected to the loads shown in Fig. 13P1. The members of the truss are made from the steel ($E = 210$ GPa). Repeat with the truss of Fig. 13P2.

Ans. 1 $u_v^{(3)} = 1.13$ mm↑ Ans. 2 $u_v^{(3)} = 0.494$ mm↓

3. Using the unit load method, compute the vertical component of translation of joint 4 of the truss of Fig. 13P2 resulting from the external actions shown in the figure and from a 20 °C increase of temperature of the members of its bottom chord. The members of the truss are made from steel (E = 210 GPa, $\alpha = 10^{-5}$/°C). Ans. $u_v^{(4)} = 1.22$ mm↓

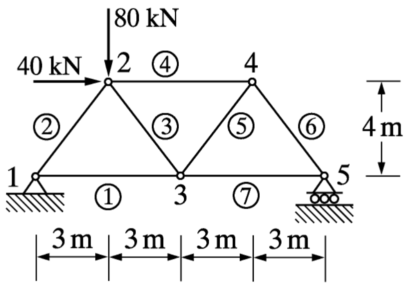
4. Using the unit load method, compute the vertical component of translation of joint 2 of the truss of Fig. 13P1, due to a 20-mm settlement of the right-hand support. Verify your results by considering the geometry of the deformed truss.

Ans. $u_v^{(2)} = 10$ mm↓



Member	Cross-sectional area
1,2,5,6	3 x 10³ mm²
3,4	2 x 10³ mm²
7	6 x 10³ mm²

Figure 13P1



Member	Cross-sectional area
1,3,4	8 x 10³ mm²
2,6	6 x 10³ mm²
5,7	10 x 10³ mm²

Figure 13P2

5. Using the unit load method compute the deflection of point A of the simply supported beam subjected to the external forces shown in Fig. 13P5. The beam has a constant cross section [$I_2 = 369.70(10^6)$ mm⁴] and is made from an isotropic, linearly elastic material (E = 200 GPa). Disregard the effect of the shear deformation of the beam.

$$\text{Ans.}\quad u_v^{(A)} = \frac{M_2^* b}{12EI_2L}\left(6bL - \frac{13b^2}{4} - 2L^2\right)$$

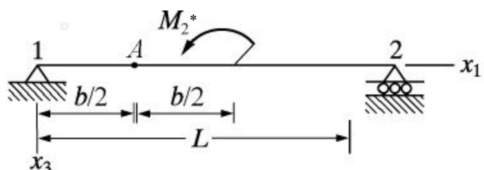


Figure 13P5

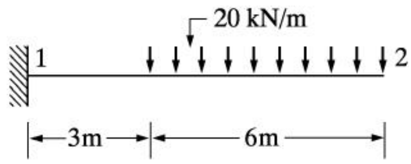


Figure 13P6

6. Using the unit load method compute the rotation of the unsupported end of the cantilever beam subjected to the actions shown in Fig. 13P6. The beam has a constant cross section [$I_2 = 369.70(10^6)$ mm⁴] and it is made from an isotropic, linearly elastic material (E = 200 GPa). Disregard the effect of shear deformation of the beam.

Ans. $\theta_2^{(2)} = 0.00003165$ rad clockwise

7. and **8.** Using the unit load method, compute the horizontal movement of support 4 of the frame loaded as shown in Fig. 13P7. The members of the frame have the same constant cross section [$I_2 = 369.70(10^6)$ mm⁴] and are made of the same material (E = 210 GPa). Disregard the effect shear and axial deformation of the members of the frame. Repeat with the horizontal movement of point 1 of the structure of Fig. 13P8.

Ans. 7 334.68 mm→ Ans. 8 $u_v^{(2)} = 6.64$ mm←

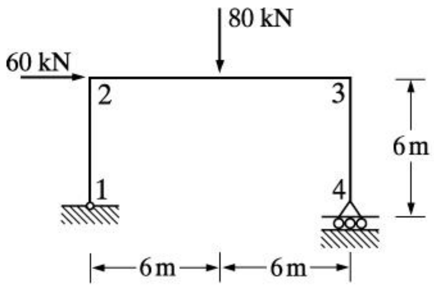


Figure 13P7

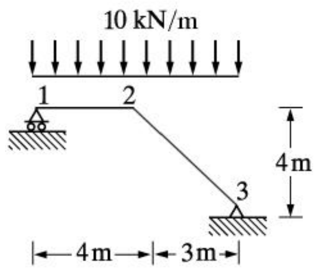
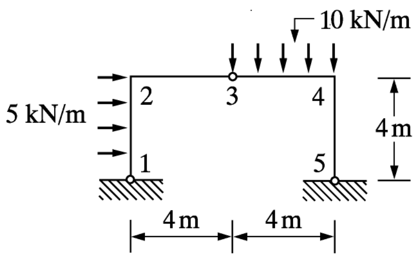


Figure 13P8

9. Using the unit load method, compute the vertical movement of point 3 and the slope of the elastic curve of member (2, 3) at point 3 of the frame subjected to the loading shown in Fig. 13P7. The members of the structure have the same constant cross section ($I_2 = 369.7 \times 10^6 \text{mm}^4$) and are made from the same material ($E = 210$ GPa) disregard the effect of shear and axial deformation of the members of the structure

Ans. $u_v^{(2)} = 4.47$ mm↓ $\theta^{(2,3)} = 0.001288$ rad clockwise



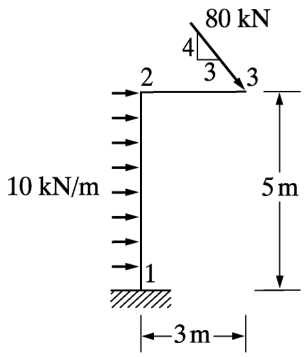


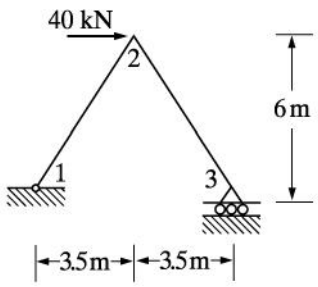
Figure 13P9

Figure 13P10

10. Using the unit load method, compute the horizontal component of translation of point 3 of the frame loaded as shown in Fig. 13P10. Disregard the effect of axial and shear deformation of the members of the frame. The members of the frame have the same constant rectangular cross section (depth = 600 mm width = 200 mm) and are made from steel ($E = 210$ GPa).

Ans. $u_v^{(3)} = 6.85$ mm↓

11. Using the unit load method, compute the horizontal component of translation of joint 2 of the structure loaded as shown in Fig. 13P11. Disregard the effect of shear and axial



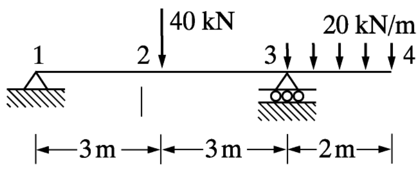


Figure 13P11

Figure 13P12

deformation of the members of the structure. The members of the structure have the same constant cross section [I_2 = 369.70 (10^6) mm^4] and are made from steel (E = 210 GPa).

Ans. $u_h^{(2)}$ = 21.47 mm→

12. Using the unit load method, compute the deflection at point 4 of the beam subjected to the external actions shown in Fig. 13P12. The beam has a constant rectangular cross section of width b = 40 mm and depth h = 200 mm and it is made from an isotropic, linearly elastic material (E = 200 GPa, ν = 1/3). Include the effect of shear deformation of the members of the beam.

Ans. $u_V^{(4)}$ = 18.86 mm

13. Using the unit load method, compute the horizontal component of translation of point 2 and the rotation at point 3 of the frame of Fig. 13P11 due to a temperature differential at its exterior (T_e = 5ºC) and interior (T_i = 45ºC) surfaces. The temperature during construction was T_o = 5ºC. The members of the frame are made from the same material (E = 210 GPa, $\alpha = 10^{-5}$/ºC) and have the same constant cross section [A = 16.2(10^3) mm^2, I = 564.8(10^6) mm^4, h = 475 mm].

Ans. $\Delta_1^{(2)}$ = 18.25 mm →
$\theta_3^{(3)}$ = 0.005849 rad counterclockwise

14. Using the unit load method, compute for point A of the beam of Fig. 13P5 the deflection due to a 20-mm settlement of support 2. Verify your results by considering the geometry of the deformed beam.

15. Using the unit load method, compute the horizontal component of translation of point 3 of the frame of Fig. 13P7 due to a temperature differential at the exterior (T_e = 10ºC) and interior (T_i = 30ºC). The temperature during construction was T_o = 20ºC. The members of the frame are made from the same material (E = 210 GPa, $\alpha = 10^{-5}$/ºC) and have the same constant cross section [A = 16.2(10^3) mm^2, I = 564.8(10^6) mm^4, h = 475 mm].

Ans. $u_h^{(3)}$ = 22.73 mm →

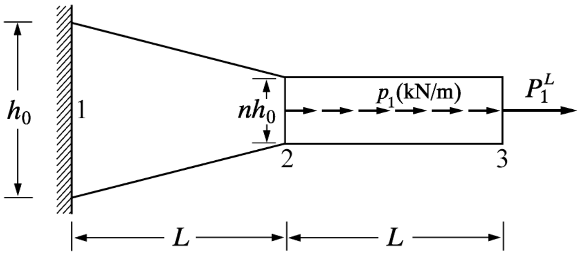


Figure 13P16

16. Using the modified Gallerkin method, establish an approximate expression for the axial component of translation $u_1(x_1)$ and the internal force $N(x_1)$ of the structure of Fig. 13P16 subjected to the axial centroidal forces shown in Fig. 13P16. The structure is made from steel (E = 210 GPa). Use S = 4, n = 0.5, L = 2m, A_0 = 4cm^2. Choose $\phi_s = \sin\left(\frac{s\pi x_1}{4L}\right)$. Evaluate $u_1(2L)$ and $N(0)$.

17. Using the modified Gallerkin method, obtain approximations of the form (13.54) for the deflection of point 2 of the beam resulting from the load shown in Fig. 13P17. The cross sections of the beam are rectangular and have a constant width $b = 300$ mm. The beam is made from steel ($E = 210$ GPa). Disregard the effect of shear deformation of the beam and use $S = 6$. Ans. approximate value $u_v^{(2)} = 18.75$ mm↓

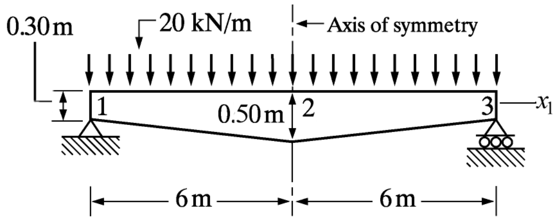


Figure 13P17

18. Consider a simply supported beam of length L and resting on an elastic foundation of modulus k. The beam is subjected to a uniform load p_3(kN/m). The beam is made from an isotropic, linearly elastic material of modulus of elasticity E.

(a) Establish the weighted residual form of the boundary value problem for computing the deflection $u_3(x_1)$ of the beam.

(b) Establish the modified weighted residual form of the boundary value problem for computing the deflection $u_3(x_1)$ of the beam.

(c) Discretize the boundary value problem using the modified Gallerkin weighted residual form, with $S = 6$.

(d) Assume that $L = 8$m, $k = 24$ MPa, $p_3 = 20$ kN/m, $E = 200$ GPa, $I_2 = 240(10^6)$ mm^4 and find an approximate expression for $u_3(x_1)$ and $M_2(x_1)$ using the results of question (c).

19. Using the second theorem of Castagliano, compute the vertical component of translation of joint 3 of the truss loaded as shown in Fig. 13P2. The members of the truss are made form steel ($E = 210$ GPa).

20. Using the second theorem of Castigliano, compute the horizontal component of translation of joint 4 of the truss loaded as shown in Fig. 13P1. The members of the truss are made from steel ($E = 210$ GPa). Ans. $u_h^{(4)}$=2.60 mm

21. and **22.** Using the second theorem of Castagliano, compute the deflection and rotation of point A of the beam loaded as shown in Fig. 13P5. The beam is made from steel ($E = 210$ GPa) and has a constant cross section [$I_2 = 369.70(10^6)$ mm^4]. Disregard the effect of shear deformation of the beam. Repeat with the rotation of the unsupported end of the beam of Fig. 13P6.

23. Obtain an approximate expression for the deflection of the beam of Fig. 13P17, using the Ritz method with $S = 6$.

Chapter 14

Analysis of Statically Indeterminate Framed Structures

14.1 The Basic Force or Flexibility Method

The reactions and internal actions of statically determinate structures can be determined from the given loads using the equations of equilibrium alone. However, the reactions and/or internal actions of statically indeterminate structures cannot be determined using only the equations of equilibrium. Additional equations are required which are established by imposing the requirement that the deformed configuration of the structure be continuous and compatible with the constraints imposed by its supports.

In this chapter, we limit our attention to statically indeterminate structures to the first degree, subjected only to external actions so as to emphasize the basic steps involved in the analysis of statically indeterminate structures using the basic force or flexibility method.

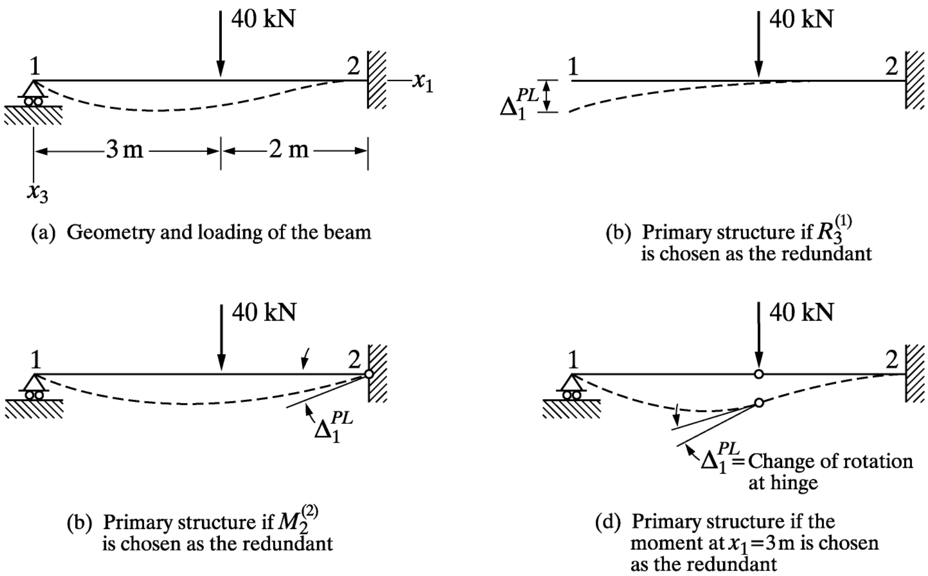


Figure 14.1 Primary structures.

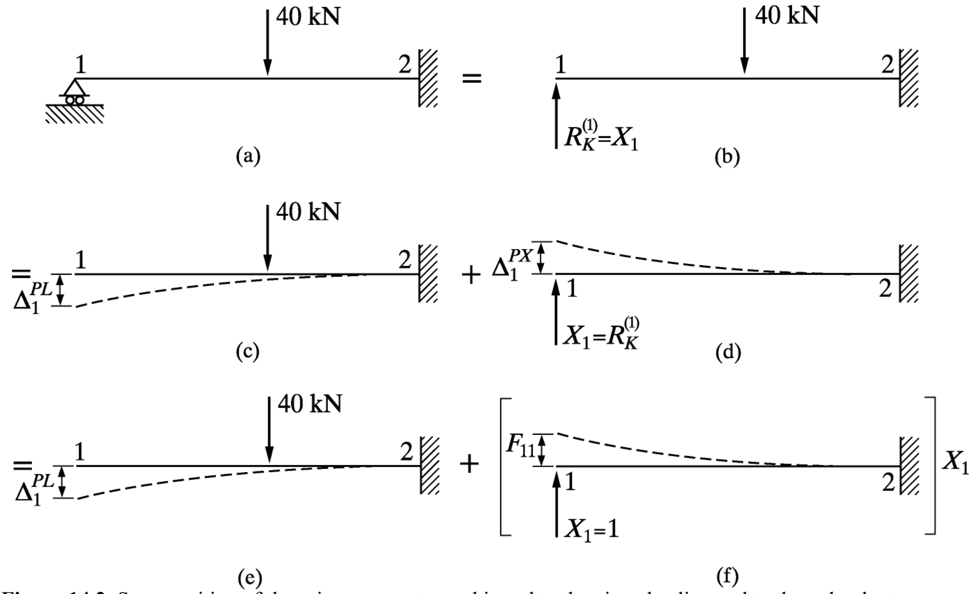


Figure 14.2 Superposition of the primary structure subjected to the given loading and to the redundant.

Let us consider the beam of Fig. 14.1a, and let us choose one of the reactions $R_3^{(1)}$, $R_3^{(2)}$, $M_2^{(2)}$, or an internal action (say the moment at $x_1 = 3$ m) as *the redundant*. The statically determinate beam resulting from the actual beam by removing the constraint which induces the chosen redundant is referred to as the *primary structure*. If the reaction $R_3^{(1)}$ is chosen as the redundant, the primary structure is the cantilever beam shown in Fig. 14.1b. If the moment $M_2^{(2)}$ is chosen as the redundant, the primary structure is the simply supported beam shown in Fig. 14.1c. Moreover, if the moment at $x_1 = 3$ m is chosen as the redundant, the primary structure is the beam of Fig. 14.1d. Notice that the reactions of a primary structure represent a set of statically admissible reactions of the actual structure corresponding to a value of zero for the chosen redundant.

Referring to Fig. 14.2 or 14.3 we can readily see that the internal forces, bending moments and the translation (deflection) at any point of the beam can be obtained by superimposing the corresponding quantities of the primary structure subjected to the given loading and those of the primary structure subjected only to the redundant. In Fig. 14.2 the redundant is the unknown reaction $R_3^{(1)}$ of the beam. In Fig. 14.3 the redundant is the unknown pair of equal and opposite moments which are applied, the one to the left and the other to the right of the hinge at point *A*. Thus, for the beam of Fig. 14.2a, we may write that the translation (deflection) $\Delta_1^{(s)}$ at point 1 of the actual structure whose magnitude we know (it is zero) is equal to the sum of the translation (deflection) Δ_1^{PL} of point 1 of the primary structure, subjected to the given loading, and the deflection Δ_1^{PX} of point 1 of the primary structure, subjected to the redundant. Hence

$$\Delta_1^{(s)} = 0 = \Delta_1^{PL} + \Delta_1^{PX} \tag{14.1}$$

Moreover, as shown in Fig. 14.2f, the internal actions and the deflection of the primary structure subjected to the redundant are equal to those of the primary structure subjected to a unit value of the redundant multiplied by the value of the unknown redundant. Thus,

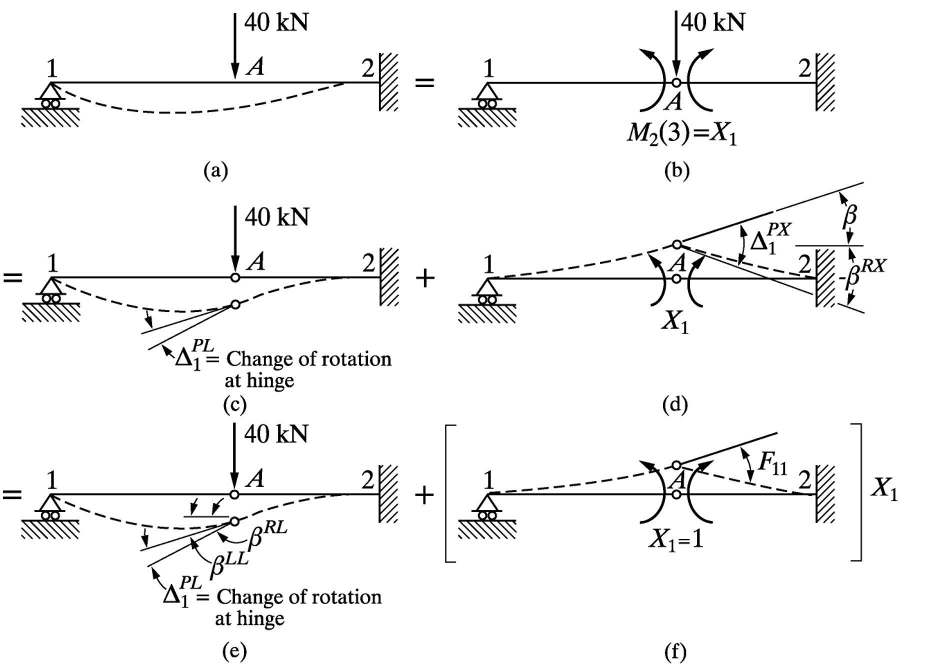


Figure 14.3 Superposition of the primary structure subjected to the given loading and to the redundant.

relation (14.1) can be rewritten as

$$\Delta_1^{(s)} = 0 = \Delta_1^{PL} + X_1 F_{11} \tag{14.2}$$

where F_{11} is the deflection of point 1 of the primary structure subjected to $X_1 = 1$. It is called the flexibility coefficient corresponding to the redundant X_1. Equation (14.2) is called the *compatibility equation* of the beam. The deflections Δ_1^{PL} and F_{11} of point 1 of the primary structure can be computed using the unit load method presented in Section 13.5. Using the values of these deflections, we obtain the redundant X_1 from relation (14.2).

Notice that if we choose as the redundant the pair of equal and opposite internal moments at point B, the slope of the tangent to the elastic curve of the primary structure at the point just to the left of the hinge at point B is not equal to that just to the right. In this case referring to Figs. 14.3e and 11.3d we have

$$\begin{aligned} \Delta_1^{PL} &= \beta^{RL} - \beta^{LL} \\ \Delta_1^{PX} &= \beta^{RX} - \beta^{LX} \end{aligned} \tag{14.3}$$

Where $\beta^{RL}\left(\beta^{LL}\right)$is the angle that the tangent to the elastic curve of the primary structure at the point just to the right (left) of the hinge at point B makes with the horizontal, when the primary structure is subjected to the given loading, $\beta^{RX}\left(\beta^{LX}\right)$ is the corresponding angle when the beam is subjected only to the redundant.

An internal action B of the structure is equal to the sum of the corresponding internal actions A^{PL} in the primary structure, subjected to the actual loading, and the corresponding internal action of the primary structure, subjected to the redundant. That is,

$$A = A^{PL} + A^{PX} X_1 \tag{14.4}$$

where A^{PX} represents the internal action in the primary structure subjected to $X_1 = 1$ corresponding to B. Thus, when the redundant is computed, any internal action in the structure can be determined using relation (14.4).

On the basis of the foregoing discussion, in order to analyze a statically indeterminate structure to the first degree subjected to external actions, we adhere to the following steps:

STEP 1 We select the redundant, and we form the primary structure by removing the constraint which induces the redundant.

STEP 2 We compute the displacement Δ_1^{PL} of the primary structure subjected to the given loading, using the unit load method (13.35).

STEP 3 We compute the displacement F_{11} of the primary structure subjected to $X_1 = 1$ using the unit load method (13.35).

STEP 4 We compute the redundant using the compatibility equation (14.2).

STEP 5 We compute the internal actions in the structure either by using relation (14.4) or by considering the equilibrium of appropriate parts of the structure.

In the sequel, the basic force method is applied to the following two examples.

Example 1 The beam of Fig. a has constant cross section and is made from an isotropic, linearly elastic material. Compute its reactions and plot its shear and moment diagrams.

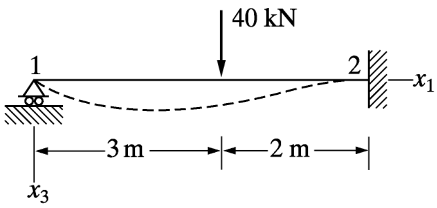


Figure a Geometry and loading of the beam.

Solution

STEP 1 The beam under consideration is statically indeterminate to the first degree. We choose as the redundant the reaction at point 1. Consequently, the primary structure is a cantilever beam.

STEP 2 We compute the deflection Δ_1^{PL} of point 1 of the primary structure subjected to the given loading (see Fig. b). For this purpose we do the following:

1. We compute the internal moment acting on the cross sections of the primary structure subjected to the given loading.
2. We compute the internal moment acting on the cross sections of the primary structure subjected to $X_1 = 1$.

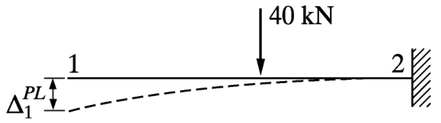


Figure b Primary structure subjected to the given loading.

The free-body diagram of the primary structure subjected to the given external force is shown in Fig. c; referring to this figure the distribution of moment in the primary beam is

$$M_2(x_1') = -80 + 40x_1' \qquad \text{for } 0 \le x_1' \le 2$$

$$M_2(x_1') = 0 \qquad \text{for } 2 \le x_1' \le 5 \tag{a}$$

The free-body diagram of the primary structure subjected to $X_1 = 1$ kN is shown in Fig. d; referring to this figure the distribution of moment in the primary beam is

$$\tilde{M}_2(x_1') = 5 - x_1' \qquad 0 \le x_1' \le 5 \tag{b}$$

Substituting relations (a) and (b) into (13.35), we get

$$\Delta_1^{PL} = \int_0^2 \frac{M_2\tilde{M}_2}{EI_2} dx_1' = \frac{1}{EI_2}\int_0^2 (-80 + 40x_1')(5 - x_1')dx_1' = -\frac{1040}{3EI_2} \tag{c}$$

STEP 3 We compute the flexibility coefficient F_{11} of the primary structure. As shown in Fig. e the flexibility coefficient F_{11} is equal to the deflection of end 1 of the primary structure subjected to $X_1 = 1$. Substituting relation (b) in (13.35), we have

$$F_{11} = \int_0^5 \frac{\tilde{M}_2\tilde{M}_2}{EI_2} dx_1' = \frac{1}{EI_2}\int_0^5 (5 - x_1')^2 dx_1' = \frac{125}{3EI_2} \tag{d}$$

STEP 4 We compute the redundant X_1 by superimposing the results of steps 2 and 3. Since end 1 of the actual structure does not move Δ_1^s is equal to zero. Substituting relations (c) and (d) into (14.2), we obtain

$$X_1 = -\frac{\Delta_1^{PL}}{F_{11}} = \frac{1040}{125} = 8.32 \text{ kN} \tag{e}$$

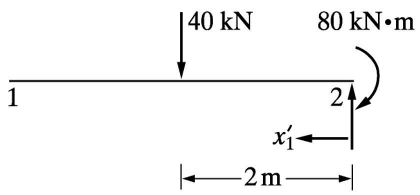


Figure c Free-body diagram of the primary structure subjected to the given force.

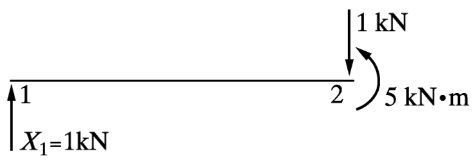


Figure d Free-body diagram of the primary structure subjected to $X_1 = 1$.

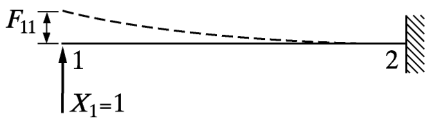


Figure e Flexibility coefficient F_{11} of the primary structure.

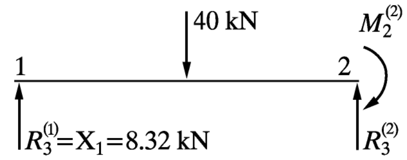


Figure f Free-body diagram of the beam subjected to the given loading.

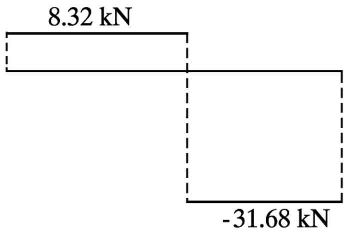


Figure g Shear diagram for the beam of Fig. a.

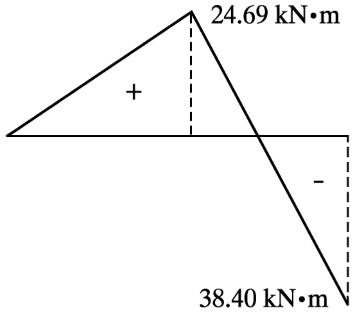


Figure h Moment diagram for the beam of Fig. a.

STEP 5 We compute the reactions at the end 2 of the beam by considering its equilibrium. Thus referring to Fig. f and using the value of X_1 established in step 4, we get

$$\sum F_3 = 0 \qquad R_3^{(2)} = 40 - 8.32 = 31.68 \text{ kN}$$

$$\sum M_2 = 0 \qquad M_2^{(2)} = -8.32(5) + 40(2) = 38.40 \text{ kN}\cdot\text{m}$$

The shear and moment diagrams for the beam of Fig. a are shown in Figs. g and h, respectively.

Example 2 Compute the internal forces $N^{(e)}$ in the members of the statically indeterminate truss shown in Fig. a. The members of the truss have the same constant cross section and are made from the same material.

Notice that the internal forces in the members of the truss will be the same whether or not members 3 and 6 are joined together at point 5. This becomes apparent by assuming that members 3 and 6 are joined at point 5 and considering the equilibrium of forces at

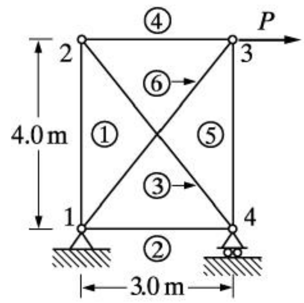


Figure a Geometry and loading of the truss.

joint 5. The force in member 1, 5 must be equal to that of member 5, 3 while the force in member 2, 5 must be equal to that in member 5, 4.

Solution

STEP 1 The truss under consideration is statically indeterminate to the first degree. We choose the force in member 6 as the redundant ($N^{(6)} = X_1$) and form the primary structure by cutting member 6.

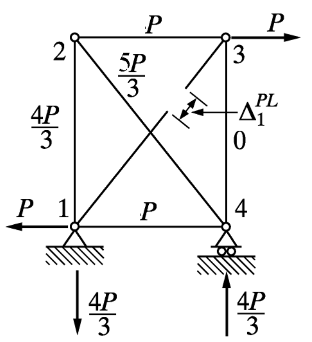


Figure b Primary structure subjected to the given loading.

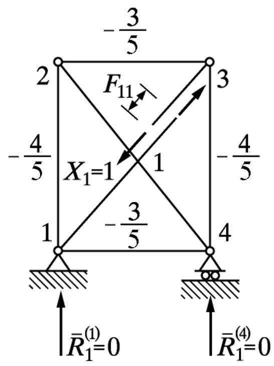


Figure c Primary structure subjected to a pair of equal and opposite unit forces.

STEP 2 We compute the relative movement Δ_1^{PL} of the ends of the cut of member 6 of the primary structure subjected to the given loading (see Fig. b). For this purpose the following quantities must be computed:

1. The internal forces $N^{PL(e)}$(e = 1, 2, ..., 6) in the members of the primary structure subjected to the given loading (see Fig. b).

2. The internal forces $\tilde{N}^{(e)}$(e = 1, 2, ..., 6) in the members of the primary structure subjected to a pair of equal and opposite unit forces X_1 = 1 (see Fig. c).

The results are given in Table a. Using relation (13.15) and referring to Table a we have

Table a Computation of the internal forces in the members of the truss.

Member	L_i	$N^{PL(e)}$	$\tilde{N}^{(e)}$	$N^{PL(e)}\tilde{N}^{(e)}L_e$	$\tilde{N}^{(e)}\tilde{N}^{(e)}L_e$	$N^{(e)} = N^{PL(e)} + \tilde{N}^{(e)}X_1$
1	4.0	1.333P	−0.8	−4.2667P	2.56	0.5833P
2	3.0	1.00 P	−0.6	−1.8000P	1.08	0.4375P
3	5.0	−1.667P	1.0	−8.3333P	5.00	−0.7292P
4	3.0	1.00 P	−0.6	−1.8000P	1.08	0.4375P
5	4.0	0	−0.8	0	2.56	−0.7500P
6	5.0	0	1.0	0	5.00	0.9375P
Total				−16.2P	17.28	

$$\Delta_1^{PL} = \sum_{e=1}^{6}\left[\int_0^L \frac{N\tilde{N}}{EA}dx_1\right]^{(e)} = \frac{1}{EA}\sum_{e=1}^{6}[N^{PL(e)}\tilde{N}^{(e)}L_e] = -\frac{16.2P}{EA}$$

STEP 3 We compute the flexibility coefficient F_{11} of the truss corresponding to the chosen redundant. The flexibility coefficient is equal to the relative movement of the cut ends of member 6 of the primary structure subjected to $X_1 = 1$ (see Fig. c). Using relation (13.35) and referring to the Table a, we have

$$F_{11} = \sum_{e=1}^{6}\left[\int_0^L \frac{\tilde{N}\tilde{N}}{EA}dx_1\right]^{(e)} = \frac{1}{EA}\sum_{e=1}^{6}[\tilde{N}^{(e)}\tilde{N}^{(e)}L_e] = \frac{17.28}{EA}$$

STEP 4 We compute the redundant by superimposing the results of steps 2 and 3. Since in the actual structure, the relative movement of the cut ends of member 6 is zero ($\Delta_1^s = 0$), from relation (14.2) we get

$$X_1 = -\frac{\Delta_1^{PL}}{F_{11}} = \frac{15P}{16} = \frac{16.20P}{17.28} = 0.9375P$$

STEP 5 We compute the internal forces in the members of the truss using relation (14.4). That is

$$N^{(e)} = N^{PL(e)} + \tilde{N}^{(e)}X_1$$

The results are tabulated in Table a and in Fig. d.

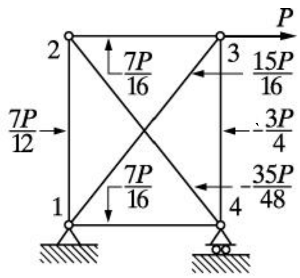


Figure d Internal forces in the members of the truss subjected to the given force.

14.2 Computation of Components of Displacement of Points of Statically Indeterminate Structures

In this section we apply the unit load method to compute the components of displacement of points of statically indeterminate structures. That is, we employ the following relation:

$$d + \sum_{s=1}^{S}\tilde{R}^{(s)}\Delta^{(s)} = \sum_{k=1}^{NM}\left[\int_0^L\left[\frac{N\tilde{N}}{AE} + \frac{M_1\tilde{M}_1}{KG} + \frac{M_2\tilde{M}_2}{EI_2} + \frac{M_3\tilde{M}_3}{EI_3}\right]\right]$$

$$+ \int_0^L \left[\alpha \tilde{N} \Delta T_c + \alpha \tilde{M}_2 \frac{\delta T_3}{h_3} + \alpha \tilde{M}_3 \frac{\delta T_2}{h_2} \right] dx_1 \Bigg]^{(k)} \tag{14.5}$$

where d is either a component of translation or a component of rotation of a point of the structure. The symbols $\tilde{N}$ and $\tilde{M}_i$ ($i = 1, 2, 3$) denote a set of statically admissible (not necessarily the actual) distributions of the internal axial force and the components of moment, respectively, in the members of the structure subjected to an auxiliary loading. This auxiliary loading consists of a unit load applied at the point of the structure where the displacement is desired. If we want to compute the component of translation of a point of a structure in the direction of the unit vector $\mathbf{i_n}$, the auxiliary loading is a unit force acting in the direction of the unit vector $\mathbf{i_n}$. If we want to compute the component of rotation of a point of a structure about an axis specified by the unit vector $\mathbf{i_m}$, the auxiliary loading is a unit moment whose vector is in the direction of the unit vector $\mathbf{i_m}$. The symbol $\Delta^{(s)}$ ($s = 1, 2, ..., S$) denotes the given components of displacements (translations or rotations) of the supports of the structure. The symbol $\tilde{R}^{(s)}$ ($s = 1, 2, ..., S$) denotes the statically admissible components of the reactions (forces and moments) at the supports of the structure corresponding to the given components of displacements $\Delta^{(s)}$, when the structure is subjected to the auxiliary loading. $R^{(s)}$ is considered positive when it acts in the direction of $\Delta^{(s)}$.

In what follows, we compute components of displacement of a statically indeterminate truss and a statically indeterminate frame using the method of virtual work.

Example 3 Compute the horizontal and vertical components of translation of joint 3 of the statically indeterminate truss shown in Fig. a. The members of the truss have the same constant cross section and are made from the same material.

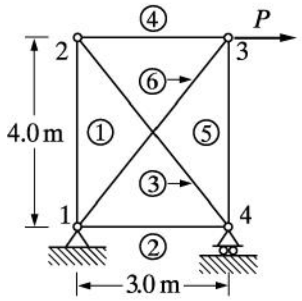


Figure a Geometry and loading of the truss

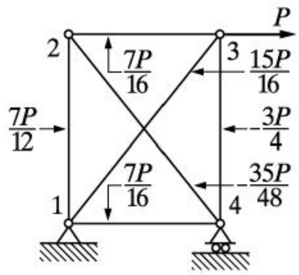


Figure b Internal forces in the members of the truss subjected to the given force.

Solution The internal forces in the members of this truss have been established in Example 3 of Section 14.2. They are shown in Fig. b.

In order to compute the horizontal component of translation of joint 3, we consider the truss subjected at this joint to a horizontal unit force (see Fig. c). This truss is statically indeterminate to the first degree. That is, it has one member more than the minimum required for not forming a mechanism. Thus, we can arbitrarily select the value of the internal force in one member of the truss and compute a statically admissible set of internal forces in its other member. For example, we can set the force in member 3 equal to zero. By considering the equilibrium of joint 2, we can see (see Fig. c) that the forces in members 1 and 4 must vanish. Moreover, by considering the equilibrium of joint 3, we an establish the forces in members 6 and 5. The results are shown in Fig. c. Thus, using the internal forces shown in Figs. b and c, we find that relation (14.5) gives

$$u_n^{(3)} = \frac{1}{AE}\left[\frac{4}{3}\left(\frac{3P}{4}\right)4 + \frac{5}{3}\left(\frac{15}{16}\right)5\right] = \frac{189P}{16AE} \qquad \text{to the right}$$

In order to compute the vertical component of translation of joint 3, we consider the truss subjected at this joint to a vertical unit force (see Fig. d). We choose the convenient, statically admissible distribution of internal forces corresponding to this loading shown in Fig. d. Using the internal forces in the members of the truss shown in Figs. b and d, we find that relation (14.5) gives

$$u_v^{(3)} = \left(-\frac{3P}{4}\right)\frac{4}{AE} = -\frac{3P}{AE} = \frac{3P}{AE} \qquad \text{downward}$$

The minus sign indicates that the displacement is in the direction opposite to that of the unit force applied to the truss in Fig. d.

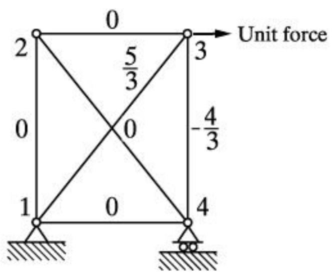


Figure c Statically admissible internal forces in the members of the truss subjected to a unit horizontal force.

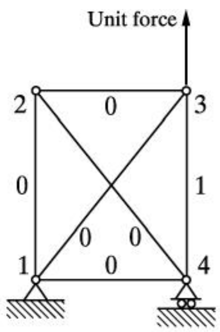


Figure d Statically admissible internal forces in the members of the truss subjected to a unit vertical force.

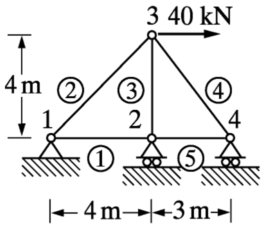
14.3 Problems

Note: In these problems disregard the effect of shear deformation of the members of the structures.

1. and **2**. Using the basic force method, compute the internal forces in the members of the truss resulting from the external forces shown in Fig. 14P1. The members of the truss are made from the same material (E = 210 GPa). Repeat with the trusses of Figs. 14P2 and 11P3.

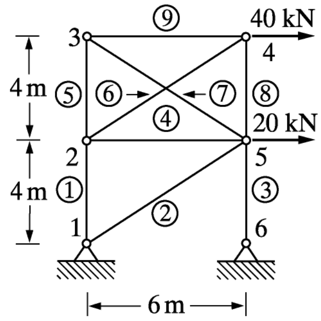
Ans. 1 $N^{(1)}$ = 16.12 kN tension $N^{(2)}$ = 33.77 kN tension $N^{(3)}$ = 2.380 kN compression
$N^{(4)}$ = 26.87 kN compression

Ans. 2 $N^{(1)}$ = 26.67 kN tension $N^{(2)}$ = 72.0 kN tension $N^{(3)}$ = 66.67 kN compression
$N^{(4)}$ = 52.70 kN compression $N^{(5)}$ = 8.93 kN tension $N^{(6)}$ = 32.75 kN tension



Member	Cross sectional area
1,3,5	3×10^3 mm^2
2	4×10^3 mm^2
4	6×10^3 mm^2

Figure 14P1



Member	Cross sectional area
1,5	3×10^3 mm^2
8,3	6×10^3 mm^2
All others	2×10^3 mm^2

Figure 14P2

3. and **4**. Using the basic force method, establish the reactions of the beam resulting from the external actions shown in Fig. 14P3. The members of the beam have the same constant cross section and are made of the same material (E = 210 GPa). Choose the reaction at support 2 as the redundant. Plot the shear and moment diagrams for the beam. Repeat with the beams of Fig. 14P4.

Ans. 3 $R_v^{(2)} = 2.5P$, $M_2^{(1)} = 0$ Ans. 4 $R_v^{(1)} = 0.9687P$, $R_v^{(2)} = 1.0625P$, $R_v^{(3)} = 0.9688P$

5. Using the basic force method, establish the reactions of the beam of Fig. 14P4. Choose the internal moment at point 2 as the redundant. Plot the shear and moment diagrams for the beam. Ans. see Problem 4

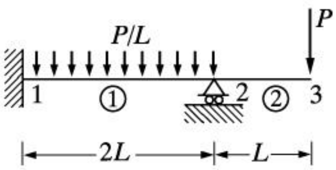


Figure 14P3

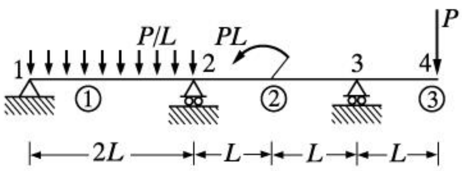


Figure 14P4

6. Using the basic force method, establish the reactions of the beam of Fig. 14P3. Choose the internal moment of point 1 as the redundant. Plot the shear and moment diagrams for the beam. Ans. see Problem 3

7. Using the basic force method, establish the reactions of the structure resulting from the external forces shown in Fig. 14P7. The members of the structure have the same constant cross section and are made from the same material. Choose the reaction at support 3 as the redundant. Plot the shear and moment diagrams for the structure. Disregard the effect of the axial deformation of the members of the structure.

Ans. $R_v^{(1)} = 36.25$, $R_h^{(1)} = 0$, $R_v^{(2)} = 65.50$ kN, $R_v = 118.75$ kN

8. Using the basic force method, establish the reactions of the beam resulting from the external forces shown in Fig. 14P8. The beam is made from a standard steel shape with cover plates extending 2 m on each side of support 2. Plot the shear and moment diagrams for the beam.

Ans. $R_v^{(1)} = 2.07$ kN, $R_v^{(2)} = 47.20$ kN, $R_v^{(3)} = 10.73$ kN

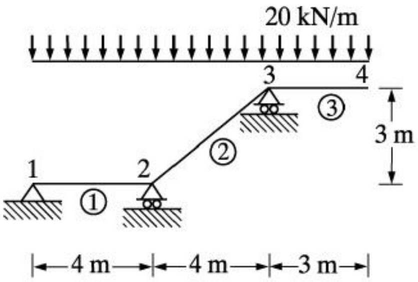


Figure 14P7

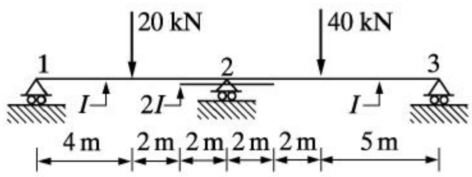


Figure 14P8

9. Using the basic force method, compute the reactions of the steel (E = 210 GPa) structure subjected to the external force shown in Fig. 14P9. The area of the cross section of the steel cable is $A_c = 800$ mm^2. The moment of inertia of the beam is $I = 369.7$ (10^6) mm^4. Disregard the effect of the axial deformation of beam 1, 3, 4. Plot the shear and moment diagrams for the beam.

Ans. $R_v^{(1)} = 76.40$ kN, $R_h^{(1)} = 235.20$ kN, $M_2^{(1)} = 58.41$ kN·m

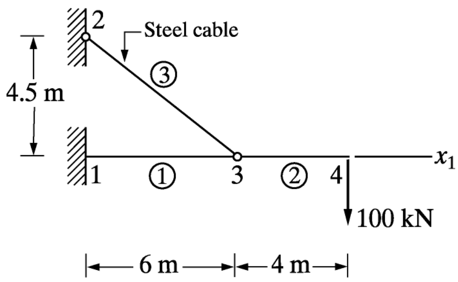


Figure 14P9

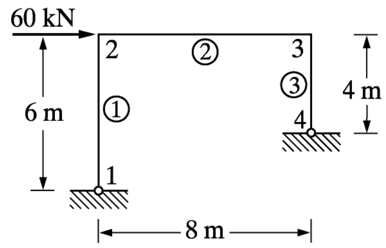


Figure 14P10

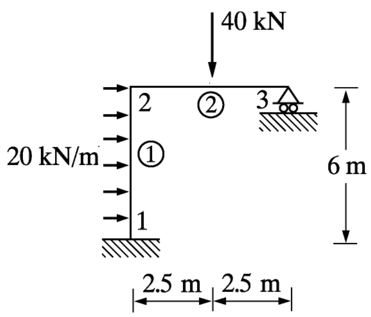


Figure 14P11

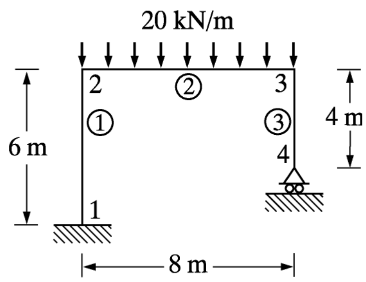


Figure 14P12

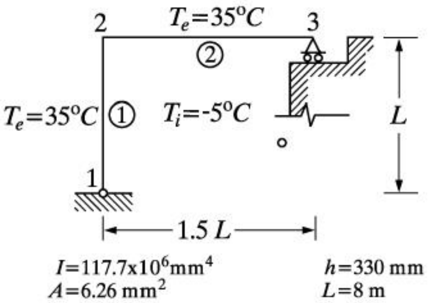
10. to **12**. Using the basic force method, plot the shear and moment diagrams for the frame subjected to the external actions shown in Fig. 14P10. The members of the frame are made from the same material and have the same constant cross section. Disregard the effect of axial deformation of the members of the frame. Repeat with the frames of Figs. 14P11 and 14P12.

Ans. 10 $R_v^{(1)} = 34.87$ kN, $R_h^{(1)} = 19.46$ kN, $R_h^{(4)} = 40.54$ kN
Ans. 11 $R_v^{(1)} = 7.28$ kN, $R_h^{(1)} = 160.0$ kN, $M_2^{(1)} = 121.83$ kN·m

Ans. 12 $R_v^{(1)} = 86.15$ kN, $R_h^{(1)} = 0$, $M_2^{(1)} = 49.23$kN·m

13. and **14**. Using the basic force method, compute the reactions of the structure and plot its shear and moment diagrams resulting from the temperature distribution shown in Fig. 14P14. The temperature during construction was T_o = 15°C. The members of the structure are made of the same material (E = 210 GPa, $\alpha = 10^{-5}$/°C), and have the same constant cross section. Repeat with the structures of Fig. 14P14.

Ans. 13 $R_v^{(1)} = 2.91$ kN, $M_2^{(1)} = 34.92$ kN·m
Ans. 14 $R_v^{(1)} = 5.11$ kN, $R_v^{(2)} = -8.94$ kN, $R_v^{(3)} = 3.83$ kN



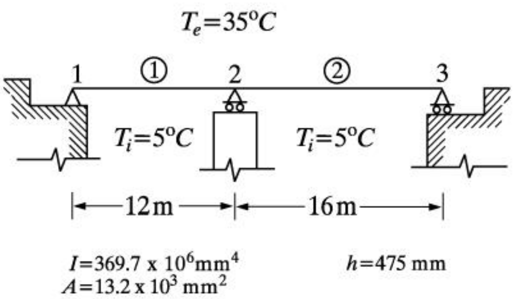
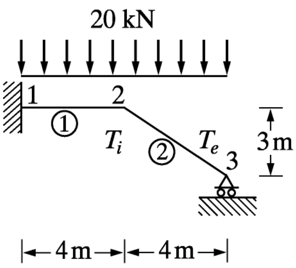


Figure 14P13 **Figure 14P14**

15. and **16**. Using the basic force method, for the structure shown in Fig. 14P13, establish the reactions and plot the shear and moment diagrams resulting from settlement of support 1 of 20 mm. The members of the structure are made from the same material (E = 210 MPa) and have the same constant cross section whose properties are given in Fig. 14P13. Disregard the effect of axial deformation of the members of the structure. Repeat with the structures of Fig. 14P14.

Ans. 15 $R_V^{(3)} = 2.860$ kN ↑ $M_2^{(1)} = 3.433$ kN·m
Ans. 16 $R_V^{(1)} = 1.155$ kN ↓ $R_V^{(2)} = 2.021$ kN·m ↑



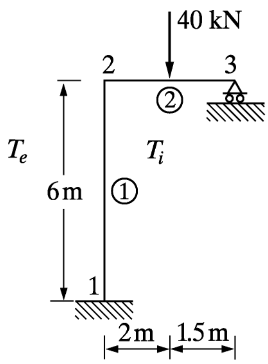


Figure 14P17 **Figure 14P18**

17. and **18**. Using the basic force method, analyze the frame shown in Fig. 14P17 for each of the following loading cases and plot its shear and moment diagrams:

(a) The external actions shown in Fig. 14P17.
(b) A settlement of 20 mm of support 1.
(c) A temperature of the external surface of its members of T_e = 35°C and of the

internal surface of its members of $T_i = -5°C$. The temperature during construction was 15°C.

The members of the frame are made from the same material ($E = 210$ GPa, $\alpha = 10^{-5}/°C$) and have the same constant cross section [$I = 162.7\ (10^6)\ mm^4$, $h = 360$ mm]. Disregard the effect of shear and axial deformation of the members of the frame. Repeat for the structure of Fig. 14P18.

Ans. 17 (a) $R_v^{(3)} = -59.09$ kN, $R_v^{(1)} = 100.91$ kN, $M_2^{(1)} = 167.36$ kN·m (b) $R_v^{(1)} = -3.88$ kN, $R_v^{(3)} = 3.88$ kN, $M_2^{(1)} = 31.04$ kN·m
(c) $R_v^{(3)} = -7.13$ kN, $R_v^{(1)} = 7.13$ kN, $M_2^{(1)} = 58.64$ kN·m

Ans. 18 (a) $R_v^{(1)} = 18.28$ kN, $R_h^{(1)} = 0$, $R_v^{(3)} = 21.72$ kN, $M_2^{(1)} = 3.99$ kN·m
(b) $R_v^{(1)} = 7.78$ kN, $R_h^{(1)} = 0$, $M_2^{(1)} = 27.24$ kN·m (c) $R_v^{(1)} = 11.73$ kN, $R_h^{(1)} = 0$, $M_2^{(1)} = 41.05$ kN·m

19. Using the unit load method, compute the horizontal component of translation of joint 3 of the truss of Fig. 14P2. Ans. $u_h^{(2)} = 3.5$ mm

20. Using the unit load method, compute the vertical component of translation of the joint 4 of the truss of Fig. 14P2. Ans. $u_v^{(3)} = 0.875$ mm

21. and **22**. Using the unit load method, compute the deflection of joint 3 of the beam of Fig. 14P3. Repeat with joint 4 of the beam of Fig. 14P4. Ans. 21 $u_v^{(3)} = 2PL/3EI_2$

23. Using the unit load method, compute the deflection of point 4 of the structure of Fig. 14P9. Ans. $u_v^{(4)} = 102.16$ mm, $\theta_2^{(4)} = 0.0235$ rad.

24. Using the unit load method, compute the horizontal component of translation of joint 3 of the frame of Fig. 14P12 [$I_2 = 369.70(10^6)mm^4$, $E = 210$ GPa]. Ans. $u_h^{(3)} = 11.4$ mm

25. Using the unit load method, compute the horizontal component of translation of joint 3 of the frame of Problem 13.

Ans. 25 $u_h^{(3)} = 6.46$ mm

Chapter 15

The Finite Element Method

15.1 Introduction

The weighted residual methods for constructing approximate solutions of boundary value problems presented in Sections 13.8 to 13.12, have the following serious shortcomings especially when applied to two- or three-dimensional boundary value problems:

1. There is not a systematic procedure available for choosing the functions ϕ_0 and ϕ_s (s = 1, 2, ..., S) of the approximate solution (13.54) on whose choice the accuracy of the solution depends. A poor choice of the trial functions ϕ_s (s = 1, 2, ..., S) may produce an ill-conditioned stiffness matrix [S] and thus, it may be difficult to solve the resulting set of linear algebraic equations (13.72).

2. It is not easy to find functions ϕ_0 and ϕ_s (s = 1, 2, ..., S) which satisfy all the essential boundary conditions of two- or three-dimensional problems having boundaries of complex geometry.

Some of these difficulties are avoided by using the *finite element method*, which is presented in this chapter. This method was originally developed from physical considerations as a method for analyzing framed structures known as the *direct stiffness method* (see Section 15.8). However, it was soon recognized that it is a very effective tool for the approximate solution of a great variety of boundary value problems.

In Sections 15.2 to 15.6, we present the finite element method as it applies to one dimensional, second order, linear boundary value problems. The application of the finite element method to those problems is of limited practical interest because usually exact solutions can be easily established for them, or in cases that they cannot, approximate solutions can be established using less sophisticated methods. Our purpose in these sections, is to introduce the fundamental concepts of the finite element method using these boundary value problems.

15.2 The Finite Element Method for One-Dimensional, Second Order, Linear Boundary Value Problems as a Modified Galerkin Method

In the weighted residual methods presented in Sections 13.8 to 13.12, each of the

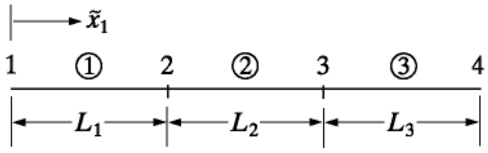


Figure 15.1 One-dimensional domain subdivided into three two-node elements.

functions $\bar{\phi}_0(\bar{x}_1)$† and $\bar{\phi}_s(\bar{x}_1)$ (s = 1, 2, ..., S) of the approximate solution (13.54) is defined with respect to the global† coordinate $\bar{x}_1$ by one expression which is valid throughout the domain of the problem. In the finite element method the domain Ω $(0 \leq \bar{x}_1 \leq L)$ of the problem is divided into a number (say n) of non-overlapping linear subdomains Ω_e $(e = 1, 2, ..., n)$ called *finite elements*. Moreover, certain key points called *nodes or nodal points* are chosen at which the values of the state variable and flux are computed. At least the ends of the elements of a one-dimensional domain must be chosen as its nodes. However, other points along the length of an element may be chosen as nodes. In Fig. 15.1 a one-dimensional domain is divided into three two-node elements numbered consecutively†† from 1 to 3. Element numbers are enclosed in a circle. The nodes of the domain are also numbered consecutively from 1 to 4.

As in the weighted residual methods, presented in Sections 13.8 to 13.12, in the finite element method the solution of one-dimensional boundary value problems is approximated by relation (13.54). However, in the finite element method the function $\bar{\phi}_0(\bar{x}_1)$ is taken equal to zero. That is, the approximation (13.54) to the state variable $\bar{X}(\bar{x}_1)$ becomes

$$\tilde{X}(\bar{x}_1) = \sum_{s=1}^{S} \tilde{c}_s \bar{\phi}_s(\bar{x}_1) = [\bar{\phi}] \left\{\tilde{c}_s\right\} \tag{15.1a}$$

where

$\tilde{c}_s$ (s = 1, 2, ..., S) = constants which as we show later, have physical meaning.

$\bar{\phi}_s(\bar{x}_1)$ (s = 1, 2, ..., S) = *global functions known as trial or basis functions*.

S = total number of nodes in the domain.

$$[\bar{\phi}] = [\bar{\phi}_1, \bar{\phi}_2, ..., \bar{\phi}_S] \tag{15.1b}$$

In the finite element method, the trial function $\bar{\phi}_s(\bar{x}_1)$ (s = 1, 2, ..., S) are defined in an element-wise fashion using a different expression in each element. Howev er, these expressions must meet a continuity requirement in order that the integrals in relations (13.79) and (13.84) exist. That is, the trial functions used for one-dimensional, second order, linear boundary value problems must be at least continuous throughout the domain of the problem. More specifically, the trial function $\bar{\phi}_s(\bar{x}_1)$ must have the following

† A bar over a symbol representing a function indicates that this symbol represents the function over the entire domain of the problem. A bar over a coordinate indicates that the coordinate is global. That is, it specifies all the points of a domain.

†† The numbering of the nodes of a domain could affect significantly the efficiency of the computations performed in obtaining a solution. The best way to number one-dimensional domains is consecutively starting from their one end.

properties:

(a) It must be continuous functions throughout the domain of the problem.
(b) It must vanish on all nodes except node s.
(c) It must be equal to unity on node s.
(d) It must vanish on all elements except those adjacent to node s.
(e) In each of the elements neighboring node s it is chosen to be a polynomial.

On the basis of the afore described properties of trial functions, at node n the trial function $\bar{\phi}_n$ is equal to unity while all other trial functions vanish. Thus, referring to relation (15.1) and denoting by $\bar{x}_1^{(n)}$ the global coordinate of node n, the approximate value of the function $\bar{X}(\bar{x}_1)$ at node n is equal to

$$\tilde{X}(\bar{x}_1^{(n)}) = \sum_{s=1}^{S} \tilde{c}_s \bar{\phi}_s = \tilde{c}_n \tag{15.2}$$

That is, the undetermined coefficient $\tilde{c}_n$ in relation (15.1) has physical meaning. It represents the approximate value of the state variable $X(\bar{x}_1)$ at node n. This property of the trial functions permits the direct satisfaction of the essential boundary conditions of a problem. For example, if the function $\tilde{X}(\bar{x}_1)$ is specified at $\bar{x}_1 = 0$, the coefficient $\tilde{c}_1$ is set equal to the specified value $\bar{X}(0)$. If the function $\tilde{X}(\bar{x}_1)$ is specified at $\bar{x}_1 = L$, the coefficient $\tilde{c}_S$ is set equal to the specified value $\tilde{X}(L)$.

The simplest set of trial functions for one-dimensional, second order, linear boundary value problems which meet all the requirements described above, consist of element-wise linear functions which referring to Fig. 15.2 have the following form:

$$\bar{\phi}_1 = \begin{cases} 1 - \dfrac{\bar{x}_1}{L_1} & \text{for } 0 \le \bar{x}_1 \le \bar{x}_1^{(1)} \\ 0 & \text{for } \bar{x}_1 \ge \bar{x}_1^{(1)} \end{cases}$$

$$\bar{\phi}_s = \begin{cases} \dfrac{\bar{x}_1 - \bar{x}_1^{(s-1)}}{L_{(s-1)}} & \text{for } \bar{x}_1^{(s-1)} \le \bar{x}_1 \le \bar{x}_1^{(s)} \\ 1 - \dfrac{\bar{x}_1 - \bar{x}_1^{(s)}}{L_s} & \text{for } \bar{x}_1^{(s)} \le x_1 \le x_1^{(s+1)} \\ 0 & \text{for } \bar{x}_1^{(s+1)} \le \bar{x}_1 \le \bar{x}_1^{(s-1)} \end{cases} \tag{15.3}$$

$$s = 2, 3....., (S-1).$$

$$\bar{\phi}_S = \begin{cases} \dfrac{\bar{x}_1 - \bar{x}_1^{(S-1)}}{L_{(S-1)}} & \text{for } \bar{x}_1^{(S-1)} \le \bar{x}_1 \le \bar{x}_1^{(S)} \\ 0 & \text{for } \bar{x}_1 \le \bar{x}_1^{(S-1)} \end{cases}$$

where

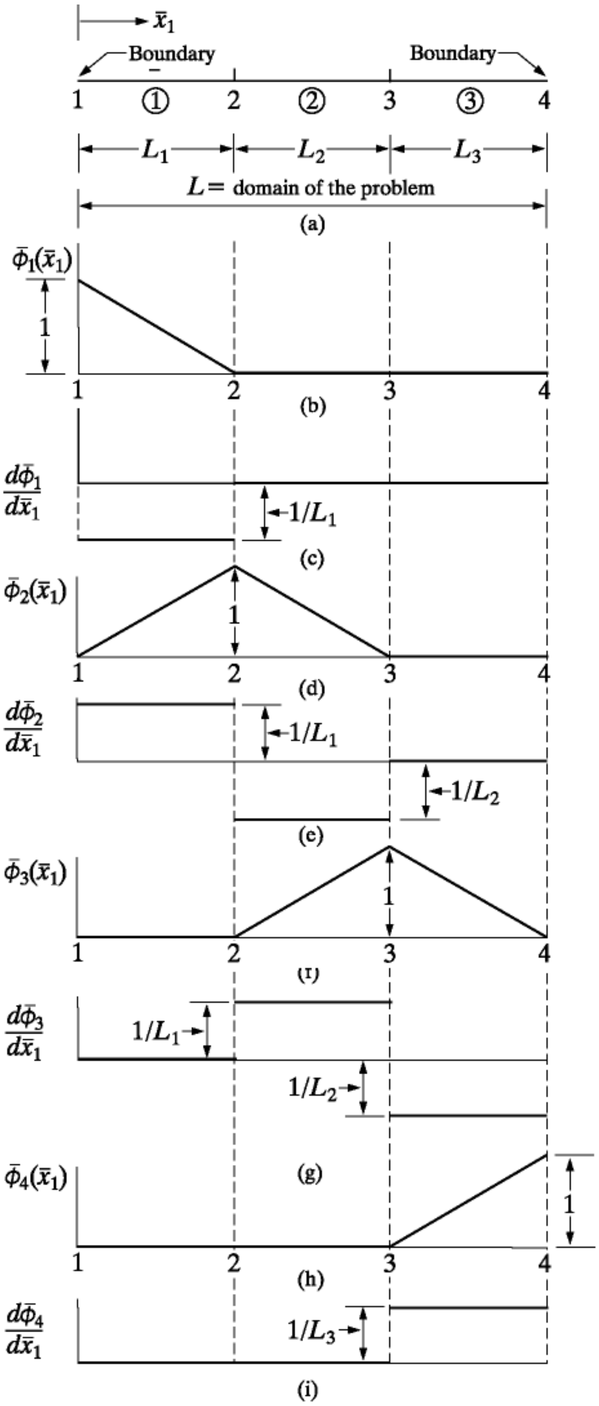


Figure 15.2 The simplest trial functions and their derivatives for one-dimensional, second order, linear boundary value problems.

$\bar{x}_1^{(s)}$ = global coordinate of node s.

L_s or L_{s-1} = length of element s or $s - 1$, respectively.

The trial functions (15.3) and their first derivatives are plotted in Fig 15.2 for a domain $0 \leq \bar{x}_1 \leq L$ which is subdivided into three elements. They are polynomials of the first degree on the elements on which they do not vanish.

Notice that the first derivatives of the trial functions (15.3) are continues in the domain of each element. *Consequently, a state variable whose first derivative has a simple discontinuity at a point inside the domain of an element cannot be approximated properly in the domain of this element by relation (15.1) with the trial functions (15.3)*. For this reason, the points of a domain where the first derivative of the state variable has a simple discontinuity or jump are usually chosen as nodes. For example, when a line member subjected to axial centroidal forces, is made from two isotropic, linear elastic materials, in series, its modulus of elasticity $E(\bar{x}_1)$ and its coefficient of thermal expansion $\alpha(\bar{x}_1)$ have a simple discontinuity or jump at the interface of the two materials. As can be deduced from relation (8.61) the component of translation $\bar{u}_1(\bar{x}_1)$ of such a member is expressed by different expressions in each material. Consequently, its first derivative has a simple discontinuity of jump at the point of change of material. Similarly, when the cross-section of a line member subjected to axial centroidal forces changes abruptly at a point, its cross-sectional area has a simple discontinuity or jump at that point and the first derivative of the component of translation $\bar{u}_1(\bar{x}_1)$ has also a simple discontinuity or jump at that point. *For this reason, we choose as nodes all the points* of abrupt change of the material and of the cross-sectional area of a member. However, often it is not practical to choose as nodes all the points of Dirac delta-type discontinuity of the source function $J(\bar{x}_1)$†. At these points the internal force $\bar{N}(\bar{x}_1)$ (see Fig. 8.19) and the first derivative of the component of translation $\bar{u}_1(\bar{x}_1)$ have also a simple discontinuity or jump [see relation (8.58)]. *Consequently, relation (15.1) with (15.3) does not approximate properly the state variable in the domain of elements wherein the source function* $J(\bar{x}_1)$ *has a Dirac delta-type discontinuity. Nevertheless, the values of the state variable and the flux at the nodes of domains subdivided into elements of constant modulus of elasticity and cross-sectional area, obtained using the approximation (15.1) with (15.3), are exact (see example of Section 15.7).*

15.2.1 Discretization of One-Dimensional, Second Order, Linear, Boundary Value Problems

In order to add physical meaning to our presentation, we focus our discussion in Sections 15.2.1 to 15.6 to the boundary value problem for computing the component of translation $u_1(\bar{x}_1)$ of a member subjected to axial centroidal forces. For this problem, we write relation (15.1) as

$$\tilde{u}_1(\bar{x}_1) = \sum_{s=1}^{S} \tilde{\Delta}_s \bar{\phi}_s(\bar{x}_1) = [\bar{\phi}_s]\{\tilde{\Delta}_s\} \tag{15.4a}$$

where $\tilde{\Delta}_s$, is equal to the axial component of translation of node s and

† The source function $J(\bar{x}_1)$ has a Dirac delta-type discontinuity at a point where a concentrated axial centroidal force is applied [see relation (13.49)].

$$\left\{\tilde{\Delta}\right\} = \begin{Bmatrix} \tilde{\Delta}_1 \\ \tilde{\Delta}_2 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\Delta}_S \end{Bmatrix} \tag{15.4b}$$

The coefficients $\tilde{\Delta}_s$ (s = 1, 2, 3, ..., S) of the approximate solution (15.4a) can be established by substituting relation (15.4a) in the modified weighted residual equation (13.79) with the Gallerkin assumption (13.75) or using the Rayleigh–Ritz method presented in Section 13.17.2. For example, for the one member structure of Fig. 8.23a, substituting the approximate solution (15.4) into relations (13.79) and using relations (13.75) and (13.80), we have

$$\begin{aligned} & - \int_0^L EA \frac{d\bar{\phi}_r}{d\bar{x}_1} \left(\sum_{s=1}^{S} \tilde{\Delta}_s \frac{d\bar{\phi}_s}{d\bar{x}_1} - H_1 \right) d\bar{x}_1 \\ & + \int_0^L \bar{\phi}_r \left[p_1(\bar{x}_1)\, \Delta(\bar{x}_1 - c_1) + \sum_{n=1}^{n_1} P_1^{(n)} \delta(\bar{x}_1 - a_{1n}) \right] d\bar{x}_1 \\ & - (\bar{\phi}_r \tilde{N})_{\bar{x}_1 = 0} + (\bar{\phi}_r P_1^L)_{\bar{x}_1 = L} = 0 \qquad r = 1, 2, ..., S \end{aligned} \tag{15.5}$$

where $\tilde{\Delta}_s$ is an approximation to the axial component of translation of node s. Relations (15.5) represent a set of S linear algebraic equations which can be written as

$$[S]\left\{\tilde{\Delta}\right\} = \{F\} \tag{15.6a}$$

or

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdot\cdot\cdot\cdot\cdot & S_{1S} \\ S_{21} & S_{22} & S_{23} & \cdot\cdot\cdot\cdot\cdot & S_{2S} \\ S_{31} & S_{32} & S_{33} & \cdot\cdot\cdot\cdot\cdot & S_{3S} \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot & \cdot \\ S_{S1} & S_{S2} & S_{S3} & \cdot\cdot\cdot\cdot\cdot & S_{SS} \end{bmatrix} \begin{Bmatrix} \tilde{\Delta}_1 \\ \tilde{\Delta}_2 \\ \tilde{\Delta}_3 \\ \cdot \\ \cdot \\ \tilde{\Delta}_S \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \cdot \\ \cdot \\ F_S \end{Bmatrix} \tag{15.6b}$$

where $[S]$ is the *stiffness matrix*. The stiffness coefficients S_{rs}, are given as

$$S_{rs} = \int_0^L E(\bar{x}_1) A(\bar{x}_1) \frac{d\bar{\phi}_r}{d\bar{x}_1} \frac{d\bar{\phi}_s}{d\bar{x}_1} d\bar{x}_1 \qquad r,s = 1, 2, ..., S \tag{15.7}$$

{F} is the *load vector*. For the one-member structure of Fig 8.23a, we have

$$F_r = \int_0^L E(\bar{x}_1)A(\bar{x}_1)\frac{d\bar{\phi}_r}{d\bar{x}_1}H_1\,d\bar{x}_1$$

$$+\int_0^L \bar{\phi}_r\left[p_1(\bar{x}_1)\Delta(\bar{x}_1 - c_1) + \sum_{n=1}^{n_1} P_1^{(n)}\delta(\bar{x}_1 - a_{1n})\right]d\bar{x}_1 + \tilde{R}_1^0\delta_{1r} + P_1^L\delta_{Sr} \tag{15.8a}$$

$$r = 1, 2, ..., S$$

where δ_{1r} and δ_{Sr} are Kroneker $\delta's$ defined as

$$\delta_{1r} = \begin{cases} 0 & if\ r \neq 1 \\ 1 & if\ r = 1 \end{cases} \qquad \delta_{Sr} = \begin{cases} 0 & if\ r \neq S \\ 1 & if\ r = S \end{cases} \tag{15.8b}$$

In obtaining relations (15.8) we took into account that referring to relation (15.3) the term $(\bar{\phi}_r N)_{\bar{x}_1 = 0} = -\tilde{R}_1^0$ contributes only to F_1 while the term $(\bar{\phi}_r N)_{\bar{x}_1 = L} = P_1^L$ contributes only to F_S.

Referring to Fig. 15.2 and to relations (15.3) and (15.7) we see that in the finite element method the stiffness matrix [S] for the domain of a problem has the following properties:

1. It is symmetric
2. A number of its coefficients vanish. That is,

$$S_{rs} \neq 0 \text{ only if } r = s \text{ or } (s \pm 1) \tag{15.9}$$

Moreover, referring to relations (15.3) and Fig.15.2 we see that the only non-zero parts of the integrals in relations (15.8) are those taken over the elements, which are adjacent to node r. That is, only the loads acting on the elements adjacent to node r and the forces (fluxes) acting on this node affect the term F_r of the load vector $\{F\}$.
Notice that in order for the integral in relations (15.7) and the first integral in relation (15.8a) to exist, the global trial functions $\bar{\phi}_r$ (r = 1, 2, .., S) must be at least of class C^0. That is, they must be continuous in the domain of the problem. However, their first derivatives could have a simple discontinuity or jump at some points of the domain. The trial functions (15.3) meet this requirement.

15.3 Element Shape Functions

We consider a typical two-node element (say element e) of the domain of a one-dimensional, second order, linear boundary value problem and we denote by j and k the global numbers assigned to the nodes at its left and right ends, respectively. Moreover, we denote by $\bar{x}_1^{ej}$ and $\bar{x}_1^{ek}$ the global coordinates of nodes j and k, respectively, of element e. Furthermore, we choose a local coordinate x_1 measured from the end j of the element in the direction $\vec{jk}$. Thus, referring to Fig. 15.3, we have

$$x_1^e = \bar{x}_1 - \bar{x}_1^{ej} \tag{15.10}$$

$$L_e = \bar{x}_1^{ek} - \bar{x}_1^{ej} \tag{15.11}$$

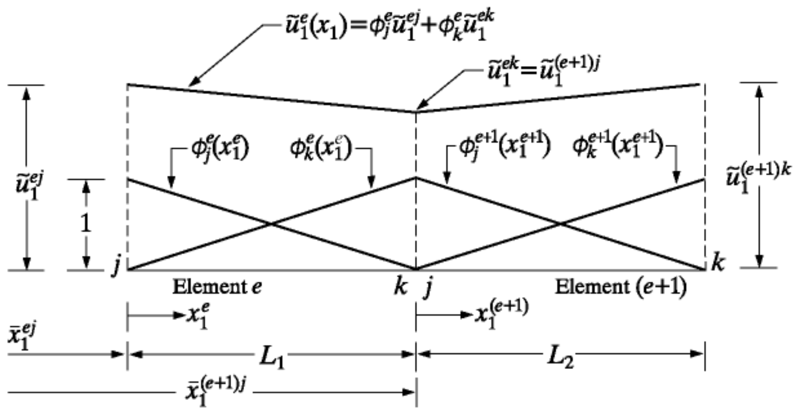


Figure 15.3 Shape functions for elements e and e+1 and approximation of the state variable over these elements.

From Fig. 15.2 we see that only the trial functions associated with nodes j and k of an element do not vanish on this element. We call the parts of these trial functions which extend over the length of element e the *shape* or *interpolation functions for element e* and we denote them by ϕ_j^e and ϕ_k^e (see Fig. 15.3). The shape functions for element e corresponding to the trial functions specified by relations (15.3) can be written (see Fig 15.3) as

$$\phi_j^e = 1 - \frac{x_1}{L_e} = \frac{\bar{x}_1^{ek} - \bar{x}_1}{L_e} \qquad \phi_k^e = \frac{x_1}{L_e} = \frac{\bar{x}_1 - \bar{x}_1^{ej}}{L_e} \tag{15.12a}$$

Thus,

$$\frac{d\phi_j^e}{dx_1} = -\frac{1}{L_e} \qquad \frac{d\phi_k^e}{dx_1} = \frac{1}{L_e} \tag{15.12b}$$

Consider an element (say element e) of length L_e of a member and assume the nodes at its ends are numbered as j and k while their global coordinates are denoted by $\bar{x}_1^{ej}$ and $\bar{x}_1^{ek}$, respectively. Recall that on element e all trial functions $\bar{\phi}_s(\bar{x}_1)$ vanish except $\bar{\phi}_j(\bar{x}_1^{ej} + x_1^e) = \phi_j^e(x_1^e)$ and $\bar{\phi}_k(\bar{x}_1^{ej} + x_1^e) = \phi_k^e(x_1^e)$. Using relation (15.4a), the state variable $\bar{u}_1^e(\bar{x}_1)$ of the boundary value problem for computing the axial component of translation of a member subjected to axial centroidal forces is approximated over element e as

$$\tilde{u}_1^e(x_1) = \bar{u}_1(x_1 + \bar{x}_1^{ej}) = (\bar{\phi}_j\tilde{\Delta}_j + \bar{\phi}_k\tilde{\Delta}_k) = \phi_j^e\tilde{u}_1^{ej} + \phi_k^e\tilde{u}_1^{ek}$$

or

$$\tilde{u}_1^e(x_1) = [\phi^e]\,\{\tilde{D}^e\} \tag{15.13a}$$

where

$$[\phi^e] = [\phi_j^e,\ \phi_k^e] \tag{15.13b}$$

$$\{\tilde{D}^e\} = \begin{Bmatrix} \tilde{u}_1^{ej} \\ \tilde{u}_1^{ek} \end{Bmatrix} \tag{15.13c}$$

$\tilde{u}_1^{ej}$ and $\tilde{u}_1^{ek}$ are approximate values of the state variable $u_1(x_1)$ at $x_1 = 0$ and $x_1 = L_e$, respectively. Relation (15.13a), is known as the *interpolation equation* for element *e*. Substituting relations (15.12) into (15.13a) we get

$$\tilde{u}_1^e(x_1) = \tilde{u}_1^{ej} + (\tilde{u}_1^{ek} - \tilde{u}_1^{ej})\frac{x_1}{L_e} \tag{15.14}$$

That is, in the domain of an element, the state variable $\tilde{u}_1^e(x_1)$ is approximated by a straight line. Consequently, two of its values and therefore two nodes are required in order to specify $\tilde{u}_1^e(x_1)$ in the domain of an element.

The shape functions (15.12) are polynomials of the first degree. They represent the simplest set of shape functions which can be employed in approximating the state variable of one-dimensionals second order, linear boundary value problems in the domain of an element. Polynomials of higher degree (say n) are used as shape functions in approximating the solution of such problems. In this case, however, the state variable is approximated in the domain of an element by a curve of degree n and each element must have $n + 1$ nodes. For example, when second order polynomials are used as shape functions, the state variable $u_1(x_1)$ is approximated by a parabola in the domain of an element and three values of the state variable must be known in order to specify it in the domain of an element. Consequently, each element must have three nodes.

When the interpolation or shape functions are polynomials, the approximate solution in (15.13a) converges to the actual as the size of the elements decreases provided that the shape functions ϕ_j^e and ϕ_k^e are such that relation (15.13a) gives a constant value of the state variable throughout the domain of each element when its nodal values $\tilde{u}_1^{ej}$ and $\tilde{u}_1^{ek}$ are identical $(\tilde{u}_1^{ej} = \tilde{u}_1^{ek} = C)$. Physically, this implies that the shape functions (15.12) can accommodate rigid-body motion of the element. This requirement imposes the following restriction on the shape functions:

$$\tilde{u}_1(x_1) = C = \tilde{u}_1^{ej}\phi_j^e + \tilde{u}_1^{ek}\phi_k^e = C\,(\phi_j^e + \phi_k^e)$$

Thus,

$$\phi_j^e + \phi_k^e = 1 \tag{15.15}$$

It can readily be shown that the shape functions specified by relations (15.12) satisfy relation (15.15). That is, they can accommodate rigid-body motion.

When the exact solution of a boundary value problem is the sum of polynomials, relation (15.13a) represents the exact expression for its state variable provided that the highest degree of the shape functions included in relation (15.13a) exceeds or equals that of the polynomials of the exact solution.

In what follows we show that for elements which have constant modulus of elasticity (E_e) and cross-sectional area (A_e) and are not subjected to external disturbances along their length $[J^e(x_1) = 0,\ H_1^e = 0]$ relation (15.13a) is exact. For such elements equation (8.61) reduces to

$$E_e A_e \frac{d^2 u_1^e}{dx_1^2} = 0 \tag{15.16}$$

In order to establish the state variable u_1^e as a function of the local coordinate x_1 of an element involving its values at the end points of this element, we use the following boundary conditions:

$$\begin{aligned} u_1^e(0) &= u_1^{ej} \\ u_1^e(L) &= u_1^{ek} \end{aligned} \tag{15.17}$$

Integrating equation (15.16) twice, we obtain

$$u_1^e(x_1) = c_{10} + c_{11}x_1 = [1,\ x_1]\begin{Bmatrix} c_{10} \\ c_{11} \end{Bmatrix} \tag{15.18}$$

The constants c_{10} and c_{11} are evaluated by requiring that $u_1^e(x_1)$ satisfies the boundary conditions (15.17). That is,

$$\begin{Bmatrix} u_1^e(0) \\ u_1^e(L_e) \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L_e \end{bmatrix}\begin{Bmatrix} c_{10} \\ c_{11} \end{Bmatrix} = \begin{Bmatrix} u_1^{ej} \\ u_1^{ek} \end{Bmatrix} \tag{15.19}$$

From relation (15.19), we get

$$\begin{Bmatrix} c_{10} \\ c_{11} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L_e \end{bmatrix}^{-1}\begin{Bmatrix} u_1^{ej} \\ u_1^{ek} \end{Bmatrix} = \frac{1}{L_e}\begin{bmatrix} L_e & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^{ej} \\ u_1^{ek} \end{Bmatrix} \tag{15.20}$$

Substituting relation (15.20) into (15.18) and referring to relations (15.12a), we have

$$\begin{aligned} u_1^e(x_1) &= \frac{1}{L_e}[1,\ x_1]\begin{bmatrix} L_e & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1^{ej} \\ u_1^{ek} \end{Bmatrix} = \left[\left(1 - \frac{x_1}{L_e}\right)\ \frac{x_1}{L_e}\right]\begin{Bmatrix} u_1^{ej} \\ u_1^{ek} \end{Bmatrix} \\ &= [\phi_j^e\ \ \phi_k^e]\begin{Bmatrix} u_1^{ej} \\ u_1^{ek} \end{Bmatrix} = [\phi^e]\ \{D^e\} \end{aligned} \tag{15.21}$$

Comparing relation (15.21) with (15.13a) we see that *the approximate expression (15.13a) for the component of translation $u_1(x_1)$ of an element is identical to the exact expression (15.21) when the element is made from one material, has constant cross section (constant E_e and A_e) and is not subjected to external loads along its length.* When an element has constant E_e and A_e but is subjected to loads along its length relation (15.13a) represents an approximate expression for its component of translation $u_1^e(x_1)$. *Thus, we may conclude that the finite element method gives the exact expression for the component of translation $u_1^e(x_1)$ and the internal force $N^e(x_1)$ for members subjected to axial centroidal forces, when they can be subdivided into elements each of which is made from one material, has a constant cross section and is not subjected to external loads along its length.*

15.4 Assembly of the Stiffness Matrix for the Domain of One-Dimensional, Second Order, Linear Boundary Value Problems from the Stiffness Matrices of Their Elements

In the finite element method the stiffness coefficients S_{rs} (r, s = 1, 2, ..., S) and the terms F_r (r = 1, 2 ,..., S) of the load vector {F} for a domain are computed as the sum of the *contributions of each of its elements.* That is, referring to relation (15.7), and denoting by *j* and *k* the global numbers assigned to the nodes at the ends of element e, we have

$$S_{rs} = \int_0^L \bar{E}\bar{A} \frac{d\bar{\phi}_r}{d\bar{x}_1} \frac{d\bar{\phi}_s}{d\bar{x}_1} d\bar{x}_1 = \sum_{e=1}^{NE} \int_{\bar{x}_1^{ej}}^{\bar{x}_1^{ek}} E_e A_e \frac{d\bar{\phi}_r}{d\bar{x}_1} \frac{d\bar{\phi}_s}{d\bar{x}_1} d\bar{x}_1 = \sum_{e=1}^{NE} S_{rs}^e \tag{15.22}$$

Where *NE* is the total number of elements of the domain; $\bar{x}_1^{ej}$ and $\bar{x}_1^{ek}$ are the global coordinates of nodes *j* and *k*, respectively, of element e; S_{rs}^e [$e = 1, 2, \ldots, NE$} represents the contribution of element *e* to the stiffness coefficient S_{rs} for the domain. Referring to relation (15.22), we have

$$S_{rs}^e = \int_{\bar{x}_1^{ej}}^{\bar{x}_1^{ek}} E_e A_e \frac{d\bar{\phi}_r}{d\bar{x}_1} \frac{d\bar{\phi}_s}{d\bar{x}_1} d\bar{x}_1 \qquad r,s = 1,2,\ldots,S \tag{15.23}$$

Substituting relations (15.3) into (15.23) we see that

$$S_{rs}^e = 0 \qquad \text{if } r,s \neq j \text{ or } k \tag{15.24}$$

Thus, the contributions of the three elements of the domain of Fig. 15.2a to its stiffness matrix are

$$[S^{(1)}] = \begin{bmatrix} S_{11}^{(1)} & S_{12}^{(1)} & 0 & 0 \\ S_{21}^{(1)} & S_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [S^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{22}^{(2)} & S_{23}^{(2)} & 0 \\ 0 & S_{32}^{(2)} & S_{33}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [S^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_{33}^{(3)} & S_{34}^{(3)} \\ 0 & 0 & S_{43}^{(3)} & S_{44}^{(3)} \end{bmatrix} \tag{15.25}$$

Notice that many of the terms of the matrices [S^e] (e = 1, 2, 3) vanish. We call the non-vanishing coefficients of the matrix [S^e] the *stiffness coefficients of element e* and we denote them by K_{rs}^e. Moreover, we call the matrix of the non-vanishing coefficients of the matrix [S^e] the *stiffness matrix* for element *e* and we denote it by [K^e]. Thus, the stiffness matrices of the elements of the domain of Fig.15.2a are

$$[K^{(1)}] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{bmatrix} \quad [K^{(2)}] = \begin{bmatrix} K_{22}^{(2)} & K_{23}^{(2)} \\ K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \quad [K^{(3)}] = \begin{bmatrix} K_{33}^{(3)} & K_{34}^{(3)} \\ K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \tag{15.26}$$

Notice that the indices of the stiffness coefficients of each element correspond to the global numbers of the nodes at the ends of the element. Referring to relation (15.22) it is apparent that the stiffness coefficients S_{rs} for the domain of a problem are obtained from the stiffness coefficients of its elements by the following relation:

$$S_{rs} = \sum_{e=1}^{NE} K_{rs}^{e} \qquad r, s = 1, 2, ..., S \tag{15.27}$$

In the above relation the stiffness coefficients for an element vanish if their indices do not correspond to the numbers of the nodes at the ends of the element. Substituting relations (15.26) into (15.27) we obtain the stiffness coefficients for the domain of Fig. 15.2a and we use them to form its stiffness matrix. That is,

$$[S] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & (K_{22}^{(1)} + K_{22}^{(2)}) & K_{23}^{(2)} & 0 \\ 0 & K_{32}^{(2)} & (K_{33}^{(2)} + K_{33}^{(3)}) & K_{34}^{(3)} \\ 0 & 0 & K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} \tag{15.28}$$

From relation (15.28) we see that the stiffness matrix for a domain is sparsely populated and banded. The solution of the stiffness equations (15.6) requires less computer time and storage when the stiffness matrix is sparse and banded than when it is complete. The numbering of the nodes of a domain affects the size of the bandwidth of its stiffness matrix. In obtaining relation (15.28) the nodes were numbered consecutively from 1 to 4 as shown in Fig.15.2. This is the best way to number one-dimensional domains.

Referring to Fig. 15.2 from relations (15.23) the stiffness coefficients for element e are

$$K_{jj}^{e} = \int_0^{L_e} E_e(x_1)A_e(x_1)\left(\frac{d\phi_j^e}{dx_1}\right)^2 dx_1$$

$$K_{jk}^{e} = K_{kj}^{e} = \int_0^{L_e} E_e(x_1)A_e(x_1)\frac{d\phi_j^e}{dx_1}\frac{d\phi_k^e}{dx_1} dx_1$$

$$K_{kk}^{e} = \int_0^{L_e} E_e(x_1)A_e(x_1)\left(\frac{d\phi_k^e}{dx_1}\right)^2 dx_1 \tag{15.29}$$

Where j and k represent the global numbers of the nodes at the ends of element e. Relations (15.29) may be rewritten as

$$[K^e] = \begin{bmatrix} K_{jj}^{e} & K_{jk}^{e} \\ K_{kj}^{e} & K_{kk}^{e} \end{bmatrix} = \int_0^{L_e}\left[E_e(x_1)A_e(x_1)\frac{d\,[\phi^e]^T}{dx_1}\frac{d\,[\phi^e]}{dx_1}\right]dx_1 \tag{15.30}$$

where the matrix $[\phi^e]$ is defined by relation (15.13a).

It is apparent that the stiffness matrix for an element of a member subjected to axial centroidal forces and to a change of temperature ΔT_c depends only on its geometry and on its material properties. It is independent of the loads acting along the length of the element. For an element having constant E_e and A_e, substituting relations (15.12) into (15.30), we get

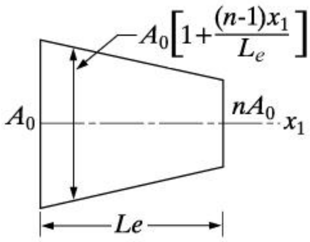


Figure 15.4 Geometry of a tapered element.

$$[K^e] = \frac{E_e A_e}{L_e^2} \int_0^{L_e} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} [-1,\ 1] dx_1 = \frac{E_e A_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{15.31}$$

The stiffness matrix for line elements with constant A_e and E_e given by formula (15.31) is exact to within the accuracy of the theory of mechanics of materials.

A formula for the stiffness matrix of the tapered element of Fig. 15.4 can be established by substituting the expression for their cross sectional area given in Fig.15.4 and relations (15.12b) and (15.13b) into (15.30) and integrating. That is,

$$\begin{aligned}
[K^e] = \begin{bmatrix} K_{jj}^e & K_{jk}^e \\ K_{kj}^e & K_{kk}^e \end{bmatrix} &= \int_0^{L_e} EA_0 \left[1 + \frac{(n-1)\,x_1}{L}\right] \frac{d[\phi^e]^T}{dx_1} \frac{d[\phi^e]}{dx_1} dx_1 \\
&= \int_0^{L_e} \frac{EA_0}{L_e^2} \left[1 + \frac{(n-1)\,x_1}{L_e}\right] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx_1 = \frac{EA_0\,(1+n)}{2L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned} \tag{15.32}$$

The stiffness matrix (15.32) is exact to within the accuracy of the theory of mechanics of materials.

Elements with variable cross section can be approximated by elements of constant cross section whose stiffness matrix is computed using relation (15.31). This will introduce an error in the stiffness matrix of the element and as a result in the solution of the problem. However this error will diminish as the number of elements, to which a domain is subdivided, increases.

Once the stiffness matrices for the elements of a domain are established its stiffness matrix can be assembled using relations (15.27).

15.5 Construction of the Load Vector for the Domain of One-Dimensional, Second Order, Linear Boundary Value Problems

Consider a node r of a one-dimensional domain and the two adjacent to it elements e and $(e + 1)$ (see Fig. 15.3). As we mention in Section 15.2.1 only the concentrated forces acting directly on node r, and the loads (external forces, change of temperature) acting on elements e and $(e + 1)$ contribute to the term F_r of the load vector $\{F\}$. Thus, for the structure of Fig. 8.23a, using the element shape functions (15.12) relations (15.8a) can be rewritten as

$$F_r = P_r^E + P_r^G + R_1^0 \delta_{1r} + P_1^L \delta_{Sr} \tag{15.33}$$

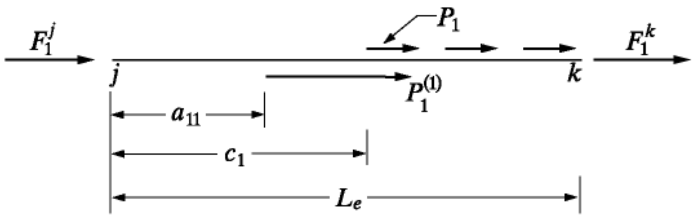


Figure 15.5 Free-body diagram of element e subjected to axial centroidal forces. The nodal forces F_1^{ej} and F_1^{ek} are considered positive as shown.

where

$P_r^G =$ given concentrated force acting on node r.

$P_r^E =$ *equivalent nodal force for node r.* It represents the contribution of the loads acting on elements e and (e + 1) of a one-dimensional domain, on the term F_r of its load vector. It is equal to the sum of the contributions V_1^{ek} of the loads acting on element e and of the contribution $V_1^{(e+1)j}$ of the loads acting on element $e + 1$. That is,

$$P_r^E = V_1^{ek} + V_1^{(e+1)j} \tag{15.34}$$

where referring to relation (15.8a), we have

$$V_1^{(e+1)j} = \int_0^{L_{e+1}} E_{e+1}(x_1)A_{e+1}H_1^{e+1}(x_1)\,\frac{d\phi_j^{e+1}}{dx_1}\,dx_1 + \int_0^{L_{e+1}} J_1^{e+1}(x_1)\phi_j^{e+1}(x_1)\,dx_1 \tag{15.35a}$$

$$V_1^{ek} = \int_0^{L_e} E_e A_e H_1^e\,\frac{d\phi_k^e}{dx_1}\,dx_1 + \int_0^{L_e} J_1^e(x_1)\phi_k^e\,dx_1 \tag{15.35b}$$

J_1^e and J_1^{e+1} represent the external axial centroidal forces acting along the length of elements e and $e + 1$, respectively, except those acting on the nodes of these elements. For example, for the element subjected to the axial centroidal forces shown in Fig.15.5, we have

$$\begin{aligned} H_1^e &= 0 \\ J_1^e(x_1) &= p_1\Delta(x_1 - c_1) + P_1^{(1)}\delta(x_1 - a_{11}) \end{aligned} \tag{15.36}$$

Substituting relations (15.36) and (15.12) into (15.35) and using relations (G.3) and (G.11) of Appendix G, we obtain

$$\begin{aligned} V_1^{ej} &= \int_0^{L_e}[\,p_1\Delta(x_1 - c_1) + P_1^{(1)}\delta(x_1 - a_{11})\,]\left(1 - \frac{x_1}{L_e}\right)dx_1 \\ &= p_1\int_{c_1}^{L_e}\left(1 - \frac{x_1}{L_e}\right)dx_1 + P_1^{(1)}\left(1 - \frac{a_{11}}{L_e}\right) = \frac{p_1(L - c_1)^2}{2L_e} + P_1^{(1)}\left(1 - \frac{a_{11}}{L_e}\right) \end{aligned} \tag{15.37a}$$

$$V_1^{ek} = \int_0^{L_e} [\ p_1\Delta(x_1 - c_1) + P_1^{(1)}\delta(x_1 - a_{11})\ ]\ \frac{x_1}{L_e}dx_1 = \frac{p_1(L_e^2 - c_1^2)}{2L_e} + \frac{P_1^{(1)}a_{11}}{L_e} \tag{15.37b}$$

On the basis of the foregoing presentation the load vector $\{F\}$ for a problem can be written as

$$\{F\} = \{P^E\} + \{P^G\} + \{R\} \tag{15.38}$$

where

$\{P^E\}$ = matrix of equivalent loads. Its terms are given by relation (15.34).

$\{P^G\}$ = matrix of given concentrated forces (fluxes) applied to the nodes of the domain.

$\{R\}$ = matrix of reactions. Its terms are either zero or the unknown resultant forces (fluxes) applied to the end nodes of the domain at which the state variable is specified, that is, at the nodes at which essential boundary conditions are specified.

As can be deduced from relations (15.34) and (15.35), the contribution of the loads acting on a line element of constant cross section made from one material to the matrix of equivalent actions of the structure is exact. Consequently, the stiffness equations (15.6) of a one-dimensional domain subdivided into elements of constant A_e and E_e are exact to within the accuracy of the theory of mechanics of materials

15.6 Direct Computation of the Contrbution of an Element to the Stiffness Matrix and the Load Vector of the Domain of One-Dimensional, Second Order, Linear Boundary Value Problems

Consider an element of a member (say element e) subjected to given loads inducing in it only a distribution of axial centroidal force $N^e(x_1)$ and assume that the component of translation $u_1^e(x_1)$ is specified at both ends of the element. That is,

$$\begin{aligned} u_1^e(0) &= u_1^{ej} \\ u_1^e(L) &= u_1^{ek} \end{aligned} \tag{15.39}$$

Substituting relations (15.13), into the modified weighed residual equation (13.79) using relations (13.75) and (13.80) and recalling that for the element under consideration $(\tilde{N}^e)_{x_1 = L_e} = P_1^L$ and $r = j$ and k, we have

$$\begin{aligned} &-\int_0^{L_e} E_e A_e \frac{d[\phi^e]^T}{dx_1}\left(\frac{d[\phi^e]}{dx_1}\{D^e\} - H_1^e\right)dx_1 \\ &+ \int_0^{L_e}[\phi^e]^T J_1^e dx_1 - \Big[\ [\phi^e]^T\ \tilde{N}^e\ \Big]_{x_1 = 0} + \Big[\ [\phi^e]^T\ \tilde{N}^e\ \Big]_{x_1 = L_e} = 0 \end{aligned} \tag{15.40}$$

where

$\{\tilde{D}^e\}$ = an approximation to the local matrix of nodal displacements for element e defined by relation (15.13c).

The matrix of shape functions $[\phi^e]$ is defined by relation (15.13b).

Referring to relations (15.12) and taking into account that $\tilde{N}^e(0) = -\tilde{F}_1^{ej}$ and $\tilde{N}^e(L) = \tilde{F}_1^{ek}$ (see Fig. 15.5) the last two terms of equation (15.40) can be written as

$$-\left[[\phi^e]^T \tilde{N}^e \right]_{x_1 = 0} = -\begin{bmatrix} \left(1 - \dfrac{x_1}{L_e}\right) \tilde{N}^e \\ \dfrac{x_1 \tilde{N}^e}{L_e} \end{bmatrix}_{x_1 = 0} = \begin{Bmatrix} -\tilde{N}^e(0) \\ 0 \end{Bmatrix} = \begin{Bmatrix} \tilde{F}_1^{ej} \\ 0 \end{Bmatrix} \tag{15.41a}$$

and

$$\left[[\phi^e]^T \tilde{N}^e \right]_{x_1 = L_e} = \begin{bmatrix} \left(1 - \dfrac{x_1}{L_e}\right) \tilde{N}^e \\ \dfrac{x_1 \tilde{N}^e}{L_e} \end{bmatrix}_{x_1 = L_e} = \begin{Bmatrix} 0 \\ \tilde{N}^e(L) \end{Bmatrix} = \begin{Bmatrix} 0 \\ \tilde{F}_1^{ek} \end{Bmatrix} \tag{15.41b}$$

Thus,

$$-\left[[\phi^e]^T \tilde{N}^e \right]_{x_1 = 0} + \left[[\phi^e]^T \tilde{N}^e \right]_{x_1 = L} = \begin{Bmatrix} \tilde{F}_1^{ej} \\ \tilde{F}_1^{ek} \end{Bmatrix} = \{\tilde{A}^e\} \tag{15.42}$$

where

$\tilde{F}_1^{ej}$ and $\tilde{F}_1^{ek}$ = approximations to the axial centroid forces acting at the end j and k of the element, respectively. They are considered positive if they act in the direction of the positive x_1 axis.

$\{\tilde{A}^e\}$ = approximation to the local matrix of nodal actions of the element.

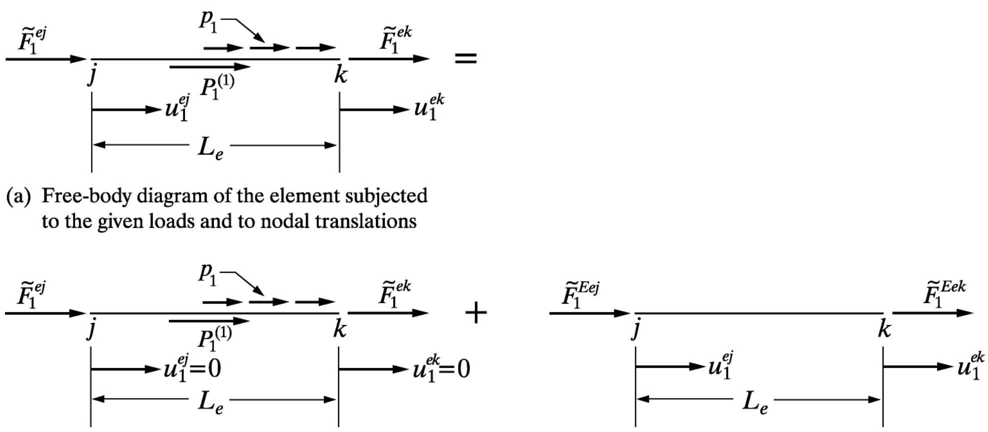
Substituting relation (15.42) into equation (15.40) and using relation (15.30), we obtain

$$\{\tilde{A}^e\} = \begin{bmatrix} K_{jj}^e & K_{jk}^e \\ K_{kj}^e & K_{kk}^e \end{bmatrix} \begin{Bmatrix} \tilde{u}_1^{ej} \\ \tilde{u}_1^{ek} \end{Bmatrix} + \{\tilde{A}^{Re}\} = \left[K^e\right] \{\tilde{D}^e\} + \{\tilde{A}^{Re}\} \tag{15.43}$$

where

$[\tilde{K}^e]$ = an approximation to the local stiffness matrix for element e defined by relation (15.30).

$$\{\tilde{A}^{Re}\} = \begin{Bmatrix} \tilde{F}_1^{Rej} \\ \tilde{F}_1^{Rek} \end{Bmatrix} = -\int_0^{L_e} \left[[\phi^e]^T J_1^e + E_e A_e H_1^e \frac{d}{dx_1} [\phi^e]^T \right] dx_1 = -\begin{Bmatrix} \tilde{V}_1^{ej} \\ \tilde{V}_1^{ek} \end{Bmatrix} \tag{15.44}$$



(a) Free-body diagram of the element subjected to the given loads and to nodal translations

(b) Free-body diagram of the element with its ends fixed subjected to the given loads

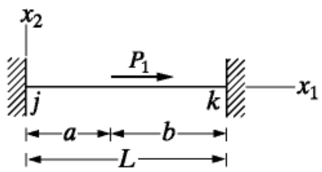
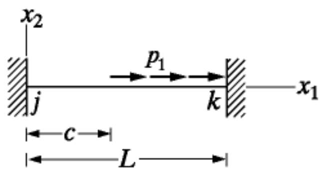
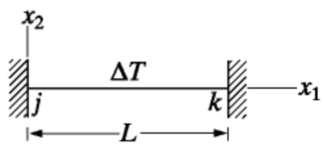
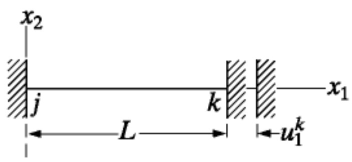
(b) Free-body diagram of the element subjected to the nodal actions required to produce the nodal translations of Fig. 15.6a

Figure 15.6 Superposition of the element subjected to the given loads with its ends fixed and the element subjected only to the nodal actions required to produce its actual nodal translations.

$\tilde{V}_1^{ej}$ and $\tilde{V}_1^{ek}$ = contributions of the loads acting on element e to the equivalent forces at nodes j and k, respectively [(see relations 15.35)].

Equation (15.43) expresses the nodal actions of an element as a linear combination of its nodal displacements. It is clear that once equation (15.43) is established for an element, its local stiffness matrix and the contribution of the loads acting along its length to the matrix of equivalent actions of the domain are known. That is, referring to

Table 15.1 Restraining actions for elements of constant cross section subjected to axial centroidal forces and temperature change $\Delta T_c(\delta T_2 = \delta T_3 = 0)$.

 $F_1^{Rj} = -\frac{P_1 b}{L}$ $F_1^{Rk} = -\frac{P_1 a}{L}$	 $F_1^{Rj} = -\frac{P_1(L^2 - c^2)}{2L}$ $F_1^{Rk} = -\frac{P_1(L - c)^2}{2L}$
	

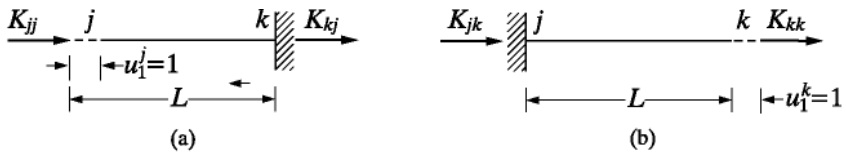


Figure 15.7 Physical significance of the local stiffness coefficients $K_{mnl}(m, n = 1, 2)$ for an element of a domain of a one-dimensional, second order, linear boundary value problem..

relations (15.34) and (15.44) the equivalent action P_r^e at node r located between elements e and $(e + 1)$ is equal to

$$P_r^E = (V_1^{ek} + V_1^{(e+1)j}) = -F_1^{Rek} - F_1^{R(e+1)j} \tag{15.45}$$

In what follows we illustrate the physical meaning of the terms of equation (15.43).

If the ends of the element are fixed, the matrix $\{D^e\}$ is a zero matrix and equation (15.43) reduces to

$$\{\tilde{A}^e\} = \{\tilde{A}^{Re}\} \tag{15.46}$$

This indicates that $\{\tilde{A}^{Re}\}$ is an approximation to the matrix of local nodal actions of the element subjected to the given loads with its ends fixed. We call $\{\tilde{A}^{Re}\}$ the matrix of fixed-end actions of element e where we call F_1^{Rej} and F_1^{Rek} the fixed-end forces of element.

From relation (15.45) we see that the equivalent force at node r defined by relation (15.34) with (15.35) is equal and opposite to the sum of the fixed-end forces acting at the ends of the elements connected to node r. The fixed-end forces for elements of constant cross section subjected to certain loads of practical interest can be computed and tabulated. In structural analysis such tables are used to compute the equivalent actions acting on the nodes of a structure (see Table 15.1).

If an element is not subjected to loads along its length but only to nodal displacements, $\{\tilde{A}^{Re}\}$ is a zero matrix and equation (15.43) reduces to

$$\{\tilde{A}^e\} = \{\tilde{A}^{Ee}\} = \begin{Bmatrix} \tilde{F}_1^{Eej} \\ \tilde{F}_1^{Eek} \end{Bmatrix} = \begin{bmatrix} K_{jj}^e & K_{jk}^e \\ K_{kj}^e & K_{kk}^e \end{bmatrix} \begin{Bmatrix} \tilde{u}_1^{ej} \\ \tilde{u}_1^{ek} \end{Bmatrix} = [K^e]\,\{\tilde{D}^e\} \tag{15.47}$$

Equation (15.47) is called the *approximate local stiffness equation for element e. From relation (15.47) we see that $\{\tilde{A}^{Ee}\}$ represents an approximation to the matrix of nodal forces which must be applied at the ends j and k of element e in order to displace them by amounts equal to their actual nodal translations u_1^{ej} and u_1^{ek}, respectively.*

When the ends of an element are displaced by $u_1^{ej} = 1$ and $u_1^{ek} = 0$, relation (15.47) gives

$$\tilde{F}_1^{Eej} = K_{jj}^e \qquad \tilde{F}_1^{Eek} = K_{kj}^e \tag{15.48}$$

That is, as shown in Fig. 15.7a, the stiffness coefficients K_{ij}^e and K_{kj}^e are the axial forces which must be placed at the ends j and k of element e, respectively, in order to shorten it by one unit. Inasmuch as the element is in equilibrium, referring to Fig 15.7a, we have

$$K^e_{kj} = -K^e_{ij} \tag{15.49}$$

Similarly, as shown in Fig. 15.7b, the stiffness coefficients K^e_{kk} and K^e_{jk} are the axial forces which must be placed at the ends k and j of element e, respectively, in order to elongate it by one unit. Moreover, from the equilibrium of the element, we get

$$K^e_{jk} = -K^e_{kk} \tag{15.50}$$

Finally, referring to Fig. 15.7 and employing the Betti-Maxwell reciprocal theorem (see Section 13.15), we obtain

$$K^e_{jk} = K^e_{kj} \tag{15.51}$$

That is, as shown in Section 15.4, the local stiffness matrix of an element of a one-dimensional domain is symmetric.

15.7 Approximate Solution of Linear Boundary Value Problems Using the Finite Element Method

In the previous sections we present a procedure for establishing the stiffness equations (15.6) for one-dimensional, second order, linear boundary value problems, using the finite element method. Similar procedures are employed for establishing the stiffness equations for any linear boundary value problem using the finite element method. In this section we present an outline of these procedures as they apply to the boundary value problems considered in this book. Moreover, we describe the steps that must be taken in order to solve the stiffness equations for a domain. Thus, in the finite element method we adhere to the following steps:

STEP 1 The modified weighted residual equation is established for the boundary value problem under consideration, following a procedure similar to the one described in Sections 13.9 to 13.12.

STEP 2 The domain of the problem is subdivided into a finite number of subdomains called *elements* and their ends and possibly other key points along their length are chosen, as *nodal points* or *nodes* (see Section 15.2). For example, we choose as nodal points for one-dimensional elements, their two end points†. Moreover, we choose as nodal points for the planar triangular element of Fig. 15.8b its three vertices††.

STEP 3 Approximate expressions for the nodal values of the local components of the fluxes are established for each element. That is,

$$\{\tilde{A}^e\} = [K^e]\,\{\tilde{D}^e\} + \{\tilde{A}^{Re}\} \tag{15.52}$$

where $[K^e]$ is called the *local stiffness matrix of element e* while the terms of the matrix $\{\tilde{A}^e\}$ represent an approximation to the values of the local components of the nodal fluxes of element e. That is,

For two-node elements of planar trusses

† In addition to their two end points, we could choose as nodal points of one-dimensional elements a finite number of points along their length. This possibility is not explored further in this book.

†† In addition to their vertices we can choose as nodal points of triangular elements a finite number of points along their sides and inside their area.

$$\{\tilde{A}^{e}\}^{T} = \left[\tilde{F}_1^{ej} \quad \tilde{F}_1^{ek}\right] \tag{15.53a}$$

For two-node elements of planar beams or frames in the x_1,x_2 plane

$$\{\tilde{A}^{e}\}^{T} = \left[\tilde{F}_1^{ej}\ \tilde{F}_2^{ej}\ \tilde{M}_3^{ej} \quad \tilde{F}_1^{ek}\ \tilde{F}_2^{ek}\ \tilde{M}_3^{ek}\right] \tag{15.53b}$$

For two-node elements of space beams or frames

$$\{\tilde{A}^{e}\}^{T} = \left[\tilde{F}_1^{ej}\ \tilde{F}_2^{ej}\ \tilde{F}_3^{ej}\ \tilde{M}_1^{ej}\ \tilde{M}_2^{ej}\ \tilde{M}_3^{ej} \quad \tilde{F}_1^{ek}\ \tilde{F}_2^{ek}\ \tilde{F}_3^{ek}\ \tilde{M}_1^{ek}\ \tilde{M}_2^{ek}\ \tilde{M}_3^{ek}\right] \tag{15.53c}$$

For the three-node planar triangular element of Fig. 15.8b

$$\{\tilde{A}^{e}\}^{T} = \left[\tilde{F}_1^{ei}\ \tilde{F}_2^{ei} \quad \tilde{F}_1^{ej}\ \tilde{F}_2^{ej} \quad \tilde{F}_1^{ek}\ \tilde{F}_2^{ek}\right] \tag{15.53d}$$

For three-node triangular elements of thin plates in the x_1,x_2 plane

$$\{\tilde{A}^{e}\}^{T} = \left[\tilde{Q}_3^{ei}\ \tilde{M}_1^{ei}\ \tilde{M}_2^{ei} \quad \tilde{Q}_3^{ej}\ \tilde{M}_1^{ej}\ \tilde{M}_2^{ej} \quad \tilde{Q}_3^{ek}\ \tilde{M}_1^{ek}\ \tilde{M}_2^{ek}\right] \tag{15.53e}$$

For the correct formulation of a boundary value problem certain quantities must be specified on its boundary. As discussed in Section 13.7 some of these quantities are called essential while the others are called natural. The matrix $\{\tilde{D}^{e}\}$ is called the matrix of nodal displacements of element e. Its terms represent an approximation to the values of the essential quantities at all nodes of element e. That is,

For two-node elements of planar trusses

$$\{\tilde{D}^{e}\}^{T} = \left[\tilde{u}_1^{ej} \quad \tilde{u}_1^{ek}\right] \tag{15.54a}$$

For two-node elements of planar beams or frames in the x_1,x_2 plane

$$\{\tilde{D}^{e}\}^{T} = \left[\tilde{u}_1^{ej}\ \tilde{u}_2^{ej}\ \tilde{\theta}_3^{ej} \quad \tilde{u}_1^{ek}\ \tilde{u}_2^{ek}\ \tilde{\theta}_3^{ek}\right] \tag{15.54b}$$

For two-node elements of a space beams or frames

$$\{\tilde{D}^{e}\}^{T} = \left[\tilde{u}_1^{ej}\ \tilde{\theta}_1^{ej}\ \tilde{u}_2^{ej}\ \tilde{\theta}_3^{ej}\ \tilde{u}_3^{ej}\ \tilde{\theta}_2^{ej} \quad \tilde{u}_1^{ek}\ \tilde{\theta}_1^{ek}\ \tilde{u}_2^{ek}\ \tilde{\theta}_3^{ek}\ \tilde{u}_3^{ek}\ \tilde{\theta}_2^{ek}\right] \tag{15.54c}$$

For the three-node planar triangular element of Fig. 15.9b

$$\{\tilde{D}^{e}\}^{T} = \left[\tilde{u}_1^{ei}\ \tilde{u}_2^{ei} \quad \tilde{u}_1^{ej}\ \tilde{u}_2^{ej} \quad \tilde{u}_1^{ek}\ \tilde{u}_2^{ek}\right] \tag{15.54d}$$

For three-node triangular elements of thin plates in the x_1,x_2 plane

$$\{\tilde{D}^{e}\}^{T} = \left[\tilde{u}_3^{ei}\ \tilde{\theta}_1^{ei}\ \tilde{\theta}_2^{ei} \quad \tilde{u}_3^{ej}\ \tilde{\theta}_1^{ej}\ \tilde{\theta}_2^{ej} \quad \tilde{u}_3^{ek}\tilde{\theta}_1^{ek}\ \tilde{\theta}_2^{ek}\right] \tag{15.54e}$$

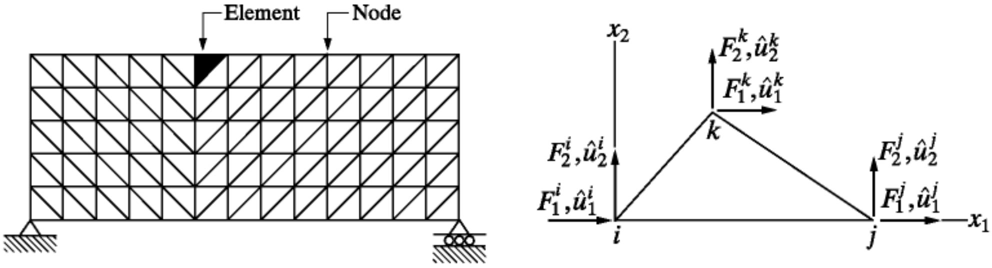
Relation (15.52) is established as follows:

1. Within each element the state variables are approximated by

$$\{\tilde{u}^{e}\} = \left[\phi\right]\{\tilde{D}^{e}\} \tag{15.55}$$

where, within an element, the terms of the matrix $\{\tilde{u}^{e}\}$ represent an approximation to the state variables. That is,

For two-node elements of planar trusses



(a) Beam subdivided into triangular elements

(b) Nodal actions and nodal displacement of a straight-edge triangular element

Fig 15.8 Plate in a state of generalized plane stress subdivided into planar triangular finite elements.

$$\{\tilde{u}^{e}\} = \tilde{u}_1^{e}(x_1) \tag{15.56a}$$

For two-node elements of planar beams or frames in the x_1,x_2 plane

$$\{\tilde{u}^{e}\}^T = \left[\tilde{u}_1^{e}(x_1) \quad \tilde{u}_2^{e}(x_1)\right] \tag{15.56b}$$

For two-node elements of a space beams or frames

$$\{\tilde{u}^{e}\}^T = \left[\tilde{u}_1^{e}(x_1) \quad \tilde{u}_2^{e}(x_1) \quad \tilde{u}_3^{e}(x_1) \quad \tilde{\theta}_1^{e}(x_1)\right] \tag{15.56c}$$

For the planar triangular element of Fig. 15.9b

$$\{\tilde{u}^{e}\}^T = \left[\tilde{u}_1^{e}(x_1x_2)\,\tilde{u}_2^{e}(x_1x_2)\right] \tag{15.56d}$$

For three-node triangular elements of thin plates in the x_1x_2 plane

$$\{\tilde{u}^{e}\} = \tilde{u}_3^{e}(x_1,x_2) \tag{15.56e}$$

$[\phi]$ is a rectangular matrix whose terms are polynomials of the local coordinates of the element. It is called the *matrix of element shape functions*. Its terms are called the *trial* or *shape functions* for the element.

2.The approximation to the state variables (15.55) for each element is substituted into the modified weighted residual form of the boundary value problem under consideration and it is assumed that only essential boundary conditions are specified at the ends of the element. This yields relation (15.52)

STEP 4 The local stiffness matrix $[K^e]$ and the local matrix $\{\tilde{A}^{Re}\}$ of each element are transformed to global.

STEP 5 The stiffness equations (15.6) for the domain are constructed from the global properties of its elements established in step 4. Using relation (15.38) the stiffness equations (15.6) can be rewritten as

$$[S]\,\{\tilde{\Delta}\} = \{\tilde{P}^{E}\} + \{P^{G}\} + \{\tilde{R}\} \tag{15.57}$$

$\{\tilde{\Delta}\}$ = matrix of nodal displacements. Its terms represent an approximation to the value of the global components of the essential quantities at the nodes of the domain. The essential quantities are those which must be specified at a boundary of the domain at which only essential boundary conditions are specified. For example, the essential quantity for a bar subjected to axial centroidal forces is the axial component of translation u_1 while the essential quantities for a beam subjected to bending about its x_2 axis without twisting are u_3 and θ_2.

$\{\tilde{P}^E\}$ = an approximation of the global matrix of equivalent actions.

$\{P^G\}$ = global components of the fluxes (reactions) acting at the nodes of the domain.

$\{\tilde{R}\}$ = an approximation to the matrix of global components of the unknown fluxes (reactions) at the nodes of the boundary of the domain where essential boundary conditions are specified.

In order to construct the stiffness equation (15.57) for a domain the following must be done:

1. The stiffness matrix $[S]$ of the domain is assembled from the global stiffness matrices of its elements established in step 4. In order to accomplish this the indices of the global stiffness coefficients of each element of the domain are chosen so as to correspond to those of the terms of the matrix $\{\tilde{\Delta}\}$. The stiffness coefficients for the domain are then computed of the basis of relations (15.27).

2. The matrix of equivalent actions $\{\tilde{P}^E\}$ of the domain is constructed from the global matrices $\{\tilde{A}^{Re}\}$ [see relation (15.52)] of its elements using relation (15.45).

3. The matrices $\{P^G\}$ and $\{\tilde{R}\}$ are constructed from the data of the problem.

STEP 6 The stiffness equations are solved to obtain the global components of the unknown quantities.

The unknown quantities in the stiffness equation (15.57) for a boundary value problem are the reactions and the unknown components of the essential quantities at the nodes of the domain. Notice that for every known component of an essential quantity at a node of the boundary of the domain there is a corresponding unknown reaction. Thus, the number of unknown quantities in the stiffness equations for a problem does not change when its boundary conditions change. Moreover, the number of unknown quantities in the stiffness equations for the domain of a problem is equal to the number of stiffness equations. However, the stiffness matrix $[S]$ for a domain is singular, and, consequently, the stiffness equations cannot be solved directly to yield the unknown quantities. In order to be able to solve the stiffness equations for a problem its essential boundary conditions must be incorporated in them. This is accomplished by rearranging the rows and columns of the stiffness equation (15.57) as follows:

1. The rows of equation (15.57) involving reactions are moved to the bottom.
2. The columns of the stiffness matrix in Equation (15.57) which are multiplied by a specified value of the components of displacement at the nodal points of the boundary of the domain are moved to the right.

An algorithm can be written for moving the rows and columns of the stiffness equations (15.57) for a domain. The resulting stiffness equations can be partitioned as follows:

$$\left\{\begin{matrix}\{P^{EF}\}\\ \{P^{ES}\}\end{matrix}\right\} + \left\{\begin{matrix}\{P^{GF}\}\\ \{0\}\end{matrix}\right\} + \left\{\begin{matrix}\{0\}\\ \{\tilde{R}^S\}\end{matrix}\right\} = \begin{bmatrix}[S^{FF}] & [S^{FS}]\\ [S^{SF}] & [S^{ss}]\end{bmatrix}\left\{\begin{matrix}\{\Delta^F\}\\ \{\Delta^s\}\end{matrix}\right\} \tag{15.58}$$

The terms of matrix $\{P^{EF}\}$ are the computed values of the equivalent actions applied to the nodes of the domain excluding those of the nodes of the boundary of the domain which correspond to specified essential boundary conditions. The terms of matrix $\{P^{ES}\}$ are the computed values of equivalent loads corresponding to specified essential boundary conditions. The terms of the matrix $\{\tilde{R}^S\}$ are the known values of the fluxes (reactions) at the nodal points of the boundary of the domain at which essential boundary conditions are specified. The terms of the matrix $\{\tilde{\Delta}^F\}$ represent an approximation to the values of the unknown global components of the essential quantities at the nodes of the domain. The terms of the matrix $\{\Delta^S\}$ are the values of the specified essential quantities at the nodes of the boundary of the domain. Relations (15.58) can be expanded to yield:

$$\{P^{EF}\} + \{P^{GF}\} = [S^{FF}]\{\tilde{\Delta}^F\} + [S^{FS}]\{\Delta^S\} \tag{15.59a}$$

$$\{P^{ES}\} + \{\tilde{R}^S\} = [S^{SF}]\{\tilde{\Delta}^F\} + [S^{SS}]\{\Delta^S\} \tag{15.59b}$$

In Sections 15.4 and 15.5 it is shown that the stiffness equations for one-dimensional domains are exact, when the domain is divided into elements made from one material and having constant cross sections. This implies that the components of the essential quantities at the nodes of a domain obtained from relation (15.59a) are exact if the domain is subdivided into elements made from one material and having constant cross section. Similarly, we can deduce that the reactions of a domain obtained from a relation (15.59b) are exact provided that the domain is subdivided into elements made from one material and having constant cross section. Notice that inasmuch as the stiffness matrix $[S]$ is symmetric, the following relation is valid:

$$[S^{FS}] = [S^{SF}]^T \tag{15.60}$$

The matrix $[S^{FF}]$ is called the *basic stiffness matrix for the domain*. Its terms depend on the stiffness matrices for the elements of the domain and on the boundary conditions. The matrix $[S^{FF}]$ is square and symmetric, and if the boundary value problem is property formulated, it is nonsingular. Thus, relation (15.59a) can be solved to yield the unknown values of the essential quantities $\{\tilde{\Delta}^F\}$ at the nodes of the domain. These can be substituted in relation (15.59b) and the resulting relations can be solved to yield the unknown reactions $\{\tilde{R}^S\}$ of the problem.

STEP 7 Approximations to the state variables and to the corresponding resultant fluxes in each element of the domain are computed. This is accomplished by substituting in relation (15.55) the values of the essential quantities at the nodes of each element established in step 6. This gives approximate expressions for the state variables in the elements. We use these expressions to obtain approximate expressions for the resultant fluxes of the elements.

Notice that in the solution of one-dimensional, boundary value problems no error is introduced in steps 2 and 5. However, this is not so in the solution of two- or three-dimensional boundary value problems. The state variables and the fluxes of boundary-value problems are required to satisfy the laws which govern their solution at every point of the interelement boundaries and the specified boundary condition at every point of the boundary of their domain. The two nodal points of each one-dimensional element constitute its entire boundary. Consequently, in steps 2 and 5 no error is introduced in the solution of one dimensional boundary value problems. However, the finite elements may not match completely the boundary of two- or three-dimensional domains, and, consequently, an error is introduced in step 2. Moreover, the nodal points of two- and three- dimensional elements, represent only a few points of their boundary (see Fig. 15.23). Thus in Step 5 the state variables and the fluxes of two- or three-dimensional boundary value problems are required to satisfy the laws which govern their solution only at a few points of the interelement boundaries. Consequently, in step 5 an error is introduced in the solution of two and three-dimensional boundary value problems. The errors introduced in steps 2 and 5 are called *discretization errors* and they decrease as the size of the elements decreases. In most finite-element formulations of boundary value problems, involving two- or three-dimensional domains, the discretization errors are reduced by an appropriate choice of nodal points and shape functions of their elements. That is, they are chosen in a way that some of the laws which govern the solution of a boundary value problem are automatically satisfied at all points of the boundary of its elements when they are satisfied at their nodal points.

In what follows we illustrate the solution of an one-dimensional, second order, linear boundary value problems using the finite element method by an example.

Example 1 Consider the fixed at both ends bar shown in Fig. a consisting of three members of constant but different cross sections. The bar is subjected to concentrated external axial centroidal forces as shown in Fig.a. The bar is a statically indeterminate structure to the first degree. Using the finite element method, compute the reactions, the axial component of translation $u_1(x_1)$ and the internal axial centroidal force $N(x_1)$ in the bar.

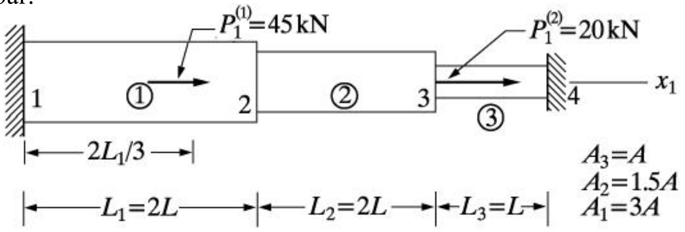


Figure a Geometry and loading of the bar.

Solution

STEP 1 The modified weighted residual equation for this boundary value problem is (13.79)

STEP 2 We subdivide the bar into three unequal elements of constant cross section. (See Fig. a). This is the smallest number of elements into which the bar of Fig. a can be subdivided, because it is necessary to have a node at each point of discontinuity of the cross sectional area of the bar.

STEP 3 We establish expressions for the nodal values of the local component of the flux $[N^e(x_1)]$ as a linear combination of its nodal displacements $\left[u_1^{ej} \text{ and } u_1^{ek}\right]$. For the elements of the bar under consideration, this has been done in Section 15.4. Thus, referring to relation (15.31) and to Fig. a the stiffness matrix $[K^e]$ of each element of the bar is

$$[K^{(1)}] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} \end{bmatrix} = \frac{E_1 A_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^{(2)}] = \begin{bmatrix} K_{22}^{(2)} & K_{23}^{(2)} \\ K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} = \frac{E_2 A_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{(a)}$$

$$[K^{(3)}] = \begin{bmatrix} K_{33}^{(3)} & K_{34}^{(3)} \\ K_{43}^{(3)} & K_{44}^{(3)} \end{bmatrix} = \frac{E_3 A_3}{L_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

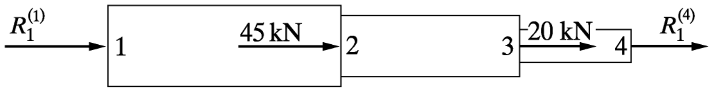


Figure b Free-body diagram of the bar.

Moreover, referring to Fig. a and to relation (15.44) we establish the restraining (fixed-end) nodal actions (forces) for each element of the bar. Referring to Fig.a, we have

$$J_1^{(1)} = P_1^{(1)}\delta\left(x_1 - \frac{2L}{3}\right) \qquad J_1^{(2)} = 0 \qquad J_1^{(0)} = 0$$

Substituting the above relations into (15.35) using relations (15.12) and integrating, we get

$$V_1^{1j} = \int_0^{L_1} J_1^{(1)}\phi_j^{(1)}dx_1 = \int_0^{L_1} P_1^{(1)}\delta\left(x_1 - \frac{2L}{3}\right)\left(1 - \frac{x_1}{L_1}\right)dx_1 = \frac{P_1^{(1)}}{3} = 15 \text{ kN}$$

$$V_1^{1k} = \int_0^{L_1} J_1^{(1)}\phi_k^{(1)}dx_1 = \int_0^{L_1} P_1^{(1)}\delta\left(x_1 - \frac{2L}{3}\right)\frac{x_1}{L_1}dx_1 = \frac{2P_1^{(1)}}{3} = 30 \text{ kN} \tag{b}$$

$$V_1^{2j} = V_1^{2k} = V_1^{3j} = V_1^{3k} = 0 \tag{c}$$

Referring to relations (15.44), we have

$$\{A^{Re}\}^T = -[V_1^{ej} \;\; V_1^{ek}] \tag{d}$$

Notice that the matrix $\{A^{Re}\}$ can be obtained by referring to Table 15.1.

As discussed in Section 15.4 the stiffness matrix for elements of constant A_e and E_e obtained on the basis of formula (15.31) is exact to within the accuracy of the theory of mechanics of materials. The same is true for the matrix of fixed-end actions $\{A^{Re}\}$ obtained from relations (15.44) with (15.37) .

STEP 4 In this problem the local axes of the elements of the bar coincide with its global axes. Thus, we do not have to transform the local stiffness matrix and the local matrix $\{A^{Re}\}$ of the elements of the bar to global.

STEP 5 We write the stiffness equations for the bar. In order to accomplish this, we assemble the stiffness matrix for the bar from the stiffness matrices (a) of its elements. Moreover, we construct the load vector for the bar from the fixed-end actions of its elements and the given concentrated forces acting on its nodes. Substituting relations (a) into relation (15.27), we obtain

$$\begin{aligned}
S_{11} &= K_{11}^{(1)} = \frac{E_1A_1}{L_1} = \frac{EA}{L} \\
S_{12} &= S_{21} = K_{12}^{(1)} = K_{21}^{(1)} = -\frac{E_1A_1}{L_1} = -\frac{EA}{L} \\
S_{13} &= S_{31} = 0 \\
S_{22} &= K_{22}^{(1)} + K_{22}^{(2)} = \frac{E_1A_1}{L_1} + \frac{E_2A_2}{L_2} = \frac{EA}{L} + \frac{1.5EA}{2L} = \frac{3.5EA}{2L}
\end{aligned} \tag{e}$$

$$
\begin{aligned}
S_{23} &= S_{32} = K_{23}^{(2)} = K_{32}^{(2)} = -\frac{E_2A_2}{L_2} = -\frac{1.5EA}{2L} \\
S_{33} &= K_{33}^{(2)} + K_{33}^{(3)} = \frac{E_2A_2}{L_2} + \frac{E_3A_3}{L_3} = \frac{1.5EA}{2L} + \frac{EA}{L} = \frac{3.5EA}{2L} \\
S_{34} &= S_{43} = K_{34}^{(3)} = -\frac{E_3A_3}{L_3} = -\frac{EA}{L} \\
S_{44} &= K_{44}^{(3)} = \frac{E_3A_3}{L_3} = \frac{EA}{L}
\end{aligned}
\tag{e}
$$

Thus, the stiffness matrix for the bar of Fig. a is

$$
[S] = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1.75 & -0.75 & 0 \\ 0 & -0.75 & 1.75 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{f}
$$

Referring to relations (b), (c) and (15.34), we obtain

$$
\{\tilde{P}^E\} = \begin{Bmatrix} 15 \\ 30 \\ 0 \\ 0 \end{Bmatrix} \tag{g}
$$

Moreover, referring to Fig. b, we get

$$
\{P^G\} = \begin{Bmatrix} 0 \\ 0 \\ 20 \\ 0 \end{Bmatrix} \qquad \{\tilde{R}\} = \begin{Bmatrix} \tilde{R}_1^0 \\ 0 \\ 0 \\ \tilde{R}_1^L \end{Bmatrix} \tag{h}
$$

The stiffness equations for the bar are obtained by substituting relations (f), (g) and (h) into (15.57). Thus, we get

$$
\frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1.75 & -0.75 & 0 \\ 0 & -0.75 & 1.75 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix} = \begin{Bmatrix} 15 \\ 30 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 20 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \tilde{R}_1^0 \\ 0 \\ 0 \\ \tilde{R}_1^L \end{Bmatrix} \tag{i}
$$

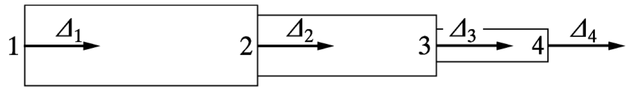


Figure c Components of translation of the nodes of the bar.

where Δ_i $(i = 1, 2, 3, 4)$ is the axial component of translation of node i of the bar (see Fig. c). Notice that since the stiffness matrix $[S]$ and the matrix of equivalent loads $\{P^E\}$ for a domain subdivided into elements having constant modules of elasticity E_e and cross sectional area A_e are exact to within the accuracy of the theory of mechanics of materials, the stiffness equations (i) are exact.

STEP 6 We compute the values of the axial components of translation of the nodes of the bar and its reactions $\tilde{R}_1^0$ and $\tilde{R}_1^L$. In order to accomplish this we introduce into the stiffness equations (i) the essential boundary conditions of the bar. That is, we rearrange the rows of the stiffness equations (i) by moving the rows which include the unknown reactions to the bottom. Moreover, we rearrange the columns of the stiffness matrix $[S]$ by moving to its right the columns which are multiplied by the specified (in this problem vanishing) axial components of translation of the supports of the bar. Furthermore, we partition the resulting stiffness equations for the bar as indicated in relation (15.58). Thus, the stiffness equations (i) can be rewritten as

$$\frac{EA}{L}\left[\begin{array}{cc|cc} 1.75 & -0.75 & -1 & 0 \\ -0.75 & 1.75 & 0 & -1 \\ \hline -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} \Delta_2 \\ \Delta_3 \\ \hline \Delta_1 \\ \Delta_4 \end{array}\right\} = \left\{\begin{array}{c} 30 \\ 0 \\ \hline 15 \\ 0 \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 20 \\ \hline 0 \\ 0 \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 0 \\ \hline \tilde{R}_1^0 \\ \tilde{R}_1^L \end{array}\right\} \tag{j}$$

Referring to relation (15.58) and (j), we have

$$\{P^{EF}\} = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix} \qquad \{P^{ES}\} = \begin{Bmatrix} 15 \\ 0 \end{Bmatrix} \qquad \{P^{GF}\} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

$$\{\tilde{R}^S\} = \begin{Bmatrix} \tilde{R}_1^0 \\ \tilde{R}_1^L \end{Bmatrix} \qquad \{\tilde{\Delta}^F\} = \begin{Bmatrix} \tilde{\Delta}_2 \\ \tilde{\Delta}_3 \end{Bmatrix} \qquad \{\Delta^S\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{k}$$

$$[S^{FF}] = \frac{EA}{L}\begin{bmatrix} 1.75 & -0.75 \\ -0.75 & 1.75 \end{bmatrix} \qquad [S^{FS}] = [S^{SF}]^T = \frac{EA}{L}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We compute the components of translation of the nodes of the bar. Substituting relations (k) into (15.59a), we obtain

$$\{\tilde{\Delta}^F\} = \begin{Bmatrix} \tilde{\Delta}_2 \\ \tilde{\Delta}_3 \end{Bmatrix} = \frac{L}{EA}\begin{bmatrix} 1.75 & -0.75 \\ -0.75 & 1.75 \end{bmatrix}^{-1}\begin{Bmatrix} 30 \\ 20 \end{Bmatrix} = \frac{L}{EA}\begin{Bmatrix} 27 \\ 23 \end{Bmatrix} \tag{l}$$

We compute the reactions of the supports of the bar. Substituting the computed values of the components of translations of the nodes of the bar (l) into relation (15.59b), we get

$$\begin{Bmatrix} \tilde{R}_1^0 \\ \tilde{R}_1^L \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} 27 \\ 23 \end{Bmatrix} - \begin{Bmatrix} 15 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -42 \\ -23 \end{Bmatrix} \text{ kN} \tag{m}$$

Notice that, as we discussed previously, the stiffness equations for a domain subdivided into elements of constant E_e and A_e are exact. Consequently, the values of the components of translation (j) of the nodes of the bar and the reactions (m) of the supports of the bar are exact, to within the accuracy of the theory of mechanics of materials.

STEP 7 We compute the axial component of translation $u_1(x_1)$ and the internal force $N(x_1)$ of each element of the bar. Referring to relation (l) the matrices of nodal displacements of the elements of the bar are

$$\{\tilde{D}^{(1)}\} = \frac{L}{EA}\begin{Bmatrix} 0 \\ 27 \end{Bmatrix} \qquad \{\tilde{D}^{(2)}\} = \frac{L}{EA}\begin{Bmatrix} 27 \\ 23 \end{Bmatrix} \qquad \{\tilde{D}^{(3)}\} = \frac{L}{EA}\begin{Bmatrix} 23 \\ 0 \end{Bmatrix} \tag{n}$$

Substituting relations (n) and (15.12) into (15.13), we get

$$\begin{aligned}
\tilde{u}_1^{(1)}(x_1) &= [\phi^{(1)}]\{\tilde{D}^{(1)}\} = \left[\left(1 - \frac{x_1}{2L}\right), \frac{x_1}{2L}\right] \frac{L}{EA}\begin{Bmatrix} 0 \\ 27 \end{Bmatrix} = \frac{27x_1}{2EA} \\
\tilde{u}_1^{(2)}(x_1) &= [\phi^{(2)}]\{\tilde{D}^{(2)}\} = \left[\left(1 - \frac{x_1}{2L}\right), \frac{x_1}{2L}\right] \frac{L}{EA}\begin{Bmatrix} 27 \\ 23 \end{Bmatrix} = \frac{27L - 2x_1}{EA} \\
\tilde{u}_1^{(3)}(x_1) &= [\phi^{(3)}]\{\tilde{D}^{(3)}\} = \left[\left(1 - \frac{x_1}{L}\right), \frac{x_1}{L}\right] \frac{L}{EA}\begin{Bmatrix} 23 \\ 0 \end{Bmatrix} = \frac{23(L - x_1)}{EA}
\end{aligned} \tag{o}$$

We compute the internal force $N(x_1)$ in each element of the bar. Substituting relations (o) into (8.58), we obtain

$$\begin{aligned}
\tilde{N}^{(1)}(x_1) &= EA_1 \frac{d\tilde{u}_1^{(1)}}{dx_1} = 27 \text{ kN} \\
\tilde{N}^{(2)}(x_1) &= EA_2 \frac{d\tilde{u}_1^{(2)}}{dx_1} = -3 \text{ kN} \\
\tilde{N}^{(3)}(x_1) &= EA_3 \frac{d\tilde{u}_1^{(3)}}{dx_1} = -23 \text{ kN}
\end{aligned} \tag{p}$$

Element 1 has a constant cross section and is subjected to an external concentrated force at a point along its length. Thus, relation (15.13a) represents an approximation to the component of translation $u_1(x_1)$ of this element. Elements 2 and 3 have a constant cross section and are not subjected to external disturbances along their length. Thus, relation (15.13a) is an exact expression for the component of translation of these elements. Therefore, $u_1^{(2)}(x_1)$, $u_1^{(3)}(x_1)$, $N^{(2)}(x_1)$ and $N^{(3)}(x_1)$ as given by the last two of relations (o) and (p) are exact, while $u_1^{(1)}(x_1)$ and $N^{(1)}(x_1)$ as given by the first of relations (o) and (p) are approximate. It can be shown that their exact values are

$$u_1^{(1)}(x_1) = \frac{1}{EA_1}\left[-45\left(x_1 - \frac{2L_1}{3}\right)\Delta\left(x_1 - \frac{2L_1}{3}\right) + 42x_1\right]$$

$$N^{(1)}(x_1) = -45\,\Delta\left(x_1 - \frac{2L_1}{3}\right) + 42 \qquad \text{(q)}$$

The free-body diagrams of the three elements of the bar are shown in Fig. d.

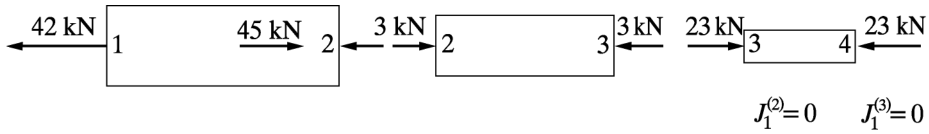


Figure d Free-body diagrams of the elements of the bar.

15.8 Application of the Finite Element Method to the Analysis of Framed Structures

The finite element method can be used to analyze any statically determinate or indeterminate framed structure (beam, truss or frame). In this case it is known as the *direct stiffness* or *displacement method* for analyzing framed structures. This method has been developed in the last 50 years and is best suited for writing programs for analyzing framed structures by computers. Practically all major general programs for analyzing framed structures have been written using the direct stiffness method because of its simplicity, generality and cost effectiveness. In this section we present the finite element method as it applies to the analysis of framed structures and we apply it to a few simple examples using a desk calculator. However, this method is not suitable for analyzing framed structures by a hand calculator or by a desk calculator, especially if their static indeterminancy is smaller than their kinematic indeterminancy. That is, the number of unknown components of displacements of their joints is bigger than the number of redundant reactions and/or internal actions.. The examples presented in this section can be easier solved using the flexibility method described in Chapter 14.

15.8.1 Elements, Nodes and Degree of Freedom of Framed Structures

In the finite element method a structure is subdivided into *elements* whose ends are imagined as being connected to a number of points called *nodes*. Thus an element of a framed structure extends between two nodes and it is either a member or a segment of a member of the structure. The nodes of a structure are its joints, its supports, the free ends of its members and any other point which we have a reason to choose along the length of its members. As a rule we choose the smallest number of nodes required for the analysis of a structure because the more nodes we choose, the more simultaneous algebraic equations we have to solve. For instance, the smallest number of nodes for the beam of Fig. 15.9b is the sum of the points of support (1, 3 and 6), the two points (4 and 5) on each side of the internal rollers. We do not have to choose as nodes the points where external concentrated actions are applied.

The nodes and elements of a structure are numbered consecutively, and the number of each element is placed in a circle, as shown in Fig. 15.9. Moreover, we label the ends of each element by *j* and *k* (*j* being the end of the element connected to the node having the smallest number). It is preferable that the nodes of framed structures are numbered

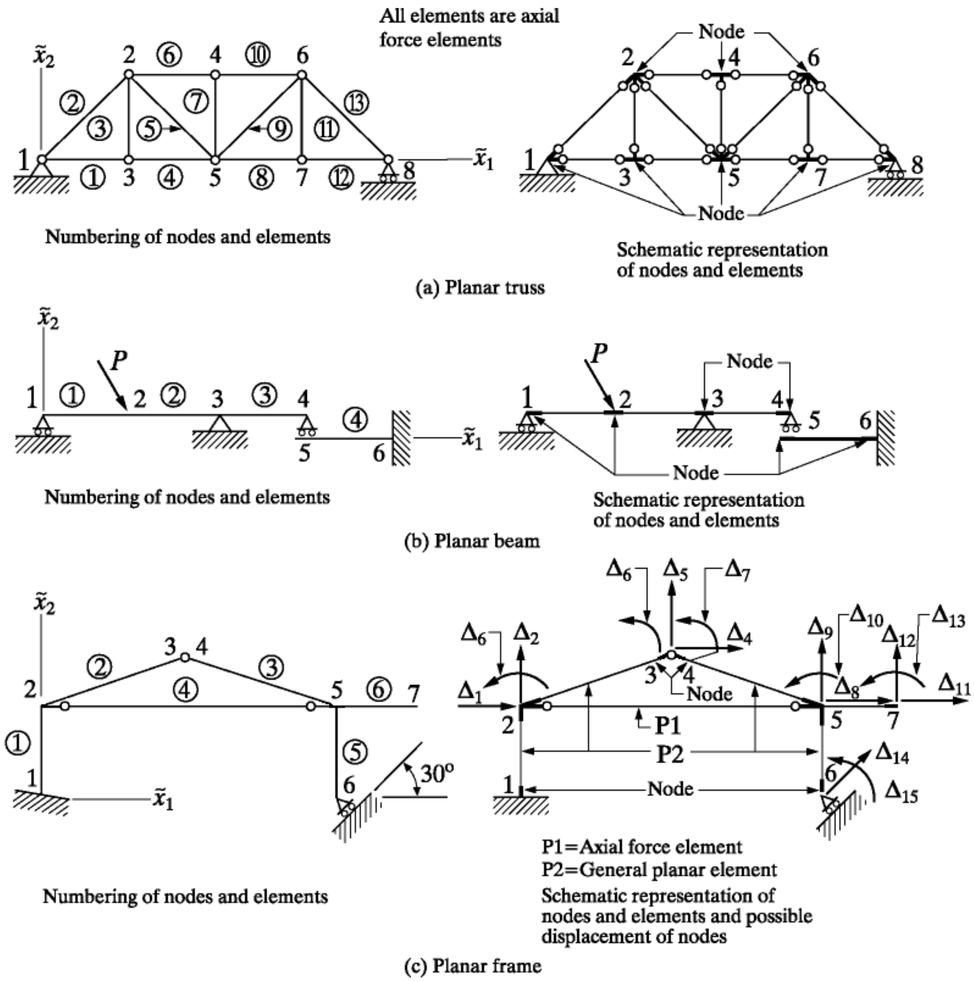


Figure 15.9 Numbering the nodes and elements of framed structures.

so that the difference between the numbers of the nodes at the ends of each element is as small as possible.

After the nodes and the elements of a structure are numbered, their connectivity can be expressed, for example, as shown in Table 15.2 for the structure of Fig. 15.9c.

Table 15.2 Connectivity of the elements and nodes of the structure of Fig. 15.9c.

Element Number	Node Number End *j*	Node Number End *k*
1	1	2
2	2	3
3	4	5
4	2	5
5	5	6
6	5	7

We classify the elements of planar framed structures as follows:

1. Elements whose cross sections are subjected only to an internal axial centroidal force. Each end of these elements is pinned to other elements or to a support. We consider the pins at the ends of these elements as being part of them. We refer to these elements as *axial force elements*.

2. Elements whose cross sections are subjected to an internal axial centroidal force, a transverse force and an internal moment whose vector is normal to the plane of the structure. The transverse force acting on an element lies on a plane which is parallel to the plane of the structure and contains the sheer center of the cross sections of the element. We refer to these elements as *general planar elements*. One end of an element of this type is rigidly connected to other elements or to a support, while the other end is either free or supported in some way (rigidly, with pins, rollers, etc.) to other elements or to a support. We consider each end of these elements as being rigidly connected to a node. That is, if one end of an element of this type is connected to other elements or to a support by a connection which is not rigid (pin, rollers), we consider this connection as part of the node. (See nodes 1, 3, 4 and 5 of the beam of Fig. 15.9b and nodes 3, 4 and 6 of the frame of Fig. 15.9c.) Thus, we have two or more nodes adjacent to an action release mechanism (see Section 8.6) which we call connected nodes. For example, there is a node on each side of the hinge at the apex of the frame of Fig. 15.9c and on each side of the internal rollers of the beam of Fig. 15.9b.

With the above classification the elements of a planar truss are axial force elements (see Fig. 15.9a); the elements of a planar beam are general planar elements (see Fig. 15.9b) while the elements of a planar frame can be either general planar elements or axial force elements (see Fig. 15.9c).

Depending on the type of the internal action release mechanism, one or more components of the relative motion of the connected nodes vanish. For instance, nodes 4 and 5 of the beam of Fig. 15.9b can rotate and translate in any direction. However, their components of translation in the direction of the $\tilde{x}_2$ axis are equal. Moreover, nodes 3 and 4 of the frame of Fig. 15.9c can rotate and translate in any direction. However, their components of translation are equal.

When a structure is subjected to loads, some of its nodes undergo translations and/or rotations which are not known, while others undergo translations and rotations which are known. For instance, the components of translation and rotation of a fixed support are zero. We refer to the components of translation and rotation of a node as its *components of displacement*. The degree of freedom of a body is equal to the smallest number of independent components of displacement of its particles required for the specification of its configuration. However, in this text we call the number of unknown components of displacement of the nodes of a framed structure its *degree of freedom* or its *degree of kinematic indeterminacy*.

In general, the displacement of an unrestrained node of a planar frame has two components of translation with respect to a set of two rectangular axes lying in the plane of the frame, and one component of rotation whose vector is normal to the plane of the frame. Moreover, the nodes of a truss do not rotate since they are assumed to be connected to the nodes by pins, and, consequently, they cannot transfer a moment to the nodes. Thus, the displacement of an unrestrained (free) node of a planar truss has two components of translation. For example, the translation of each of nodes 2, 3, 4 and 5 of the simple planar truss shown in Fig. 15.10 has a horizontal and a vertical component, while the translation of node 6 has only a horizontal component because this node cannot move in the vertical direction. Thus, the simple planar truss of Fig. 15.10 is kinematically indeterminate to the ninth degree.

As discussed in Section 8.2, the deformed configuration of a cross section of a general planar element in the x_1x_2 plane is completely specified if its components of translation u_1, u_2 and its components of rotation θ_3 are known. However, in the classical theory of

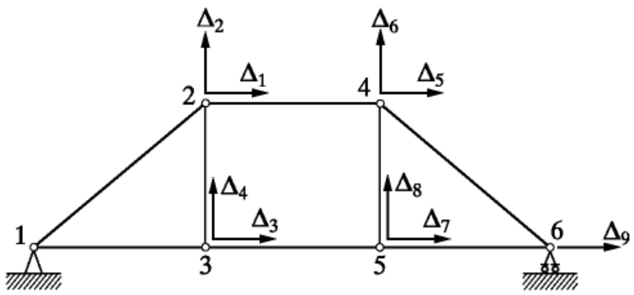


Figure 15.10 Degree of freedom of a planar truss.

beams employed in this text the component of rotation θ_3 of the cross sections of an element can be established if the component of translation $u_2(x_1)$, is a known function of its axial coordinate [see relation (9.27b)]. Thus, the deformed configuration of a general planar element in the $x_1 x_2$ plane is completely specified if the components of translation $u_1(x_1)$ and $u_2(x_1)$ are known functions of the axial coordinate of the element. We call the components of displacement which are needed in order to specify the deformed configuration of an element its *state variables*. The state variables of the types of elements which we are considering are

axial force element	$u_1(x_1)$
general planar element in the x_1x_2 plane	$u_1(x_1)$, $u_2(x_1)$

The state variables of a general planar element are not coupled. Each one of them can be established as a function of the axial coordinate of the element by solving one of the boundary-value problems described in Section 8.12.

15.8.2 Global and Local Axes of Reference

We refer each planar framed structure to a right-handed rectangular system of axes (cartesian axes) $\bar{x}_1$, $\bar{x}_2$ called the *global axes* of the structure. Moreover, we refer each element of a structure to a right-handed rectangular system of axes x_1, x_2, called its *local axes*. As local axes of an element, we choose the set of axes whose origin is the centroid of its cross section at its end *j;* its x_1 axis is directed along the axis of the element from its end *j* to its end *k*; its x_2 and x_3 axes are the principal centroidal axes of the cross section at the end *j* of the element (see Fig. 15.11).

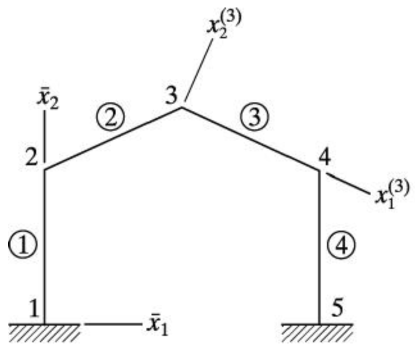


Figure 15.11 Global axes of a planar frame and local axes of element 3.

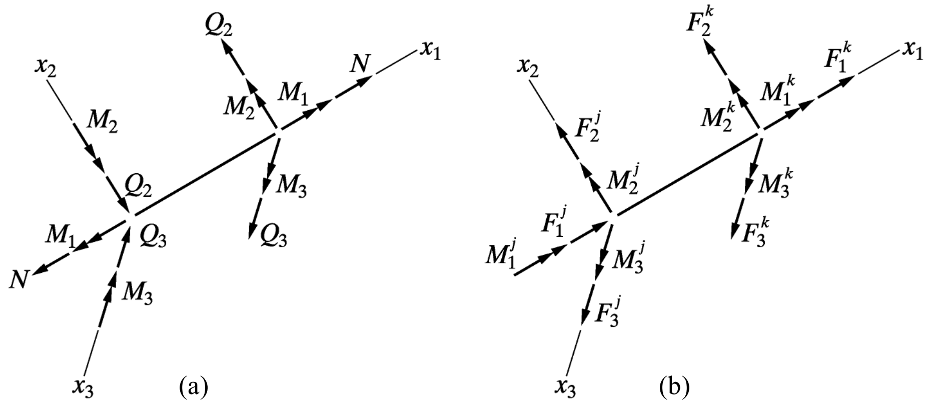


Figure 15.12 Sign conventions for internal actions.

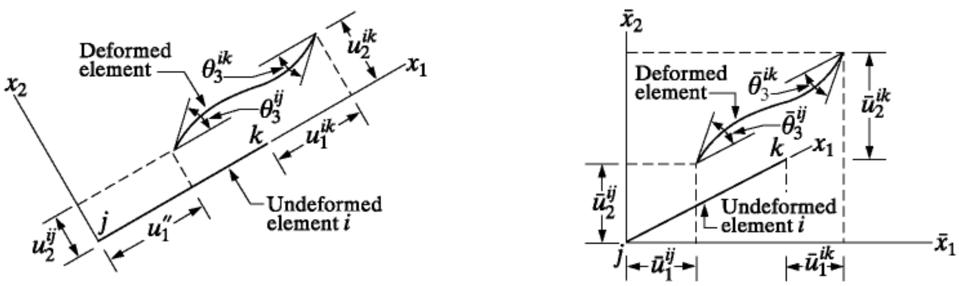
15.8.3 Nodal Actions and Nodal Displacements of an Element

When we analyze structures using the finite element method (direct stiffness or displacement method) we denote the components of the internal force and moment acting on the end q ($q = j$ or k) of an element by F_m^q and M_m^q ($m = 1, 2, 3$), respectively, when referred to local axes and by $\tilde{F}_m^q$ and $\tilde{M}_m^q$, respectively, when referred to global axes. Moreover, we consider as positive† the components of force and moment acting on the ends of an element if their sense coincides with the positive sense of the corresponding local axes x_1, x_2, x_3 (see Fig. 15.12b). That is, the sign convention for the nodal actions of an element used in this chapter is different than the one employed in the previous chapters (see Fig. 15.12a). Thus,

$$N(0) = -F_1^j \tag{15.61a}$$

$$Q_i(0) = -F_i^j \qquad i = 2, 3 \tag{15.61b}$$

$$M_i(0) = -M_i^j \qquad i = 1, 2, 3 \tag{15.61c}$$



(a) Local components of displacement

(b) Global components of display

Figure 15.13 Positive local and global components of nodal displacements of an element of a planar structure.

†The sign convention for the nodal actions (see Fig. 15.12b) used in this chapter is different than the one used in the previous chapters (see Fig. 15.12a).

$$Q_i(L) = F_i^k \qquad i = 2, 3 \tag{15.61d}$$

$$M_i(L) = M_i^k \qquad i = 1, 2, 3 \tag{15.61e}$$

The component of the internal forces and moments acting at the ends of an element are called *components of its nodal actions*.

The components of translations and rotations of the ends of an element are called *components of its nodal displacements*. The positive components of displacement of the ends of an element of a planar structure are shown in Fig. 15.13.

15.8.4 Computation of the Local Shape Functions for a General Planar Element

The matrix of state variables $\{u^e\}$ and the matrix of the nodal displacements $\{D^e\}$ of general planar element e are given by relations (15.56b) and (15.54b), respectively. The matrix of state variables $\{u^e\}$ is approximated on element e as

$$\{\tilde{u}^e\} = \begin{Bmatrix} \tilde{u}_1^e(x_1) \\ \tilde{u}_2^e(x_1) \end{Bmatrix} = \begin{bmatrix} \phi_j & 0 & 0 & \phi_k & 0 & 0 \\ 0 & \phi_{2j}^u & \phi_{2j}^\theta & 0 & \phi_{2k}^u & \phi_{2k}^\theta \end{bmatrix} \begin{Bmatrix} \tilde{u}_1^{ej} \\ \tilde{u}_2^{ej} \\ \tilde{\theta}_3^{ej} \\ \tilde{u}_1^{ek} \\ \tilde{u}_2^{ek} \\ \tilde{\theta}_3^{k} \end{Bmatrix} = [\phi]\,\{\tilde{D}^e\} \tag{15.62}$$

where $\tilde{u}_2^{ej}$ and $\tilde{u}_2^{ek}$ are approximate values of the state variable $u_2(x_1)$ at $x_1 = 0$ and $x_1 = L_e$, respectively. $\tilde{\theta}_3^{ej}$ and $\tilde{\theta}_3^{ek}$ are approximate values of the rotation $\theta_3(x_1)$ at $x_1 = 0$ and $x_1 = L_e$, respectively. Relation (15.62) is known as the *interpolation equation for element e*. The shape functions ϕ_j and ϕ_k are given by relations (15.12), while the shape functions $\phi_j^u\ \phi_j^\theta$ and $\phi_k^u\ \phi_k^\theta$ are chosen so that at the nodes of each element relation (15.62) gives the approximate values of the components of translation $u_1^e(x_1)$ and $u_2^e(x_1)$. That is,

$$\tilde{u}_2^e(0) = \tilde{u}_2^{ej} \qquad\qquad \tilde{u}_2^e(L) = \tilde{u}_2^{ek}$$

$$\tilde{\theta}_3^e(0) = \left.\frac{d\tilde{u}_2^e}{dx_1}\right|_{x_1=0} = \tilde{\theta}_3^{ej} \qquad\qquad \tilde{\theta}_3^e(L) = \left.\frac{d\tilde{u}_3^e}{dx_1}\right|_{x_1=L} = \tilde{\theta}_3^{ek} \tag{15.63}$$

The effect of axial deformation of general planar elements is in general small, and it is usually disregarded when analyzing framed structures by hand calculations. In this case, we do not include the axial components of the nodal forces of an element in its matrix $\{A^e\}$ and the axial components of its nodal displacements in the matrix $\{D^e\}$. That is, relation (15.62), reduces to

$$\tilde{u}_2^e(x_1) = [\phi_j^u \quad \phi_j^\theta \quad \phi_k^u \quad \phi_k^\theta] \begin{Bmatrix} \tilde{u}_2^{ej} \\ \tilde{\theta}_3^{ej} \\ \tilde{u}_2^{ek} \\ \tilde{\theta}_3^{ek} \end{Bmatrix} \tag{15.64a}$$

and

$$\tilde{\theta}_3(x_1) = \frac{d\tilde{u}_2^e}{dx_1} = \frac{d\phi_j^u}{dx_1}\tilde{u}_2^{ej} + \frac{d\phi_j^\theta}{dx_1}\tilde{\theta}_3^{ej} + \frac{d\phi_k^u}{dx_1}\tilde{u}_2^{ek} + \frac{d\phi_k^\theta}{dx_1}\tilde{\theta}_3^{ek} \tag{15.64b}$$

In order that each shape function has the properties described by relations (15.63) it must involve at least four constants. Consequently, it must be at least a polynomial of the third degree. That is, the simplest polynomials which can be chosen as shape functions for one-dimensional, fourth order, linear boundary value problems, as the one under consideration, have the following form:

$$\begin{aligned}
\phi_j^u(x_1) &= A_{j0}^u + A_{j1}^u x_1 + A_{j2}^u x_1^2 + A_{j3}^u x_1^3 \\
\phi_k^\theta(x_1) &= A_{k0}^\theta + A_{k1}^\theta x_1 + A_{k2}^\theta x_1^2 + A_{k3}^\theta x_1^3 \\
\phi_j^\theta(x_1) &= A_{j0}^\theta + A_{j1}^\theta x_1 + A_{j2}^\theta x_1^2 + A_{j3}^\theta x_1^3 \\
\phi_k^u(x_1) &= A_{k0}^u + A_{k1}^u x_1 + A_{k2}^u x_1^2 + A_{k3}^u x_1^3
\end{aligned} \tag{15.65}$$

The constants A_{qi}^u and A_{qi}^θ ($q = j \text{ or } k$) (i = 0, 1, 2, 3) are evaluated by requiring that the element shape functions satisfy relations (15.63). That is,

$$\phi_j^u(0) = 1 = A_{j0}^u \qquad \left.\frac{d\phi_j^u}{dx_1}\right|_{x_1=0} = 0 = A_{j1}^u$$

$$\phi_j^u(L_e) = 0 = 1 + L_e^2 A_{j2}^u + L_e^3 A_{j3}^u \qquad \left.\frac{d\phi_j^u}{dx_1}\right|_{x_1=L_e} = 0 = 2L_e A_{j2}^u + 3L_e^2 A_{j3}^u$$

$$\phi_j^\theta(0) = 0 = A_{j0}^\theta \qquad \left.\frac{d\phi_j^\theta}{dx_1}\right|_{x_1=0} = 1 = A_{j1}^\theta$$

$$\phi_j^\theta(L_e) = 0 = L_e + L_e^2 A_{j2}^\theta + L_e^3 A_{j3}^\theta \qquad \left.\frac{d\phi_j^\theta}{dx_1}\right|_{x_1=L_e} = 0 = 1 + 2L_e A_{j2}^\theta + 3L_e^2 A_{j3}^\theta$$

$$\phi_k^u(0) = 0 = A_{k0}^u \qquad \left.\frac{d\phi_k^u}{dx_1}\right|_{x_1=0} = 0 = A_{k1}^u$$

$$\phi_k^u(L_e) = 1 = L_e^2 A_{k2}^u + L_e^3 A_{k3}^u \qquad \left.\frac{d\phi_k^u}{dx_1}\right|_{x_1=L_e} = 0 = 2L_e A_{k2}^u + 3L_e^3 A_{k3}^u$$

$$\phi_k^\theta(0) = 0 = A_{k0}^\theta \qquad \left.\frac{d\phi_k^\theta}{dx_1}\right|_{x_1=0} = A_{k1}^\theta$$

$$\phi_k^\theta(L_e) = 0 = L_e^2 A_{k2}^\theta + L_e^3 A_{k3}^\theta \qquad \left.\frac{d\phi_k^\theta}{dx_1}\right|_{x_1=L_e} = -1 = 2L_e A_{k2}^\theta + 3L_e^2 A_{k3}^\theta \tag{15.66}$$

Substituting the values of the constants obtained from relations (15.66) into relations (15.65) we get the following element shape functions for the one-dimensional, fourth order, linear boundary value problem for computing the transverse component of translation $u_2(x_1)$:

$$\begin{aligned} \phi_j^u(x_1) &= 1 - \frac{3x_1^2}{L_e^2} + \frac{2x_1^3}{L_e^3} \qquad & \phi_j^\theta(x_1) &= x_1 - \frac{2x_1^2}{L_e} + \frac{x_1^3}{L_e^2} \\ \phi_k^u(x_1) &= \frac{3x_1^2}{L_e^2} - \frac{2x_1^3}{L_e^3} \qquad & \phi_k^\theta(x_1) &= -\frac{x_1^2}{L_e} + \frac{x_1^3}{L_e^2} \end{aligned} \tag{15.67}$$

It can be shown that the element shape functions for the boundary value problem for computing the transverse component of translation $u_3(x_1)$ are

$$\begin{aligned} \phi_j^u(x_1) &= 1 - \frac{3x_1^2}{L_e^2} + \frac{2x_1^3}{L_e^3} \qquad & \phi_j^\theta(x_1) &= -x_1 + \frac{2x_1^2}{L_e} - \frac{x_1^3}{L_e^2} \\ \phi_k^u(x_1) &= \frac{3x_1^2}{L_e^2} - \frac{2x_1^3}{L_e^3} \qquad & \phi_k^\theta(x_1) &= \frac{x_1^2}{L_e} - \frac{x_1^3}{L_e^2} \end{aligned} \tag{15.68}$$

Elements with more than three nodes are employed for one-dimensional, fourth order, linear boundary value problems. However, their shape functions are polynomials of a degree higher than the third.

When the interpolation or shape functions are polynomials, the approximate solution (15.62) converges to the actual as the seize of the elements decreases provided that the shape functions ϕ_j^u, ϕ_j^θ, ϕ_k^u and ϕ_k^θ can accommodate rigid-body motion of the element. That is, provided that relation (15.62) satisfies the following requirements:

1. It gives a constant value of the state variable $u_2(x_1)$ throughout the domain of each element when its nodal values u_2^{ej} and u_2^{ek} are identical ($u_2^{ej} = u_2^{ek} = C$) while θ_3^{ej}, and θ_3^{ek} vanish. That is,

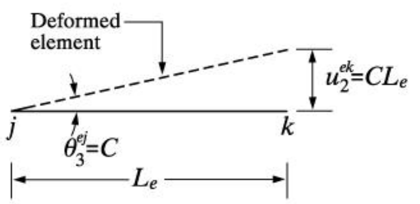


Figure 15.14 Rigid-body rotation of an element.

$$\tilde{u}_2(x_1) = C = \tilde{u}_2^{ej}\phi_j^u + \tilde{u}_2^{ek}\phi_k^u = C\,(\phi_j^u + \phi_k^u)$$

Thus,

$$\phi_j^u + \phi_k^u = 1 \tag{15.69}$$

It can be readily shown that the shape functions specified by relations (15.67) satisfy requirement (15.69).

2. It gives a constant value of $\theta_3(x_1) = -du_3/dx_1$ throughout the domain of the element when $\theta_3^{ej} = \theta_3^{ek} = C$, $u_2^{ej} = 0$ and $u_2^{ek} = CL_e$ (see Fig. 15.14). That is, using relations (15.64a) we have

$$\theta_3^e(x_1) = \frac{du_3^e}{dx_1} = C = \left[\frac{d\phi_j^u}{dx_1}u_2^{ej} + \frac{d\phi_j^\theta}{dx_1}\theta_3^{ej} + \frac{d\phi_k^u}{dx_1}u_2^{ek} + \frac{d\phi_j^\theta}{dx_1}\theta_3^{ek}\right]$$

$$= C\left[\frac{d\phi_j^\theta}{dx_1} - \frac{d\phi_k^u}{dx_1}L_e + \frac{d\phi_k^\theta}{dx_1}\right]$$

Thus,

$$\frac{d\phi_j^\theta}{dx_1} - L_e\frac{d\phi_k^u}{dx_1} + \frac{d\phi_k^\theta}{dx_1} = 1 \tag{15.70}$$

It can be readily shown that the shape functions (15.67) satisfy requirement (15.70). Moreover, it can be shown that relation (15.62) with (15.67) represents the exact solution of the boundary value problem for computing the component of translation $u_2(x_1)$ of elements which have constant modules of elasticity E_e and cross sectional area A_e and are not subjected to loads along their length. However, for elements with constant E_e and A_e which are subjected to loads along their length, relation (15.62) represents an approximate solution. *Thus, we may conclude that the finite element method gives the exact expression for the component of translation displacement $u_2(x_1)$, the internal shearing force $Q_2(x_1)$ and the internal bending moment $M_3(x_1)$ for beams and frames, when they can be subdivided into elements each of which is made from one material, has a constant cross section and is not subjected to loads along their lengths.*

15.8.5 Direct Computation of the Contribution of a General Planar Element to the Stiffness Matrix and the Load Vector of a Planar Framed Structure in the $x_1 x_2$ Plane

Consider a general planar element (say element e) of planar beams or frames in the $x_1 x_2$ plane. The element is subjected to given loads inducing in it only a distribution of shearing force $Q_2^e(x_1)$, and bending moment $M_3^e(x_1)$. Assume that the component of translation (deflection) $u_2^e(x_1)$ and the component of rotation $\theta_3(x_1)$ are specified at both ends of the element

$$\begin{aligned} u_2(0) &= u_2^{ej} \qquad & u_2(L) &= u_2^{ek} \\ \theta_3(0) &= \theta_3^{ej} \qquad & \theta_3(L) &= \theta_3^{ek} \end{aligned} \tag{15.71}$$

Substituting relation (15.64a) into (9.32b) and the resulting relation into the modified weighed residual equation (13.80), with $\epsilon_1 = \epsilon_2 = 0$ and using the Gallerkin assumption

(13.75), for the element under consideration, we obtain

$$-\int_0^{L_e} E_e I_2^e \frac{d^2\phi_r}{dx_1^2}\left(\frac{d^2[\phi]}{dx_1^2}\{\tilde{D}^e\} + H_2^e\right) dx_1$$

$$+\int_0^{L_e} \phi_r J_3^e dx_1 + \left[\phi_r \frac{d\tilde{M}_2}{dx_1}\right]_{x_1=0}^{x_1=L} - \left[\frac{d\phi_r}{dx_1}\tilde{M}_3\right]_{x_1=0}^{x_1=L} = 0 \qquad (15.72)$$

$$r = 1,2,3,4$$

where

$$\phi_1 = \phi_j^u, \quad \phi_2 = \phi_j^\theta, \quad \phi_3 = \phi_k^u, \quad \phi_4 = \phi_k^\theta$$

Relation (15.72) may be rewritten as

$$-\int_0^{L_e} E_e I_2^e \frac{d^2[\phi]^T}{dx_1^2}\left(\frac{d^2[\phi]}{dx_1^2}\{\tilde{D}^e\} + H_2^e\right) dx_1$$

$$+\int_0^{L_e} [\phi]^T J_3^e dx_1 + \left[[\phi]^T \frac{d\tilde{M}_2}{dx_1}\right]_{x_1=0}^{x_1=L} - \left[\frac{d[\phi]^T}{dx_1}\tilde{M}_3\right]_{x_1=0}^{x_1=L} = 0 \qquad (15.73a)$$

The matrix of shape functions $[\phi]$ is defined by relation (15.67). Referring to relations (15.67) and taken into account that Q_2=$-dM_3/dx_1$, we have

where

$$\left[[\phi]^T \frac{d\tilde{M}_3}{dx_1}\right]^{x_1=L} = -\begin{Bmatrix}0\\0\\1\\0\end{Bmatrix}\tilde{Q}_2^{ek}(L) = -\begin{Bmatrix}0\\0\\\tilde{F}_2^{ek}\\0\end{Bmatrix}$$

$$\left[[\phi]^T \frac{d\tilde{M}_3}{dx_1}\right]_{x_1=0} = -\begin{Bmatrix}1\\0\\0\\0\end{Bmatrix}\tilde{Q}_2^{ej}(0) = -\begin{Bmatrix}\tilde{F}_2^{ej}\\0\\0\\0\end{Bmatrix}$$

$$-\left[\frac{d[\phi]^T}{dx_1}\tilde{M}_3\right]^{x_1=L} = -\begin{Bmatrix}0\\0\\0\\1\end{Bmatrix}\tilde{M}_3^{ek}(L) = \begin{Bmatrix}0\\0\\0\\\tilde{M}_3^{ek}\end{Bmatrix} \qquad (15.73b)$$

$$-\left[\frac{d[\phi]^T}{dx_1}\tilde{M}_3\right]_{x_1=0} = \begin{Bmatrix}0\\1\\0\\0\end{Bmatrix}\tilde{M}_3^{ej}(0) = -\begin{Bmatrix}0\\\tilde{M}_3^{ej}\\0\\0\end{Bmatrix}$$

$\tilde{F}_2^{ej}$ and $\tilde{F}_2^{ek}$ = approximations to the shearing forces acting at the ends j and k of element e, respectively. They are considered positive if they act in the direction of the positive x_2 axis (see Fig. 15.12b).

$\tilde{M}_3^{ej}$ and $\tilde{M}_3^{ek}$ = approximations to the bending moments acting at the ends j and k of the element, respectively. They are considered positive if their vector acts in the direction of the positive x_3 axis (see Fig. 15.12b).

Substituting relation (15.73b) into (15.73a), we get

$$\{A^{\tilde{e}}\} = [\tilde{K}^e]\{\tilde{D}^e\} + \{\tilde{A}^{Re}\} \tag{15.74}$$

where

$$\{\tilde{A}^e\} = \begin{Bmatrix} \tilde{F}_2^{ej} \\ \tilde{M}_3^{ej} \\ \tilde{F}_2^{ek} \\ \tilde{M}_3^{ek} \end{Bmatrix} \tag{15.75}$$

$\{\tilde{D}^e\}$ = an approximation to the matrix of nodal displacements for element e defined by relation (15.54b).

$[\tilde{K}^e]$ = the local stiffness matrix for element e. Referring to relation (15.73) it is equal to

$$[K^e] = \int_0^{L_e} \left[E_e(x_1) I_3^e(x_1) \frac{d^2[\phi]^T}{dx_1^2} \frac{d^2[\phi]}{dx_1^2} \right] dx_1 \tag{15.76}$$

It can be shown that the terms of the matrix $[\tilde{A}^{Re}]$ are equal and opposite to the contributions, of the loads acting on element e, to the equivalent actions at the nodes of the structure where the ends j and k of element e are connected.

It is clear that once equation (15.74) is established for an element its local stiffness matrix and the contribution of the loads acting along its length to the matrix of equivalent actions of the domain are known. That is, denoting by e and $(e + 1)$ the two elements adjacent to node r, the equivalent actions $\{P_r^E\}$ at node r are equal to

$$\{P_r^{Ee}\} = -\begin{Bmatrix} \tilde{F}_2^{ek} + \tilde{F}_2^{(e+1)j} \\ \tilde{M}_3^{ek} + \tilde{M}_3^{(e+1)j} \end{Bmatrix} \tag{15.78}$$

If the effect of the axial deformation is negligible, the stiffness matrix for a general planar element is obtained by substituting relations (15.67) into (15.76). That is,

$$[K^e] = \frac{E_e I_2^e}{L_e^3} \begin{bmatrix} 12 & -6L_e & -12 & -6L_e \\ -6L_e & 4L_e^2 & 6L_e & 2L_e^2 \\ -12 & 6L_e & 12 & 6L_e \\ -6L_e & 2L_e^2 & 6L_e & 4L_e^2 \end{bmatrix} \tag{15.79}$$

The stiffness matrix of a prismatic general planar beam element made from one material, when the effect of axial deformation is not negligible, is obtained by combining relation (15.79) and (15.31). That is,

$$[K^e] = \begin{bmatrix} \frac{A_e E_e}{L_e} & 0 & 0 & -\frac{A_e E_e}{L_e} & 0 & 0 \\ 0 & \frac{12E_e I_3^e}{L_e^3} & \frac{6E_e I_3^e}{L_e^2} & 0 & -\frac{12E_e I_3^e}{L_e^3} & \frac{6E_e I_3^e}{L_e^2} \\ 0 & \frac{6E_e I_3^e}{L_e^2} & \frac{4E_e I_3^e}{L_e} & 0 & -\frac{6E_e I_3^e}{L_e^2} & \frac{2E_e I_3^e}{L_e} \\ -\frac{A_e E_e}{L_e} & 0 & 0 & \frac{A_e E_e}{L_e} & 0 & 0 \\ 0 & -\frac{12E_e I_3^e}{L_e^3} & -\frac{6E_e I_3^e}{L_e^2} & 0 & \frac{12E_e I_3^e}{L_e^3} & -\frac{6E_e I_3^e}{L_e^2} \\ 0 & \frac{6E_e I_3^e}{L_e^2} & \frac{2E_e I_3^e}{L_e} & 0 & -\frac{6E_e I_3^e}{L_e^2} & \frac{4E_e I_3^e}{L_e} \end{bmatrix} \tag{15.80}$$

where
L_e = length of element.
I_3^e = moment of inertia of cross section of element about its x_3 local axis.
E_e = modulus of elasticity of the material from which element is made.

Moreover, the stiffness matrix for an axial force element of a frame if the effect of axial deformation is not negligible, is

$$[K] = \frac{A_e E_e}{L_e} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15.81}$$

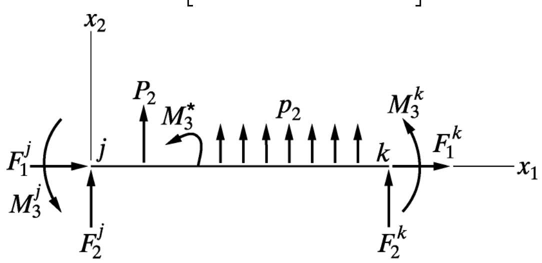


Figure 15.15 Free-body diagram of an element subjected to bending about the x_3 principal centroidal axis.

15.8.6 Physical Meaning of Relation (15.74)

In this section we illustrate the physical meaning of relation (15.74). If the ends of the element are fixed the matrix $\{\tilde{D}^e\}$ is a zero matrix and equation (15.74) reduces to

$$\{\tilde{A}^e\} = \{\tilde{A}^{Re}\} \tag{15.82}$$

This indicates that $\{\tilde{A}^{Re}\}$ *is an approximation to the matrix of local nodal actions (the reactions) of the element subjected to the given loads with its ends fixed. We call* $\{\tilde{A}^{Re}\}$ *the matrix of fixed-end actions of the element.*

If an element is subjected only to nodal displacements, $\{\tilde{A}^{Re}\}$ is a zero matrix and when the effect of axial deformation is included, relation (15.74) reduces to

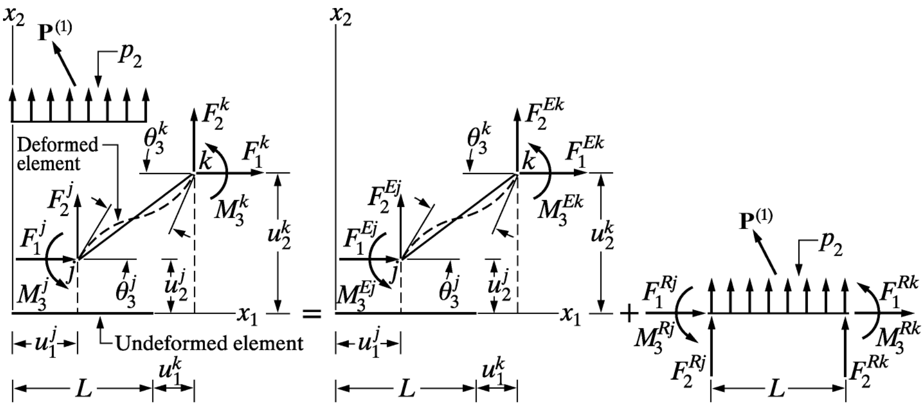
$$\{\tilde{A}^{Ee}\} = \begin{Bmatrix} \tilde{F}_1^{ej} \\ \tilde{F}_2^{ej} \\ \tilde{M}_3^{ej} \\ \tilde{F}_1^{ek} \\ \tilde{F}_2^{ek} \\ \tilde{M}_3^{ek} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \tilde{u}_1^{ej} \\ \tilde{u}_2^{ej} \\ \tilde{\theta}_3^{ej} \\ \tilde{u}_1^{ek} \\ \tilde{u}_2^{ek} \\ \tilde{\theta}_3^{ek} \end{Bmatrix} \tag{15.83}$$

Equation (15.83) is the *approximate local stiffness equation for element e when theeffect of axial deformation is not negligible. The terms of the matrix* $\{\tilde{A}^{Ee}\}$ *represent an approximation to the nodal actions which must be applied to the ends j and k of element e in order to displace them by amounts equal to its actual nodal translations and rotations.*

Thus, as shown in Fig. 15.16, in relation (15.74), the nodal actions of an element are expressed as the sum of the nodal actions when the element is subjected only to its nodal displacements and its nodal actions when it is subjected, to the given loading with its ends fixed. Consider an element subjected to the loading shown in Fig. 15.16a. The external actions acting on this element are equal to the sum of the external actions acting on the element subjected to loadings of Fig. 15.16b and c. Moreover, each component of nodal displacement of the element subjected to the loads shown in Fig. 15.16a is equal to the sum of the corresponding component of nodal displacement of the element subjected to the loads shown in Figs. 15.16b and c. Consequently, since the principle of superposition is valid for the structures which we are considering, the internal actions acting on any cross section of the element subjected to the loads shown in Fig. 15.16b and the components of displacement of any of its points are equal to the sum of the corresponding quantities of the element subjected to the loads shown in Figs. 15.6b and c.

The fixed-end actions for elements of constant cross section subjected to loads of practical interest can be computed and tabulated (see Table 15.3). In structural analysis such tables are used to compute the matrix $\{A^{Re}\}$.

The physical significance of the stiffness coefficients of the second column of the stiffness matrix of relation (15.83) for a general planar element can be established by considering such an element subjected only to the components of nodal actions which are required to induce the following nodal displacements:



(a) Element subjected to the given loading and its actual nodal displacements

(b) Element subjected on to its actual nodal displacements

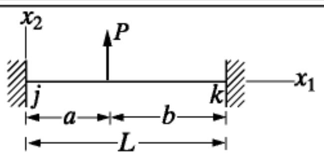
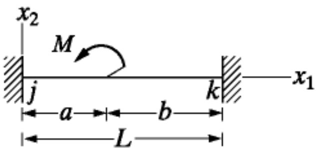
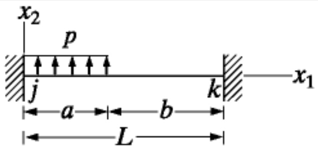
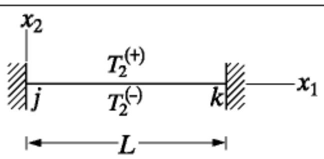
(c) Element subjected to the given loading acting along its length with its ends fixed

Figure 15.16 Superposition of the response of a general planar element subjected only to its nodal displacements, and that of the element subjected to the given loads acting along its length with its ends fixed.

$$u_2^j = 1 \qquad u_1^j = \theta_3^j = u_1^k = u_2^k = \theta_3^k = 0 \tag{15.84}$$

In this case the stiffness equations (15.83) reduce to

Table 15.3 Restraining actions for elements of constant cross section subjected to axial centroidal forces and temperature change $\delta T_3 (\Delta T_c = \delta T_2 = 0)$.

$F_2^{Rj} = -\frac{Pb^2}{L^2}\left(3 - \frac{2b}{L}\right)$ $M_3^{Rj} \; -\frac{Pab^2}{L^2}$		$F_2^{RK} = -\frac{Pa^2}{L^2}\left(3 - \frac{2a}{L}\right)$ $M_3^{RK} = \frac{Pa^2 b}{L^2}$
$F_2^{Rj} = \frac{6Mab}{L^3}$ $M_3^{Rj} = \frac{Mb}{L}\left(2 - \frac{3b}{L}\right)$		$F_2^{RK} = \frac{-6Mab}{L^3}$ $M_3^{RK} = \frac{Ma}{L}\left(2 - \frac{3a}{L}\right)$
$F_2^{Rj} = -\frac{pa}{2}\left(2 - \frac{2a^2}{L^2} + \frac{a^3}{L^3}\right)$ $M_3^{Rj} = -\frac{pa^2}{12}\left(6 - \frac{8a}{L} + \frac{3a^2}{L^2}\right)$		$F_2^{RK} = -\frac{pa^3}{2L^2}\left(2 - \frac{a}{L}\right)$ $M_3^{RK} = \frac{pa^3}{12L}\left(4 - \frac{3a}{L}\right)$
$M_3^{Rj} = -\frac{EI\alpha\delta T_2}{h_2}$		$M_3^{RK} = \frac{EI\alpha\delta T_2}{h_2}$

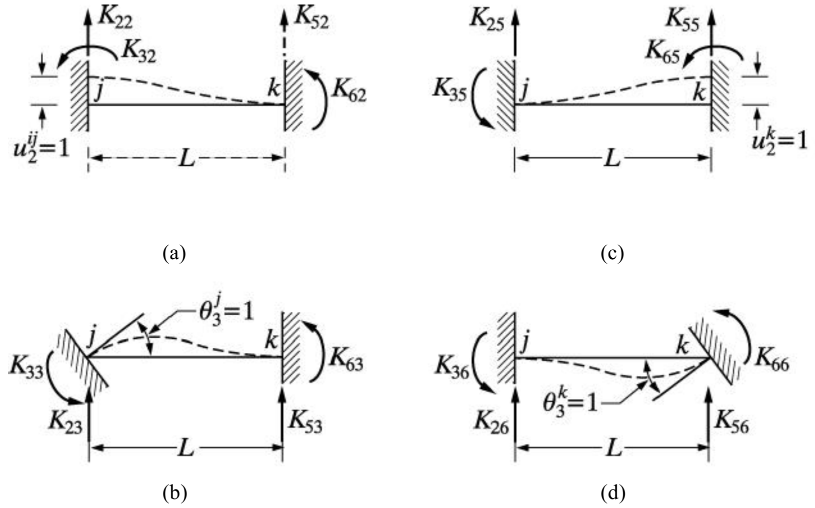


Figure 15.17 Physical significance of the local stiffness coefficients $K_{mn}(m, n = 1, 2, ..., 6)$ of a general planar element.

$$\{\tilde{A}^{Ee}\} = \begin{Bmatrix} 0 \\ \tilde{F}_2^{ej} \\ \tilde{M}_3^{ej} \\ 0 \\ \tilde{F}_2^{ek} \\ \tilde{M}_3^{ek} \end{Bmatrix} = \begin{Bmatrix} K_{12} \\ K_{22} \\ K_{32} \\ K_{42} \\ K_{52} \\ K_{62} \end{Bmatrix} \tag{15.85}$$

Thus, as shown in Fig. 15.17a the stiffness coefficients K_{22}, K_{32}, K_{52} and K_{62} represent the nodal actions which must be applied to an element in order to induce the nodal displacements specified by relation (15.84).

The physical significance of the stiffness coefficients of the third column of the stiffness matrix of a general planar element can be established by considering an element subjected only to the components of nodal actions which are required to induce the following nodal displacements:

$$\theta_3^j = 1 \qquad u_1^j = u_2^j = u_1^k = u_2^k = \theta_3^k = 0 \tag{15.86}$$

In this case the stiffness equations (15.83) reduce to

$$\{\tilde{A}^{Ee}\} = \begin{Bmatrix} 0 \\ \tilde{F}_2^{ej} \\ \tilde{M}_3^{ej} \\ 0 \\ \tilde{F}_2^{ek} \\ \tilde{M}_3^{ek} \end{Bmatrix} = \begin{Bmatrix} K_{13} \\ K_{23} \\ K_{33} \\ K_{43} \\ K_{53} \\ K_{63} \end{Bmatrix} \tag{15.87}$$

Thus, as shown in Fig. 15.17b the stiffness coefficients K_{23}, K_{33}, K_{53} and K_{63} represent the nodal actions which must be applied on an element in order to induce the nodal displacements specified by relation (15.86).

The physical significance of the remaining non-vanishing stiffness coefficients for a general planar element is illustrated in Fig. 15.17c and d.

15.8.7 Transformation of the Matrices of Nodal Actions of an Element

In this section, we give the transformation relations between the local and global matrices of nodal actions of the elements of the various types of structures that we are considering.

Elements of a Planar Truss

Consider an element of a planar truss and choose the x_1, x_2 and $\bar{x}_1$, $\bar{x}_2$ axes in the plane of the truss. Referring to Fig. 15.18, we have

$$\begin{aligned} F_1^j &= \bar{F}_1^j \cos\phi_{11} + \bar{F}_2^j \sin\phi_{11} \\ F_1^k &= \bar{F}_1^k \cos\phi_{11} + \bar{F}_2^k \sin\phi_{11} \end{aligned} \tag{15.88}$$

Hence,

$$\{A\} = \begin{Bmatrix} F_1^j \\ F_1^k \end{Bmatrix} = \begin{bmatrix} \cos\phi_{11} & \sin\phi_{11} & 0 & 0 \\ 0 & 0 & \cos\phi_{11} & \sin\phi_{11} \end{bmatrix} \begin{Bmatrix} \bar{F}_1^j \\ \bar{F}_2^j \\ \bar{F}_1^k \\ \bar{F}_1^k \end{Bmatrix} = [\Lambda_{PT}]\{\bar{A}\} \tag{15.89}$$

Moreover,

$$\{\bar{A}\} = \begin{Bmatrix} \bar{F}_1^j \\ \bar{F}_2^j \\ \bar{F}_1^k \\ \bar{F}_1^k \end{Bmatrix} = \begin{bmatrix} \cos\phi_{11} & 0 \\ \sin\phi_{11} & 0 \\ 0 & \cos\phi_{11} \\ 0 & \sin\phi_{11} \end{bmatrix} \begin{Bmatrix} F_1^j \\ F_1^k \end{Bmatrix} = [\Lambda_{PT}]^T\{A\} \tag{15.90}$$

where

$$[\Lambda_{PT}] = \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 & 0 \\ 0 & 0 & \lambda_{11} & \lambda_{12} \end{bmatrix} = \begin{bmatrix} \cos\phi_{11} & \sin\phi_{11} & 0 & 0 \\ 0 & 0 & \cos\phi_{11} & \sin\phi_{11} \end{bmatrix} \tag{15.91}$$

Elements of a Planar Beam or a Planar Frame

Consider an element of a planar beam or a planar frame and choose the x_1, x_2 and $\bar{x}_1$, $\bar{x}_2$ axes in the plane of the structure. Noting that M_3^q is equal to $\bar{M}_3^q$ ($q = j$ or k), and referring to relations (1.36) and (1.37) the matrices of nodal actions $\{A^q\}$ and $\{\bar{A}^q\}$($q = j$ or k) of the element under consideration are related by the following transformation relations:

$$\{A^q\} = \begin{Bmatrix} F_1^q \\ F_2^q \\ M_3^q \end{Bmatrix} = [\hat{\Lambda}_{PF}] \begin{Bmatrix} \bar{F}_1^q \\ \bar{F}_2^q \\ \bar{M}_3^q \end{Bmatrix} = [\hat{\Lambda}_{PF}]\{\bar{A}^q\} \qquad q = j \text{ or } k \tag{15.92}$$

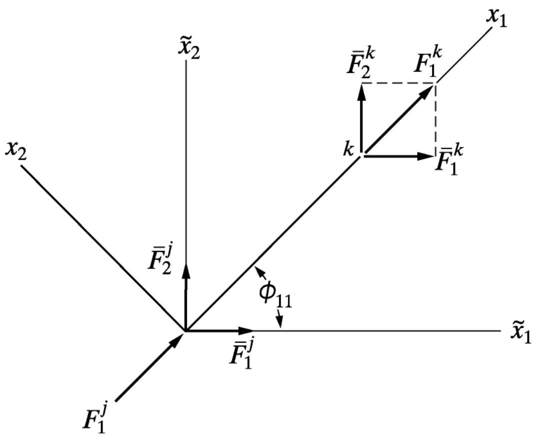


Figure 15.18 Free-body diagram of an element of a truss.

and

$$\{\bar{A}^q\} = [\hat{\Lambda}_{PF}]^T\{A^q\} \tag{15.93}$$

where

$$[\hat{\Lambda}_{PF}] = \begin{bmatrix} [\Lambda_P] & [0] \\ [0] & 1 \end{bmatrix} \tag{15.94}$$

The matrix $\{\Lambda_P\}$ is the transformation matrix for the element given by relations (1.37). The matrices of nodal actions $\{A\}$ and $\{\bar{A}\}$ of an element of a planar beam or a planar frame are related by the following transformation relations:

$$\{A\} = \begin{Bmatrix} F_1^j \\ F_2^j \\ M_3^j \\ -- \\ F_1^k \\ F_2^k \\ M_3^k \end{Bmatrix} = \begin{Bmatrix} \{A^j\} \\ \{A^k\} \end{Bmatrix} = [\Lambda_{PF}]\begin{Bmatrix} \{\bar{A}^j\} \\ \{\bar{A}^k\} \end{Bmatrix} = [\Lambda_{PF}]\{\bar{A}\} \tag{15.95}$$

and

$$\{\bar{A}\} = [\Lambda_{PF}]^T\{A\} \tag{15.96}$$

where

$$[\Lambda_{PF}] = \begin{bmatrix} [\hat{\Lambda}_{PF}] & [0] \\ [0] & [\hat{\Lambda}_{PF}] \end{bmatrix} = \begin{bmatrix} [\Lambda_P] & [0] & [0] & [0] \\ [0] & 1 & [0] & [0] \\ [0] & [0] & [\Lambda_P] & [0] \\ [0] & [0] & [0] & 1 \end{bmatrix} \tag{15.97}$$

The matrix $[\Lambda_P]$ is the transformation matrix for the element given by relation (1.37).

Generalization of the Results

On the basis of the foregoing presentation, the relation between the matrices of nodal actions $\{A\}$ and $\{\bar{A}\}$ of an element of a structure may be written as

$$\{A\} = [\Lambda][\bar{A}\} \tag{15.98a}$$

and

$$\{\bar{A}\} = [\Lambda]^T[A\} \tag{15.98b}$$

where depending on the type of the structure the matrix $\{\Lambda\}$ is one of the matrices $[\Lambda_{PT}]$ or $[\Lambda_{PF}]$ given by relation (15.91) or (15.97), respectively. From relations (15.98), it is apparent that

$$[\Lambda][\Lambda^T] = [I] \tag{15.99}$$

where $[I]$ is the unit matrix.

15.8.8 Transformation of the Matrices of Nodal Displacements of an Element

It is apparent that the transformation relations between the local and global matrices of nodal displacements are analogous to those between the nodal and global matrices of nodal actions. That is, in general, referring to relations (15.98) the relations between the local and global matrices of nodal displacements can be written as

$$\{D\} = [\Lambda]\{\bar{D}\} \tag{15.100a}$$

and

$$\{\bar{D}\} = [\Lambda]^T\{D\} \tag{15.100b}$$

where depending on the type of structure, the matrix $[\Lambda]$ is one of the matrices $[\Lambda_{PT}]$ or $[\Lambda_{PF}]$ given by relation (15.91) or (15.97), respectively.

15.8.9 Transformation of the Local Stiffness Matrix to Global

In this section we express the stiffness equation (15.47) or (15.83) for an element of a framed structure in global form. That is,

$$\{\bar{A}^E\} = [\bar{K}]\{\bar{D}\} \tag{15.101}$$

For this purpose consider relation (15.83) for an element of a framed structure. That is,

$$[A^E] = [K][D] \tag{15.102}$$

Substituting relation (15.100a) into (15.102), we obtain

$$[A^E] = [\overset{\sqcap}{K}][\bar{D}] \tag{15.103}$$

We call the matrix $[\overset{\sqcap}{K}]$ the *hybrid stiffness matrix for the element.* It transforms the global components of nodal displacements of an element to the local components of its nodal actions. It is given as

$$[\overset{\sqcap}{K}] = [K][\Lambda] \tag{15.104}$$

where depending on the type of the structure the matrix $[\Lambda]$ is one of the matrices $[\Lambda_{PT}]$ or $[\Lambda_{PF}]$ given by relation (15.91) or (15.97), respectively. Premultiplying each side of relation (15.102) by $[\Lambda]^T$ and using relation (15.103) and (15.104), we obtain

$$\{\bar{A}^E\} = [\Lambda]^T\{A^E\} = [\Lambda]^T[\overset{\sqcap}{K}]\{\bar{D}\} = [\Lambda]^T[K][\Lambda][\bar{D}] \tag{15.105}$$

Referring to relations (15.101) and (15.105) we may conclude that

$$[\bar{K}] = [\Lambda]^T[K][\Lambda] = [\Lambda]^T[\overset{\sqcap}{K}] \tag{15.106}$$

Equation (15.106) is employed in obtaining the global stiffness matrix of an element from its local stiffness matrix.

The global matrix of nodal actions of an element of a framed structure subjected to given loads can be expressed as

$$\{\bar{A}\} = \{\bar{A}^E\} + \{\bar{A}^R\} = [\bar{K}]\{\bar{D}\} + \{\bar{A}^R\} \tag{15.107}$$

15.8.10 Restrained Structure — Structure Subjected to Equivalent Actions

Consider the frame subjected to the loading shown in Fig. 15.19a. The external actions acting on this frame are equal to the sum of the corresponding external actions acting on the frame subjected to the loads shown in Fig. 15.19b and c. Consequently, since the principle of superposition is valid for the structures which we are considering (see Section 3.13), the internal actions at any cross section of an element and the components of displacement of any point of the frame, loaded as shown in Fig. 15.19a, are equal to the sum of the corresponding quantities of the frame loaded as shown in Fig. 15.19b and c. Notice that the values of the external actions $\bar{S}_1^{(2)}$, $\bar{S}_2^{(2)}$, $\bar{S}_3^{(2)}$, $\bar{S}_1^{(3)}$, $\bar{S}_2^{(3)}$ and $\bar{S}_3^{(3)}$ can be chosen so that the components of translation and of rotation of nodes 2 and 3 of the frame loaded as shown in Fig. 15.19b vanish. For this choice of the external actions $\bar{S}_1^{(2)}$, $\bar{S}_2^{(2)}$, $\bar{S}_3^{(2)}$, $\bar{S}_1^{(3)}$, $\bar{S}_2^{(3)}$ and $\bar{S}_3^{(3)}$, the frame of Fig. 15.19b becomes kinematically determinate. That is, all the components of displacement of its nodes vanish. In this case the structure of Fig. 15.19b is called the *restrained structure*. The external actions $\bar{S}_1^{(2)}$, $\bar{S}_2^{(2)}$, $\bar{S}_3^{(2)}$, $\bar{S}_1^{(3)}$, $\bar{S}_2^{(3)}$ and $\bar{S}_3^{(3)}$ acting on the nodes of the restrained structure are called the *restraining actions*. Moreover, the external actions $\bar{S}_1^{(2)}$, $\bar{S}_2^{(2)}$, $\bar{S}_3^{(2)}$, $\bar{S}_1^{(3)}$, $\bar{S}_2^{(3)}$, and $\bar{S}_3^{(3)}$ acting on the nodes of the structure of Fig. 15.19c are called the *equivalent actions*. They are equal and opposite to the restraining actions.

The restrained structure is comprised of elements which are either fixed at both ends or pinned at both ends. Thus, the external loads (external actions, change of temperature) acting along the length of an element of the restrained structure affect only this element.

Notice that by considering the equilibrium of a node of the restrained structure, it can

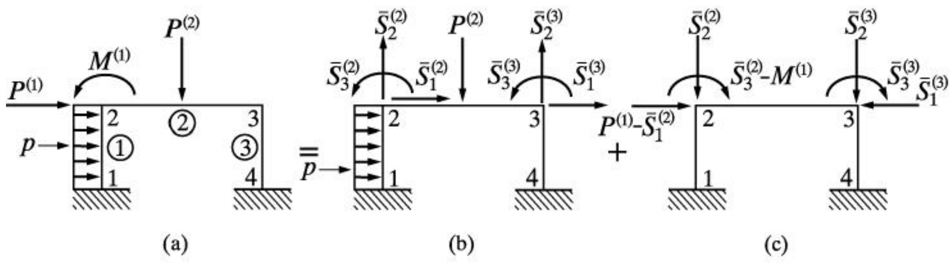


Figure 15.19 Principle of superposition.

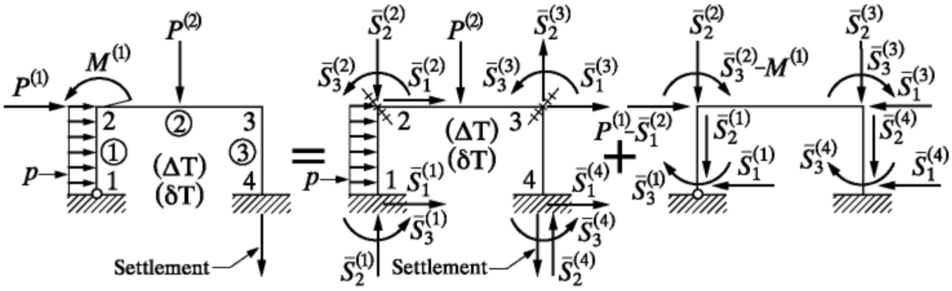


Figure 15.20 Superposition of the restrained structure and the structure subjected to the equivalent actions and to the given concentrated actions acting on its nodes.

be shown that the global components of the restraining actions acting on this node are equal to the sum of the corresponding global components of the fixed-end actions of the ends of the elements connected to this node. Moreover, notice that for uniformity of treatment we consider the reactions of the supports of the restrained structure as restraining actions (see Fig. 15.20). Referring to Fig. 15.20, it is apparent that the superposition of the components of displacement of nodes 1, 2 and 3 of the restrained structure (see Fig. 15.20b) and those of the structure subjected to the equivalent actions and to the given concentrated actions acting on its nodes (see Fig. 15.20c) must yield the corresponding components of displacement of nodes 1, 2 and 3, of the structure subjected to the given loading (see Fig. 15.20a). *Inasmuch as the components of displacement of nodes* 1, 2 *and* 3 *of the restrained structure are zero, the components of displacement of nodes* 1, 2 *and* 3 *of the structure, subjected to the equivalent actions and to the given concentrated actions acting on its nodes, must be equal to the corresponding components of displacement of the nodes of the structure subjected to the given loading.* Moreover, the superposition of the reactions and the equivalent actions acting at the supports of the structure, subjected to the equivalent actions and to the given concentrated actions acting on its nodes (see Fig. 15.20c) and the corresponding restraining actions of the restrained structure (see Fig. 15.20b), yields the corresponding reactions of the actual structure subjected to the given loads. *Thus, it is apparent that the nodal displacements and the reactions of the structure subjected to the equivalent actions and to the given concentrated actions acting on its nodes are equal to those of the structure subjected to the given loads.* Consequently, the stiffness equation (15.57) for the structure subjected to the given loads is identical to those of the structure subjected to the equivalent actions and the given concentrated actions acting on its nodes. However, the components of displacement (translations and rotations) and the internal actions of elements subjected to loads along their length are not equal to the corresponding quantities of the structure subjected to the equivalent actions and to the given concentrated actions acting on its nodes. The components of displacement and the internal actions of points along the length of such elements obtained on the basis of the finite element method represent an approximation to their actual components of displacement and their internal actions, respectively. The correct values of the components of displacement and of the internal actions of points along the length of an element subjected to loads along its length are obtained by superimposing those of the structure subjected to the equivalent actions and the given concentrated actions acting on its nodes and those of the restrained structure.

15.8.11 Analysis of Framed Structures

When we analyze a framed structure using the finite element method (direct stiffness method), we adhere to the following steps:

STEP 1 We subdivide the structure into a number of elements. Usually we choose each member of constant cross section as an element.

STEP 2 We compute the local matrix of fixed-end actions $\{A^R\}$ of each element of the structure. The terms of this matrix represent the local components of nodal actions of the element subjected to the given loads with its ends fixed. Moreover, using relation (15.98b), we transform the local matrix $\{A^R\}$ to global.

STEP 3 We establish the matrix of equivalent actions $\{P^E\}$ to be applied on the nodes of the structure from the global components of the fixed-end actions of its elements. Moreover, we form the matrix $\{P^G\}$ of the given concentrated actions acting on the nodes of the structure from the data of the problem.

STEP 4 We establish the local stiffness matrix $[K]$ for each element of the structure using relations (15.31) or (15.79) or (15.80) and (15.81).

STEP 5 We compute the hybrid $[\overset{\sqcap}{K}]$ and the global $[\bar{K}]$ stiffness matrices for each element of the structure using relations (15.104) and (15.106), respectively.

STEP 6 We assemble the stiffness matrix for the structure from the global stiffness matrices of its elements (see Section 15.4). Moreover, we form the stiffness equation (15.57) for the structure.

STEP 7 We compute the components of displacements of the nodes of the structure and its reactions. To accomplish this we do the following:

1. We rearrange the rows and columns of the stiffness equations for the structure in order to incorporate its boundary conditions in them (see Section 15.7).

2. We partition the modified stiffness matrix of the structure as shown in relation (15.58).

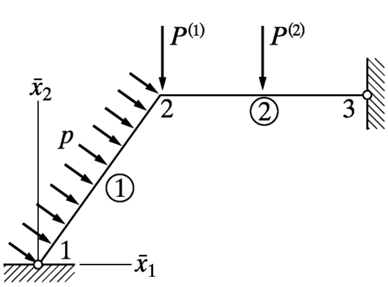
3. We use relations (15.59a) and (15.59b) to compute the components of displacement of the nodes and the reactions of the supports, respectively, of the structure subjected to the equivalent actions, to the given concentrated action acting on its nodes and to the given components of translation and rotation of its supports. They are equal to those of the structure subjected to the given loads including the given components of translation and rotation of its supports.

STEP 8 From the components of displacement of the nodes of the structure computed in step 7, we compute the global components of nodal displacements of each element of the structure. From these, we compute the local components of nodal actions of each element of the structure. That is, substituting relation (15.103) into (15.52) we obtain

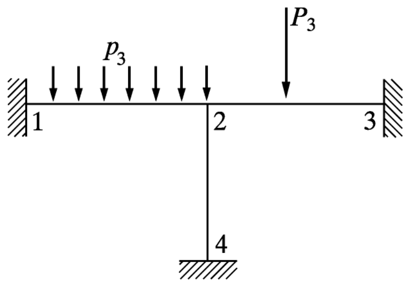
$$\{A\} = [\overset{\sqcap}{K}]\{\bar{D}\} + \{A^R\} \tag{15.108}$$

where $[\overset{\sqcap}{K}]$ is the hybrid stiffness matrix of the element defined by relation (15.104). It transforms the global components of nodal displacements of an element to the local components of its nodal actions when subjected only to its nodal displacements.

A computer program can be written for analyzing any framed structure statically determinate or indeterminate using the finite element method (direct stiffness method) presented in this section. However, when analyzing framed structures using only a desk calculator, this method should be employed only for statically indeterminate framed structures of a degree of static indeterminancy higher than the number of unknown components of displacements of their joints. For example, the frame of Fig. 15.21a is statically undetermined to the second degree but its joints have four unknown components



(a) Frame that preferably should be analyzed using the flexibility method presented in Chapter 14



(b) Frame that preferably should be analyzed using the direct stiffness method

Figure 15.21 Preferred method for analyzing statically indeterminate framed structures when only a desk calculator is available.

of displacement. Thus, its analysis will involve substantially less algebra if the flexibility method presented in Chapter 14 is employed instead of the direct stiffness method presented in this section. Moreover, the structure of Fig. 15.21b is statically undetermined to the sixth degree but has only three unknown components of displacement of its joints. Thus, its analysis would involve less algebra if the direct stiffness method is employed instead of the flexibility method.

Example 2 Using the finite element method (direct stiffness method), compute the components of displacements of the nodes, the reactions and the internal forces in the elements of the truss subjected to the forces shown in Fig. a. The elements of the truss are made of the same isotropic, linearly elastic material and have the same constant cross section (AE = 20,000 kN).

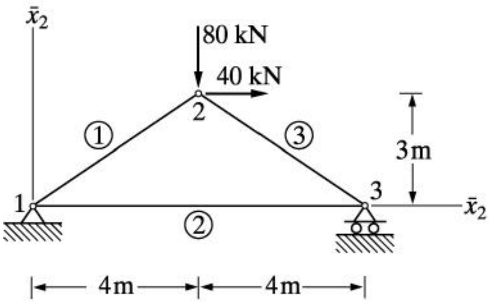


Figure a Geometry and loading of the truss.

Solution

STEP 1 We choose as elements the three members of the truss.

STEPS 2 and 3 The truss under consideration is subjected to loads acting only on its joints. Thus,

$$\{P^E\} = 0 \qquad \text{(a)}$$

Moreover, referring to Fig. a we have

$$\{P^G\}^T = [\ 0 \ \ 0 \ \ 40 \ \ -80 \ \ 0 \ \ 0\ ] \qquad \text{(b)}$$

STEP 4 We compute the local stiffness matrix for each element of the truss. That is, referring to relations (15.31), we have

$$[K^1] = \frac{EA}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [K^2] = \frac{EA}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [K^3] = \frac{EA}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{c}$$

STEP 5 We compute the hybrid and the global stiffness matrices for each element of the truss. Substituting relations (c) and (15.91) into (15.104), we get

$$[\check{K}^1] = [K^1][\Lambda_{PT}] = \frac{EA}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} = \frac{EA}{5}\begin{bmatrix} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{bmatrix}$$

$$[\check{K}^2] = [K^2][\Lambda_{PT}] = \frac{EA}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \frac{EA}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \tag{d}$$

$$[\check{K}^3] = [K^3][\Lambda_{PT}] = \frac{EA}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.8 & -0.6 & 0 & 0 \\ 0 & 0 & 0.8 & -0.6 \end{bmatrix} = \frac{EA}{5}\begin{bmatrix} 0.8 & -0.6 & -0.8 & -0.6 \\ -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix}$$

Substituting relations (d) and (15.91) into (15.105), we obtain

$$[\bar{K}^1] = \frac{1}{5}\begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 0 & 4 \\ 0 & 3 \end{bmatrix}\frac{EA}{5}\begin{bmatrix} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{bmatrix} = \frac{EA}{125}\begin{bmatrix} 16 & 12 & -16 & -12 \\ 12 & 9 & -12 & -9 \\ -16 & -12 & 16 & 12 \\ -12 & -9 & 12 & 9 \end{bmatrix}$$

$$[\bar{K}^2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{EA}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} = \frac{EA}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{e}$$

$$[\bar{K}^3] = \frac{1}{5}\begin{bmatrix} 4 & 0 \\ -3 & 0 \\ 0 & 4 \\ 0 & -3 \end{bmatrix}\frac{EA}{5}\begin{bmatrix} 0.8 & -0.6 & -0.8 & -0.6 \\ -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix} = \frac{EA}{125}\begin{bmatrix} 16 & -12 & -16 & 12 \\ -12 & 9 & 12 & -9 \\ -16 & 12 & 16 & -12 \\ 12 & -9 & -12 & 9 \end{bmatrix}$$

STEP 6 We assemble the stiffness matrix for the truss from the global stiffness matrices for its elements and we form the stiffness equations for the truss. In order to accomplish this we choose the indices of the stiffness coefficients for each element of the truss so as to correspond to those of the components of displacements of the nodes of the truss to which the element is connected. Referring to Fig. b the global stiffness equation (15.102) for the elements of the truss may be expressed as

$$[\bar{A}^1] = [\bar{K}^1][\bar{D}^1] = \begin{bmatrix} \bar{K}_{11}^1 & \bar{K}_{12}^1 & \bar{K}_{13}^1 & \bar{K}_{14}^1 \\ \bar{K}_{21}^1 & \bar{K}_{22}^1 & \bar{K}_{23}^1 & \bar{K}_{24}^1 \\ \bar{K}_{31}^1 & \bar{K}_{32}^1 & \bar{K}_{33}^1 & \bar{K}_{34}^1 \\ \bar{K}_{41}^1 & \bar{K}_{42}^1 & \bar{K}_{43}^1 & \bar{K}_{44}^1 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix}$$

$$[\bar{A}^2] = [\bar{K}^2][\bar{D}^2] = \begin{bmatrix} \bar{K}_{11}^2 & \bar{K}_{12}^2 & \bar{K}_{15}^2 & \bar{K}_{16}^2 \\ \bar{K}_{21}^2 & \bar{K}_{22}^2 & \bar{K}_{25}^2 & \bar{K}_{26}^2 \\ \bar{K}_{51}^2 & \bar{K}_{52}^2 & \bar{K}_{55}^2 & \bar{K}_{56}^2 \\ \bar{K}_{61}^2 & \bar{K}_{62}^2 & \bar{K}_{65}^2 & \bar{K}_{66}^2 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix} \tag{f}$$

$$[\bar{A}^3] = [\bar{K}^3][\bar{D}^3] = \begin{bmatrix} \bar{K}_{33}^3 & \bar{K}_{34}^3 & \bar{K}_{35}^3 & \bar{K}_{36}^3 \\ \bar{K}_{43}^3 & \bar{K}_{44}^3 & \bar{K}_{45}^3 & \bar{K}_{46}^3 \\ \bar{K}_{53}^3 & \bar{K}_{54}^3 & \bar{K}_{55}^3 & \bar{K}_{56}^3 \\ \bar{K}_{63}^3 & \bar{K}_{64}^3 & \bar{K}_{65}^3 & \bar{K}_{66}^3 \end{bmatrix} \begin{Bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}$$

The stiffness matrix for the truss has the following form:

$$[\hat{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \tag{g}$$

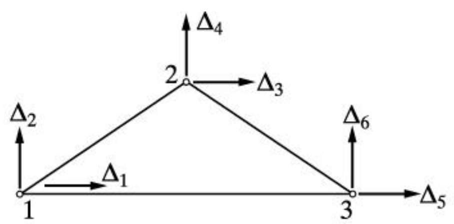


Figure b Numbering of the components of displacements of the nodes of the truss.

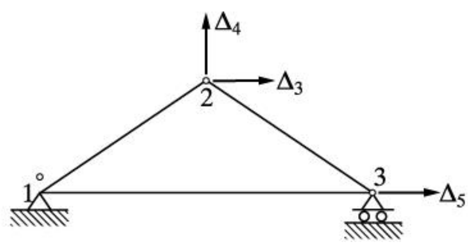


Figure c Degrees of freedom of the nodes of the truss.

where the stiffness coefficient S_{pq} for the truss is equal to the sum of the stiffness coefficients $\bar{K}_{pq}$ for all the elements of the truss. Thus, referring to relations (e) to (g) and using relation (15.27), we have

$$
\begin{aligned}
S_{11} &= \bar{K}^1_{11} + \bar{K}^2_{11} = \frac{EA}{125}\left(16 + \frac{125}{8}\right) = \frac{31.625EA}{125} \\
S_{12} &= \bar{K}^1_{12} + \bar{K}^2_{12} = \frac{EA}{125}(12 + 0) = \frac{12EA}{125} \\
S_{13} &= \bar{K}^1_{13} = -\frac{16EA}{125} \\
S_{14} &= \bar{K}^1_{14} = -\frac{12EA}{125} \\
S_{15} &= \bar{K}^2_{15} = \frac{EA}{8}(-1) = -\frac{15.625EA}{125} \\
S_{16} &= \bar{K}_{16} = 0 \\
&\;\;\vdots \\
S_{66} &= \bar{K}^2_{66} + \bar{K}^3_{66} = \frac{9EA}{125}
\end{aligned}
\tag{h}
$$

Substituting relation (h) into (g), we obtain the stiffness matrix $[\hat{S}]$ for the truss. That is,

$$
[\hat{S}] = \frac{EA}{125}\begin{bmatrix}
31.625 & 12 & -16 & -12 & -15.625 & 0 \\
12 & 9 & -12 & -9 & 0 & 0 \\
-16 & -12 & 32 & 0 & -16 & 12 \\
-12 & -9 & 0 & 18 & 12 & -9 \\
-15.625 & 0 & -16 & 12 & 31.625 & -12 \\
0 & 0 & 12 & -9 & -12 & 9
\end{bmatrix}
\tag{i}
$$

Notice that the determinant of the matrix $[\hat{S}]$ vanishes. This becomes apparent by noting that the sum of rows 2 and 4 of this determinant is the negative of row 6. This indicates that the stiffness matrix is singular. This was anticipated because in establishing the matrix $[\hat{S}]$ we did not take into account the supports of the structure and, thus, it can move as a rigid body.

Using relation (i) and referring to Figs. a and b the stiffness equations for the truss are

$$\begin{Bmatrix} 0 \\ 0 \\ 40 \\ -80 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ 0 \\ 0 \\ 0 \\ R_2^{(3)} \end{Bmatrix} = \frac{EA}{125}\begin{bmatrix} 31.625 & 12 & -16 & -12 & -15.625 & 0 \\ 12 & 9 & -12 & -9 & 0 & 0 \\ -16 & -12 & 32 & 0 & -16 & 12 \\ -12 & -9 & 0 & 18 & 12 & -9 \\ -15.625 & 0 & -16 & 12 & 31.625 & -12 \\ 0 & 0 & 12 & -9 & -12 & 9 \end{bmatrix}\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix} \tag{j}$$

STEP 7 We compute the components of displacements of the nodes of the truss and its reactions. In order to accomplish this we take into account the boundary conditions of the truss. That is, we rearrange the rows of the stiffness equation (j) in order to move to the bottom those corresponding to the reactions of the supports of the truss. Moreover, we rearrange the columns of the stiffness matrix $[\hat{S}]$ in order to move to the right the columns which are multiplied by the vanishing components of displacements of the supports of the truss (see Fig. c). Furthermore, we partition the resulting stiffness equations as indicated in relation (15.58). Thus,

$$\begin{Bmatrix} 40 \\ -80 \\ 0 \\ --- \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ --- \\ R_1^{(1)} \\ R_2^{(1)} \\ R_2^{(3)} \end{Bmatrix} = \frac{EA}{125}\left[\begin{array}{ccc|ccc} 32 & 0 & -16 & -16 & -12 & 12 \\ 0 & 18 & 12 & -12 & -9 & -9 \\ -16 & 12 & 31.625 & -15.625 & 0 & -12 \\ \hline -16 & -12 & -15.625 & 31.625 & 12 & 0 \\ -12 & -9 & 0 & 12 & 9 & 0 \\ 12 & -9 & -12 & 0 & 0 & 9 \end{array}\right]\begin{Bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ --- \\ \Delta_1 \\ \Delta_2 \\ \Delta_6 \end{Bmatrix} \tag{k}$$

Referring to relations (15.59) and (k), we have

$$\{P^{EF}\} = \begin{Bmatrix} 40 \\ -80 \\ 0 \end{Bmatrix} \qquad \{P^{ES}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \{R\} = \begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ R_2^{(3)} \end{Bmatrix} \qquad \{\Delta^S\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{S^{FF}\} = \frac{EA}{125}\begin{bmatrix} 32 & 0 & -16 \\ 0 & 18 & 12 \\ -16 & 12 & 31.625 \end{bmatrix} \qquad \{S^{SF}\} = \frac{EA}{125}\begin{bmatrix} -16 & -12 & -15.625 \\ -12 & -9 & 0 \\ 12 & -9 & -12 \end{bmatrix} \tag{l}$$

We compute the components of displacements of the nodes of the truss. Substituting relation (l) into (15.59a), we obtain

$$\{\Delta^F\} = \begin{Bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \end{Bmatrix} = \frac{125}{EA}\begin{bmatrix} 32 & 0 & -16 \\ 0 & 18 & 12 \\ -16 & 12 & 31.625 \end{bmatrix}^{-1}\begin{Bmatrix} 40 \\ -80 \\ 0 \end{Bmatrix} = \frac{1}{EA}\begin{Bmatrix} 449.58 \\ -946.67 \\ 586.67 \end{Bmatrix} = \begin{Bmatrix} -22.470 \\ -47.333 \\ 29.333 \end{Bmatrix} \text{ mm} \tag{m}$$

We compute the reactions of the truss. Using relation (l) and (m) from relations (15.59b), we get

$$\{R\} = \begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ R_2^{(3)} \end{Bmatrix} = \frac{1}{125}\begin{bmatrix} -16 & -12 & -15.625 \\ -12 & -9 & 0 \\ 12 & -9 & -12 \end{bmatrix}\begin{Bmatrix} 449.58 \\ -946.67 \\ 586.67 \end{Bmatrix} = \begin{Bmatrix} -40.0 \\ 25.0 \\ 55.0 \end{Bmatrix} \text{ kN} \tag{n}$$

The reactions of the truss are shown in Fig. d. Their values can be checked by considering the equilibrium of the truss.

STEP 8 We compute the local components of internal forces in the elements of the truss. Referring to Fig. b and relation (m), the global components of nodal displacements of the elements of the truss are

$$\{\bar{D}^1\} = \begin{Bmatrix} 0 \\ 0 \\ \Delta_3 \\ \Delta_4 \end{Bmatrix} = \frac{1}{EA}\begin{Bmatrix} 0 \\ 0 \\ 449.58 \\ -946.67 \end{Bmatrix} \qquad \{\bar{D}^2\} = \begin{Bmatrix} 0 \\ 0 \\ \Delta_5 \\ 0 \end{Bmatrix} = \frac{1}{EA}\begin{Bmatrix} 0 \\ 0 \\ 586.67 \\ 0 \end{Bmatrix}$$

$$\{\bar{D}^3\} = \begin{Bmatrix} \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ 0 \end{Bmatrix} = \frac{1}{EA}\begin{Bmatrix} 449.58 \\ -946.67 \\ 586.67 \\ 0 \end{Bmatrix} \tag{o}$$

We substitute relations (o) and (d) into (15.103), to get

$$\{A^1\} = \frac{EA}{5}\begin{bmatrix} 0.8 & 0.6 & -0.8 & -0.6 \\ -0.8 & -0.6 & 0.8 & 0.6 \end{bmatrix}\frac{1}{EA}\begin{Bmatrix} 0 \\ 0 \\ 449.58 \\ -946.67 \end{Bmatrix} = \begin{Bmatrix} 41.67 \\ -41.67 \end{Bmatrix}$$

$$\{A^2\} = \frac{EA}{8}\begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}\frac{1}{EA}\begin{Bmatrix} 0 \\ 0 \\ 586.67 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -73.33 \\ 73.33 \end{Bmatrix}$$

$$\{A^3\} = \frac{EA}{5}\begin{bmatrix} 0.8 & -0.6 & -0.8 & 0.6 \\ -0.8 & 0.6 & 0.8 & -0.6 \end{bmatrix}\frac{1}{EA}\begin{Bmatrix} 449.58 \\ -946.67 \\ 586.67 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 91.67 \\ -91.67 \end{Bmatrix}$$

The results are shown in Fig. d. They can be checked by considering the equilibrium of the joints of the truss. It is apparent that the finite element method is not suitable for analyzing trusses, using a desk calculator only, because in this case the analysis of a simple statically determinate truss involves lengthy computations. The reason for this is that the truss of Fig. a has six unknown components of displacement while it is statically determinate.

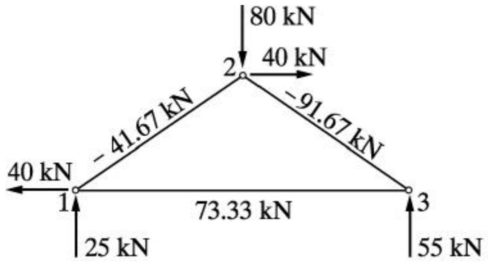


Figure d Results for the truss of Fig. a.

Example 3 Using the finite element method (direct stiffness method), compute the components of displacement of the nodes of the frame of Fig. a subjected to the external actions shown in this figure, as well as to a change of temperature of its elements and to a settlement of 20 mm of support 1. Moreover, compute the reactions of the frame and the nodal actions of its elements. The temperature of the top and bottom fibers of the elements of the frame is $T_t = 25°C$ and $T_b = -15°C$, respectively. The temperature during construction was $T_0 = 5°C$; thus $\Delta T_C = 0°C$. The elements of the frame are made of the same isotropic, linearly elastic material ($E = 210$ kN/mm^2, $\alpha = 10^{-5}$/°C) and have the same constant cross section ($A = 16 \times 10^3$ mm^2, $I_3 = 400 \times 10^6$ mm^4, $h = 420$ mm).

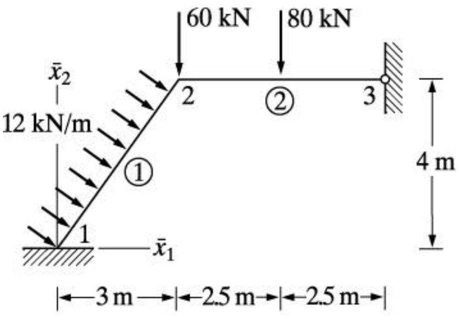


Figure a Geometry and loading of the frame.

Solution We establish the internal actions of the elements of the frame of Fig. a by superimposing (a) the corresponding nodal actions of the elements of the restrained structure subjected to the given actions except the given concentrated external actions acting on its nodes (b) the structure subjected to the equivalent actions and the given concentrated actions acting on its nodes and the settlement of its support 1 (see Fig. b).

STEP 1 We choose as elements the two members of the frame. When we use the finite element method to analyze a framed structure, we actually analyze the structure subjected to the equivalent actions and the given concentrated actions acting on its nodes and the settlement of its supports (see Fig. bc). The components of nodal displacement and the reactions of this structure are equal to the corresponding quantities of the structure subjected to the given loads. However, since the elements that we have chosen are subjected to loads along their length, the components of displacement and the internal actions of the elements of the structure of Fig. bc are an approximation to those of the

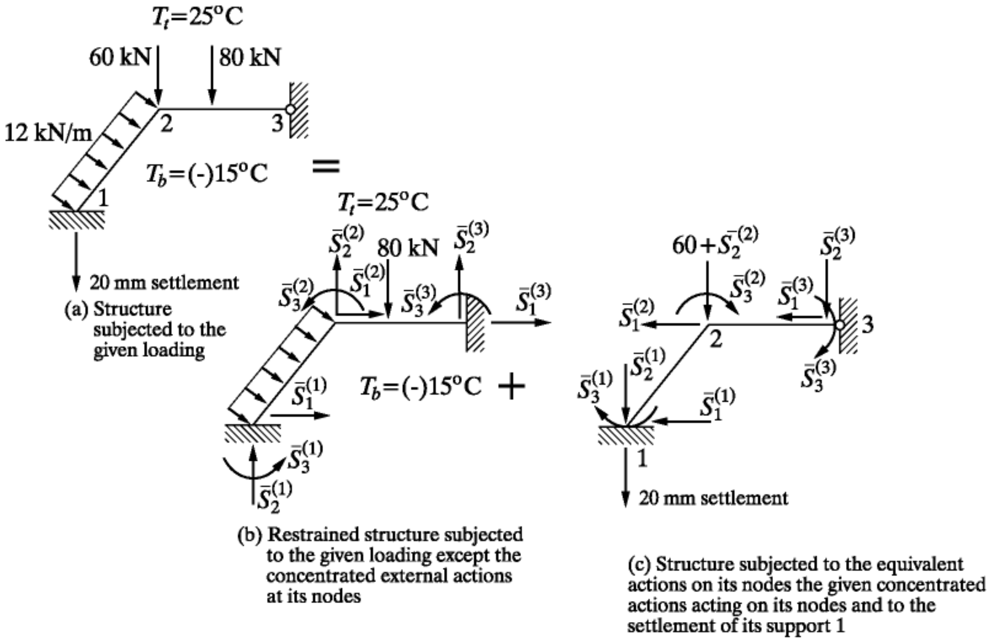


Figure b Superposition of the restrained structure and the structure subjected to the equivalent actions and the settlement of support 1.

structure subjected to the given loads. The results could improve by choosing smaller elements. However, in the analysis of framed structures, we can obtain exact results for the internal actions and the components of displacement of their elements by superimposing the corresponding qualities of the corresponding elements of the restrained structure to those of the structure subjected to the equivalent actions (see Fig. b).

STEP 2 We establish the fixed-end actions of the elements of the structure subjected to the given actions except the given concentrated external actions acting on its nodes, by referring to Table 15.3. They are shown in Fig. c. Moreover, we establish the fixed-end

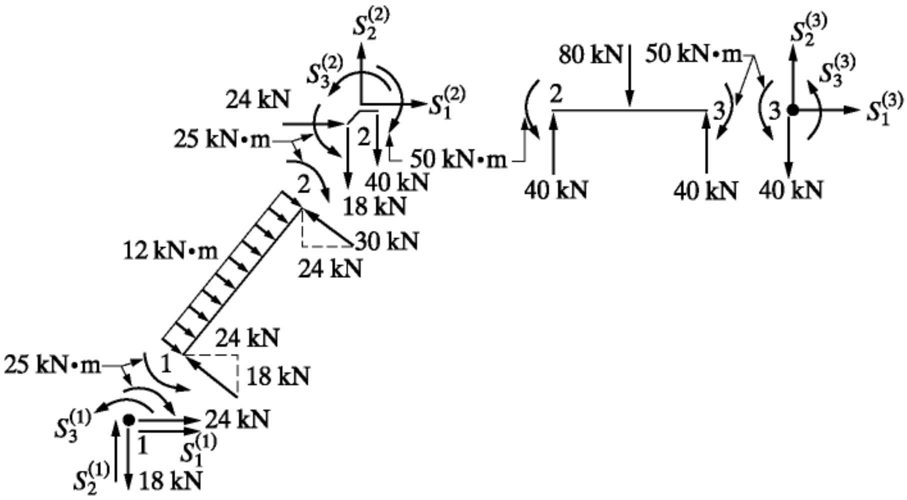


Figure c Free-body diagrams of the elements and nodes of the restrained structure subjected to the given actions except the given concentrated actions acting on the nodes of the structure.

actions of the elements of structure subjected to the given temperature change by referring to Table 15.3. Referring to Fig. c the local matrices of fixed-end actions of the elements of the structure are

$$\{A^{R1}\} = \left\{\begin{array}{c} 0 \\ 30 \\ 25 \\ \hdashline 0 \\ 30 \\ -25 \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 0 \\ -80 \\ \hdashline 0 \\ 0 \\ 80 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 30 \\ -55 \\ \hdashline 0 \\ 30 \\ 55 \end{array}\right\} \qquad \{A^{R2}\} = \left\{\begin{array}{c} 0 \\ 40 \\ 50 \\ \hdashline 0 \\ 40 \\ -50 \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 0 \\ -80 \\ \hdashline 0 \\ 0 \\ 80 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 40 \\ -30 \\ \hdashline 0 \\ 40 \\ 30 \end{array}\right\} \tag{a}$$

We transform the local matrix of fixed-end actions of each element of the structure to global. The local axes of the elements of the frame are shown in Fig. d; referring to this figure we form the transformation matrix for each element of the frame [see relations (15.97) and (1.37)], and we substitute it in relation (15.98b) to obtain its global matrix of fixed-end actions. That is,

$$\{\bar{A}^{R1}\} = \left\{\begin{array}{c} \{\bar{A}^{R1j}\} \\ \hdashline \{\bar{A}^{R1k}\} \end{array}\right\} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & -0.8 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left\{\begin{array}{c} 0 \\ 30 \\ -55 \\ \hdashline 0 \\ 30 \\ 55 \end{array}\right\} = \left\{\begin{array}{c} -24 \\ 18 \\ -55 \\ \hdashline -24 \\ 18 \\ 55 \end{array}\right\} \tag{b}$$

$$\{\bar{A}^{R2}\} = \{A^{R2}\}$$

STEP 3 We form the matrix of equivalent actions from the global matrices of fixed-end actions of the elements of the frame. That is, referring to relations (b), we have

$$\{P^{E}\}^{T} = \left[\ 24\ \ -18\ \ \ 55\ \middle|\ 24\ \ -58\ \ -25\ \middle|\ 0\ \ -40\ \ -30\ \right] \tag{c}$$

Morever, we form the matrix of the given actions $\{P^{G}\}$ acting on the nodes of the structure. Referring to Fig. a we have

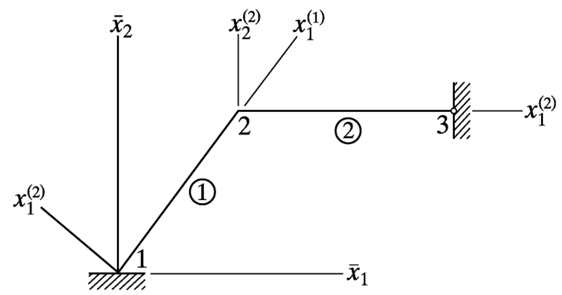


Figure d Global axes of the frame and local axes of its elements.

$$\{P^G\}^T = [\ 0 \quad 0 \quad 0 \quad 0 \ -60 \quad 0 \quad 0 \quad 0 \quad 0\ ] \tag{d}$$

Thus,

$$[\{P^E\} + \{P^G\}]^T = [\ 24 \ -18 \quad 55 \mid 24 \ -118 \ -25 \mid 0 \ -40 \ 30\ ] \tag{e}$$

The structure subjected to the equivalent actions is shown in Fig. e.

STEP 4 We compute the local stiffness matrix for each element of the structure including the effect of axial deformation. That is, referring to relation (15.80), we have

$$[K^1] = [K^2] = \frac{EI}{125}\begin{bmatrix} 1000 & 0 & 0 & -1000 & 0 & 0 \\ 0 & 12 & 30 & 0 & -12 & 30 \\ 0 & 30 & 100 & 0 & -30 & 50 \\ -1000 & 0 & 0 & 1000 & 0 & 0 \\ 0 & -12 & -30 & 0 & 12 & -30 \\ 0 & 30 & 50 & 0 & -30 & 100 \end{bmatrix} \tag{f}$$

STEP 5 We compute the hybrid and the global stiffness matrices of each element of the structure. The transformation matrix $[\Lambda^1]$ of element 1 is given by relation (15.97). That is,

$$[\Lambda^1] = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{g}$$

Substituting relation (f) and (g) into (15.104), we get

$$[\overset{\sqcap}{K}{}^1] = \frac{EI}{125}\begin{bmatrix} 1000 & 0 & 0 & -1000 & 0 & 0 \\ 0 & 12 & 30 & 0 & -12 & 30 \\ 0 & 30 & 100 & 0 & -30 & 50 \\ -1000 & 0 & 0 & 1000 & 0 & 0 \\ 0 & -12 & -30 & 0 & 12 & -30 \\ 0 & 30 & 50 & 0 & -30 & 100 \end{bmatrix}\begin{bmatrix} 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
= \frac{EI}{125}\begin{bmatrix} 600.0 & 800.0 & 0 & -600.0 & -800.0 & 0 \\ -9.6 & 7.2 & 30 & 9.6 & -7.2 & 30 \\ -24.0 & 18.0 & 100 & 24.0 & -18.0 & 50 \\ -600.0 & -800.0 & 0 & 600.0 & 800.0 & 0 \\ 9.6 & -7.2 & -30 & -9.6 & 7.2 & -30 \\ -24.0 & 18.0 & 50 & 24.0 & -18.0 & 100 \end{bmatrix} \tag{h}
$$

$$
[\bar{K}^2] = [K^2] = \frac{EI}{125}\begin{bmatrix} 1000 & 0 & 0 & -1000 & 0 & 0 \\ 0 & 12 & 30 & 0 & -12 & 30 \\ 0 & 30 & 100 & 0 & -30 & 50 \\ -1000 & 0 & 0 & 1000 & 0 & 0 \\ 0 & -12 & -30 & 0 & 12 & -30 \\ 0 & 30 & 50 & 0 & -30 & 100 \end{bmatrix}
$$

Substituting relation (g) and (h) into (15.106), we get

$$
[\bar{K}^1] = \frac{EI}{125}\begin{bmatrix} 367.68 & 474.24 & -24 & -367.68 & -474.24 & -24 \\ 474.24 & 644.32 & 18 & -474.24 & -644.32 & 18 \\ -24.00 & 18.00 & 100 & 24.00 & -18.00 & 50 \\ -367.68 & -474.24 & 24 & 367.68 & 474.24 & 24 \\ -474.24 & -644.32 & -18 & 474.24 & 644.32 & -18 \\ -24.00 & 18.00 & 50 & 24.00 & -18.00 & 100 \end{bmatrix} \tag{i}
$$

and

$$
[\bar{K}^2] = [K^2] \tag{j}
$$

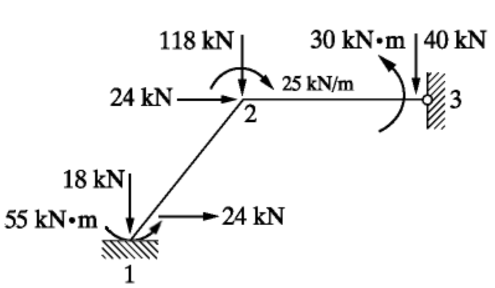


Figure e Structure subjected to the equivalent actions.

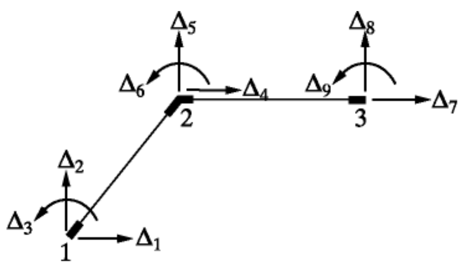


Figure f Numbering of the components of displacements of the nodes of the frame.

STEP 6 We assemble the stiffness matrix for the structure from the global stiffness matrices for its elements. To accomplish this we choose the indices of the stiffness coefficients for each element of the structure so as to correspond to those of the components of displacement of the nodes of the structure to which the element is connected. Referring to Fig. c the global stiffness relations for the elements of the structure may be written as

$$\{\bar{A}^1\} = [\bar{K}^1]\{\bar{D}^1\} = \begin{bmatrix} \bar{K}^1_{11} & \bar{K}^1_{12} & \bar{K}^1_{13} & \bar{K}^1_{14} & \bar{K}^1_{15} & \bar{K}^1_{16} \\ \bar{K}^1_{21} & \bar{K}^1_{22} & \bar{K}^1_{23} & \bar{K}^1_{24} & \bar{K}^1_{25} & \bar{K}^1_{26} \\ \bar{K}^1_{31} & \bar{K}^1_{32} & \bar{K}^1_{33} & \bar{K}^1_{34} & \bar{K}^1_{35} & \bar{K}^1_{36} \\ \bar{K}^1_{41} & \bar{K}^1_{42} & \bar{K}^1_{43} & \bar{K}^1_{44} & \bar{K}^1_{45} & \bar{K}^1_{46} \\ \bar{K}^1_{51} & \bar{K}^1_{52} & \bar{K}^1_{53} & \bar{K}^1_{54} & \bar{K}^1_{55} & \bar{K}^1_{56} \\ \bar{K}^1_{61} & \bar{K}^1_{62} & \bar{K}^1_{63} & \bar{K}^1_{64} & \bar{K}^1_{65} & \bar{K}^1_{66} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix}$$

$$\{\bar{A}^2\} = [\bar{K}^2]\{\bar{D}^2\} = \begin{bmatrix} \bar{K}^2_{44} & \bar{K}^2_{45} & \bar{K}^2_{46} & \bar{K}^2_{47} & \bar{K}^2_{48} & \bar{K}^2_{49} \\ \bar{K}^2_{54} & \bar{K}^2_{55} & \bar{K}^2_{56} & \bar{K}^2_{57} & \bar{K}^2_{58} & \bar{K}^2_{59} \\ \bar{K}^2_{64} & \bar{K}^2_{65} & \bar{K}^2_{66} & \bar{K}^2_{67} & \bar{K}^2_{68} & \bar{K}^2_{69} \\ \bar{K}^2_{74} & \bar{K}^2_{75} & \bar{K}^2_{76} & \bar{K}^2_{77} & \bar{K}^2_{78} & \bar{K}^2_{79} \\ \bar{K}^2_{84} & \bar{K}^2_{85} & \bar{K}^2_{86} & \bar{K}^2_{87} & \bar{K}^2_{88} & \bar{K}^2_{89} \\ \bar{K}^2_{94} & \bar{K}^2_{95} & \bar{K}^2_{96} & \bar{K}^2_{97} & \bar{K}^2_{98} & \bar{K}^2_{99} \end{bmatrix} \begin{Bmatrix} \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_7 \\ \Delta_8 \\ \Delta_9 \end{Bmatrix} \tag{k}$$

The stiffness matrix for the structure of Fig. a has the following form:

$$[\hat{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} \\ S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} \\ S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} \\ S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99} \end{bmatrix} \tag{l}$$

where the stiffness coefficient S_{ij} for the structure is obtained from the global stiffness coefficients $\bar{K}_{ij}$ for its elements. Thus, referring to relations (i) to (l) and using relation (15.27) the stiffness matrix for the structure of Fig. a is

$$
[\hat{S}] = \frac{EI}{125}\begin{bmatrix}
367.68 & 474.24 & -24 & -367.68 & -474.24 & -24 & 0 & 0 & 0 \\
474.24 & 644.32 & 18 & -474.24 & -644.32 & 18 & 0 & 0 & 0 \\
-24.00 & 18.00 & 100 & 24.00 & -18.00 & 50 & 0 & 0 & 0 \\
-367.68 & -474.24 & 24 & 1367.68 & 474.24 & 24 & -1000 & 0 & 0 \\
-474.24 & -644.32 & -18 & 474.24 & 656.32 & 12 & 0 & -12 & 30 \\
-24.00 & 18.00 & 50 & 24.00 & 12.00 & 200 & 0 & -30 & 50 \\
0 & 0 & 0 & -1000.00 & 0 & 0 & 1000 & 0 & 0 \\
0 & 0 & 0 & 0 & -12.00 & -30 & 0 & 12 & -30 \\
0 & 0 & 0 & 0 & 30.00 & 50 & 0 & -30 & 100
\end{bmatrix} \tag{m}
$$

Referring to relations (e) and (m) and to Figs. a and f the stiffness equations for the frame of Fig. a are

$$
\left\{\begin{array}{c} 24 \\ -18 \\ 55 \\ \hline 24 \\ -118 \\ -25 \\ \hline 0 \\ -40 \\ -30 \end{array}\right\} + \left\{\begin{array}{c} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \\ \hline 0 \\ 0 \\ 0 \\ \hline R_1^{(3)} \\ R_2^{(3)} \\ 0 \end{array}\right\} = \frac{EI}{125}\begin{bmatrix}
367.68 & 474.24 & -24 & -367.68 & -474.24 & -24 & 0 & 0 & 0 \\
474.24 & 644.32 & 18 & -474.24 & -644.32 & 18 & 0 & 0 & 0 \\
-24.00 & 18.00 & 100 & 24.00 & -18.00 & 50 & 0 & 0 & 0 \\
-367.68 & -474.24 & 24 & 1367.68 & 474.24 & 24 & -1000 & 0 & 0 \\
-474.24 & -644.32 & -18 & 474.24 & 656.32 & 12 & 0 & -12 & 30 \\
-24.00 & 18.00 & 50 & 24.00 & 12.00 & 200 & 0 & -30 & 50 \\
0 & 0 & 0 & -1000.00 & 0 & 0 & 1000 & 0 & 0 \\
0 & 0 & 0 & 0 & -12.00 & -30 & 0 & 12 & -30 \\
0 & 0 & 0 & 0 & 30.00 & 50 & 0 & -30 & 100
\end{bmatrix}\left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \hline \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \hline \Delta_7 \\ \Delta_8 \\ \Delta_9 \end{array}\right\} \tag{n}
$$

STEP 7 We compute the components of displacement of the nodes of the frame and its reactions. In order to accomplish this we take into account the boundary conditions of the frame. That is, we rearrange the rows of the stiffness equations (n) in order to move to the bottom the equations involving the reactions of the frame. Moreover, we rearrange the columns of the stiffness matrix in order to move to its right the columns which are multiplied by the vanishing components of displacements of the supports of the frame. Furthermore, we partition the resulting stiffness equations as indicated in relation (15.58). Thus, referring to Fig. f, we have

$$
\left\{\begin{array}{c} 24 \\ -118 \\ -25 \\ -30 \\ \hline 24 \\ -18 \\ 55 \\ 0 \\ -40 \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \\ R_1^{(3)} \\ R_2^{(3)} \end{array}\right\} = \frac{EI}{125}\left[\begin{array}{cccc|ccccc}
1367.68 & 474.24 & 24 & 0 & -367.68 & -474.24 & 24 & -1000 & 0 \\
474.24 & 656.32 & 12 & 30 & -474.24 & -644.32 & -18 & 0 & -12 \\
24.00 & 12.00 & 200 & 50 & -24.00 & 18.00 & 50 & 0 & -30 \\
0 & 30.00 & 50 & 100 & 0 & 0 & 0 & 0 & -30 \\
\hline
-367.68 & -474.24 & -24 & 0 & 367.68 & 474.24 & -24 & 0 & 0 \\
-474.24 & -644.32 & 18 & 0 & 474.24 & 644.32 & 18 & 0 & 0 \\
24.00 & -18.00 & 50 & 0 & -24.00 & 18.00 & 100 & 0 & 0 \\
-1000.00 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 \\
0 & -12.00 & -30 & -30 & 0 & 0 & 0 & 0 & 12
\end{array}\right]\left\{\begin{array}{c} \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ \Delta_9 \\ \hline \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_7 \\ \Delta_8 \end{array}\right\} \tag{o}
$$

Referring to relation (15.58) and (o), we have

$$\{P^{EF}\} = \begin{Bmatrix} 24 \\ -118 \\ -25 \\ -30 \end{Bmatrix} \qquad \{P^{ES}\} = \begin{Bmatrix} 24 \\ -18 \\ 55 \\ 0 \\ -40 \end{Bmatrix} \qquad \{R\} = \begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \\ R_1^{(3)} \\ R_2^{(3)} \end{Bmatrix} \qquad \{\Delta^S\} = \begin{Bmatrix} 0 \\ -0.020 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[S^{FF}] = \frac{EI}{125}\begin{bmatrix} 1367.68 & 474.24 & 24 & 0 \\ 474.24 & 656.32 & 12 & 30 \\ 24.00 & 12.00 & 200 & 50 \\ 0 & 30.00 & 50 & 100 \end{bmatrix}$$

$$[S^{FS}] = \frac{EI}{125}\begin{bmatrix} -367.68 & -474.24 & 24 & -1000 & 0 \\ -474.24 & -644.32 & -18 & 0 & -12 \\ -24.00 & 18.00 & 50 & 0 & -30 \\ 0 & 0 & 0 & 0 & -30 \end{bmatrix} \tag{p}$$

$$[S^{SF}] = \frac{EI}{125}\begin{bmatrix} -367.68 & -474.24 & -24 & 0 \\ -474.24 & -644.32 & 18 & 0 \\ 24.00 & -18.00 & 50 & 0 \\ -1000.00 & 0 & 0 & 0 \\ 0 & -12.00 & -30 & -30 \end{bmatrix}$$

$$[S^{SS}] = \frac{EI}{125}\begin{bmatrix} 367.68 & 474.24 & -24 & 0 & 0 \\ 474.24 & 644.32 & 18 & 0 & 0 \\ -24.00 & 18.00 & 100 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

Substituting from relations (p) into relation (15.59a), we obtain

$$\{\Delta^F\} = [S^{FF}]^{-1}[\{P^{EF}\} - [S^{FS}]\{\Delta^S\}] = \frac{125}{EI}\begin{bmatrix} 1367.68 & 474.24 & 24 & 0 \\ 474.24 & 656.32 & 12 & 30 \\ 24.00 & 12.00 & 200 & 50 \\ 0 & 30.00 & 50 & 100 \end{bmatrix}^{-1}\begin{Bmatrix} 24 \\ -118 \\ -25 \\ 30 \end{Bmatrix} \tag{q}$$

$$-\frac{EI}{125}\begin{bmatrix} -367.68 & -474.24 & 24 & -1000 & 0 \\ -474.24 & -644.32 & -18 & 0 & -12 \\ -24.00 & 18.00 & 50 & 0 & -30 \\ 0 & 0 & 0 & 0 & -30 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.020 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{q}$$

$$= \frac{125}{EI}\left[\begin{Bmatrix} 0.103735 \\ -0.244736 \\ -0.075565 \\ -0.188797 \end{Bmatrix} - \begin{Bmatrix} 0.058791 \\ 13.329256 \\ -1.161961 \\ -3.417797 \end{Bmatrix}\right] = \frac{125}{EI}\begin{Bmatrix} 0.04494 \\ -13.57399 \\ 1.08640 \\ 3.22900 \end{Bmatrix}$$

Moreover, substituting from relations (p) into relation (15.59b) we get

$$\begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \\ R_1^{(3)} \\ R_2^{(3)} \end{Bmatrix} + \begin{Bmatrix} 24 \\ -18 \\ 55 \\ 0 \\ 40 \end{Bmatrix} = [S^{SF}]\{\Delta^F\} + [S^{SS}]\{\varDelta^S\}$$

$$= \begin{bmatrix} -367.68 & -474.24 & -24 & 0 \\ -474.24 & -644.32 & 18 & 0 \\ 24.00 & -18.00 & 50 & 0 \\ -1000.00 & 0 & 0 & 0 \\ 0 & -12.00 & -30 & -30 \end{bmatrix}\begin{Bmatrix} 0.04494 \\ -13.57399 \\ 1.08640 \\ 3.22900 \end{Bmatrix} \tag{r}$$

$$+ \frac{EI}{125}\begin{bmatrix} 367.68 & 474.24 & -24 & 0 & 0 \\ 474.24 & 644.32 & 18 & 0 & 0 \\ -24.00 & 18.00 & 100 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.020 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 6394.732 \\ 8744.236 \\ 299.730 \\ -44.940 \\ 33.426 \end{Bmatrix} + \begin{Bmatrix} -6373.786 \\ -8659.661 \\ -241.920 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 20.946 \\ 84.575 \\ 57.810 \\ -44.940 \\ 33.426 \end{Bmatrix}$$

Thus,

$$\begin{Bmatrix} R_1^{(1)} \\ R_2^{(1)} \\ R_3^{(1)} \\ R_1^{(3)} \\ R_2^{(3)} \end{Bmatrix} = \begin{Bmatrix} -3.05 \\ 102.57 \\ 2.81 \\ -44.94 \\ 73.43 \end{Bmatrix}$$

STEP 8 We compute the internal actions in the elements of the frame. Referring to Fig. f and relation (q), the global components of nodal displacements of the elements of the frame are

$$\{\bar{D}^1\} = \begin{Bmatrix} 0 \\ \Delta_2 \\ 0 \\ --- \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{Bmatrix} = \frac{125}{EI}\begin{Bmatrix} 0 \\ -\frac{EI}{125}(0.02) \\ 0 \\ ------- \\ 0.04494 \\ -13.57399 \\ 1.08640 \end{Bmatrix} = \frac{125}{EI}\begin{Bmatrix} 0 \\ -13.44000 \\ 0 \\ ------- \\ 0.04494 \\ -13.57399 \\ 1.08640 \end{Bmatrix}$$

$$\{\bar{D}^2\} = \begin{Bmatrix} \Delta_4 \\ \Delta_5 \\ \Delta_6 \\ --- \\ 0 \\ 0 \\ \Delta_9 \end{Bmatrix} = \frac{125}{EI}\begin{Bmatrix} 0.04494 \\ -13.57399 \\ 1.08640 \\ ------- \\ 0 \\ 0 \\ 3.22900 \end{Bmatrix} \tag{s}$$

Substituting relations (t) and (s) into relation (15.108), we obtain

$$\{A^1\} = \begin{bmatrix} 600.0 & 800.0 & 0 & -600.0 & -800.0 & 0 \\ -9.6 & 7.2 & 30 & 9.6 & -7.2 & 30 \\ -24.0 & 18.0 & 100 & 24.0 & -18.0 & 50 \\ -600.0 & -800.0 & 0 & 600.0 & 800.0 & 0 \\ 9.6 & -7.2 & -30 & -9.6 & 7.2 & -30 \\ -24.0 & 18.0 & 50 & 24.0 & -18.0 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ -13.44000 \\ 0 \\ ------- \\ 0.04494 \\ -13.57399 \\ 1.08640 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 30 \\ -55 \\ --- \\ 0 \\ 30 \\ 55 \end{Bmatrix} = \begin{Bmatrix} 80.24 \\ 63.99 \\ 2.82 \\ ----- \\ -80.24 \\ -3.99 \\ 167.15 \end{Bmatrix}$$

$$\{A^2\} = \begin{bmatrix} 1000 & 0 & 0 & -1000 & 0 & 0 \\ 0 & 12 & 30 & 0 & -12 & 30 \\ 0 & 30 & 100 & 0 & -30 & 50 \\ -1000 & 0 & 0 & 1000 & 0 & 0 \\ 0 & -12 & -30 & 0 & 12 & -30 \\ 0 & 30 & 50 & 0 & -30 & 100 \end{bmatrix} \begin{Bmatrix} 0.04494 \\ -13.5740 \\ 1.0864 \\ \hdashline 0 \\ 0 \\ 3.2290 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 40 \\ -30 \\ \hdashline 0 \\ 40 \\ 30 \end{Bmatrix} = \begin{Bmatrix} 44.94 \\ 6.57 \\ -187.13 \\ \hdashline -44.94 \\ 73.43 \\ 0 \end{Bmatrix} \tag{t}$$

15.9 Approximate Solution of Scalar Two-Dimensional, Second Order, Linear Boundary Value Problems Using the Finite Element Method

In this section we apply the finite element method to scalar two-dimensional, second order, linear boundary value problems. The domain of these problems is a plane surface and its boundary is a planar curve.

15.9.1 The Strong or Classical Form of Scalar Two-Dimensional, Second Order, Linear Boundary Value Problems

Consider the boundary value problem for establishing a scalar function $X(x_2, x_3)$ (state variable) which has the following attributes:

1. It satisfies in a two-dimensional domain Ω ended by a closed planar curve Γ the following differential equation:

$$\frac{\partial}{\partial x_2}\left(a\frac{\partial X}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(b\frac{\partial X}{\partial x_3}\right) + cX + f = 0 \tag{15.109}$$

where a, b, c and f are known functions of x_2 and x_3.
2. It satisfies the following boundary conditions:

$$X = X_1 \quad \text{on the portion } \Gamma_1 \text{ of } \Gamma \tag{15.110a}$$

$$a\lambda_{n2}\frac{\partial X}{\partial x_2} + b\lambda_{n3}\frac{\partial X}{\partial x_3} = g \quad \text{on the portion } \Gamma\text{–}\Gamma_1 \text{ of } \Gamma \tag{15.110b}$$

where $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ is the unit vector normal to the boundary Γ; X_1 is a function of x_2 and x_3 specified on the portion Γ_1 of Γ; g is a function of x_2 and x_3 specified on the portion $\Gamma - \Gamma_1$ of Γ. The boundary condition (15.110a) is *essential* while the boundary condition (15.110b) is *natural*. A point on the boundary Γ can be specified by the following parametric equations:

$$x_2 = x_2(x_s) \qquad x_3 = x_3(x_s) \tag{15.111}$$

where x_s is the length from an arbitrary reference point on the boundary Γ to the point (x_2, x_3), measured along the boundary Γ. The values of a function $f(x_2, x_3)$ on Γ can be expressed as follows:

$$f(x_s) \equiv f[x_2(x_s), x_3(x_s)] \tag{15.112}$$

If $a = b = 1$ and $c = 0$, equation (15.109) reduces to the following form known as the *Poisson equation:*

$$\frac{\partial^2 X}{\partial x_2^2} + \frac{\partial^2 X}{\partial x_3^2} + f = 0 \tag{15.113}$$

If $f = 0$, equation (15.113) reduces to the following form known as the *Laplace equation*

$$\frac{\partial^2 X}{\partial x_2^2} + \frac{\partial^2 X}{\partial x_3^2} = 0 \tag{15.114}$$

The boundary value problem for establishing the function $X(x_2, x_3)$ which satisfies the differential equation (15.114) at the points of a domain and the boundary condition (15.110a) is known as the *Dirichlet problem*. The boundary value problem for establishing the function $X(x_2, x_3)$ which satisfies the differential equation (15.114) and the boundary condition (15.110b) is known as the *Neumann problem*. The solution of the Dirichlet problem is unique while the solution of the Neumann problem is determined only to within an additive constant.

Special cases of the boundary value problems (15.109) and (15.110) arise in many fields.

15.9.2 Approximations to the Solution of Scalar Two-Dimensional, Second Order, Linear Boundary Value Problems by Trial Functions

Consider a scalar two-dimensional, second order, linear boundary value problem whose strong form involves the determination of the function $X(x_2, x_3)$ which satisfies the differential equation (15.109) and the boundary conditions (15.110). We construct approximate solutions for this boundary value problem of the following form:

$$\tilde{X}(x_2, x_3) = \phi_0(x_2, x_3) + \sum_{s=1}^{S} c_s \phi_s(x_2, x_3) \tag{15.115}$$

where

$c_s (s = 1, 2, ..., S)$	= undetermined coefficients also known as degrees of freedom.
$\phi_0(x_2, x_3)$	= continuous function of x_2 and x_3 chosen to satisfy the boundary conditions in as big a portion of the boundary as possible.
$\phi_s(x_2, x_3)$ $(s = 1, 2, ..., S)$	= linearly independent functions known as *trial* or *basis functions.* They are chosen to satisfy the homogeneous part of the boundary conditions on the portion of the boundary where they are satisfied by the function $\phi_0(x_2, x_3)$.

The selection of the function $\phi_0(x_2, x_3)$ and the trial function $\phi_s(x_2, x_3)$ $(s = 1, 2, ..., S)$ affects the accuracy of the approximate solution of a boundary value problem. In order to insure that as s increases the approximate solution (15.115) converges to the actual solution of the boundary value problem, the trial functions must be a sequence of functions from a *complete set of functions* starting from the lowest order up to the order S without missing an intermediary term (see Section 13.8). Moreover, the approximate solution (15.115) should not vanish at points where the actual solution does not vanish. Furthermore, the speed of convergence of the approximate solution improves if the function $\phi_0(x_2, x_3)$ and the trial functions $\phi_s(x_2, x_3)$ $(s = 1, 2, ..., S)$ satisfy the symmetry conditions, if any, of the problem.

15.9.3 The Weighted Residual Form of Scalar Two-Dimensional, Second Order, Linear Boundary Value Problems

Following a procedure similar to the one adhered to in sections 13.9 and 13.10 and assuming that the solution (15.115) satisfies at least the essential boundary conditions of the problem we obtain the following weighting residual equations for the scalar, two dimensional, second order, linear boundary value problem (15.109) with (15.110):

$$\iint_{\Omega} W_r \left[\frac{\partial}{\partial x_2}\left(a \frac{\partial \tilde{X}}{\partial x_2} \right) + \frac{\partial}{\partial x_3}\left(b \frac{\partial \tilde{X}}{\partial x_3} \right) + c\tilde{X} + f \right] d\Omega$$
$$+ \varepsilon \int_{\Gamma - \Gamma_1} W_{rb} \left(a\lambda_{n2} \frac{\partial \tilde{X}}{\partial x_2} + b\lambda_{n3} \frac{\partial \tilde{X}}{\partial x_3} - g \right) d\Gamma = 0 \qquad r = 1, 2, ..., S \tag{15.116a}$$

where

$$\epsilon = \begin{cases} 0 & \text{when the approximate solution (15.115) satisfies the boundary condition (15.110b).} \\ 1 & \text{when the approximate solution (15.115) does not satisfy the boundary condition (15.110b).} \end{cases} \tag{15.116b}$$

The line integral is taken in the counterclockwise direction; $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ is the unit vector normal to the element $d\Gamma$ of the boundary. The weighting functions $W_r(x_1)$ and $W_{rb}(x_1)$ are such that the integrals of equations (15.116a) do not become infinite.

It is apparent that the exact solution of the boundary value problem under consideration satisfies the weighted residual equation (15.116a). Moreover, it can be shown that the function $X(x_2, x_3)$ which satisfies the essential boundary conditions and equations (15.116a) for every set of functions $W_r(x_1)$ and $W_{rb}(x_1)$ is the solution of the boundary value problem under consideration. Thus the weighted residual form of the boundary value problem for computing the function $X(x_2, x_3)$ can be stated as follows: *"Find the function $X(x_2, x_3)$ which satisfies the essential boundary conditions and the weighted residual equation (15.116b) for every set of weighting functions $W_r(x_1)$ and $W_{rb}(x_1)$."*

In the weighted residual methods the functions ϕ_0 (x_2, x_3) and the shape functions ϕ_s (x_2, x_3) $(s = 1, 2, ..., S)$ are usually chosen to satisfy all the boundary conditions of the problem. The parameters c_s $(s = 1, 2, ..., S)$ of the approximate solution (15.115) of the boundary value problem (15.109) with (15.110) are established by substituting relation (15.115) into its weighted residual equations (15.116a) and solving the resulting linear algebraic equations. There are several weighted residual methods available in the literature. They defer only in the choice of the weighting function $W_r(x_2 x_3)$. The most extensively used of these methods is the Gallerkin method in which the weighting function is taken as

$$W_r(x_2, x_3) = \phi_r(x_2, x_3) \tag{15.117}$$

15.9.4 The Modified Weighted Residual (Weak) Form of Scalar, Two-Dimensional, Second Order, Linear Boundary Value Problems

We can obtain the modified weighted residual (weak) form of the boundary value problems (15.109) to (15.110) by adhering to the following steps:

STEP 1 We write its weighted residual equation (15.116a) as

$$\iint_\Omega\left[\left[\frac{\partial}{\partial x_2}\left(W_r\, a\,\frac{\partial\tilde{X}}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(W_r\, b\,\frac{\partial\tilde{X}}{\partial x_3}\right)\right]-a\frac{\partial W_r}{\partial x_2}\frac{\partial\tilde{X}}{\partial X_2}-b\frac{\partial W_r}{\partial x_3}\frac{\partial\tilde{X}}{\partial x_3}\right.$$

$$\left.+\,cW_r\tilde{X}+fW_r\right]dx_2dx_3+\epsilon\int_{\Gamma-\Gamma_1}W_{rb}\left(a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}-g\right)d\Gamma=0$$

$$r=1,2,\ldots,S \qquad (15.118)$$

where the parameter ϵ is defined by relation (15.116b) and the line integral is taken in the counterclockwise direction.

STEP 2 We apply Green's theorem of the plane [see relations (6.19)] to the first two terms of the integral over the domain $\Omega$of relation (15.118), to obtain

$$\iint_\Omega\left[-\left(a\frac{\partial W_r}{\partial x_2}\frac{\partial\tilde{X}}{\partial x_2}+b\frac{\partial W_r}{\partial x_3}\frac{\partial\tilde{X}}{\partial x_3}\right)+cW_r\tilde{X}+W_rf\right]dx_2dx_3$$

$$+\int_\Gamma W_r\left(a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}\right)d\Gamma \qquad (15.119)$$

$$+\,\epsilon\int_{\Gamma-\Gamma_1}W_{rb}(a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{dx_3}-g)d\Gamma=0 \qquad r=1,2,\ldots,S$$

where the parameter ϵ is defined by relation (15.116b) and the line integrals are taken in the counterclockwise direction.

STEP 3 We simplify relation (15.119) by limiting the choice of the weighting functions as follows

$$W_{rb}=-W_r \qquad \text{on } \Gamma-\Gamma_1 \qquad (15.120)$$

If the approximate solution (15.115) does not satisfy the natural boundary condition ϵ = 1. In this case using relation (15.120) the last two terms of relation (15.119) reduce to

$$\int_\Gamma W_r\left(a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}\right)d\Gamma+\epsilon\int_{\Gamma-\Gamma_1}W_{rb}\left(a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}-g\right)d\Gamma$$

$$=\int_{\Gamma_1}W_r\left(a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}\right)d\Gamma+\int_{\Gamma-\Gamma_1}W_rgd\Gamma \qquad r=1,2,\ldots,S \qquad (15.121)$$

If the approximate solution (15.115) satisfies the natural boundary condition $\epsilon=0$ and $a\lambda_{n2}\partial\tilde{X}/\partial x_2+b\lambda_{n3}\partial\tilde{X}/\partial x_3=g$ on the portion of the boundary $\Gamma-\Gamma_1$. Thus, relation (15.121) is still valid. Substituting relation (15.121) into(15.119), we obtain

$$\iint_{\Omega}\left[-\left(a\frac{\partial W_r}{\partial x_2}\frac{\partial \tilde{X}}{\partial x_2}+b\frac{\partial W_r}{\partial x_3}\frac{\partial \tilde{X}}{\partial x_3}\right)+cW_r\tilde{X}\right]dx_2dx_3+\iint_{\Omega}W_r f\,dx_2dx_3$$

$$+\int_{\Gamma-\Gamma_1}W_r g\,d\Gamma+\int_{\Gamma_1}W_r\left(a\lambda_{n2}\frac{\partial \tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial \tilde{X}}{\partial x_3}\right)d\Gamma=0 \qquad (15.122)$$

$$r = 1, 2, \ldots, S$$

Notice that, if the boundary conditions are all essential $\epsilon = 0$ and $\Gamma = \Gamma_1$ and relation (15.119) reduces to (15.122).

Relation (15.122) is the *modified weighted residual equations* for the scalar, two dimensional, second order, linear boundary value problem (15.109) with (15.110). In order to ensure that the integrals on the left side of relation (15.122) exist, we impose the requirement that $W_r(x_2, x_3)$ and $\tilde{X}(x_2, x_3)$ are continuous functions of the space coordinates in the domain Ω

From our presentation in this section it is clear that the solution $X(x_2, x_3)$ of the boundary value problem under consideration satisfies relation (15.122). Moreover, we can deduce that a function $X(x_2, x_3)$ which satisfies the essential boundary conditions of the problem and the integral equation (15.122) for every weighting function $W_r(x_2, x_3)$ is the solution of the boundary value problem (15.109) with (15.110). Hence, the modified weighted residual (*weak*) form of the boundary value problem under consideration involves the determination of the function $X(x_2, x_3)$ which satisfies the essential boundary conditions of the problem and the integral equation (15.122) for every continuous function $W_r(x_2, x_3)$.

15.9.5 The Finite Element Method for Scalar, Two-Dimensional Second Order, Linear Boundary Value Problems, as a Modified Weighted Residual Method

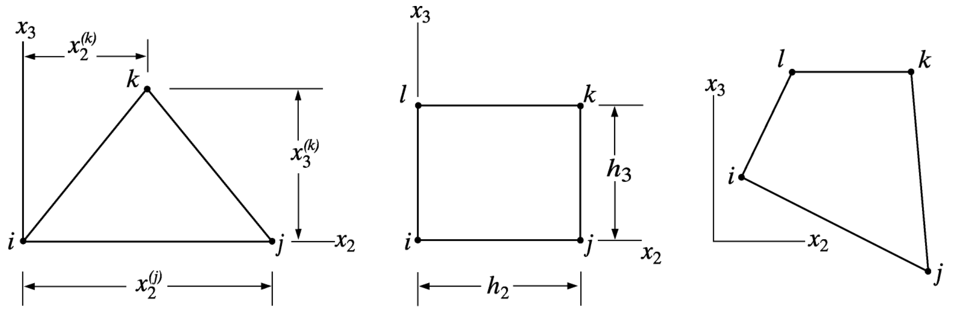
In the finite element method the domain Ω of the problem under consideration is divided into a number (say n) of non-overlapping two-dimensional subdomains Ω_e (e = 1, 2, ..., n) called *finite elements*. We choose elements which are well suited for modeling irregular domains and yet are simple enough to minimize computational effort. For two-dimensional domains such elements are the straight edge and curved edge triangular and the straight edge and curved edge quadrilateral. In this book we consider only straight edge elements (see Fig. 15.22).

If the boundary Γ or portion of the boundary of a domain is a curve, as in Fig. 15.23, there will be some *modeling error* since the boundary of the finite element mesh consisting of straight edge triangular and/or quadrilateral elements cannot be made to coincide with the curved portions of the boundary Γ of the domain. However, as the size of the elements located close to the curved portions of the boundary decreases, the accuracy, by which the boundary of the mesh approximates that of the domain, increases and the modeling error decreases. Certain key points of each element are chosen as nodes. They must include at least the vertices of each element and could include other points on its boundary or within its domain. As in the case of one-dimensional elements, the number of nodes in two-dimensional elements depends on the order of the shape functions used.

In the finite element method the state variable $X(x_2, x_3)$, of the boundary value problem under consideration, is approximated as follows:

$$\tilde{X}(\bar{x}_2, \bar{x}_3) = \sum_{s=1}^{S}\tilde{X}_s\,\bar{\phi}_s(\bar{x}_2, \bar{x}_3)=[\bar{\phi}]\{\tilde{X}\} \qquad (15.123a)$$

where



(a) Three-node, straight-sided triangular

(b) Four-node straight-sided rectangular

(c) Four node straight-sided quadrilateral

Figure 15.22 Finite elements for two-dimensional domains.

S = total number of nodes in the domain

$\bar{\phi}_s(x_2,x_3)$ $(s = 1, 2, ..., S)$ = global shape, trial, interpolation or basis functions

$$[\bar{\phi}] = [\bar{\phi}_1,\ \bar{\phi}_2, \ldots,\ \bar{\phi}_s] \tag{15.123b}$$

$$\{\tilde{X}_s\}^T = [\tilde{X}_1,\ \tilde{X}_2, \ldots,\ \tilde{X}_S] \tag{15.123c}$$

The shape functions $\bar{\phi}_n$ are chosen to have the following properties:

1. Are a continuous function throughout the domain of the problem.
2. Vanish on all nodes except node n.
3. Are equal to unity on node n.
4. Vanish on all elements except those containing node n.
5. Have continuous first derivatives inside the domain of each element where it does not vanish.

From the above properties we see that at node n all the shape functions vanish except $\bar{\phi}_n$ which is equal to unity. Thus, referring to relation (15.123a) the approximate value of $\tilde{X}(\bar{x}_1)$ at node n is equal to

$$\tilde{X}_1(\bar{x}_2\ ,\ \bar{x}_3)\ =\ \sum_{s=1}^{s} \tilde{X}_s \bar{\phi}_s\ =\ \tilde{X}_n \tag{15.124}$$

Consequently, the coefficient $\tilde{X}_n$ in relation (15.123a) has physical meaning. It represents an approximation to the value of the state variable at node n. This property of the shape functions permits the direct satisfaction of the essential boundary conditions of the problem. In order to accomplish this, each of the coefficients $\tilde{X}_s$ which is associated with a node that is located on the portion Γ_1 of the boundary of the domain where essential boundary conditions are specified, is set equal to the specified value of the state variable $\tilde{X}(\bar{x}_2, \bar{x}_3)$ at node n. The simplest set of shape functions which meet all the above requirements are linear as shown in Fig 15.23.

The coefficients $\tilde{X}_s$ (s = 1, 2, ..., S) of the approximate solution (15.123a) can be established by substituting it in the modified weighted residual (weak) equation (15.122), and using the Gallerkin assumption (15.117). This gives a set of S linear algebraic

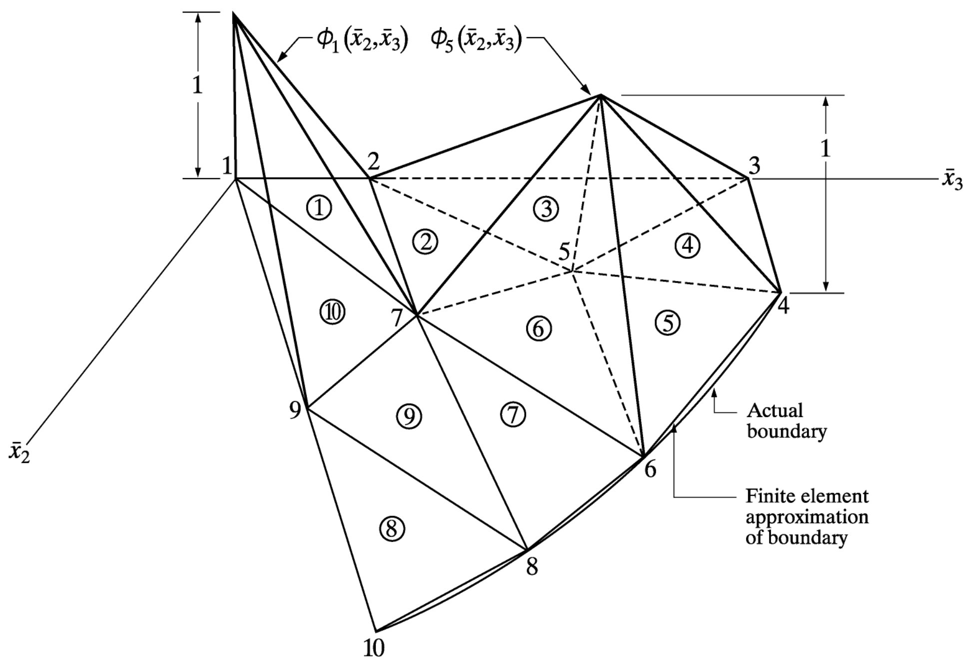


Figure 15.23 Shape functions for a scalar, two-dimensional, second order, linear boundary value problem.

equations which can be written as

$$[S]\,\{\tilde{X}\} = \{F\} \tag{15.125}$$

where

$\{\tilde{X}\}$= matrix of approximate values of the state variable at the nodes of the domain.
$[S]$ = the global stiffness matrix of the domain. Its terms are equal to

$$S_{rs} = \iint_{\Omega}\left[\left[a\frac{\partial \bar{\phi}_r}{\partial \bar{x}_2}\frac{\partial \bar{\phi}_s}{\partial \bar{x}_2} + b\frac{\partial \bar{\phi}_r}{\partial \bar{x}_3}\frac{\partial \bar{\phi}_s}{\partial \bar{x}_3}\right] + c\bar{\phi}_s\,\bar{\phi}_r\right]d\Omega \qquad r, s = 1, 2, \ldots, S \tag{15.126}$$

$\{F\}$ = the global load vector of the domain. Its terms are equal to

$$F_r = \iint_{\Omega}\bar{\phi}_r f\,d\Omega + \int_{\Gamma-\Gamma_1}\bar{\phi}_r g\,d\Gamma + \int_{\Gamma_1}\bar{\phi}_r\left[a\lambda_{n2}\frac{\partial \tilde{X}}{\partial \bar{x}_2} + b\lambda_{n3}\frac{\partial \tilde{X}}{\partial \bar{x}_3}\right]d\Gamma \qquad r = 1, 2, \ldots, S \tag{15.127}$$

Relations (15.125) are the stiffness equations for the boundary value problem described in Section 15.9.1.

Referring to relations (15.126) we see that for the scalar, two-dimensional, second order, linear boundary value problems under consideration the global shape functions $\bar{\phi}_s(\bar{x}_2, \bar{x}_3)(s = 1, 2, \ldots, S)$ must be at least of class C°; that is, they must be at least continuous over the domain of the problem.

In the finite element method the global stiffness matrix $[S]$ and the load vector $\{F\}$ of a domain are established directly from the contributions of its elements. Moreover, following the procedure described in Section 15.7 the essential boundary conditions are introduced directly in the stiffness equations (15.125) and the resulting equations are solved to obtain an approximation to the values of the state variable $X(x_2, x_3)$ at the nodes inside the domain of the problem and those of the boundary at which natural boundary conditions have been specified.

An important advantage of the global shape functions used in the finite element method is that the calculations for the contributions of the elements to the stiffness matrix and the load vector of the domain can be made repetitive. That is, for each type of element a formula can be derived for the stiffness matrix and for its contributions to the load vector of the domain and used to compute the stiffness matrices and their contribution to the load vector of all the elements of this type.

15.9.6 Local Shape Functions for Two-Dimensional Elements

For two dimensional scalar problems we use global shape functions $\bar{\phi}_q(\bar{x}_2, \bar{x}_3)$ whose non-vanishing part $\phi^e_q(x_2, x_3)$ over element e is a polynomial of x_2 and x_3 of the following form

$$\begin{aligned}\phi^e_q(x_2,x_3) &= c_{q0} + c_{q1}x_2 + c_{q2}x_2 + c_{q3}x_2x_3 + c_{q4}x_2^2 + c_{q5}x_3^2 \\ &\quad + c_{q6}x_2^3 + c_{q7}x_2^2x_3 + c_{q8}x_2x_3^2 + c_{q9}x_3^3\end{aligned} \tag{15.128}$$

where $c_{qi}(i = 0, 1, 2, ...)$ are constants which are evaluated by requiring that the shape function $\phi^e_q(x_2, x_3)$ satisfies the requirements that it is equal to unity at node q and to zero at all other nodes of the element. Thus, the number of monomials retained in the expression for shape function (15.128) for an element of the domain of a scalar, two-dimensional, second order, linear boundary value problem is equal to the number of its nodes. For example, consider a triangular element with nodes only at its three vertices, which as shown in Fib. 15.22a we denote by i, j and k (we always proceed counterclockwise from node i to nodes j and k). For this element we know the value of each shape function at its three nodes. Consequently, we can evaluate only three constants in relation (15.128). Hence, for a three-node, triangular element we use the following set of linear shape functions:

$$\phi^e_q(x_2, x_3) = c_{q0} + c_{q1}x_2 + c_{q2}x_3 \qquad q = i, j, \text{or } k \tag{15.129}$$

As can be seen from Fig. 15.23 each of the shape function (15.129) for a three node straight edge triangular element is part of a continuous global shape function.

Consider a rectangular element with nodes only at its four vertices which as shown in Fig. 15.22b we denote by i, j, k and l. Inasmuch as we know the values of the shape functions for this element only at its four-nodes we can evaluate four constants in relation (15.128). Consequently, for the four node straight edge rectangular element we use as shape functions the following set of bilinear functions:

$$\phi^e_q(x_2, x_3) = c_{q0} + c_{q1}x_2 + c_{q2}x_3 + c_{q3}x_2x_3 \qquad q = i, j, k \text{ or } l \tag{15.130}$$

Relation (15.130) represents a surface whose intersections with the planes x_2 = constant and x_3 = constant are straight lines. Thus, continuity of the global shape functions at the sides of a straight edge, four-node, rectangular element is ensured provided that the shape functions of two adjacent elements are equal at their common nodes.

The shape functions of triangular elements having more than three nodes and of rectangular elements having more than four nodes, must be complete polynomials of higher order than that of the linear (15.129) or the bilinear (15.130) polynomials, respectively.

In this text we consider only straight edge, three-node, triangular and straight edge, four node, rectangular elements.

15.9.7 Shape Functions for Straight Edge, Three-Node Triangular Elements

As shown in Fig. 15.22a we denote the nodes of a three-node triangular element by i, j and k. As discussed in the previous section, the shape functions for such an element have the form (15.129). The constants c^e_{qr} ($q = i, j$ or k) ($r = 0, 1, 2$) are established by requiring that the shape functions satisfy the following relations at the nodes of the element:

$$\phi^e_q(x_2^{(p)}, x_3^{(p)}) = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases} \qquad q = i, j, \text{ or } k \tag{15.131}$$

where $x_2^{(p)}, x_3^{(p)}$ represent the local coordinates of node p. Substituting relations (15.129) into (15.131) for the shape function $\phi^e_i(x_2, x_3)$, we have

$$\begin{bmatrix} 1 & x_2^{(i)} & x_3^{(i)} \\ 1 & x_2^{(j)} & x_3^{(j)} \\ 1 & x_2^{(k)} & x_3^{(k)} \end{bmatrix} \begin{Bmatrix} c_{i0}^e \\ c_{i1}^e \\ c_{i2}^e \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{15.132}$$

From relation (15.132), we obtain

$$c_{i0}^e = \frac{x_2^{(j)}x_3^{(k)} - x_2^{(k)}x_3^{(j)}}{2\Omega_e} \qquad c_{i1}^e = \frac{x_3^{(j)} - x_3^{(k)}}{2\Omega_e} \qquad c_{i2}^e = \frac{x_2^{(k)} - x_2^{(j)}}{2\Omega_e} \tag{15.133}$$

where Ω_e is the area of element e given as

$$\Omega_e = \frac{1}{2} det \begin{bmatrix} 1 & x_2^{(i)} & x_3^{(i)} \\ 1 & x_2^{(j)} & x_3^{(j)} \\ 1 & x_2^{(k)} & x_3^{(k)} \end{bmatrix} = \frac{1}{2}\left[x_2^{(j)}x_3^{(k)} - x_3^{(j)}x_2^{(k)} - x_2^{(i)}x_3^{(k)} + x_3^{(i)}x_2^{(k)} + x_2^{(i)}x_3^{(j)} - x_3^{(i)}x_2^{(j)}\right] \tag{15.134}$$

The constants c^e_{jr} ($r = 0, 1, 2$) are obtained from relations (15.133) by changing the subscripts $j \rightarrow k$ and $k \rightarrow i$. The constants c^e_{kr} ($r = 0, 1, 2$) are obtained from relation (15.133) by changing subscripts $j \rightarrow i$ and $k \rightarrow j$. Taking this into account and substituting the values of the constants (15.133) into relation (15.129), we get

$$\begin{aligned}
\phi^e_i &= \frac{1}{2\Omega_e}\left[x_2^{(j)}x_3^{(k)} - x_2^{(k)}x_3^{(j)} + (x_3^{(j)} - x_3^{(k)})x_2 + (x_2^{(k)} - x_2^{(j)})x_3\right] \\
\phi^e_j &= \frac{1}{2\Omega_e}\left[x_2^{(k)}x_3^{(i)} - x_2^{(i)}x_3^{(k)} + (x_3^{(k)} - x_3^{(i)})x_2 + (x_2^{(i)} - x_2^{(k)})x_3\right] \\
\phi^e_k &= \frac{1}{2\Omega_e}\left[x_2^{(i)}x_3^{(j)} - x_2^{(j)}x_3^{(i)} + (x_3^{(i)} - x_3^{(j)})x_2 + (x_2^{(j)} - x_2^{(i)})x_3\right]
\end{aligned} \tag{15.135}$$

15.9.8 Shape Functions for Straight Edge, Four-Node, Rectangular Elements

Following a procedure analogous to the one adhered to in Section 15.9.7 we obtain the following expression for the shape functions for straight edge, four-node, rectangular elements:

$$\phi_k^e = \frac{x_2 x_3}{h_2^e h_3^e} \qquad \phi_l^e = \frac{(h_2^e - x_2)x_3}{h_2^e h_3^e}$$

$$\phi_i^e = \frac{(h_2^e - x_2)(h_3^e - x_3)}{h_2^e h_3^e} \qquad \phi_j^e = \frac{x_2(h_3^e - x_3)}{h_2^e h_3^e} \tag{15.136}$$

15.9.9 Direct Computation of the Contribution of a Straight Edge Element to the Stiffness Matrix and the Load Vector of the Domain of Scalar Two Dimensional Second Order Linear Boundary Value Problems

Referring to relations (15.123) and to Fig 15.24 it can be seen the state function $X(\bar{x}_2, \bar{x}_3)$ is approximated over element e as

$$\tilde{X}^e(x_2, x_3) = [\phi^e]\{\tilde{X}^e\} \tag{15.137}$$

where for a straight edge three-node triangular element

$$[\phi^e] = [\phi_i^e, \phi_j^e, \phi_k^e] \tag{15.138a}$$

$$\{\tilde{X}^e\}^T = \left\{ \tilde{X}^{ei} \; \tilde{X}^{ej} \; \tilde{X}^{ek} \right\} \tag{15.138b}$$

while for a straight-edge four node rectangular element

$$[\phi^e] = [\phi_i^e \quad \phi_j^e \quad \phi_k^e \quad \phi_l^e] \tag{15.139a}$$

$$\{\tilde{X}^e\}^T = \left\{ \tilde{X}^{ei} \; \tilde{X}^{ej} \; \tilde{X}^{ek} \; \tilde{X}^{el} \right\} \tag{15.139b}$$

For a straight edge, three-node, triangular element relation (15.136) may be rewritten as

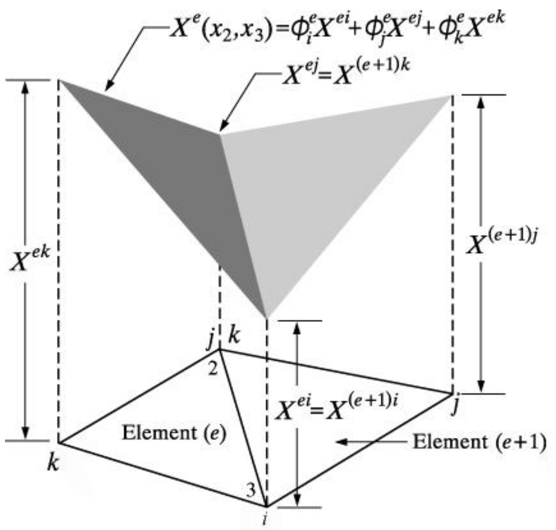


Figure 15.24 Approximation of the state variable in the domain of a three-node, triangular element.

$$\tilde{X}^e(x_2, x_3) = \phi_i^e \tilde{X}^{ei} + \phi_j^e \tilde{X}^{ej} + \phi_k^e \tilde{X}^{ek} \tag{15.140}$$

From relations (15.135) and (15.140) we see that the state variable $\tilde{X}^e(x_2, x_3)$ is approximated over a straight edge, triangular element by a planar surface (see shaded triangle in Fig. 15.24). Notice that $\tilde{X}^e(x_2, x_3)$ is a straight line along each side of the triangle. Thus, it is uniquely determined by its values at the two nodes at the end of this side. Consequently, if the values of state variable of two adjacent elements coincide at their common nodes, their values will coincide along their common line boundary (line 2.3 of Fig. 15.24). Moreover, from relations (15.136) we see that the shape functions for straight edge, four-node, rectangular elements are linear along each of their sides. This ensures that the state variable $\tilde{X}(\bar{x}_1, \bar{x}_2)$ will be continuous on the common side of two adjacent elements provided that the values of the approximations to the state variables over these two elements coincide at two points of their common side. Thus two nodes are required on each side of a straight edge, four-node, rectangular element.

Consider the straight edge, three-node, triangular element shown in Fig. 15.22a. Substituting relations (15.123a) into the modified weighted residual equation (15.122) and using the Gallerkin assumption (15.117) for the element under consideration, we obtain

$$\iint_{\Omega_e}\left[-\left(a\frac{\partial \phi_r^e}{\partial x_2}\frac{\partial[\phi^e]^T}{\partial x_2}\{\tilde{X}^e\} + b\frac{\partial \phi_r}{\partial x_3}\frac{\partial[\phi^e]^T}{\partial x_2}\{\tilde{X}^e\}\right) + c\,\phi_r[\phi^e]^T\{\tilde{X}^e\}\right]d\Omega$$

$$+\iint_{\Omega_e}\phi_r^e f\,d\Omega + \int_{(\Gamma-\Gamma_1)^e}\phi_r^e g\,d\Gamma + \int_{\Gamma_1^e}\phi_r^e\left[a\lambda_{n2}\frac{\partial \tilde{X}}{\partial x_2} + b\lambda_{n3}\frac{\partial \tilde{X}}{\partial x_3}\right]d\Gamma = 0 \tag{15.141}$$

$$r = 1, 2, \ldots, S$$

Where the first line integral in relation (15.141) is different than zero only for elements having the two nodes of one of their sides on the portion $(\Gamma - \Gamma_1)$ of the boundary of the domain. The integration is carried out along the length $(\Gamma - \Gamma_1)^e$ of this side of such an element. The second line integral of relation (15.141) is different than zero only for elements having the two nodes of one of their sides on the portion Γ_1 of the boundary of the domain of the problem. The integration is carried out along the length Γ_1^e of this side of such an element and $\mathbf{i}_n = \lambda_{n2}\mathbf{i}_2 + \lambda_{n3}\mathbf{i}_3$ is the unit vector normal to that side. The second and third integrals of relation (15.141) can be evaluated using the element shape functions and the known functions f and g. We can approximate the source function $f(x_2, x_3)$ over each element by its value at the centroid of the element f_c^e. Thus, we can move f_c^e outside the integral sign of the second integral of relation (15.141) and evaluate the remaining integral. The function $g(x_2^s, x_3^s) = g(x_s)$ represents the specified flux at the boundary of the domain. Often the side of an element may not coincide with the boundary of the domain and a modeling error is introduced. This error decreases as the size of the elements close to the boundary decreases. The function $g(x_s)$ is usually approximated by its value at the midpoint of the external boundary of an element or by its average value over the external boundary. Thus, we can move the constant value of g outside the integral sign of the third integral of relation (15.141).

On the basis of the foregoing discussion denoting by g_c^e the constant value of g over the side of element e, having two of its nodes on the boundary $(\Gamma - \Gamma_1)$, relation (15.141) can be approximated as

$$\iint_{\Omega_e}\left[-\left[a\frac{\partial[\phi^e]^T}{\partial x_2}\frac{\partial[\phi^e]}{\partial x_2}+b\frac{\partial[\phi^e]^T}{\partial x_3}\frac{\partial[\phi^e]}{\partial x_3}\right]+c[\phi^e]^T[\phi^e]\right]\{X^e\}d\Omega+f_c^e\iint_{\Omega_e}[\phi^e]^T d\Omega$$

$$+\ g_c^e\iint_{(\Gamma-\Gamma_1)^e}[\phi^e]^T g\,d\Gamma+\int_{\Gamma_1^e}[\phi^e]^T\left[a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}\right]d\Gamma=0$$

(15.142)

The constants f_c^e and g_c^e are part of the data of a problem.

The quantity $[a\lambda_{n2}\partial X/\partial x_2+b\lambda_{n3}\partial X/\partial x_3]$ of the integrand of the fourth integral of relation (15.142) is not specified on the portion Γ_1 of the boundary of the domain. *Consequently, this integral is an unknown which is established as part of the solution of the problem.* This integral is analogous to the end actions of elements of framed structures.

Relation (15.142) may be rewritten as

$$[K^e]\ \{X^e\}\ =\ \{F^e\} \tag{15.143}$$

$[K^e\}$ is the local stiffness matrix of element *e which* referring to relation (15.142) is equal to

$$[\hat{K}^e]=\iint_{\Omega_e}\left[-\left[a\frac{\partial[\phi^e]^T}{\partial x_2}\frac{\partial[\phi^e]}{\partial x_2}+b\frac{\partial[\phi^e]^T}{\partial x_3}\frac{\partial[\phi^e]}{\partial x_3}\right]+c[\phi^e]^T[\phi^e]\right]d\Omega \tag{15.144}$$

$\{F^e\}$ represents the contribution of the straight edge, three-node, triangular element e to the load vector of the domain. Referring to relation (15.142) it is equal to

$$\{F^e\}=f_c^e\iint_{\Omega^e}[\phi^e]^T\,d\Omega+g_c^e\iint_{(\Gamma-\Gamma_1)^e}[\phi^e]^T g\,d\Gamma+\int_{\Gamma_1^e}[\phi^e]^T\left[a\lambda_{n2}\frac{\partial\tilde{X}}{\partial x_2}+b\lambda_{n3}\frac{\partial\tilde{X}}{\partial x_3}\right]d\Gamma$$

(15.145)

For a straight edge, three-node, triangular element

$$[F^e]^T=[F_i^e\quad F_j^e\quad F_k^e] \tag{15.146}$$

For a straight edge, four-node, rectangular element

$$[F^e]^T=[F_i^e\quad F_j^e\quad F_k^e\quad F_l^e] \tag{15.147}$$

For a straight edge three-node triangular element, referring to relation, (15.135), we get

$$\frac{\partial\phi_q^e}{\partial x_2}=c_{q1}^e\qquad\frac{\partial\phi_q^e}{\partial x_3}=c_{q2}^e\qquad q=i,j\text{ or }k \tag{15.148}$$

Thus, referring to relation (15.138a), we obtain

$$\frac{\partial[\phi^e]}{\partial x_2}=[c_{i1}^e,\ c_{j1}^e\ ,c_{k1}^e]\qquad\frac{\partial[\phi^e]}{\partial x_3}=[c_{i2}^e,\ c_{j2}^e\ ,c_{k2}^e] \tag{15.149}$$

Substituting relations (15.149) into (15.144) and taking c = 0 we get the following approximation to the stiffness matrix for a straight edge, three-node triangular element:

$$[\hat{K}^e] = \Omega_e \left[a \begin{bmatrix} (c_{i1}^e)^2 & c_{i1}^e c_{j1}^e & c_{i1}^e c_{k1}^e \\ c_{j1}^e c_{i1}^e & (c_{j1}^e)^2 & c_{j1}^e c_{k1}^e \\ c_{k1}^e c_{i1}^e & c_{k1}^e c_{j1}^e & (c_{k1}^e)^2 \end{bmatrix} + b \begin{bmatrix} (c_{i2}^e)^2 & c_{i2}^e c_{j2}^e & c_{i2}^e c_{k2}^e \\ c_{j2}^e c_{i2}^e & (c_{j2}^e)^2 & c_{j2}^e c_{k2}^e \\ c_{k2}^e c_{i2}^e & c_{k2}^e c_{j2}^e & (c_{k2}^e)^2 \end{bmatrix} \right] \tag{15.150}$$

Similarly, for a straight edge, four-node, rectangular element, we obtain

$$[K^e] = \frac{ah_3^e}{6h_2^e} \begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{bh_2^e}{6h_3^e} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \tag{15.151}$$

15.9.10 Approximate Solutions of Scalar, Two-Dimensional, Second Order, Linear Boundary Value Problems

When we use the finite element method to construct approximate solutions for scalar, two-dimensional, second order, linear boundary value problems, we adhere to the steps described in Section 15.7. In what follows we apply the finite element method to establish approximate expressions for the components of stress τ_{12} and τ_{13} and the angle of twist α of a prismatic member, of square cross section, subjected to equal and opposite torsional moments at its ends. Our aim is to illustrate the application of the finite element method to a scalar, two-dimensional, second order, linear boundary value problem, not to obtain accurate results. For this reason, in order to avoid lengthy calculations we subdivide the domain of the problem into a very small number of finite elements.

Example 4 Using the finite element method, establish an approximate expression for the angle of twist per unit length and for the components of stress acting on the particles of a prismatic bar having the square cross section shown in Fig. a. The bar is made from an isotropic, linearly elastic material with shear modulus G and it is subjected to two equal and opposite torsional moments M_1 at its ends.

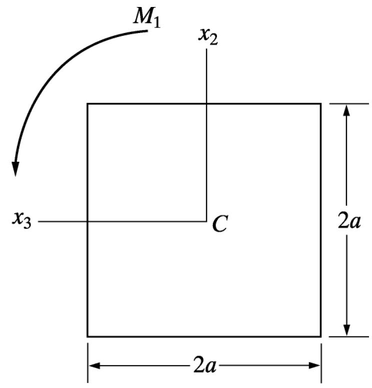


Figure a Geometry of the rectangular cross section of the bar.

Formulation

As discussed in Section 6.3, we solve this problem by first finding the stress function $\psi(x_2, x_3)$ which satisfies the differential equation (6.31) and vanishes on the boundary. That is,

$$\frac{\partial^2 \psi}{\partial x_2^2} \; \frac{\partial^2 \psi}{\partial x_3^2} = -2 \quad \text{at the points inside } \Omega \tag{a}$$

and

$$\psi = 0 \quad \text{at the points of } \Gamma \tag{b}$$

Consequently, comparing (15.109) with (a) and (15.110a) with (b) we have

$$a = b = 1 \qquad c = 0 \qquad f = 2 \qquad \Gamma = \Gamma_1 \tag{c}$$

Solution

STEP 1 We subdivide the domain of the problem (the cross section of the beam) into elements. As shown in Fig. b, the cross section of the prismatic bar has four axes of symmetry with respect to which the function $\psi(x_2, x_3)$ is symmetric while the shearing components of stress are antisymmetric. Thus it is sufficient to establish the stress distribution only on a portion of the cross section of the bar between two adjacent axes of symmetry as, for example, the portion *CAB* shown in Fig. b. We subdivide this portion of the cross section into three elements — two identical three-node, triangular and one square as shown in Fig. c. These elements are not enough to obtain accurate results. However, as we mentioned previously, our purpose is to illustrate the application of the finite element method without getting involved into excessive numerical calculations.

STEP 2 We compute an approximation to the stiffness matrix for each element of the domain. For the triangular elements of Fig. c, we have

$$\begin{aligned} x_2^{(i)} &= 0 \qquad & x_2^{(j)} &= \frac{a}{2} \qquad & x_2^{(k)} &= \frac{a}{2} \\ x_3^{(i)} &= 0 \qquad & x_3^{(j)} &= 0 \qquad & x_3^{(k)} &= \frac{a}{2} \end{aligned} \tag{d}$$

and

$$\Omega_e = \frac{a^2}{8} \tag{e}$$

Substituting relations (d) and (e) into relations (15.133), we obtain

$$\begin{aligned}
c_{i0}^{(1)} &= c_{i0}^{(3)} = \frac{x_2^{(j)}x_3^{(k)} - x_2^{(k)}x_3^{(j)}}{2\Omega_e} = 1 \qquad & c_{j0}^{(1)} &= c_{j0}^{(3)} = \frac{x_2^{(k)}x_3^{(i)} - x_2^{(i)}x_3^{(k)}}{2\Omega_e} = 0 \\
c_{i1}^{(1)} &= c_{i1}^{(3)} = \frac{x_3^{(j)} - x_3^{(k)}}{2\Omega_e} = -\frac{2}{\alpha} \qquad & c_{j1}^{(1)} &= c_{j1}^{(3)} = \frac{x_3^{(k)} - x_3^{(i)}}{2\Omega_e} = \frac{2}{\alpha} \\
c_{i2}^{(1)} &= c_{i2}^{(3)} = \frac{x_2^{(k)} - x_2^{(j)}}{2\Omega_e} = 0 \qquad & c_{j2}^{(1)} &= c_{j2}^{(3)} = \frac{x_2^{(i)} - x_2^{(k)}}{2\Omega_e} = -\frac{2}{\alpha} \\
c_{k0}^{(1)} &= c_{k0}^{(3)} = \frac{x_2^{(i)}x_3^{(j)} - x_2^{(j)}x_3^{(i)}}{2\Omega_e} = 0 & & \\
c_{k1}^{(1)} &= c_{k1}^{(3)} = \frac{x_3^{(i)} - x_3^{(j)}}{2\Omega_e} = 0 \qquad & c_{k2}^{(1)} &= c_{k2}^{(3)} = \frac{x_2^{(j)} - x_2^{(i)}}{2\Omega_e} = \frac{2}{\alpha}
\end{aligned} \tag{f}$$

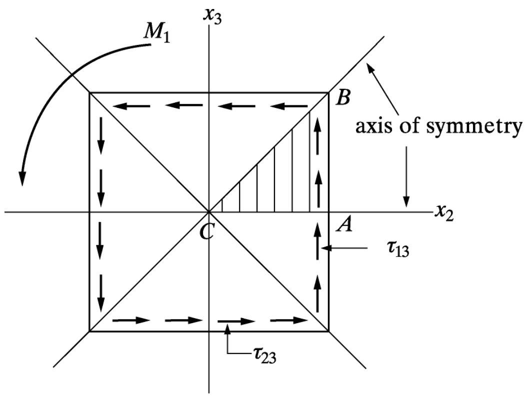


Figure b Cross section of the bar and stress distribution.

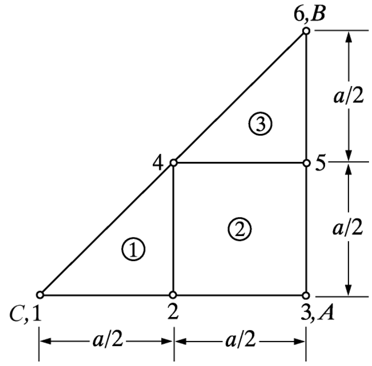


Figure c Portion CAB of the cross section of the bar subdivided into three elements.

Substituting the values of the constants (f) into relations (15.129), we get the following shape functions for the triangular elements of Fig. c:

$$\phi_i = 1 - \frac{2x_2}{a} \qquad \phi_j = \frac{2}{a}(x_2 - x_3) \qquad \phi_k = \frac{2x_3}{a} \tag{g}$$

Substituting the values of the shape functions (g) into the relation (15.144) and using relation (c), we obtain

$$[K^{(1)}] = [K^{(3)}] = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{h}$$

This matrix represents an approximation to the stiffness matrix for any orthogonal isosceles triangular element. The approximation to the stiffness matrix for the four-node square element may be established by substituting $a = b = 1$ and $h = h_2 = h_3$ in relation (15.151). Thus,

$$[K^{(2)}] = \frac{1}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix} \tag{i}$$

STEP 3 We use the approximations to the stiffness matrices of the elements of the domain to assemble an approximation to its stiffness matrix $[S]$. In order to accomplish this, referring to Fig. c, we choose the subscripts of the stiffness coefficients for each element so as to correspond to those of the components of displacements of the nodes to which the element is connected. That is,

$$[K^{(1)}] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{14}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{24}^{(1)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{44}^{(1)} \end{bmatrix} \qquad [K^{(2)}] = \begin{bmatrix} K_{22}^{(2)} & K_{23}^{(2)} & K_{25}^{(2)} & K_{24}^{(2)} \\ K_{32}^{(2)} & K_{33}^{(2)} & K_{35}^{(2)} & K_{34}^{(2)} \\ K_{52}^{(2)} & K_{53}^{(2)} & K_{55}^{(2)} & K_{54}^{(2)} \\ K_{42}^{(2)} & K_{43}^{(2)} & K_{45}^{(2)} & K_{44}^{(2)} \end{bmatrix} \tag{j}$$

$$[K^{(3)}] = \begin{bmatrix} K_{44}^{(3)} & K_{45}^{(3)} & K_{46}^{(3)} \\ K_{54}^{(3)} & K_{55}^{(3)} & K_{56}^{(3)} \\ K_{64}^{(3)} & K_{65}^{(3)} & K_{66}^{(3)} \end{bmatrix} \tag{j}$$

Referring to relations (h), (i), and (j) from relation (15.27), we get

$$[S] = \frac{1}{6}\begin{bmatrix} 3 & -3 & 0 & 0 & 0 & 0 \\ -3 & 10 & -1 & -4 & -2 & 0 \\ 0 & -1 & 4 & -2 & -1 & 0 \\ 0 & -4 & -2 & 10 & -4 & 0 \\ 0 & -2 & -1 & -4 & 10 & -3 \\ 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix} \tag{k}$$

STEP 4 We establish an approximation to the load vector of the domain. In order to accomplish this we use relation (15.145) to compute the contribution of each element to the load vector of the domain. Referring to relations (c) from relation (15.145), we have

$$\{F^e\} = 2\iint_{\Omega^e} [\phi^e]^T \, d\Omega + \{R^e\} \tag{l}$$

where

$$\{R^e\} = \int_{\Gamma^e} [\phi^e]^T \left[\lambda_{n2}\frac{\partial\psi}{\partial x_2} + \lambda_{n3}\frac{\partial\psi}{\partial x_3}\right] d\Gamma \tag{m}$$

Referring to relations (6.6) and (6.7), we get

$$\lambda_{n2}\frac{\partial\psi}{\partial x_2} + \lambda_{n3}\frac{\partial\psi}{\partial x_3} = \frac{dx_2}{dx_n}\frac{\partial\psi}{\partial x_2} + \frac{dx_3}{dx_n}\frac{\partial\psi}{\partial x_3} = \frac{d\psi}{dx_n} \tag{n}$$

Substituting relation (n) into (m) we obtain

$$\{R^e\} = \int_{\Gamma^e} [\phi^e]^T \frac{d\psi}{dx_n}\, d\Gamma \tag{o}$$

$d\psi/dx_n$ is the rate of change of $\psi(x_2, x_3)$ in the direction normal to the boundary. Since on the boundary ψ is constant, it is clear that the boundary is a contour line of the function $\psi(x_2, x_3)$. In Section 8.3 we show that the shearing stress acting at a point of a cross section of a prismatic bar subjected to equal and opposite torsional moments at its ends is tangent to the contour line of the stress function $\phi = \alpha G\psi$ passing through that point. Moreover, referring to relations (6.62) and (6.30) the shearing stress is equal to

$$\tau_{1s} = -\frac{d\phi}{dx_n} = -G\alpha\frac{d\psi}{dx_n} \tag{p}$$

For the triangular elements 1 and 3 we substitute the shape functions (g) in relation (l), to obtain

$$\begin{Bmatrix} F_i^{(e)} \\ F_j^{(e)} \\ F_k^{(e)} \end{Bmatrix} = 2\begin{Bmatrix} \int_0^{a/2}\left[\int_0^{x_3=x_2}(1-\frac{2x_2}{a})dx_3\right]dx_2 \\ \int_0^{a/2}\left[\int_0^{x_3=x_2}\frac{2}{a}(x_2-x_3)\,dx_3\right]dx_2 \\ \int_0^{a/2}\left[\int_0^{x_3=x_2}\frac{2x_3}{a}\,dx_3\right]dx_2 \end{Bmatrix} + \{R^{(e)}\} = \frac{a^2}{12}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \{R^{(e)}\} \tag{q}$$

$$e = 1, 3$$

$[R^{(1)}]$ is a zero matrix since the sides of element 1 are not part of the boundary of the domain. In evaluating the matrix $\{R^e\}$ $(e = 2, 3)$ we assume that $d\psi/dx_n$ is constant on the side of elements which are part of the boundary of the cross section of the bar. That is,

$$\{R^3\} = \begin{Bmatrix} R_i^{(3)} \\ R_j^{(3)} \\ R_k^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \left[\left(\frac{d\psi}{dx_n}\right)_5 \int_0^{a/2} \phi_j^{(3)}\Big|_{x_2=\frac{a}{2}} dx_3\right] \\ \left[\left(\frac{d\psi}{dx_n}\right)_6 \int_0^{a/2} \phi_k^{(3)}\Big|_{x_2=\frac{a}{2}} dx_3\right] \end{Bmatrix} = \frac{a}{4}\begin{Bmatrix} 0 \\ \left(\frac{d\psi}{dx_n}\right)_5 \\ \left(\frac{d\psi}{dx_n}\right)_6 \end{Bmatrix} \tag{r}$$

For the rectangular element 2, substituting relation (15.136) into (l) and using $h_2 = h_3 = h = a/2$, we get

$$\begin{Bmatrix} F_i^{(2)} \\ F_j^{(2)} \\ F_k^{(2)} \\ F_l^{(2)} \end{Bmatrix} = 2\begin{Bmatrix} \int_0^{a/2}\int_0^{a/2}\frac{(a-2x_2)(a-2x_3)}{a^2}\,dx_2dx_3 \\ \int_0^{a/2}\int_0^{a/2}\frac{2x_2(a-2x_3)}{a^2}\,dx_2dx_3 \\ \int_0^{a/2}\int_0^{a/2}\frac{4x_2x_3}{a^2}\,dx_2dx_3 \\ \int_0^{a/2}\int_0^{a/2}\frac{2x_3(a-2x_2)}{a^2}\,dx_2dx_3 \end{Bmatrix} + \{R^{(2)}\} = \frac{a^2}{8}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} + \{R^{(2)}\} \tag{s}$$

where

$$\{R^2\} = \begin{Bmatrix} R_i^{(2)} \\ R_j^{(2)} \\ R_k^{(2)} \\ R_l^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \left(\dfrac{d\psi}{dx_n}\right)_3 \displaystyle\int_0^{a/2} \phi_j^{(2)}\Big|_{x_2=\frac{a}{2}} dx_3 \\ \left(\dfrac{d\psi}{dx_n}\right)_5 \displaystyle\int_0^{a/2} \phi_k^{(2)}\Big|_{x_2=\frac{a}{2}} dx_3 \\ 0 \end{Bmatrix} = \frac{a}{4}\begin{Bmatrix} 0 \\ \left(\dfrac{d\psi}{dx_n}\right)_3 \\ \left(\dfrac{d\psi}{dx_n}\right)_5 \\ 0 \end{Bmatrix} \tag{t}$$

STEP 5 We form the approximate stiffness equations for the problem and we introduce in them its essential boundary conditions. We solve the approximate stiffness equations to obtain the unknown values of the state function $\psi_s (s = 1, 2, ..., 6)$ at the nodes of the domain and the values of the normal derivative of the state variable at the nodes which are located on the portion of the boundary where essential boundary conditions are specified. Referring to relations (k), and (q) to (t), the stiffness equations for the domain of Fig. c are

$$\frac{1}{6}\begin{bmatrix} 3 & -3 & 0 & 0 & 0 & 0 \\ -3 & 10 & -1 & -4 & -2 & 0 \\ 0 & -1 & 4 & -2 & -1 & 0 \\ 0 & -4 & -2 & 10 & -4 & 0 \\ 0 & -2 & -1 & -4 & 10 & -3 \\ 0 & 0 & 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \frac{a^2}{24}\begin{Bmatrix} 2 \\ 5 \\ 3 \\ 7 \\ 5 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ R_3 \\ 0 \\ R_5 \\ R_6 \end{Bmatrix} \tag{u}$$

where referring to relations (r) and (t), and using relation (6.62), R_3, R_5,and R_6 are equal to

$$2R_3 = R_j^{(2)} = \frac{a}{4}\left(\frac{d\psi}{dx_n}\right)_3 = -\frac{a(\tau_{1S})_3}{4\alpha G}$$

$$R_5 = R_k^{(2)} + R_j^{(3)} = \frac{a}{2}\left(\frac{d\psi}{dx_n}\right)_5 = -\frac{a(\tau_{1S})_5}{2\alpha G} \tag{v}$$

$$R_6 = R_k^{(3)} = \frac{a}{4}\left(\frac{d\psi}{dx_n}\right)_6 = -\frac{a(\tau_{1S})_6}{4\alpha G}$$

We rearrange the rows and columns of the stiffness equations (u) and we partition them as follows:

$$\frac{1}{6}\left[\begin{array}{ccc|ccc} 3 & -3 & 0 & 0 & 0 & 0 \\ -3 & 10 & -4 & -1 & -2 & 0 \\ 0 & -4 & 10 & -2 & -4 & 0 \\ \hline 0 & -1 & -2 & 4 & -1 & 0 \\ 0 & -2 & -4 & -1 & 10 & -3 \\ 0 & 0 & 0 & 0 & -3 & 3 \end{array}\right]\left\{\begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_4 \\ \hline \psi_3 \\ \psi_5 \\ \psi_6 \end{array}\right\} = \frac{a^2}{24}\left\{\begin{array}{c} 2 \\ 5 \\ 7 \\ \hline 3 \\ 5 \\ 2 \end{array}\right\} + \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline R_3 \\ R_5 \\ R_6 \end{array}\right\} \qquad \text{(w)}$$

Noting that $\psi_3 = \psi_5 = \psi_6 = 0$ we have

$$\frac{1}{6}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 10 & -4 \\ 0 & -4 & 10 \end{bmatrix}\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_4 \end{Bmatrix} = \frac{a^2}{24}\begin{Bmatrix} 2 \\ 5 \\ 7 \end{Bmatrix} \qquad \text{(x)}$$

and

$$\frac{1}{6}\begin{bmatrix} 0 & -1 & -2 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_4 \end{Bmatrix} = \frac{a^2}{24}\begin{Bmatrix} 3 \\ 5 \\ 2 \end{Bmatrix}\begin{Bmatrix} R_3 \\ R_5 \\ R_6 \end{Bmatrix} \qquad \text{(y)}$$

Solving relation (x), we get

$$\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_4 \end{Bmatrix} = \frac{a^2}{4}\begin{Bmatrix} 2.480 \\ 1.824 \\ 1.432 \end{Bmatrix} \qquad \text{(z)}$$

Substituting results (z) into relation (y), we obtain

$$\begin{Bmatrix} R_3 \\ R_5 \\ R_6 \end{Bmatrix} = -\frac{a^2}{24}\begin{Bmatrix} 7.688 \\ 14.376 \\ 0 \end{Bmatrix} \qquad \text{(za)}$$

STEP 6 We compute the angle of twist per unit length of the bar under consideration using relation (6.32). Noting that the area of the portion of the cross section shown in Fig. c is one-eighth that of the total cross section, referring to relations (6.33) and using relation (15.123a), we have

$$R_T = 2G\iint_{\Omega}\bar{\psi}dA = 8G\left[2\sum_{e=1}^{3}\iint_{\Omega_e}\psi^e dx_2 dx_3\right] = 16G\sum_{e=1}^{3}\left[\left[\iint_{\Omega_e}[\phi^e]dx_2 dx_3\right]\{\psi^e\}\right] \qquad \text{(zb)}$$

The integrals $\iint_{\Omega_e} [\phi^e] dx_2 dx_3$ $(e = 1,2,3)$ have been evaluated for the triangular elements 1 and 3 in relation (q) and for the rectangular element 2 in relation (s). Substituting the values of these integrals from relations (q) and (s) in relation (zb), we obtain

$$R_T = 2G\iint_{\Omega} [\phi^e] dA \{\psi^e\} = 16G \left[\frac{a^2}{24} [1 \;\; 1 \;\; 1] \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_4 \end{Bmatrix} + \frac{a^2}{16} [1 \;\; 1 \;\; 1 \;\; 1] \begin{Bmatrix} \psi_2 \\ 0 \\ 0 \\ \psi_4 \end{Bmatrix} \right. \tag{zc}$$

$$\left. + \frac{a^2}{24} [1 \;\; 1 \;\; 1] \begin{Bmatrix} \psi_4 \\ 0 \\ 0 \end{Bmatrix} \right] = \frac{a^2 G}{3} \left[2\psi_1 + 5\psi_2 + 7\psi_4 \right] = 2.1 G a^4$$

Substituting relation (zc) into relation (6.32), we get

$$\alpha = \frac{M_1}{R_T} = \frac{M_1}{2.1 G a^4} = \frac{0.4762 M_1}{G a^4} \tag{zd}$$

STEP 7 We compute the components of stress using relations (6.34). That is,

$$\tau_{12} = \alpha G \frac{\partial \psi}{\partial x_3} \qquad \tau_{13} = -\alpha G \frac{\partial \psi}{\partial x_2} \tag{ze}$$

Substituting relation (15.123a) into relations (ze), we get

$$\tau_{12}^e = \alpha G \frac{\partial [\phi^e]}{\partial x_3} \{\psi^e\} \qquad \tau_{13}^e = -\alpha G \frac{\partial [\phi^e]}{\partial x_2} \{\psi^e\} \tag{zf}$$

Where referring to Fig. c , we have

$$[\psi^{(1)}]^T = [\psi_1, \psi_2, \psi_4] \qquad [\psi^{(2)}]^T = [\psi_2, \psi_3, \psi_4, \psi_5] \qquad [\psi^{(3)}]^T = [\psi_4, \psi_5, \psi_6] \tag{zg}$$

For the triangular elements 1 and 3 substituting relations (g) into (15.138) and the resulting matrix into (zf) and using relations (zg), (z) and (zd), we obtain

$$\tau_{12}^{(1)} = \alpha G \left[0, \; -\frac{2}{a}, \; \frac{2}{a} \right] \{\psi^{(1)}\} = \frac{2\alpha G}{a} (\psi_4 - \psi_2) = \frac{2(0.4762)(1.432 - 1.824) M_1}{4a^3}$$
$$= -\frac{0.0933 M_1}{a^3} \tag{zh}$$

$$\tau_{13}^{(1)} = -\alpha G\left[-\frac{2}{a},\ \frac{2}{a},\ 0\right]\{\psi^{(1)}\} = -\frac{2\alpha G}{a}(\psi_2 - \psi_1)$$

$$= -\frac{2(0.4764)M_1(2.48-1.824)}{4a^3} = \frac{0.1562M_1}{a^3} \tag{zh}$$

$$\tau_{12}^{(3)} = \alpha G\left[0, -\frac{2}{a},\ \frac{2}{a}\right]\begin{Bmatrix}\psi_4\\0\\0\end{Bmatrix} = 0 \tag{zi}$$

$$\tau_{13}^{(3)} = -\alpha G\left[-\frac{2}{a},\ \frac{2}{a},\ 0\right]\begin{Bmatrix}\psi_4\\0\\0\end{Bmatrix} = \frac{2\alpha G\psi_4}{a} = \frac{(0.4762)(1.432)M_1}{2a^3} = \frac{0.341M_1}{a^3}$$

These are the average values of the components of the stress over the area of the triangular elements.

For the rectangular element 2 referring to relations (zd),(zf),(zg) and (15.139a), we get

$$\tau_{12}^{(2)} = \alpha G\left[-\frac{2(a-2x_2)}{a^2}, -\frac{4x_2}{a^2}, \frac{4x_2}{a^2}, \frac{2(a-2x_2)}{a^2}\right]\begin{Bmatrix}\psi_2\\ \psi_3=0\\ \psi_5=0\\ \psi_4\end{Bmatrix}$$

$$= \alpha G\frac{(2a-4x_2)}{a^2}(\psi_4 - \psi_2) = -0.0934\left(1 - \frac{2x_2}{a}\right)\frac{M_1}{a^3} \tag{zj}$$

$$\tau_{13}^{(2)} = -\alpha G\left[-\frac{2(a-2x_3)}{a^2}, -\frac{2(a-2x_3)}{a^2}, \frac{4x_3}{a^2}, -\frac{4x_3}{a^2}\right]\begin{Bmatrix}\psi_2\\0\\0\\ \psi_4\end{Bmatrix}$$

$$= \alpha G\left[\frac{2\psi_2}{a} - \frac{4x_3(\psi_2-\psi_4)}{a^2}\right] = \left(0.4343 - \frac{0.1867x_3}{a}\right)\frac{M_1}{a^3} \tag{zk}$$

From relation (zj) we see that, as expected, the component of stress $\tau_{12}^{(2)}$ vanishes on the line $x_2 = a/2$. This line is part of the stress free boundary of the cross section of the bar (see Fig. c). Referring to relation (zj), the maximum stress $\tau_{12}^{(2)}$ on element 2 occurs at $x_2 = 0$ and it is equal to

$$\left.(\tau_{12}^{(2)})_{\max}\right|_{x_2=0} = -0.0934\frac{M_1}{a^3} \tag{zl}$$

This is the average value of the component of stress τ_{12} over the length of line 2, 4 (see Fig. c). It could be considered as the value of the component of stress τ_{12} at the middle point of line 2, 4. From relation (zk) we see that the maximum stress $\tau_{13}^{(2)}$ on element 2 occurs at $x_3 = 0$ and it is equal to

$$\left.(\tau_{13}^{(2)})_{\max}\right|_{x_3=0} = -0.4343\frac{M_1}{a^3} \tag{zm}$$

This is the average value of the component of stress τ_{13} on line 2, 3 (see Fig. c). It could be considered as the value of the component of stress τ_{13} at the middle of line 2, 3. If we assume that the component of stress τ_{13} vanishes at point 1 and varies linearly along line 1, 3 from relation (zm), we get

$$\left.(\tau_{13}^{(2)})_{\max}\right|_{\substack{\bar{x}_2=a\\ \bar{x}_3=0}} = \frac{4}{3}\left(\frac{0.4343M_1}{a^3}\right) = \frac{0.579M_1}{a^3} \tag{zn}$$

The maximum value of stress obtained on the basis of the theory of elasticity occurs at point 3 and it is equal to

$$\left.(\tau_{13}^{(2)})_{\max}\right|_{\substack{\bar{x}_2=a\\ \bar{x}_3=0}} = \frac{0.601M_1}{a^3} \tag{zo}$$

The component of stress $\tau_{1s} = \tau_{13}$ at nodes 3, 5 and 6 of the boundary can be established by substituting relations (za) into (v). That is,

$$(\tau_{13})_3 = (\tau_{1S})_3 = -\frac{4R_3\alpha G}{a} = -\frac{4R_3(0.4762)M_1}{a^5} = \frac{4(7.688)(0.4762)M_1}{a^3} = \frac{0.610M_1}{a^3}$$

$$(\tau_{13})_5 = (\tau_{1S})_5 = -\frac{2R_5\alpha G}{a} = -\frac{2(14.376)(0.4762)M_1}{24}a^3 = \frac{0.570M_1}{a^3} \tag{zp}$$

$$(\tau_{13})_6 = (\tau_{1S})_6 = -\frac{4R_6\alpha G}{a} = 0$$

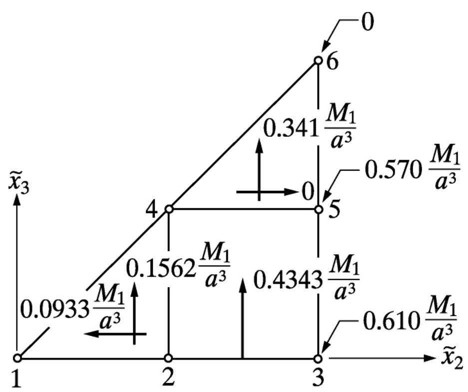


Figure c Results.

Thus, the computed value of the maximum component of stress $(\tau_{13})_3$ is approximately 1.5% in error. From the results obtained in this example we see that the components of stress τ_{12} and τ_{13} do not vary in the domain of three-node, triangular elements. Moreover, they vary only with the one coordinate in the domain of four-node, rectangular elements. The results can improve as follows:

1. By using a large number of elements. As the size of the elements decreases, the constant gradient result becomes less objectionable.
2. By using triangular elements with more than three nodes and/or rectangular elements with more than four nodes.

15.10 Problems

1. Using the finite element method (direct stiffness method), establish the component of translation $u_1(x_1)$ and the internal forces of the structure of Fig. 15P1 subjected to the axial centroidal forces shown in that figure. The structure has constant width b. Use n = 0.5. Subdivide the structure into two equal elements. (*Hint:* Use relation (15.42) to compute the local stiffness matrix of the tapered element.)

$$\text{Ans.}\quad \begin{Bmatrix} \Delta_2 \\ \Delta_3 \end{Bmatrix} = \frac{L}{1.5EA_0} \begin{Bmatrix} 2p_1L + 2P_1^L \\ 3.5p_1L + 5P_1^L \end{Bmatrix}$$

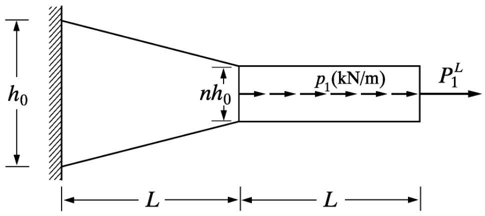


Figure 15P1

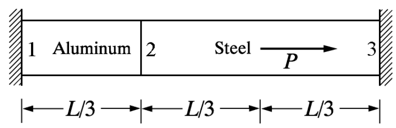


Figure 15P2

2. Using the finite element method (direct stiffness method), establish the component of translation $u_1(x_1)$ and the internal force $N(x_1)$ in the bar of Fig. 15P2, due to a uniform increase of temperature ΔT = 20°C and to a concentrated force P = 40kN. The bar has constant cross section (A = 4 cm^2) and is made of two materials in series steel (E = 210 GPa $\alpha = 1.2(10^{-5}$°C) and an aluminum alloy (E = 70GPa α = 2.3(10^5/°C).

$$\text{Ans.}\quad \begin{Bmatrix} R_1^{(1)} \\ R_1^{(3)} \end{Bmatrix} = \begin{Bmatrix} 13.67 \\ 53.67 \end{Bmatrix} \text{kN}$$

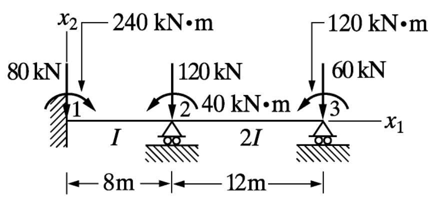


Figure 15P3

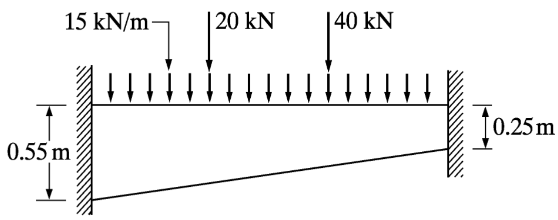


Figure 15P4

3. Using the finite element method (direct stiffness method), establish the deflection, the shearing force and the bending moment of the beam subjected to the loading shown in

Fig. 15P3. Subdivide the beam in two elements of length 8 m and 12 m.

Ans. $R_2^{(1)} = 78.125$ KN $\quad R_3^{(1)} = 235$ KN·m

4. Establish an approximate expression for the fixed-end actions of the beam of variable width b shown in Fig. 15P4 by subdividing it into three elements of equal length. Approximate each element by a parallelepiped of depth equal to that of the middle cross section of the real element. Ans. $R_2^{(1)} = 80.55$ kN $\quad R_3^{(1)} = 117.20$ kN·m

5. Given a function $X(x_1)$ which satisfies the following differential equation in the domain $1 \leq \bar{x}_1 \leq 2$:

$$\frac{d}{d\bar{x}_1}\left(\bar{x}_1 \frac{dX}{d\bar{x}_1}\right) = 2 \tag{a}$$

and the boundary conditions

$$X(1) = 2 \qquad \text{essential B.C. (b)}$$

$$\left[F = \bar{x}_1 \frac{dX}{d\bar{x}_1}\right]_{\bar{x}_1 = 2} = \frac{1}{2} \qquad \text{natural B.C. (c)}$$

where $F(\bar{x}_1)$ is the flux of the problem. Establish the modified weighted residual equation. Using the finite element method, establish an approximate solution for $X(\bar{x}_1)$. Subdivide the domain with four elements.

6. The elements of the frame of Fig. 15P6 have constant cross sections and are subjected to the following two cases of loading:

Case 1

To a uniform temperature increase ΔT. This temperature increase represents the difference between the uniform temperature to which the frame is exposed in its present state and the temperature existing during its construction.

Case 2

To a temperature T_1 at the external fibers of its elements and to a temperature $T_2 > T_1$ at their internal fibers. Assume that in this case the temperature of the axis of the elements of the frame is the same as the temperature during construction. Thus, the axis of the elements of the frame does not elongate. For each case of loading compute the equivalent actions to be applied to the nodes of the frame.

Ans. *Case* 1 $\{P^E\}^T = \left[0 \;\; -E_1A_1\alpha_1 \;\; 0 \mid -E_2A_2\alpha_2 \;\; E_1A_1\alpha_1 \;\; 0 \mid E_2A_2\alpha_2 \;\; E_3A_3\alpha_3 \;\; 0 \mid 0 \;\; -E_3A_3\alpha_3 \;\; 0\right] \Delta T$

Ans. *Case* 2 $[P_3^E] = -\alpha_1 E_1 I^{(1)}(T_2 - T_1)/h^{(1)} \qquad [P_6^E] = \left(\alpha_1 E_1 I^{(1)}/h^{(1)} - \alpha_2 E_2 I^{(2)}/h^{(2)}\right)(T_2 - T_1)$

$[P_9^E] = \left(\alpha_2 E_2 I^{(2)}/h^{(2)} + -\alpha_3 E_3 I^{(3)}/h^{(3)}\right)(T_2 - T_1) \qquad [P_{12}^E] = \alpha_3 E_3 I^{(3)}(T_2 - T_1)/h^{(3)}$

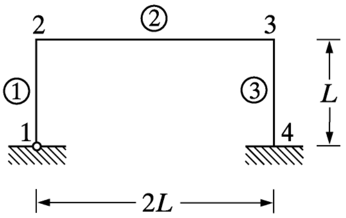


Figure 15P6

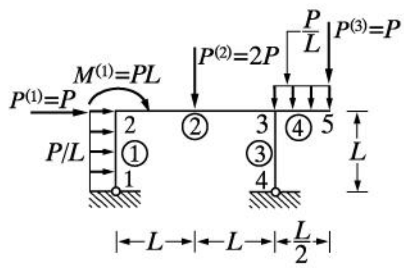


Figure 15P7

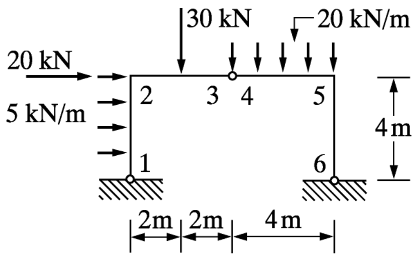


Figure 15P8

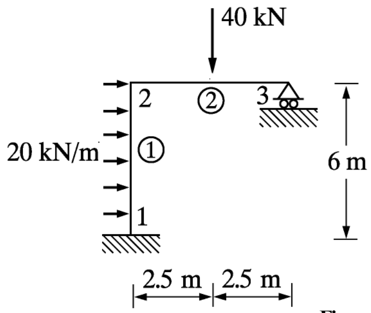


Figure 15P9

7. to **11.** The elements of the frame are made from the same isotropic, linearly elastic material and have the same cross section. Compute and show on a sketch the equivalent actions to be applied to the nodes of the frame of Fig. 15P7. Repeat with the frames of Fig. 15P8 to 15P11.

Ans. 7 $\left[\{P^E\} + \{P^G\}\right]^T = \left[\, P/2 \quad 0 \quad -PL/12 \mid 3P/2 \quad -P \quad -17PL/12 \mid 0 \quad -5P/4 \quad 23PL/12 \mid 0 \quad 0 \quad 0 \mid 0 \quad -5P/4 \quad PL/48 \,\right]$

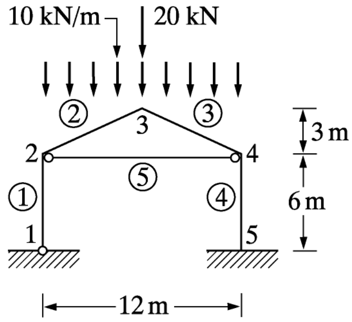


Figure 15P10

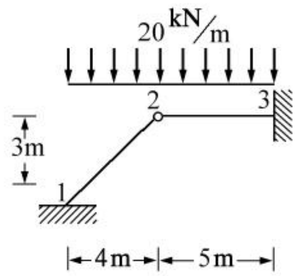


Figure 15P11

12. Assemble the stiffness matrix for the structure shown in Fig. 15P12. The elements of this structure are made from the same isotropic, linearly elastic material and have the same constant cross sections with I/A = 0.025 m^2. The elements of the structure are connected by a pin.

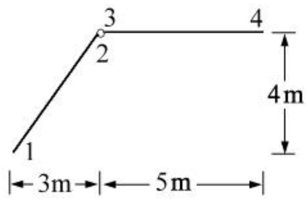


Figure 15P12

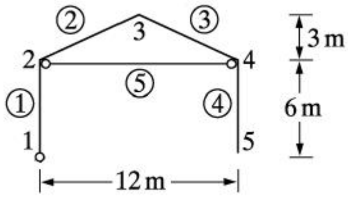
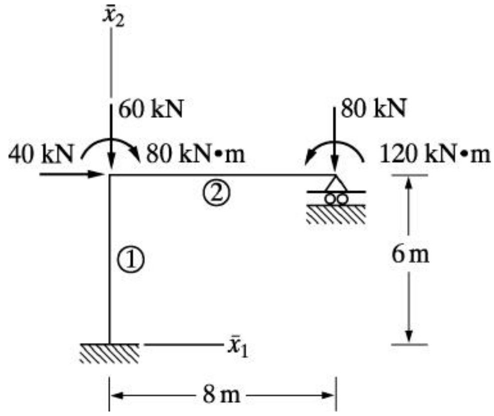


Figure 15P13

13. Assemble the stiffness matrix for the frame of Fig. 15P13. The elements of the frame are made of steel (E = 210 GPa). The area of the cross section of the pin at both ends member is $A_4 = 800$ mm^2. The other elements of the frame have a constant cross section [$A = 13.2(10^3)$ mm^2, $I = 369.7(10^6)$ mm^4].

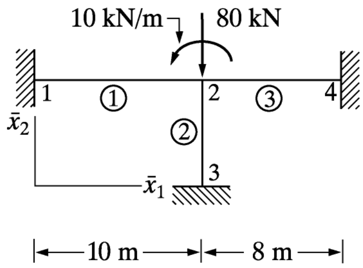
14. and 15. Consider the structure subjected to the external actions and supported as shown in Fig. 15P14. The elements of the structure are made from the same isotropic, linearly elastic material (E = 210 GPa) and the areas and moments of inertia of their cross

sections are shown in Fig. 15P14. Compute
(a) The components of displacements of the nodes of the frame
(b) The reactions of the supports of the frame
(c) The nodal actions of the elements of the frame
Repeat with the structures of Figs. 15P15.



$I^{(1)} = 117 \times 10^6\ mm^4$ $A_1 = 6.26 \times 10^3\ mm^2$
$I^{(2)} = 83.6 \times 10^6\ mm^4$ $A_2 = 5.38 \times 10^3\ mm^2$

Figure 15P14



$I^{(1)} = 369.7 \times 10^6\ mm^4$ $A_1 = 13.2 \times 10^3\ mm^2$
$I^{(2)} = I^{(3)} = 117.7 \times 10^6\ mm^4$ $A_2 = A_3 = 6.26 \times 10^3\ mm^2$

Figure 15P15

16. The structure shown in Fig. 15P10 is subjected to the force shown in that figure. The members of the structure have a constant cross section and are made from the same isotropic, linearly elastic material ($E = 210$ GPa, $A = 13.2(10^3)\ mm^2$, $I = 369.7(10^6)\ mm^4$).

Compute
(a) The components of displacement of the nodes of the beam
(b) The reactions of the supports of the beam
(c) The internal forces in the members of the beam

17. The frame shown in Fig. 15P17 is subjected to the following loading:
(a) The external actions shown in Fig. 15P17
(b) Settlement of support 1 of 20 mm
(c) Temperature of the upper or outside fibers $T_e = 35$°C and of lower or inside fibers $T_i = -5$°C. the temperature during the construction of the frame was $T_0 = 15$°C.

The elements of the frame are made of steel ($E = 210\ kN/mm^2$, $\alpha = 10^{-5}$/°C) and have the same constant cross section ($A = 13.2 \times 10^3 mm^2$, $I = 369.7 \times 10^6\ mm^4$, $h = 425$mm). Disregard the effect of axial translations of the members of the frame. Using the finite element method (direct stiffness method) compute
(a) The components of displacements of the nodes of the frame

(b) The reactions of the supports of the frame
(c) The nodal actions of the elements of the frame

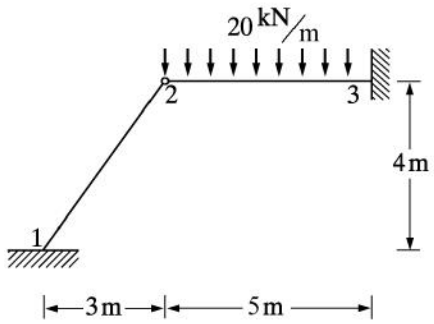


Figure 15P17

18. Using the finite element method establish the angle of twist per unit length and the shearing components of stress of a prismatic bar of rectangular cross section subjected to equal and opposite torsional moments at its ends. The bar is made from an isotropic, linearly elastic material of shear modulus, *G(*GPa*)*. The cross section of the bar has two axes of symmetry with respect to which the stress is antisymmetric. Thus use only the portion *ABCD* of the cross section and subdivide it into nine identical four-node rectangular elements of dimensions $b/3$ and $a/3$ as shown in Fig. 15P18.

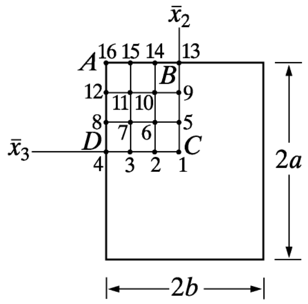


Figure 15P18

Chapter 16

Plastic Analysis and Design of Structures

16.1 Strain–Curvature Relation of Prismatic Beams Subjected to Bending without Twisting

The assumptions of the theories of mechanics of materials for line members, discussed in Section 8.2, apply to line members made from any material. Thus, relations (9.5) apply to beams made from isotropic linearly elastic–ideally plastic materials (see Fig. 3.14e). Choosing the x_2 and x_3 axis to be principal centroidal and limiting our attention to beams subjected to bending about the x_2 axis without twisting, we have

$$\hat{u}_1(x_1, x_2, x_3) = u_1(x_1) + x_3\theta_2(x_1) \tag{16.1}$$

and

$$e_{11}(x_1, x_2, x_3) = \frac{du_1}{dx_1} + x_3\frac{d\theta_2}{dx_1} \tag{16.2}$$

For the classical theory of beams, referring to relations (9.27a) the above relations may be rewritten as

$$\hat{u}_1(x_1, x_2, x_3) = u_1(x_1) - x_3\frac{du_3}{dx_1} \tag{16.3}$$

$$e_{11}(x_1, x_2, x_3) = \frac{du_1}{dx_1} - x_3\frac{d^2u_3}{dx_1^2} \tag{16.4}$$

Substituting relation (9.26a) into (16.4), we obtain

$$e_{11}(x_1, x_2, x_3) = \frac{du_1}{dx_1} + \frac{x_3}{\rho_{13}} \tag{16.5}$$

Relation (16.5) indicates that the neutral axis ($e_{11} = 0$) of the cross sections of a beam is not necessarily a centroidal axis ($x_3 = 0$). For a better understanding of relation (16.5)

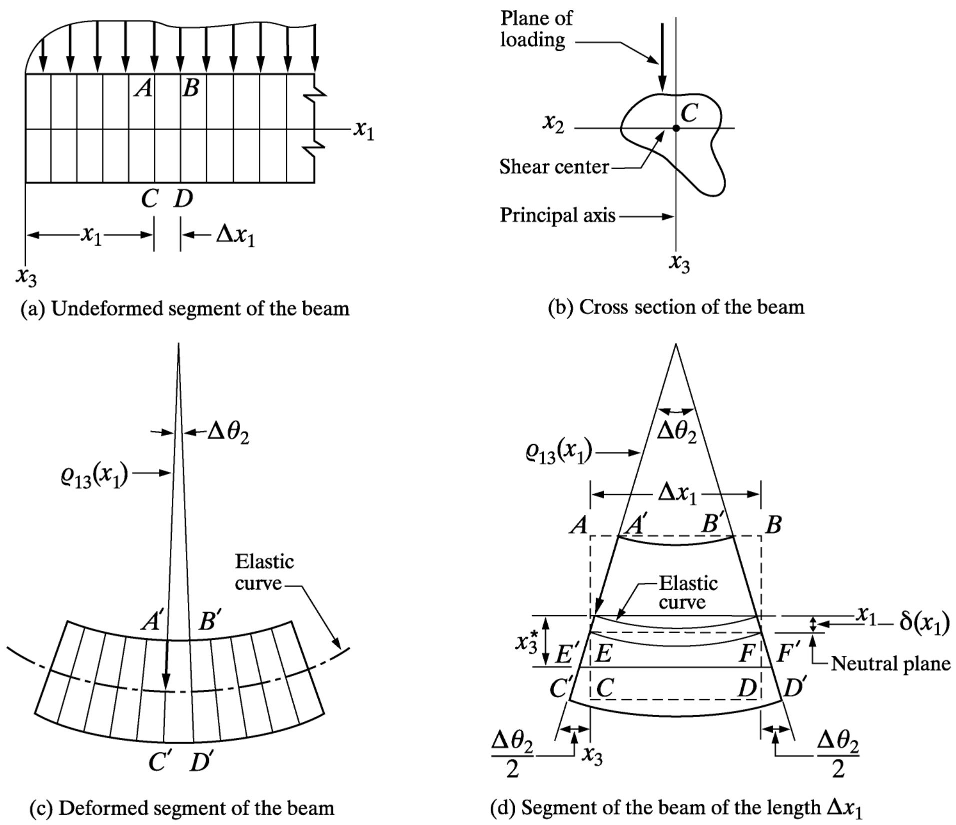


Figure 16.1 Portion of beam subjected to bending about its x_2 axis.

let us consider the element of length Δx_1 of a beam, shown in Fig. 16.1d cut by two planes normal to its axis. On the basis of the assumptions of the theories of mechanics of materials, as shown in Fig. 16.1c and d, the faces of the element remain plane after deformation and normal to the deformed centroidal axis (elastic curve) of the beam. Consequently, the radius of curvature of the elastic curve of the beam lies on the deformed face of the element. Consider the longitudinal material line *EF* of the element under consideration located at $x_3 = x_3^*$. Its length prior to deformation is denoted by Δx_1, whereas its length subsequent to deformation is

$$E'F' = \Delta x_1 + \Delta\hat{u}_1 = \Delta x_1 + EE' + FF' \tag{16.6}$$

where $\Delta\hat{u}_1$ is the elongation, due to deformation, of the longitudinal material line of the beam of undeformed length Δx_1 located at $x_3 = x_3^*$. Referring to Fig. 16.1d and denoting by $\delta(x_1)$ the distance of the neutral axis ($e_{11} = 0$) from the centroidal axis $x_2(x_3 = 0)$, we have

$$EE' = FF' = \frac{\Delta\theta_2}{2}[x_3 - \delta(x_1)] \tag{16.7}$$

$$\Delta x_1 = [\rho_{13}(x_1) + \delta(x_1)]\,\Delta\theta_2 \approx \rho_{13}\Delta\theta_2 \tag{16.8}$$

In obtaining relation (16.8) we took into account that, in general, the radius of curvature ρ_{13} is large as compared to the depth of the beam. Consequently, to our order of approximation, δ is negligible as compared to ρ_{13}. From relations (16.6) and (16.7), we have

$$\Delta\hat{u}_1 = EE' + FF' = \Delta\theta_2[x_3 - \delta(x_1)] \tag{16.9}$$

Using relations (16.8) and (16.9), the component of strain e_{11} is expressed as

$$e_{11} = \lim_{\Delta x_1 \to 0} \frac{\Delta\hat{u}_1}{\Delta x_1} \approx \frac{x_3}{\rho_{13}(x_1)} - \frac{\delta(x_1)}{\rho_{13}(x_1)} \tag{16.10}$$

This relation is equivalent to (16.5). It is used to determine the stress and displacement fields in beams made from any material. It is always possible to determine the stress and displacement fields in a beam if the stress–strain diagram of the material from which it is made is known, no matter what the shape of this diagram is.

16.2 Initiation of Yielding Moment and Fully Plastic Moment of Beams Made from Isotropic, Linearly Elastic–Ideally Plastic Materials

Consider a prismatic, statically determinate beam made from a homogeneous, isotropic, linearly elastic–ideally plastic material (see Fig. 3.14e). The beam is initially in a reference stress–free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. Subsequently, the beam is subjected to slowly increasing external forces and moments which bend it without twisting it and bring it to a second state of mechanical and thermal equilibrium at the uniform temperature T_o. In order to simplify our presentation we chose the x_2 and x_3 axes to be principal centroidal ($I_{23} = 0$) and we assume that the external actions acting on the beam bend it only about the x_2 axis ($M_3 = 0$) without twisting it ($\theta_1 = 0$). When the values of the external actions acting on the beam are sufficiently small, the components of strain of all its particles are elastic. In this case, as shown in Section 9.1 (see relation 9.6d), we have

$$\delta(x_1) = 0 \tag{16.11}$$

This implies that the neutral axis is a centroidal axis.

Substituting relation (16.11) into (16.10) and using relation (9.26a), we get

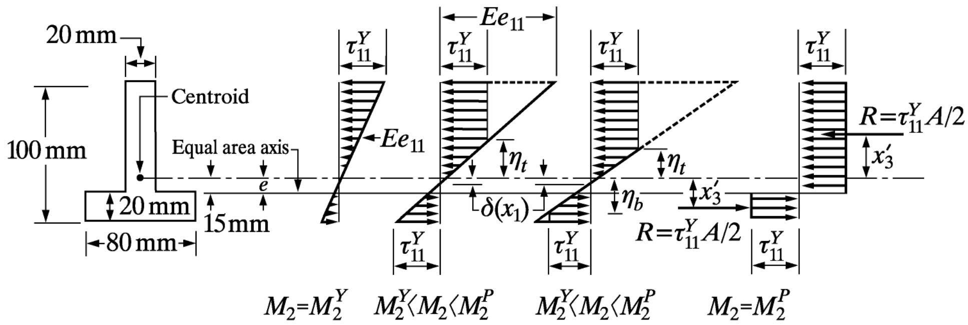
$$e_{11} = \frac{x_3}{\rho_{13}} = -x_3\frac{d^2u_3}{dx_1^2} \tag{16.12}$$

Referring to relations (8.44) for the beam under consideration ($\Delta T_c = \delta T_2 = \delta T_3 = 0$), we have

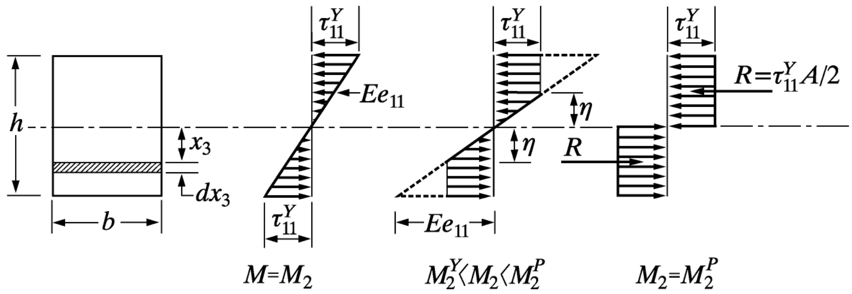
$$\tau_{11} = E\, e_{11} \tag{16.13}$$

Substituting relation (16.12) into the above and using (9.29a), we obtain

$$\tau_{11} = \frac{Ex_3}{\rho_{13}} = -Ex_3\frac{d^2u_3}{dx_1^2} = \frac{M_2x_3}{I_2} \tag{16.14}$$



(a) Cross section with one axis of symmetry



(b) Cross section with two axes of symmetry

Figure 16.2 Distribution of the normal component of stress on the cross-sections of beams made from an isotropic, linearly elastic–ideally plastic material.

In general, the internal moment M_2 acting on the cross sections of the beam is a function of x_1. Without loss of generality we assume that the internal moment assumes its maximum value at one cross section of the beam which we call the *critical cross section*. Thus, for a certain value of the external actions to which the beam is subjected, the moment acting on its critical cross section reaches the value $M_2 = M_2^Y$ which induces on the top and/or the bottom particles of the critical cross section a normal component of stress equal to the yield stress τ_{11}^Y in uniaxial tension or compression of the material from which the beam is made. We call the moment M_2^Y the *initiation of yielding* or the *elastic*

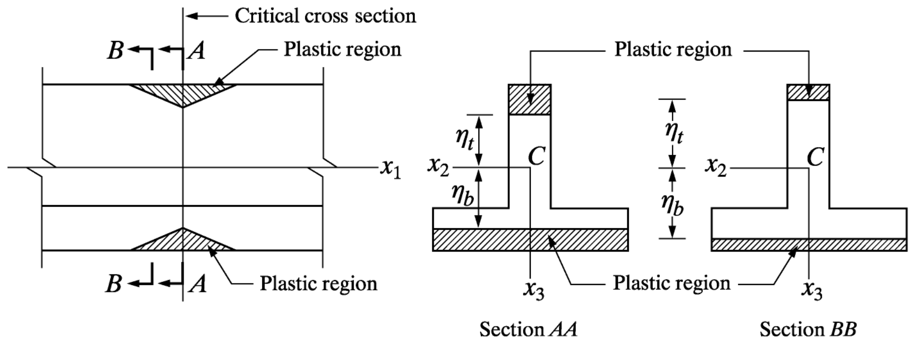


Figure 16.3 Plastic regions in the neighborhood of the critical cross section.

failure moment. For example, the normal component of stress acting on the particles of the top line of the critical cross section of the T-beam of Fig. 16.2a reaches the value of the yield stress first. For beams whose x_2 axis is an axis of symmetry of their cross sections, the normal component of stress acting on the particles of both the top and the bottom lines of their critical cross section reach the value of the yield stress simultaneously (see the beam of rectangular cross section of Fig. 16.2b).

As the value of the applied moments increases above M_2^Y, one or two plastic regions are formed in the neighborhood of the critical cross section (see Fig. 16.3). The normal component of stress acting on the particles of these regions is equal to τ_{11}^Y. The geometry of the plastic regions depends on the geometry of the cross sections of the beam. For a beam whose cross sections are symmetric with respect to the x_2 axis, as the internal moment at the critical cross section assumes values greater than M_2^Y, there exist two equal plastic regions and between them an elastic core. The normal component of stress τ_{11}^E acting on the particles of the elastic core varies linearly with x_3. That is, if we specify the elastoplastic boundary by $x_3 = \pm\eta$ (see Fig. 16.2b), we have

$$\tau_{11}^E = \frac{\tau_{11}^Y x_3}{\eta} \tag{16.15}$$

When the external actions reach values which produce at all particles of the critical cross section of the beam a normal component of stress τ_{11}, which is equal to the yield stress τ_{11}^Y, the moment acting on the critical cross section reaches its maximum value. This moment is referred to as the *fully plastic moment* and we denote it by M_2^P. The ratio of the fully plastic moment to the initiation of yielding or the elastic failure moment M_2^Y depends only on the geometry of cross sections of the beam. It is called its *shape factor*, and we denote it by *s.f.* That is,

$$s.f. = \frac{M_2^P}{M_2^Y} \tag{16.16}$$

As soon as the external actions acting on a statically determinate beam reach values which produce the fully plastic moment at its critical cross section the beam deforms to failure, while the external actions remain constant. That is, the beam rotates about its critical cross section as if there was a hinge at this cross section. We say that a *plastic* hinge has been formed at the critical cross section of the beam. Moreover, we call this type of failure *plastic collapse* and the corresponding actions acting on the beam its *collapse load*. Statically indeterminate beams can usually carry bigger actions than those producing a fully plastic moment at one of their cross sections (see Section 16.8).

Inasmuch as the resultant compression on any cross section of the beam must be equal to the resultant tension, it is obvious that at the instant the fully plastic moment is reached at a cross section, the neutral axis ($e_{11} = 0$) of this cross section is a line dividing the cross section into two equal areas. Thus, as the moment acting on any cross section increases from M_2^Y to M_2^P its neutral axis shifts from the centroidal axis x_2 to the equal area axis. If the centroidal axis x_2 happens to coincide with the equal area axis, as in the case of cross sections having the x_2 axis as an axis of symmetry, the neutral axis does not shift as the moment increases above M_2^Y. The fully plastic moment of any cross section may be computed as

$$M_2^P = \frac{\tau_{11}^Y A}{2}\left(|\bar{x}_3^t| + |\bar{x}_3^b|\right) \tag{16.17}$$

where A is the area of the cross section and $\bar{x}_3^t$ and $\bar{x}_1^b$ are the distances from the centroidal axis of the cross section of the beam to the centroids of the portions of the cross section which are located above and below the equal area axis, respectively. Thus, the fully plastic moment depends only on the geometry of the cross section of the beam and the value of the yield stress of the material from which the beam is made.

In what follows we present two examples which illustrate the computation of the fully plastic moment and the shape factor of beams.

Example 1 Compute the fully plastic moment and the shape factor of a beam of rectangular cross sections of width b and depth d.

Solution Referring to relation (16.14), the moment which produces yielding at the top and bottom particles of the critical cross section of the beam under consideration may be computed as

$$M_2^Y = \frac{2\tau_{11}^Y I_2}{d} = \frac{\tau_{11}^Y b\ d^2}{6} \tag{a}$$

Referring to relation (16.17) and noting that $\bar{x}_3^t = \bar{x}_3^b = d/4$, the fully plastic moment is given by

$$M_2^P = \frac{\tau_{11}^Y b d^2}{4} \tag{b}$$

Therefore, the shape factor is

$$s.f. = \frac{M_2^P}{M_2^Y} = 1.5 \tag{c}$$

Example 2 Compute the fully plastic moment and the shape factor for the beams whose cross section is shown in Fig. a.

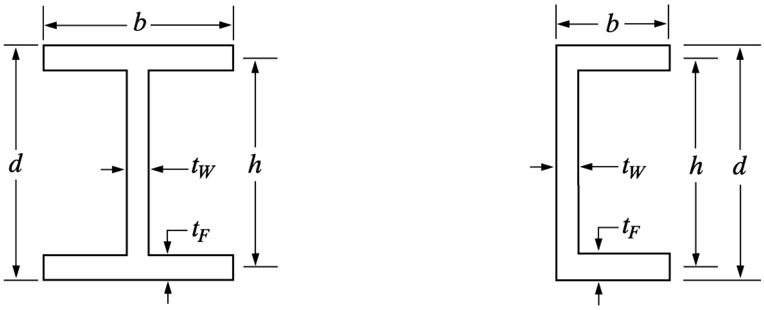


Figure a Geometry of the cross section of the beams.

Solution Referring to relation (16.14) the moment which produces yielding at the top and bottom particles of the critical cross section of the beams of Fig. a is given by

$$M_2^Y = \frac{\tau_{11}^Y}{d}\left[\frac{1}{3}bt_F^3 + bt_F(d - t_F)^2 + \frac{t_W}{6}(d - 2t_F)^3\right] \tag{a}$$

Referring to relation (16.17), the fully plastic moment of the beams of Fig. a is given by

$$M_2^P = \left[bt_F(d - t_F) + \frac{t_W}{4}(d - 2t_F)^2\right]\tau_{11}^Y \tag{b}$$

Taking as an illustration $t_F = 5$ mm, $t_W = 4$ mm, $d = 80$ mm, $b = 40$ mm, we obtain

$$M_2^Y = 16{,}942\ \tau_{11}^Y \qquad\qquad M_2^P = 19{,}900\ \tau_{11}^Y$$

Therefore, the shape factor for this beam is equal to

$$s.f. = \frac{M_2^P}{M_2^Y} = \frac{19{,}900}{16{,}942} = 1.175 \tag{c}$$

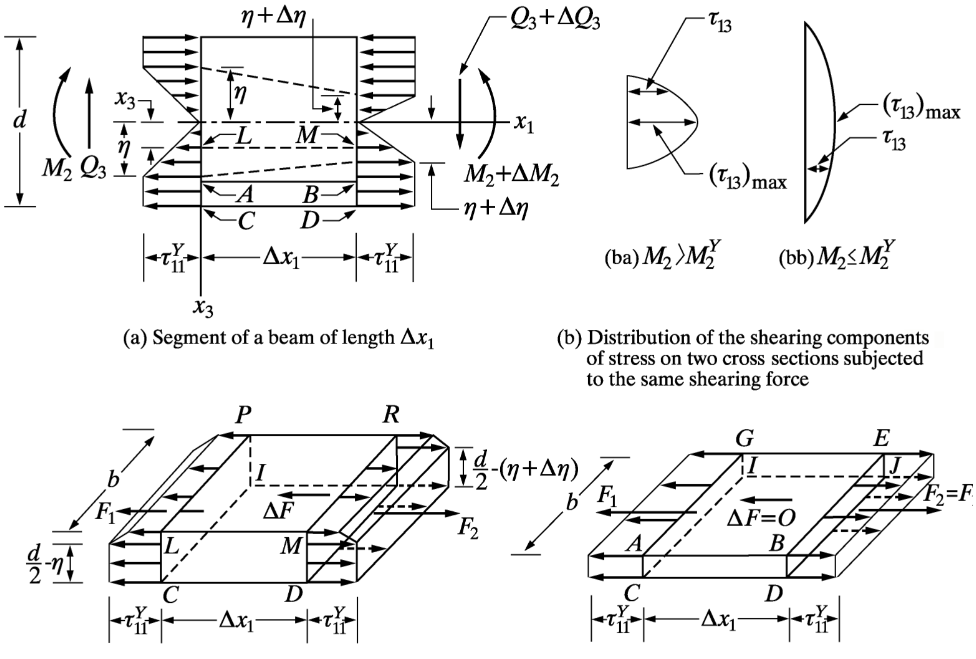
Thus, for an I-beam or a channel, the difference between M_2^P and M_2^Y is small. From physical intuition, we may deduce that as the depth of the beam increases M_2^Y approaches M_2^P.

16.3 Distribution of the Shearing Component of Stress Acting on the Cross Sections of Beams Where $M_2^Y < M_2 < M_2^P$

Consider a beam of rectangular cross section supported in some fashion and subjected to external actions which bend it about the x_2 principal centroidal axis. In Fig. 16.4a we show a segment of this beam of length Δx_1 cut from the part of the beam where the moment acting on its cross sections is bigger than M_2^Y . In Fig. 16.4d we show the free–body diagram of a piece *ABCDEG* cut from the segment of Fig. 16.4a by a plane parallel to the x_1x_2 plane at $x_3 > \eta$. It can be seen that the resultants of the normal component of stress acting on the surfaces *CAG* and *DBE* of this piece are equal. Consequently, a shearing force ΔF is not required on the surface *ABEG* in order to keep the piece under consideration in equilibrium. Thus, the shearing component of stress τ_{13} is zero at the particles of the beam at which plastic components of strain have been produced (see Fig. 16.4b).

In what follows we compute the distribution of the shearing component of stress τ_{13} acting on the particles of the elastic core ($x_3 < \eta$) of a cross section of a beam of rectangular cross section subjected to a moment $M_2(M_2^Y < M_2 < M_2^P)$. For this purpose we show in Fig. 16.4c the free–body diagram of piece *CDMLPR* cut from the segment of the beam shown in Fig. 16.4a by a plane parrallel to the x_1x_2 plane at $x_3 < \eta + \Delta\eta$. Referring to this figure, we note that in general the resultant F_1 of the distribution of the normal component of stress acting on the particles of surface *CLP* is not equal to the resultant F_2 of the distribution of the normal component of stress acting on the particles of surface *DMR*. Consequently, a shearing force ΔF is required on the surface *MLPR* equal to

$$\Delta F = F_2 - F_1 \tag{16.18}$$



(a) Segment of a beam of length Δx_1

(b) Distribution of the shearing components of stress on two cross sections subjected to the same shearing force

(c) Part of the segment of the beam of Fig. 15.4a

(d) Part of the segment of the beam of Fig. 15.4a

Figure 16.4 Distribution of shearing stress in a beam of rectangular cross sections made from an isotropic, linearly elastic–ideally plastic material.

For a beam of rectangular cross section of width b and depth d, we have

$$F_1 = b\,\tau_{11}^{Y}\left(\frac{d}{2} - \eta\right) + \frac{b\eta\tau_{11}^{Y}}{2} - \frac{b\tau_{11}^{Y}x_3^2}{2\eta}$$
$$F_2 = b\,\tau_{11}^{Y}\left(\frac{d}{2} - \eta - \Delta\eta\right) + \frac{b\tau_{11}^{Y}}{2}(\eta + \Delta\eta) - \frac{b\tau_{11}^{Y}x_3^2}{2(\eta + \Delta\eta)} \tag{16.19}$$

Substituting relations (16.19) into (16.18), we get

$$\Delta F = -\frac{b\tau_{11}^{Y}\Delta\eta}{2}\left[1 - \frac{x_3^2}{\eta(\eta + \Delta\eta)}\right] \tag{16.20}$$

Consequently, the shearing component of stress τ_{13} acting on the particles of the elastic core of a cross section of a beam of rectangular cross sections subjected to a moment M_2 $(M_2^Y < M_2 < M_2^P)$ is equal to

$$\tau_{13} = \lim_{\Delta x_1 \to 0} \frac{\Delta F}{b\Delta x_1} = -\frac{\tau_{11}^{Y}}{2}\frac{d\eta}{dx_1}\left[1 - \left(\frac{x_3}{\eta}\right)^2\right] \tag{16.21}$$

From relation (16.21) we see that the maximum value of the shearing component of stress τ_{13} occurs at $x_3 = 0$ and it is equal to

$$(\tau_{13})_{\max} = -\frac{\tau_{11}^{Y}}{2}\frac{d\eta}{dx_1} \tag{16.22}$$

It can be shown that $d\eta/dx_1$ and, consequently, the maximum shearing stress $(\tau_{13})_{\max}$ become very large as the moment M_2 acting on the critical cross section approaches M_2^P.

On the basis of the foregoing presentation, we may conclude that, when a beam is subjected to increasing transverse forces, before the upper and the lower plastic regions at the critical cross section meet, the maximum shearing stress at the centroidal axis reaches the value of the yield stress in shear and a new plastic region is formed around the centroidal axis. The establishment of the boundary of this plastic region is complicated by the fact that the value of the yield stress in shear† is affected by the presence of the normal component of stress τ_{11}. As the load increases, the three plastic regions increase until, for a certain value of the load, the critical cross section becomes fully plastic. It is evident that the value of the fully plastic moment at the critical cross section depends not only upon the geometry of the cross section and upon the value of the yield stress in tension, but also upon the value of the shearing force acting on the cross section, and upon the value of the yield stress in pure shear. If the existence of the plastic region due

† When a particle of a body made from an isotropic linearly elastic–ideally plastic material is subjected to shearing and to normal components of stress simultaneously, it yields at values of the normal component stress which are smaller than the yield stress in uniaxial tension, or at values of the shearing component of stress which are smaller than the values of the yield stress in pure shear (see Chapter 4).

to yielding in shear is taken into account, the computation of the collapse load of a beam will become cumbersome. Moreover, the size of the plastic region due to yielding in shear is rather small. For this reason, *the effect of this plastic region which is formed around the centroidal axis of a beam on its collapse load and on its deflection is disregarded.*

16.4 Location of the Elastoplastic Boundaries — Moment–Curvature Relation

For any cross section of a beam at which the moment M_2 is greater than M_2^Y and less than M_2^P, the position of the neutral axis and the dimensions of the elastic core may be obtained from the requirement that the distribution of the normal component of stress on the cross sections of the beam is such that its resultant force F_1 vanishes while its resultant moment is equal to M_2. That is,

$$F_1 = \iint_A \tau_{11} dA = 0$$
$$M_2 = \iint_A \tau_{11} x_3 dA \tag{16.23}$$

Relations (16.23) involve the functions $\eta_t(x_1)$, $\eta_b(x_1)$ and $\delta(x_1)$(see Fig. 16.3) which specify the geometry of the plastic regions and the position of the neutral axis. The functions $\eta_t(x_1)$ and $\eta_b(x_1)$ are, respectively, the distances between the centroidal axis x_2 and the elastoplastic boundary of the top and bottom plastic regions of a beam. $\delta(x_1)$ is the distance between the x_2 axis and the neutral axis. In what follows we write relations (16.23) in terms of the functions $\eta_t(x_1)$, $\eta_b(x_1)$ and $\delta(x_1)$. In the elastic region τ_{11} varies linearly with x_3 and vanishes when $x_3 = \delta$. For beams made from an isotropic, linearly elastic–ideally plastic material subjected to bending about their x_2 axis without twisting in an environment of constant temperature, the normal component of stress in the elastic region is obtained by substituting relation (16.10) in the first of the stress–strain relations (8.44). That is,

$$\tau_{11} = E\, e_{11} = \frac{E}{\rho_{13}(x_1)}[x_3 - \delta(x_1)] \tag{16.24}$$

When two plastic regions exist on a cross section of a beam from relation (16.24) we see that the following relations must be valid on the elastoplastic boundaries ($x_3 = -n_t, x_3 = n_b$):

$$\tau_{11}^Y = \frac{E}{\rho_{13}}(\eta_b - \delta) \qquad -\tau_{11}^Y = \frac{E}{\rho_{13}}(-\eta_t - \delta) \tag{16.25}$$

Consequently,

$$\eta_t + \delta = \eta_b - \delta \tag{16.26}$$

or

$$\delta = \frac{1}{2}(\eta_b - \eta_t) \tag{16.27}$$

Moreover, substituting relation (16.27) into the first relations (16.25), we have

$$\frac{E}{\rho_{13}} = \frac{2\tau_{11}^Y}{\eta_t + \eta_b} \tag{16.28}$$

Relation (16.27) indicates that the neutral axis is located at the middle of the elastic core. The normal component of stress in the plastic region can be expressed as

$$\tau_{11} = \tau_{11}^Y \frac{x_3}{|x_3|} \tag{16.29}$$

Substituting relations (16.24) and (16.29) into (16.23), we get

$$\frac{E}{\rho_{13}} \iint_{A_E} x_3 \, dA - \frac{E A_E \delta}{\rho_{13}} + \tau_{11}^Y (A_{pb} - A_{pt}) = 0 \tag{16.30}$$

$$M_2 = \frac{E}{\rho_{13}} \iint_{A_E} x_3^2 \, dA - \frac{E \delta}{\rho_{13}} \iint_{A_E} x_3 \, dA + \tau_{11}^Y \left(\iint_{A_{pb}} x_3 \, dA - \iint_{A_{pt}} x_3 \, dA \right) \tag{16.31}$$

where

A_E = area of the portion of the cross section whose particles deform elastically.

A_{pt} and A_{pb} = area of top and bottom plastic regions, respectively, of the cross section under consideration.

For given geometry of the cross sections of a beam the areas A_{pt} ,A_{pb} and A_E can be expressed in terms of η_t, η_b and δ. Using relations (16.27) and (16.28) we can eliminate δ and ρ_{13} from relations (16.30) and (16.31) to obtain a set of two equations for η_t and η_b. For beams whose cross sections are not symmetric with respect to the x_2 axis the solution of these equations is very cumbersome. It requires a trial and error approach. However, as it is discussed in the next section, in practice it is rarely required to compute η_t, η_b, δ, or $1/\rho_{13}$ for given values of M_2. Nevertheless, once η_t and η_b are established for a given value of $M_2 (M_2^Y < M_2 < M_2^P)$, they can be substituted into relations (16.27) and (16.28) to give the position of the neutral axis (δ) and the curvature ($1/\rho_{13}$) for the given value of M_2. In Fig. 16.5 we plot the moment–curvature relation. For values of the moment less than the elastic design moment (M_2^Y) the moment–curvature relation is given by relation (16.14) and it is linear. For values of the moment greater than M_2^Y the moment–curvature relation is obtained from relations (16.31) and it is non-linear. As the moment approaches its fully plastic value, the curvature becomes very large. The moment–curvature curve approaches asymptotically the fully plastic moment (see Fig.

16.5). As the moment acting on a cross section of a beam is reduced after reaching a value greater than M_2^Y the moment–curvature curve is a straight line parallel to that during loading with $M_2 < M_2^Y$.

For beams whose cross sections are symmetric with respect to the x_2 axis we have

$$\eta_t = \eta_b = \eta \tag{16.32a}$$

$$A_{pt} = A_{pb} = A_p \tag{16.32b}$$

Moreover, the x_2 axis is centroidal axis of the area A_E and consequently

$$\iint_{A_E} x_3 \, dA = 0 \tag{16.33}$$

Furthermore, relation (16.28) reduces to

$$\frac{E}{\rho_{13}} = \frac{\tau_{11}^Y}{\eta} \tag{16.34}$$

Taking relations (16.32) and (16.33) into account, relations (16.30) and (16.31) reduce to

$$\delta = 0 \tag{16.35}$$

$$\frac{E}{\rho_{13}}\iint_{A_E} x_3^2 \, dA + 2\tau_{11}^Y \iint_{A_p} x_3 \, dA = M_2 \tag{16.36}$$

Relation (16.35) indicates that for beams whose cross sections are symmetric with respect to the x_2 axis the neutral axis of its cross sections does not shift as the moment acting on it exceeds M_2^Y but is less than M_2^P. Substituting relation (16.34) into (16.36), we obtain

$$\frac{\tau_{11}^Y}{\eta}\iint_{A_E} x_3^2 \, dA + 2\tau_{11}^Y \iint_{A_p} x_3 \, dA = M_2 \tag{16.37}$$

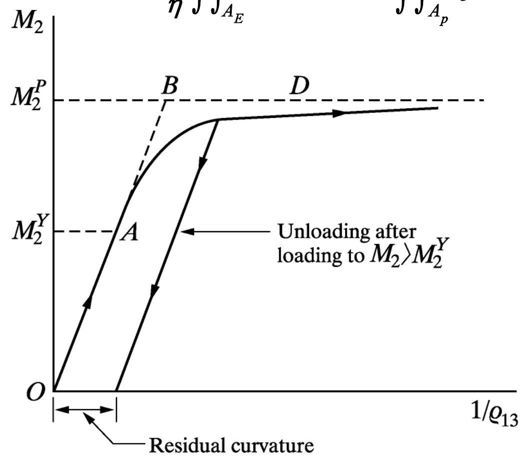


Figure 16.5 Moment–curvature relation.

Relation (16.37) can be used to compute η in terms of the moment M_2 for values of the moment greater than M_2^Y but less than M_2^P. Once the distance $\eta(x_1)$ is computed, the distribution of the normal component of stress on the cross sections of the beam which are subjected to a moment M_2 $(M_2^Y < M_2 < M_2^P)$ can be established using relation (16.15). Moreover, the radius of curvature of the beam can be established using relation (16.34). For example, for a beam of rectangular cross section, referring to Fig. 16.2b, relation (16.37) becomes

$$M_2(x_1) = \frac{2b\tau_{11}^Y}{\eta}\int_0^{\eta} x_3^2\,dx_3 + 2b\tau_{11}^Y\int_{\eta}^{h/2} x_3\,dx_3 = b\,\tau_{11}^Y\left[\left(\frac{d}{2}\right)^2 - \frac{\eta(x_1)}{3}\right] \tag{16.38}$$

Solving relation (16.38) for η, we get

$$\eta\,(x_1) = \sqrt{3}\left[\left(\frac{d}{2}\right)^2 - \frac{M_2(x_1)}{b\,\tau_{11}^Y}\right]^{1/2} \tag{16.39}$$

Referring to relation (b) of Example 1 of Section 16.2, the fully plastic moment of a beam of rectangular cross section is

$$M_2^P = \frac{\tau_{11}^Y bd^2}{4} \tag{16.40}$$

Eliminating τ_{11}^Y from relation (16.39) by using relation (16.40), we obtain

$$\eta(x_1) = \frac{d\sqrt{3}}{2}\left[1 - \frac{M_2(x_1)}{M_2^P}\right]^{1/2} \tag{16.41}$$

Substituting relation (16.41) into (16.34) we get the following expression for the curvature of the portions of a prismatic beam of rectangular cross sections at which the moment is greater than M_2^Y but less than M_2^P.

$$\frac{1}{\rho_{13}} = \frac{2\,\tau_{11}^Y}{Ed\sqrt{3}\left[1 - \dfrac{M_2(x_1)}{M_2^P}\right]^{1/2}} \tag{16.42}$$

In what follows we present an example of establishing the geometry of the plastic regions of beams of rectangular cross sections.

Example 3 Consider a simply supported beam of length L and rectangular cross sections of width b and depth d. The beam is made from an isotropic linearly elastic–ideally plastic material of yield stress τ_{11}^Y. The beam is subjected to uniformly distributed forces p_3 along its length. Establish the geometry of the plastic regions of the beam when the force p_3 is greater than the elastic failure load p_3^Y but less than the plastic collapse load

p_3^C. Determine the distribution of the normal component of stress acting to the cross sections of the beam, when subjected to the uniform force p_3 where $p_3^Y < p_3 < p_3^C$. Moreover, determine the residual stress distribution when the beam is unloaded.

Solution The moment at any cross section of the beam is equal to

$$M_2(x_1) = \frac{p_3 x_1 (L - x_1)}{2} \tag{a}$$

When p_3 is equal to the elastic failure load p_3^Y we have

$$\tau_{11}^Y = \frac{M_2^Y d}{2I_2} = \frac{6M_2^Y}{bd^2} = \frac{3p_3^Y L^2}{4bd^2}$$

or

$$p_3^Y = \frac{4bd^2 \tau_{11}^Y}{3L^2} \tag{b}$$

Substituting relation (a) into (16.41) for the region of the beam where $M_2^Y < M_2 < M_2^P$, we have

$$\eta(x_1) = \frac{d\sqrt{3}}{2}\left[1 - \frac{p_3 x_1 (L - x_1)}{2M_2^P}\right]^{\frac{1}{2}} \tag{c}$$

From relation (c) we see that the elastoplastic boundaries of the beam under consideration are hyperbolas. As the load increases the distance $\eta(x_1)$ decreases until it becomes equal to zero at the middle cross section ($x_1 = L/2$) of the beam. For this value of the load the moment acting on the middle cross section of the beam is equal to the fully plastic moment and the beam deforms to failure, while the load remains constant. That is, the beam behaves as if a hinge has developed at its middle cross section (see Fig. a). We say that the beam *collapses plastically* and we call the corresponding load p_3^C its *collapse load*. Substituting relation (16.40) into (a), for $x_1 = L/2$, we obtain

$$p_3^C = \frac{2\tau_{11}^Y bd^2}{L^2} \tag{d}$$

The elastoplastic regions extend from $x_1 = a$ to $x_1 = L - a$ where the distance a is obtained from relation (c) by setting $\eta = d/2$ and $x_1 = a$. That is,

$$a = \frac{L}{2}\left[1 \pm \sqrt{1 - \frac{16M_2^P}{3p_3 L^2}}\right] \tag{e}$$

Substituting relations (16.40) and (d) into (e) we get the following value for the distance a at plastic collapse:

$$a^C = \frac{L}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right) = 0.2113\,L \tag{f}$$

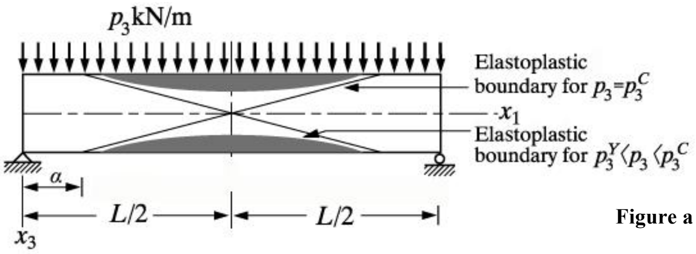


Figure a Plastic regions of the beam.

The normal component of stress acting on the cross sections of the beam where $M_2^Y < M_2 < M_2^P$ may be obtained by referring to relations (16.15) and (16.29). That is,

$$\tau_{11} = \begin{cases} \dfrac{\tau_{11}^Y x_3}{\eta} = \dfrac{\tau_{11}^Y x_3}{\dfrac{d\sqrt{3}}{2}\left[1 - \dfrac{M_2(x_1)}{M_2^P}\right]^{\frac{1}{2}}} & \text{for } 0 \le |x_3| \le |\eta| \\[2ex] \dfrac{\tau_{11}^Y x_3}{|x_3|} & \text{for } |\eta| \le |x_3| \le \left|\dfrac{d}{2}\right| \end{cases} \tag{g}$$

If, after subjecting the beam to a load greater than the elastic failure load, but less than the collapse load, the beam is unloaded, it will deform elastically. The stress distribution at any cross section during unloading may be established by subtracting from the stress distribution which existed prior to unloading, the elastic stress distribution caused by a moment equal in magnitude to the reduction of the applied moment. The residual stress distribution, therefore, which remains on a cross section of a beam when the moment acting on it is reduced to zero after reaching a value $M_2^*(M_2^Y < M_2^* < M_2^P)$, is given by

$$\tau_{11} = \begin{cases} \dfrac{\tau_{11}^Y x_3}{\dfrac{d\sqrt{3}}{2}\left[1 - \dfrac{M_2^*}{M_2^P}\right]^{\frac{1}{2}}} - \dfrac{12M_2^* x_3}{bd^3} & \text{for } 0 \le |x_3| \le |\eta| \\[2ex] \dfrac{\tau_{11}^Y \bar{x}_3}{|\bar{x}_3|} - \dfrac{12M_2^* x_3}{bd^3} & \text{for } |\eta| \le |x_3| \le \left|\dfrac{d}{2}\right| \end{cases} \tag{h}$$

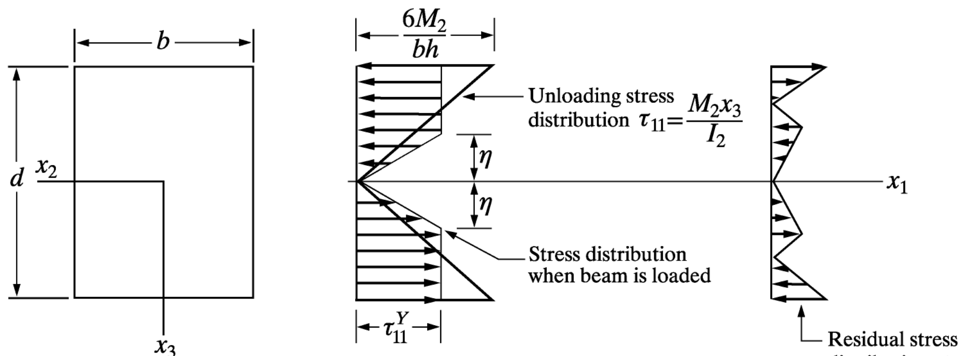


Figure b Residual stress distribution.

The residual stress distribution (h) is plotted in Fig. b.

Referring to relation (16.22), the maximum value of the shearing component of stress acting on the cross sections of the beam is equal to

$$(\tau_{13})_{max} = -\frac{\tau_{11}^{Y}}{2}\frac{d\eta}{dx_1} \tag{i}$$

Substituting relation (c) into (i), we obtain

$$(\tau_{13})_{max} = \frac{\sqrt{3}p_3(L - 2x_1)\,\tau_{11}^{Y}}{16M_2^P\left[1 - \dfrac{p_3 x_1(L - x_1)}{2M_2^P}\right]^{\frac{1}{2}}} \tag{j}$$

Using relation (c), the surface A_E of the elastic core of the cross sections of the beam may be expressed as

$$A_E = 2b\eta = \sqrt{3}bd\left[1 - \frac{p_3 x_1(L - x_1)}{2M_2^P}\right]^{\frac{1}{2}} \tag{k}$$

Substituting relation (k) into (j), using relations (16.40) and noting that $Q_3 = P_3(L-2x_1)/2$ the shearing force acting on the cross sections on the beam is equal to

$$(\tau_{13})_{max} = \frac{3Q_3}{2A_E} \tag{l}$$

where Q_3 is the shearing force acting on the cross section. Notice that the relation (l) is valid for every cross section of the beam. For cross sections subjected to $M_2 < M_2^Y$ the area A_E is equal to that of the cross section.

16.5 Computation of the Deflection of Beams Made from Isotropic, Linearly Elastic–Ideally Plastic Materials

When the loads acting on a statically determinate beam do not produce plastic deformation at any of its particles $[M_2(x_1) < M_2^Y]$, its deflection may be established by computing the internal moment as a function of the axial coordinate substituting it into the moment–curvature relation (9.32a) and integrating the resulting differential equation. When the loads acting on a statically determinate beam produce plastic deformation of the particles of a region of the beam, its moment curvature relation for the segments of the beam whose cross sections do not have particles undergoing plastic deformation may be obtained by substituting relation (9.26a) into (9.32a). Moreover, the moment–curvature relation for the regions of the beam whose cross sections have some particles which have undergone plastic deformation, can be established from relation (16.28). Thus,

$$\frac{1}{\rho_{13}} = -\frac{d^2u_3}{dx_1^2} = \begin{cases} \dfrac{M_2}{EI_2} & \text{for cross sections where } M_2 \leq M_2^Y \\ \dfrac{2\tau_{11}^Y x_3}{E(\eta_t + \eta_b)} & \text{for cross sections where } M_2^Y \leq M_2 \leq M_2^P \end{cases} \tag{16.43}$$

$\eta_t(x_1)$ and $\eta_b(x_1)$ are the distances of the top and bottom elastoplastic boundary, respectively, from the x_2 axis of the beam. They are related to the bending moment $M_2(x_1)$ by relation (16.31). For example, for a beam of rectangular cross section of width b and depth d, taking into account that $\eta_t = \eta_b = \eta$ and referring to relations (16.40) to (16.42), we have

$$\frac{1}{\rho_{13}} = -\frac{d^2u_3}{dx_1^2} = \begin{cases} \dfrac{12M_2}{Ebd^3} & \text{for cross sections where } M_2 \leq M_2^Y \quad \text{(a)} \\ \dfrac{2\tau_{11}^Y}{Eh\sqrt{3}\left[1 - \dfrac{4M_2(x_1)}{bd^2\tau_{11}^Y}\right]} & \text{for cross sections where } M_2^Y \leq M_2 \leq M_2^P \quad \text{(b)} \end{cases} \tag{16.44}$$

Relation (16.44a) may be integrated to give an expression for the deflection $u_3(x_1)$ for each portion of the beam whose cross sections do not have particles which underwent plastic deformation. Each of these expressions involves two unknown constants. Moreover, relation (16.44b) can be integrated to give an expression for the component of translation $u_3(x_1)$ for each portion of the beam whose cross sections have particles which underwent plastic deformation. Each of these expressions involves two unknown constants. The constants are evaluated by requiring that

1. The expressions for the deflection of the end segments of the beam satisfy its essential boundary conditions.
2. The expressions for the deflection of the various segments of the beam give continuous deflection $u_3(x_1)$ and rotation $\theta_2(x_1)$.
3. The actions obtained from the expressions for the deflection of the various segments of the beam are in equilibrium.

Theoretically, this procedure is straightforward. In practice, however, it is very cumbersome and it is avoided by assuming that the moment–curvature relation of a beam is linear up to the value of $M_2 = M_2^P$. That is, the moment–curvature relation in Fig. 16.5 (see p. 774) is approximated by the straight-line $OABD$ and the deflection $u_3(x_1)$ of the beam is computed as if the beam was made from an elastic material until a plastic hinge is formed. This assumption is justified by referring to Fig.16.5 and observing that the curvature becomes large only when the value of the moment M_2 is very close to M_2^P. Thus, large curvatures occur for a very small segment of the beam. Consequently, for beams subjected to loads whose values are not very close to the collapse load the approximation of the moment–curvature relation by a straight line introduces only a small error. This error also depends on the geometry of the cross sections of the beam. For beams having the cross section of Fig. a of Example 2 of Section 16.2, where b and d are

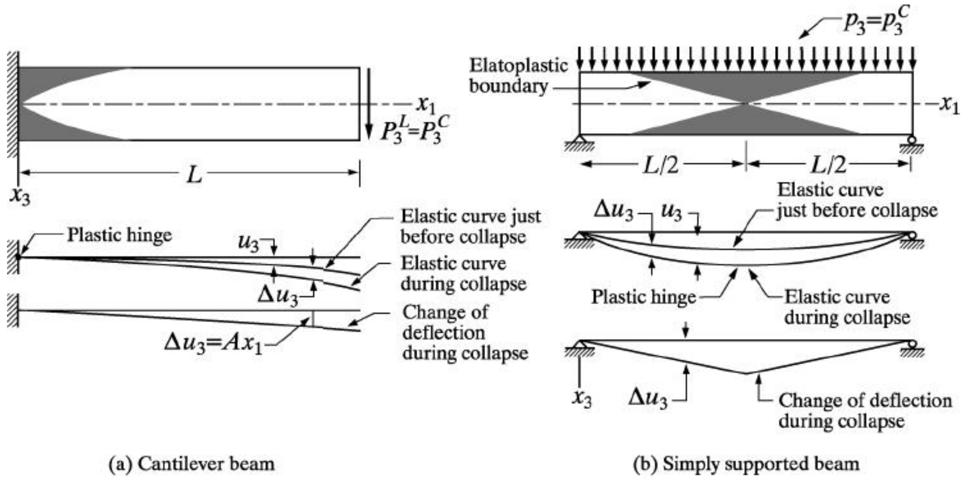


Figure 16.6 Beams subjected to collapse forces.

large while t_W and t_F are small, the elastic failure moment M_2^Y does not differ much from the fully plastic moment M_2^P and, consequently, the error due to the above approximation is small.

When the moment at the critical cross section of a statically determinate beam reaches its fully plastic value, the beam deforms to failure while the load remains constant. This implies that the moment $M_2(x_1)$ of all the cross sections of the beam remains constant during plastic collapse ($\Delta M_2 = 0$). Hence, referring to relation (16.44) we see that during plastic collapse the right side of relation (16.44) does not change and, consequently, the curvature $\rho_{13}(x_1)$ of all the cross sections of the beam remains constant except that of the critical cross section which increases without limit. Moreover, referring to relation (16.44b), the additional displacement Δu_3 of the beam must satisfy the following relation:

$$\frac{d^2 \Delta u_3}{dx_1^2} = \frac{d^2(u_3^* + \Delta u_3)}{dx_1^2} - \frac{d^2 u_3^*}{dx_1^2} = 0 \tag{16.45}$$

where u_3^* is the deflection of the beam at the instance plastic collapse begins. Consequently, the additional deflections Δu_3 of the beam is a linear function of x_1. That is, as shown in Fig. 16.6, the two parts of the beam, that to the left and that to the right of the plastic hinge, rotate as rigid bodies about the plastic hinge formed at the critical cross section.

16.6 Effect of Stress Concentrations on the Design of Line Members

Consider a member made from an ideally plastic material, having an abruptly or suddenly changing cross section and assume that it is subjected to slowly increasing equal and opposite axial centroidal forces at its ends. The distribution of the normal component of stress τ_{11} on the cross sections of the member which are located close to the abruptly or suddenly changing cross section is not uniform. For a certain value of the applied forces $P_1 = P_1^Y$ the normal component of stress acting on one or more particles of the critical cross section of the member will reach its yield value τ_{11}^Y. As the forces increase

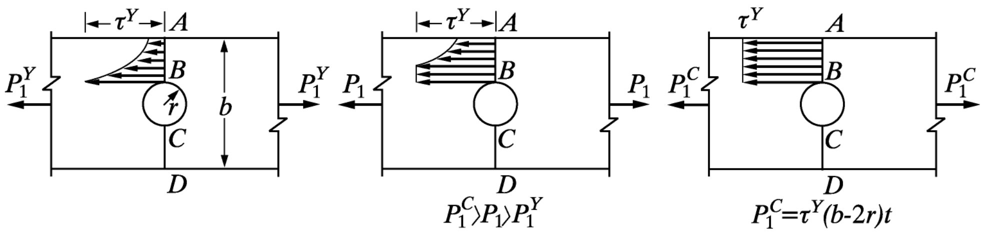


Figure 16.7 Stress distribution in a flat member of constant thickness with a circular hole.

above P_1^Y, in the neighborhood of the abruptly or suddenly changing cross section of the member, plastic regions will be formed. The normal component of stress acting on every particle of these regions will be equal to the yield stress τ_{11}^Y. Any additional increase of the applied forces increases the normal component of stress acting on the particles of the parts of the member which have not been stressed up to the elastic limit. In Fig. 16.7 the distribution of the normal component of stress on the critical cross section of a plate with a hole is shown for various values of the applied forces. It is assumed that the plate is made from an isotropic, linearly elastic–ideally plastic material. When the value of the applied forces is small, the components of strain of all the particles of the plate are elastic. For such value of the applied forces as it is shown in Example 3 of Section 7.7 the maximum normal component of stress $(\tau_{11})_{max}$ acts on the particles located at points B and C and it is equal to three times its average value (P_1/A). For a certain value P_1^Y of the applied forces, as shown in Fig. 16.7a, the value of the normal component of stress τ_{11} acting on particles B and C is equal to the yield stress τ_{11}^Y in uniaxial tension of the material from which the plate is made. As the value of the applied forces increases, the normal component of stress acting on particles in the neighborhood of points B and C reaches the yield stress τ_{11}^Y (see Fig. 16.7b), until for a certain value P_1^C of the applied forces referred to as the *collapse load*, as shown in Fig. 16.7c, the normal component of stress τ_{11} acting on all particles of the cross section $ABCD$ of the member reaches the value of the yield stress and the member continues to deform to fracture, while the applied forces remain constant. This value of the applied forces is equal to

$$P_1^C = \tau_{11}^Y A \tag{16.46}$$

where A is the net area of the cross section $ABCD$ of the member. It is clear that the effect of the local stress concentration disappears as the external forces approach P_1^C. However, this conclusion is valid only for the ideally plastic model of material behavior. If a member with an abruptly or suddenly changing cross section is made from an ideally brittle material (that is, a material which fractures without yielding) and is subjected to gradually increasing axial centroidal forces, according to the maximum normal component of stress criterion (see Section 4.6) it fractures when the applied forces reach the value for which the maximum stress (acting at points B and C) is equal to the ultimate stress in uniaxial tension of the material from which the member is made. That is, for such a member, the local stress concentration is an important design parameter.

The foregoing observations are also valid for members subjected to torsional moments and for beams. Thus, the following statements can be made:

1. Plastic deformation tends to eliminate the stress concentrations produced by abrupt or sudden changes of the cross sections of members made from a highly ductile material. Consequently, it is assumed that stress concentrations do not affect the value of the applied loads at failure of line members made from highly ductile materials, except in cases of repeated cycles of loading and unloading. For such loading a member made from a ductile material could fail as if it was made from a brittle material. This phenomenon is known as fatigue (see Section 4.5).
2. Stress concentrations are important design parameters for members of abruptly or suddenly changing cross sections when they are made from brittle materials. It is assumed that members made from highly brittle materials fracture, as soon as the values of the components of stress acting on one or more of their particles satisfy an appropriate brittle fracture criterion for the material from which the member is made, as, for example, the maximum normal component of stress criterion presented in Section 4.6.
3. The effect of stress concentrations on the load at failure of members with suddenly or abruptly changing cross sections made from materials which are neither very ductile nor very brittle can only be established experimentally.

On the basis of the foregoing discussion, the effect of stress concentrations is of considerable importance in the design of machines, airplanes and other structures having members made from ductile materials which are exposed to repeated cycles of loading and unloading or having members made from brittle materials.

16.7 Elastic and Plastic Design for Strength of Statically Determinate Structures

Consider a prismatic statically determinate member made from a homogeneous, isotropic, linearly elastic–ideally plastic material, subjected to increasing external axial centroidal forces. For certain values of the external forces, the normal components of stress τ_{11} acting on each particle of one or more cross sections of the member reach simultaneously the yield stress of the material from which the member is made and the member deforms to failure, while the external forces remain constant. We say the member collapses plastically. That is, in this case the collapse load of the structure is equal to its initiation of yielding or elastic design load.

Consider a prismatic statically determinate member of solid or hollow circular cross sections made from a homogeneous, isotropic, linearly elastic–ideally plastic material, subjected to external torsional moments whose magnitude is specified by one parameter only which we call the *load parameter*. We denote the torsional moments acting on these cross sections by M_1^Y. As the load parameter increases the magnitude of the components stress acting on the particles of the member increases. For a certain value of the load parameter, which we call the *initiation of yielding or elastic failure load parameter*, the components of stress acting on the particles of the perimeter of some cross sections of the member reach simultaneously the value of the yield stress in shear of the material from which the member is made. As the load parameter increases above its initiation of yielding value, a plastic region is formed (see Fig. 16.8). The shearing component of stress $\tau_{1\theta}$ acting on every particle of this region is equal to the yield stress in shear $\tau_{1\theta}^Y$. For a certain value of the load parameter, called the *plastic collapse load parameter*, the shearing stress acting on all the particles of one or more cross sections of the member reach the value of the yield stress in pure shear, and the member twists to failure while the external torsional moments remain constant. We say that the member *collapsed*

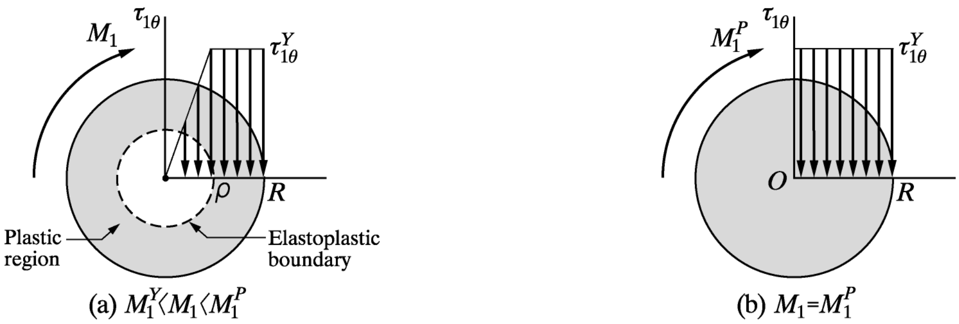


Figure 16.8 Stress distribution on a cross section of a member of circular cross section made from an isotropic, linearly elastic–ideally plastic material subjected to torsional moments.

plastically. If the member is designed on the basis of elastic design, its dimensions are established by requiring that the anticipated maximum value of the load parameter multiplied by the factor of safety is equal to its initiation of yielding or elastic design value. If the structure is designed using plastic design, its dimensions are established by requiring that the anticipated maximum value of the load parameter multiplied by the factor of safety is equal to its plastic collapse value.

Consider a statically determinate prismatic beam made from a homogeneous, isotropic, linearly elastic–ideally plastic material subjected to transverse forces and bending moments inducing bending without twisting. Assume that the magnitute of the loads is specified by one load parameter. For a certain value of the load parameter, called the initiation of yielding or elastic failure load parameter, the normal component of stress τ_{11} acting on one or more particles of a cross section of the beam reaches the value of the yield stress τ_{11}^Y in uniaxial tension or compression of the material from which the beam is made. As the load parameter increases above its initiation of yielding value, it reaches a value, called the plastic collapse load parameter, at which the moment acting on one cross section reaches the value of the fully plastic moment M_2^P and the beam deflects to failure while the external actions remain constant.

If a statically determinate beam is designed using *elastic design*, it is assumed that it fails when the load parameter multiplied by the factor of safety reaches its initiation of yielding or elastic failure value. If a statically determined beam is designed using *plastic design*, it is assumed that it fails when the load parameter multiplied by the factor of safety reaches its plastic collapse value.

In what follows we present an example.

Example 4 Design a simply supported beam to withstand the forces shown in Fig. a. The

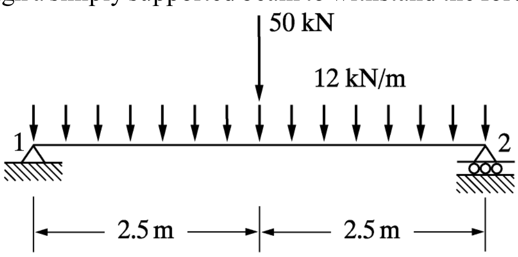


Figure a Maximum anticipated loading on the beam.

beam must have a length of 5 m and it is to be made of steel (yield stress in uniaxial tension and compression 250 MPa). Choose one of the standard wide-flange beams whose cross sectional properties are given in Appendix H so that the factor of safety for elastic failure is not less than 5. Compute the factor of safety corresponding to the chosen cross section for elastic failure and for plastic collapse.

Solution We first determine the maximum moment in the beam. Referring to Fig. b the reactions of the beam are

$$\sum M_2^{(2)} = (12)(5)(2.5) + 50(2.5) - 5R^{(1)} = 0 \quad \text{or} \quad R^{(1)} = 55\,\text{kN} \tag{a}$$

Due to symmetry of the load, we have

$$R^{(2)} = 55\,\text{kN} \tag{b}$$

Referring to Fig. c, the moment in the beam is equal to

$$M_2(x_1) = 55x_1 - \frac{12x_1^2}{2} \qquad 0 \le x_1 \le L/2 \tag{c}$$

The maximum moment occurs at $x_1 = 2.5$ m and it is equal to

$$M_2(2.5) = 100\,\text{kN}\cdot\text{m} \tag{d}$$

The allowable stress on the basis of elastic design is

$$(\tau_{11})_{\text{allowable}} = \frac{\tau_{11}^Y}{F.S.} = \frac{250}{5.0} = 50\,\text{MPa} \tag{e}$$

The maximum normal component of stress acting on the cross section at $x_1 = 2.5$ m is

$$(\tau_{11})_{\max} = \frac{M_2 d}{2I_2} = \frac{M_2}{S} \le 50(10^{-3})\frac{\text{kN}}{\text{mm}^2} \tag{f}$$

where S is the so-called *section modulus* of the beam. Thus, on the basis of elastic design we need a beam with a section modulus equal to or greater than

$$S \ge \frac{M_2}{50(10^{-3})} = \frac{100(10^3)}{50(10^{-3})} = 2{,}000(10^3)\,\text{mm}^3 \tag{g}$$

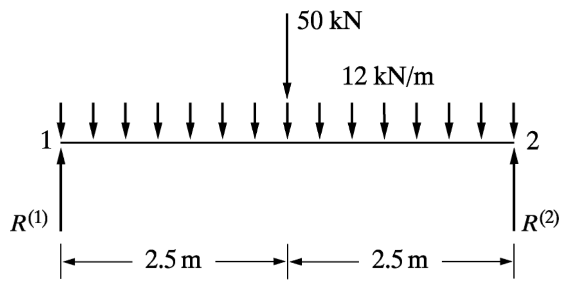


Figure b Free-body diagram of the beam.

Referring to the tables of Appendix H, we see that the lightest beam which meets the above requirement is W 530 x 92 with $S = 2{,}070\ \text{mm}^3$. The dimensions of the cross section of the beam are shown in Fig. d. The maximum stress in the beam is

$$(\tau_{11})_{max} = \frac{100(10^3)}{2{,}070(10^3)} = 48.03(10^{-3})\frac{\text{kN}}{\text{mm}^2}$$

Thus, the factor of safety of the chosen cross section on the basis of elastic design is

$$F.S. = \frac{250}{48.03} = 5.2$$

Referring to Fig. d and to relation (16.17) the plastic collapse moment is

$$M_2^P = \frac{\tau_{11}^Y A}{2}\left(\bar{x}_3^a + \bar{x}_3^b\right) = 2\tau_{11}^Y \frac{A}{2}\bar{x}_3^a$$

$$= 2(250)(10^{-3})\left[209(15.6)(266.5 - 7.8) + (266.5 - 15.6)^2\frac{10.2}{2}\right] = 582\ \text{kN}\cdot\text{m}$$

Thus, the factor of safety of the chosen cross section on the basis of plastic design is

$$F.S. = \frac{582}{100} = 5.82$$

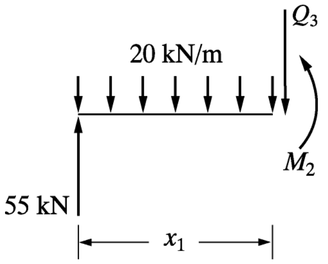


Figure c Free-body diagram of a portion of the beam.

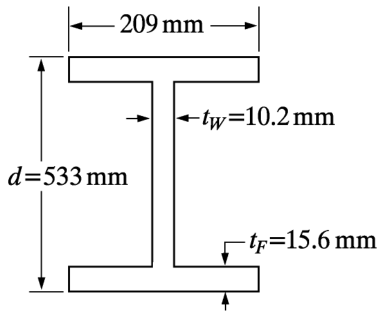


Figure d Geometry of the chosen cross section of the beam.

16.8 Plastic Analysis and Design of Planar Statically Indeterminate Beams and Frames

In Sections 16.2 to 16.5 we investigate the behavior of statically determinate beams made from an isotropic, linearly elastic–ideally plastic material when subjected to loads bigger than those required to initiate yielding of one or more of their particles. Under such loads in the neighborhood of the critical cross section (the cross section subjected to the maximum moment) of the beam plastic regions are formed, which increase as the loads are increased. For certain values of the loads the components of stress acting on every particle of the one half of the critical cross section reach the value of the yield stress in uniaxial tension for the material from which the beam is made, while the stress acting on every particle of the other half of the critical cross section reaches the value of the yield stress in uniaxial compression. We call the moment corresponding to this stress distribution the *fully plastic moment*. When the fully plastic moment acts on its critical cross section a beam behaves as if a hinge has been formed at its critical cross section. Thus, a statically determinate beam becomes a mechanism (see Section 8.6.1) and the curvature at its critical cross section increases without any limit while the load remains constant. *We say that the statically determinate beam collapses plastically*.

In this and the subsequent sections we present methods for computing the collapse load of statically indeterminate beams and frames made from an isotropic, linearly elastic–ideally plastic material having the same yield stress in uniaxial tension and compression. When the fully plastic moment acts on a cross section of statically indeterminate beams or frames, they behave as if a hinge has been formed at this cross section. However, they do not become a mechanism. Statically indeterminate beams or frames can support loads higher than those which produce a fully plastic moment at one their cross sections. They collapse plastically when plastic hinges reduce a part of them to a mechanism, that is, when a part of them can deform to failure while the load remains constant.

Consider the fixed at both ends beam shown in Fig. 16.9a made from an isotropic, linearly elastic–ideally plastic material subjected to a uniformly distributed force p_3 and assume that the x_2 and x_3 axes are axes of symmetry of the cross section of the beam. This beam is statically indeterminate to the third degree. For small values of the force p_3 the normal component of stress acting on all the particles of the beam will be less than the yield stress in uniaxial tension or compression of the material from which the beam is made and its reactions and internal actions can be computed using the force method presented in Chapter 14. It can be shown that the values of the moments at the ends O and L and at the middle point A of the beam are

$$M_2^0 = M_2^L = \frac{p_3 L^2}{12} \qquad\qquad M_2^A = \frac{p_3 L^2}{24} \tag{16.47}$$

As the load increases, it reaches a certain value p_3^Y, for which the normal component of stress acting on the top and bottom particles of the end cross sections of the beam becomes equal to the yield stress in uniaxial tension or compression for the material for which the beam is made. On the basis of elastic design, p_3^Y is the maximum value of the load that the beam can safely withstand. From the first of relations (16.47) we find

$$p_3^Y = \frac{M_2^Y 12}{L^2} \tag{16.48}$$

As the load increases above p_3^Y, for a certain value p_3^H the moment at the ends O and L of the beam becomes equal to the fully plastic moment M_2^P. *We assume that the beam behaves as if it was made from an elastic material until the fully plastic moment is developed at the ends O and L*. Thus, referring to Fig. 16.9b we find

$$p_3^H = \frac{12 M_2^P}{L^2} \tag{16.49}$$

For any value of the load greater than p_3^H the beam carries the increment of the load Δp_3 as if hinges have been formed at its ends, that is, as if it was simply supported (see Fig.

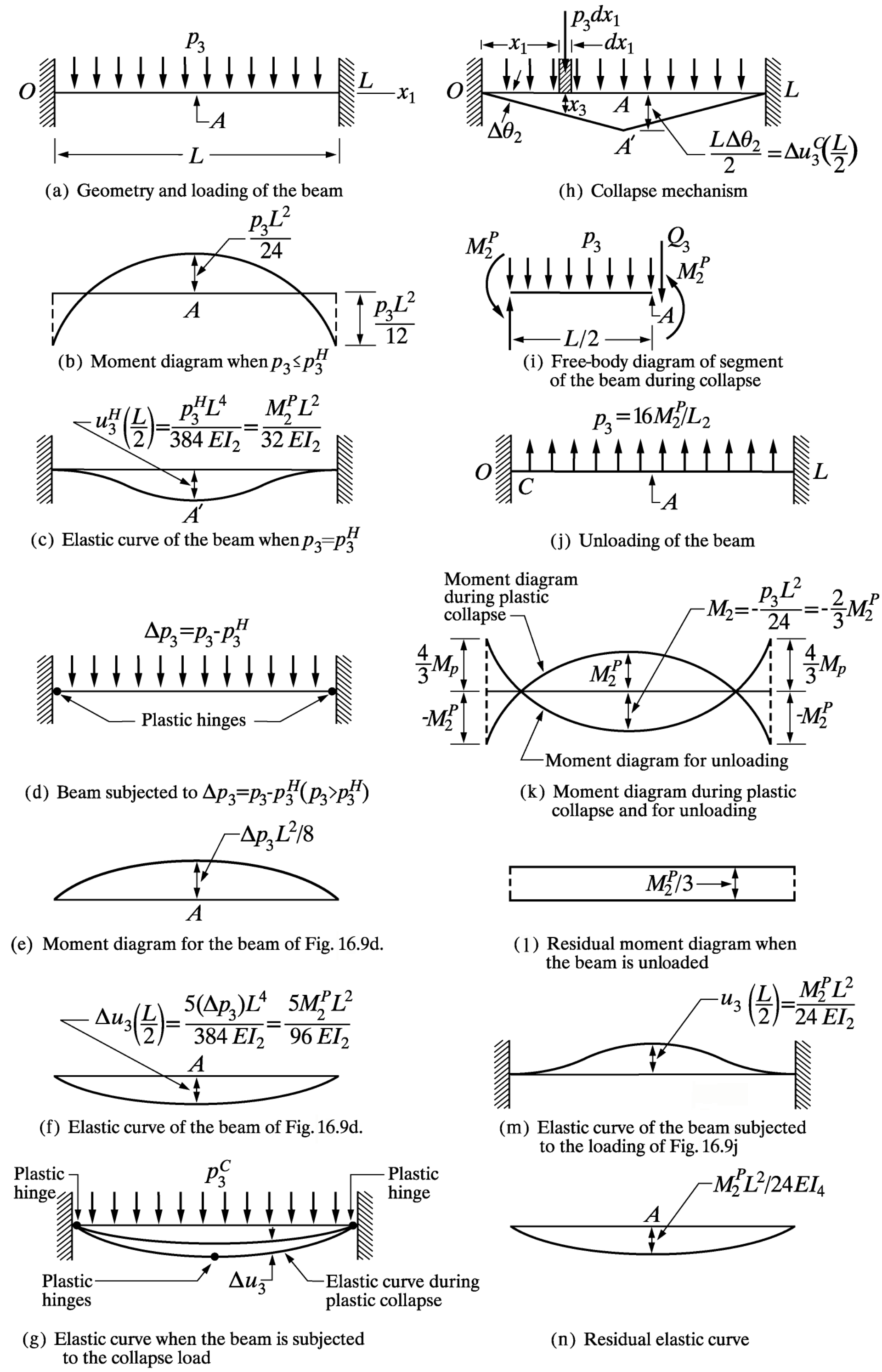


Figure 16.9 Fixed at both ends beam subjected to increasing distributed forces.

16.9d). For a certain value p_3^C of the load called the *collapse load of the beam* the moment at point A reaches its fully plastic value and the beam becomes a mechanism with plastic hinges at points O, A and L. That is, the beam continues to deflect while the load remains constant. We say that the *beam collapses plastically*. It is clear that during plastic collapse of the beam its moment remains constant throughout its length and, consequently the curvature of all its cross sections remains constant except that of its cross sections at O, A, and L at which it increases without limit. This indicates that the additional deflection Δu_3 in excess of that which exists when the load reaches the value p_3^C is a linear function of x_1. The beam rotates about the plastic hinges until it collapses. The additional deflection Δu_3 of the beam is plotted in Fig. 16.9h. This diagram is known as the *plastic collapse mechanism for the beam*.

If we denote by Δp_3^C the difference between the loads p_3^C and p_3^H, referring to Fig. 16.9b and 16.9e the moment at point A is equal to

$$M_2^{(A)} = M_2^P = \frac{p_3^H L^2}{24} + \frac{\Delta p_3^C L^2}{8} \tag{16.50}$$

Substituting relation (16.49) into (16.50), we find that

$$\Delta p_3^C = \frac{4M_2^P}{L^2} \tag{16.51}$$

Consequently using relations (16.49) and (16.51) the collapse load for the beam is equal to

$$q_3^C = p_3^H + \Delta p_3^C = \frac{16M_2^P}{L^2} \tag{16.52}$$

If we denote by α the ratio of M_2^P to M_2^Y from relations (16.48) and (16.52) we get

$$p_3^C = \frac{4\alpha}{3} p_3^Y \tag{16.53}$$

When we design a structure using plastic design we choose its dimensions so that none of its parts behave like a mechanism, when it is subjected to values of the loads equal to the maximum values to which it is anticipated that it will be subjected during its lifetime, multiplied by an appropriate factor of safety. Thus, from an elastic design point of view the beam under consideration cannot support a uniformly distributed load higher than p_3^Y while from a plastic design point of view the beam can support a uniformly distributed load as big as p_3^C which as can be seen from relation (16.53) is considerably bigger than p_3^Y.

During unloading the behavior of the beam is linearly elastic. The difference between the moment diagrams for the beam subjected to a load bigger than p_3^H from that for the

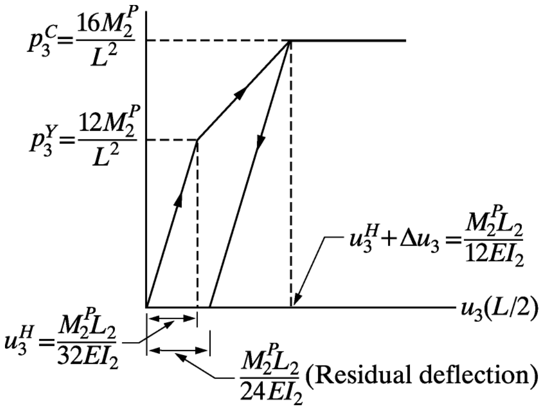


Figure 16.10 Load–deflection diagram.

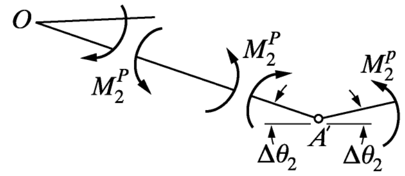


Figure 16.11 Moments acting on the plastic hinges at the instance of collapse.

beam subjected to an equal and opposite load (unloading) (see Fig. 16.9k) is the residual moment diagram. As shown in Fig. 16.9l when the beam under consideration is unloaded from a value of the load just below p_3^C, the residual moment is constant and equal to $M_2^P/3$. As shown in Fig. 16.9n the corresponding elastic curve of the beam is an arc of a circle. The load–deflection diagram for the beam is shown in Fig. 16.10.

It is important to note that the collapse load for a structure can be computed by considering its equilibrium at the instant plastic collapse begins without having to analyze the statically indeterminate beam, as we have done in this example. That is, referring to Fig. 16.9i from the equilibrium of segment *OA* of the beam, we have

$$\sum M_2^A = M_2^P - \frac{p_3^C L}{2}\left(\frac{L}{4}\right) + M_2^P = 0$$

or

$$p_3^C = \frac{16M_2^P}{L^2} \tag{16.54}$$

The collapse load for a structure can also be computed using the principle of virtual work established in Section 13.2 and choosing as the "virtual" displacement the increment Δu_3^C corresponding to its collapse mechanism. Referring to Fig. 16.9h we see that the segments *OA* and *AL* of the beam during plastic collapse move as rigid bodies. Therefore the components of strain of the particles of these segments due to the displacement Δu_3^C vanish. Taking this into account and noting that $\Delta u_3^C = x_1 \Delta\theta_2^C$ the principle virtual world (13.6) for segments *OA* and *AL* of the beam reduces to

$$W_{\text{ext forces}} = 2\int_0^{L/2} \left(\Delta\theta_2^C x_1\right) p_3^C dx_1 - 4M_2^P \Delta\theta_2^C = 2\Delta\theta_2^C p_3^C \int_0^{L/2} x_1 dx_1 - 4M_2^P \Delta\theta_2^C$$

$$=\Delta\theta_2^C \frac{L^2}{4} p_3^C - 4M_2^P \Delta\theta_2^C = 0 \tag{16.55}$$

From relations (16.55), we obtain

$$p_3^C = \frac{16M_2^P}{L^2} \tag{16.56}$$

In plastic analysis of structures we are usually interested in computing only their plastic collapse load.

16.9 Direct Computations of the Collapse Load of Beams and Frames

From the example of the previous section we may deduce that the collapse load of a structure is not affected by the presence of residual stresses. Moreover, in most cases of practical interest the value of the collapse load of a structure does not depend on the history of loading. However, the order at which plastic hinges are formed and the deformation of a structure depend on the history of loading. For example, if we first apply on half of the length of the beam of Fig. 16.9a, a load of $p_3 = 16M_2^P/L^2$ and then we subject the other half of its length to an increasing uniformly distributed load, the beam collapses when this load reaches the value of $p_3 = 16M_2^P/L^2$. That is, the beam collapses for the same value of the load as when it was subjected to an increasing load over its entire length.

On the basis of the foregoing observations when we compute the collapse load of a structure, we assume that its load is applied in such a way that it creates simultaneously all the plastic hinges required for plastic collapse of the structure. That is, we compute the collapse load of a structure assuming that it behaves elastically until it collapses. This assumption does not affect the value of its collapse load.

For the beam of Fig. 16.9a there is only one possible collapse mechanism and thus the collapse load can easily be established. More complicated structures can collapse in one of several possible collapse mechanisms. The actual collapse mechanism of the structure is the one which requires the smallest collapse load. Thus, one way to establish the actual plastic collapse load for a structure is to identify all possible collapse mechanisms and to compute the corresponding collapse loads. The smallest of these loads is the collapse load for the structure. This method is known as the kinematic method.

In what follows we present an example.

Example 5 Establish the collapse load of the frame whose geometry and loading are shown in Fig. a. The members of the frame have the same constant cross section and are made from the same isotropic, linearly elastic–ideally plastic material with a fully plastic moment M_2^P.

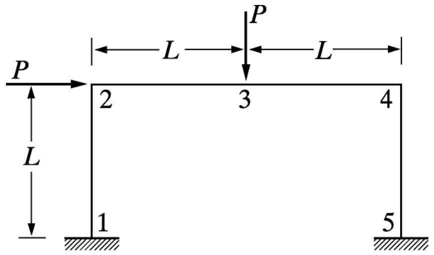


Figure a Geometry and loading of the frame.

Solution

Referring to Fig. a we see that there are five cross sections of the frame under consideration, where plastic hinges could be formed, namely, those at points 1, 2, 3, 4 and 5. We can write a number of equations of equilibrium including the moments at these five cross sections by drawing free-body diagrams of appropriate parts of the frame and considering their equilibrium. However, the frame is statically indeterminate to the third degree. That is, three of these five moments are in excess of those which can be computed using the equations of equilibrium. This implies that if we know three of these moments, we can compute the other two using the equations of equilibrium. Consequently, only two of the equations of equilibrium that we write are independent.

STEP 1 We write equations of equilibrium involving the moments acting on the cross sections at points 1, 2, 3, 4 and 5, by considering the equilibrium of appropriate parts of this frame. Referring to Fig. ba from the equilibrium of members (4, 5), (3, 4) and (2, 4), we have

$$M_2^{(5)} = M_2^{(4)} + R_h^{(5)} L \tag{a}$$

$$M_2^{(3)} = M_2^{(4)} + R_v^{(5)} L \tag{b}$$

$$M_2^{(2)} = M_2^{(4)} - PL + 2R_v^{(5)} L \tag{c}$$

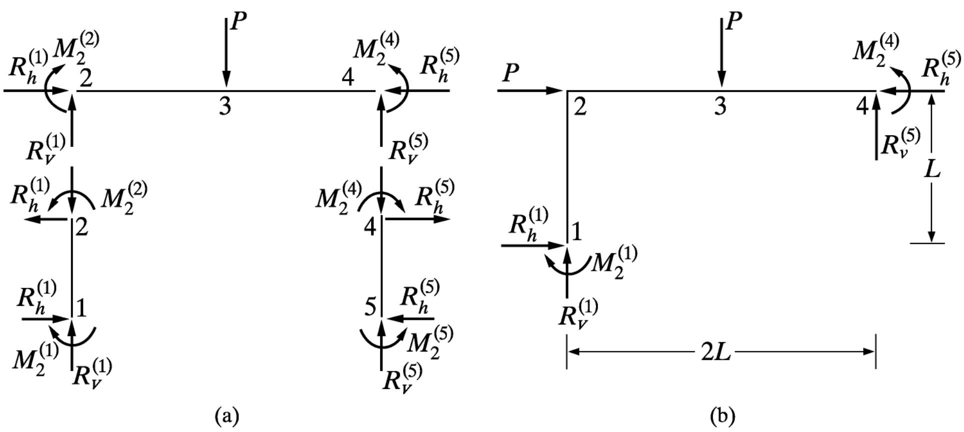
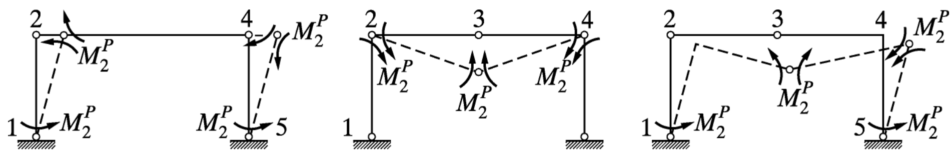


Figure b Free-body diagrams of the members of the frame.



(a) Sidesway (b) Beam (a) Combined

Figure c Possible collapse mechanisms of the frame of Fig. a.

Moreover, referring to Fig. bb from the equilibrium of the portion 1, 2, 3, 4 of the frame, we have

$$M_2^{(1)} = M_2^{(4)} + 2R_v^{(5)}L + R_h^{(5)}L - 2PL \tag{d}$$

Notice that some of the equilibrium equations that we have chosen involve the force P in addition to the moments at the five cross sections at which plastic hinges could be formed. We can establish a convenient set of two independent equations of equilibrium involving the moments at the five cross sections and the force P by eliminating $R_v^{(5)}$ and $R_h^{(5)}$ from relations (a) to (d). Thus, we get

$$PL = -M_2^{(1)} + M_2^{(2)} - M_2^{(4)} + M_2^{(5)} \tag{e}$$

$$PL = -M_2^{(2)} + 2M_2^{(3)} - M_2^{(4)} \tag{f}$$

For any value of the load the internal moments at point 1, 2, 3, 4, 5 must satisfy the above independent equilibrium equations.

STEP 2 As shown in Fig. c, there are three possible collapse mechanisms of the frame of Fig. a. We compute the collapse load for each possible collapse mechanism of the frame. Referring to Fig. ca we see that for the sidesway collapse mechanism the moments at points 1, 2, 4 and 5 must be equal to the fully plastic moment. That is,

$$M_2^{(1)} = -M_2^P \qquad M_2^{(2)} = M_2^P \qquad M_2^{(4)} = -M_2^P \qquad M_2^{(5)} = M_2^P \tag{g}$$

Notice that during plastic collapse the sign of the moments is such that all the terms of relations (e) and (f) are positive. Substituting relations (g) into (e) we obtain the following collapse load for the sidesway collapse mechanism:

$$P_3^C = \frac{4M_2^P}{L} \tag{h}$$

For plastic collapse of the beam of the frame, referring to Fig. cb, we see that the moments at points 2, 3 and 4 must be equal to the fully plastic moment. That is,

$$M_2^{(2)} = -M_2^P \qquad M_2^{(3)} = M_2^P \qquad M_2^{(4)} = -M_2^{(P)} \tag{i}$$

Substituting relations (i) into relation (f), we get the following collapse load for plastic collapse of the beam of the frame:

$$P_3^C = \frac{4M_2^P}{L} \tag{j}$$

Finally, for the combined collapse mechanism referring to Fig. cc we see that the moments at points 1, 3, 4 and 5 must be equal to the fully plastic moment. That is,

$$M_2^{(1)} = -M_2^P \qquad M_2^{(3)} = M_2^P \qquad M_2^{(4)} = -M_2^P \qquad M_2^{(5)} = M_2^P \tag{k}$$

Substituting relations (k) into relations (e) and (f), we get the following collapse load for the combined collapse mechanism of the frame:

$$P_3^C = \frac{3M_2^P}{L} \tag{l}$$

The collapse load of the combined collapse mechanism given by relation (l) is the smallest of the collapse loads of the three possible collapse mechanisms of the frame of Fig. a and, consequently, is its collapse load.

The method employed in this example to obtain the collapse load was very effective because the frame under consideration has only three possible collapse mechanisms (see Fig. c). In practice, however, there are structures which have many possible collapse mechanisms. The collapse load of such structures is established using more systematic procedures than the one used in this section. One such procedure is presented in Section 16.12.

16.10 Derivation of the Equations of Equilibrium for a Structure Using the Principle of Virtual Work

The equations of equilibrium of a structure may also be obtained by using the principle of virtual work derived in Section 13.2. This is illustrated by the following example.

Example 6 Establish the equations of equilibrium of the frame of Fig. a using the principle of virtual work.

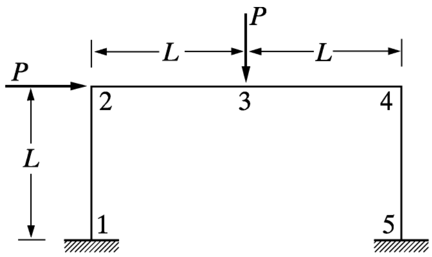


Figure a Geometry and loading of the frame.

Solution We consider the auxiliary frame shown in Fig. b which has the same geometry as the actual frame, except that it has hinges at cross sections 1, 2, 4, 5 where the pairs of positive moments $M_2^{(1)}$, $M_2^{(2)}$, $M_2^{(4)}$, $M_2^{(5)}$ act as shown in Fig. b. When this frame is subjected to the same loading as that acting on the actual frame and to the external moments $M_2^{(1)}$, $M_2^{(2)}$, $M_2^{(4)}$, $M_2^{(5)}$ which are equal to the corresponding internal moments of the actual frame, the distribution of internal moments in the members of the two frames (the auxiliary and the actual) are identical. Suppose that the auxiliary frame of Fig. b is subjected to the virtual displacement shown in Fig. c. During this displacement, the segments 12, 24 and 45 of the frame are displaced as rigid bodies, and, consequently, the components of strain of the particles of these members vanish. Thus, the principle of virtual work, (13.6) reduces to

$$W_{\text{ext forces}} = PL\theta_2^{(1)} + M_2^{(1)}\theta_2^{(1)} - M_2^{(2)}\theta_2^{(1)} + M_2^{(4)}\theta_2^{(1)} - M_2^{(5)}\theta_2^{(1)} = 0$$

or

$$PL = -M_2^{(1)} + M_2^{(2)} - M_2^{(4)} + M_2^{(5)} \tag{a}$$

We consider the auxiliary frame shown in Fig. d and subject it to the virtual displacement shown in Fig. e. During this displacement, the segments 23 and 34 of the frame are displaced as rigid bodies. Consequently, the components of strain of the particles of these segments vanish. Thus, the principle of virtual work (13.6) reduces to

$$W_{\text{ext forces}} = PL\theta^{(2)} + M^{(2)}\theta_2^{(2)} - 2M_2^{(3)}\theta_2^{(2)} + M_2^{(4)}\theta_2^{(2)} = 0$$

or

$$PL = -M_2^{(2)} + 2M_2^{(3)} - M_2^{(4)} \tag{b}$$

Equations (a) and (b) are two independent equations of equilibrium corresponding to the two elementary collapse mechanisms of Figs. c and e. These equations may be combined to give the following equation:

$$2PL = -M_2^{(1)} + 2M_2^{(3)} - 2M_2^{(4)} + M_2^{(5)} \tag{c}$$

The above equation may also be obtained using the principle of virtual work, that is, subjecting the auxiliary frame of Fig. f, to the virtual displacement shown in Fig. g and taking into account that during this displacement the components of strain of the particles of the members of the structure vanish. Thus, the principle of virtual work (13.6) gives

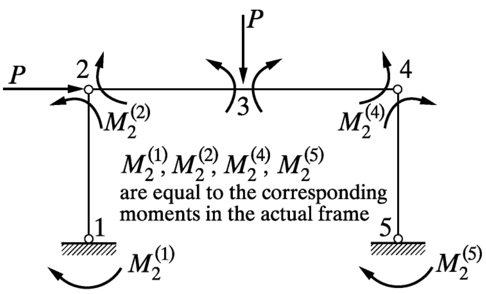


Figure b Auxiliary frame.

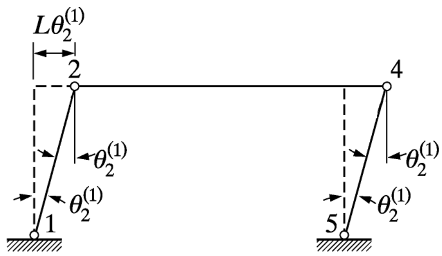


Figure c Virtual displacement of the auxiliary frame of Fig. b.

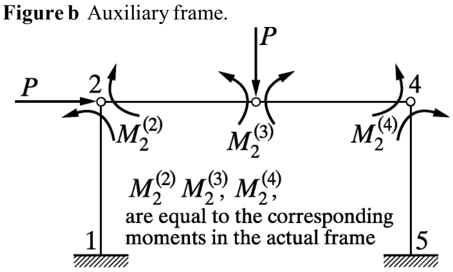


Figure d Auxiliary frame.

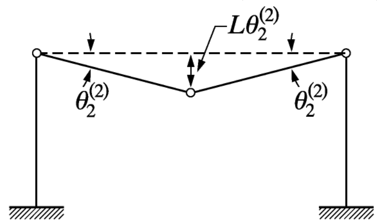


Figure e Virtual displacement of the auxiliary frame of Fig. d.

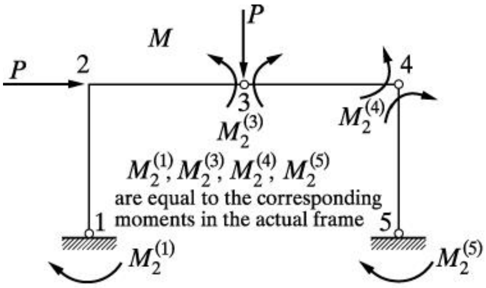


Figure f Auxiliary frame.

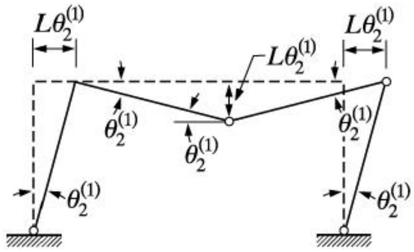


Figure g Virtual displacement of the auxiliary frame of Fig. f.

$$W_{\text{ext forces}} = 2PL\theta_2^{(1)} + M_2^{(1)}\theta_2^{(1)} - 2M_2^{(3)}\theta_2^{(1)} + 2M_2^{(4)}\theta_2^{(1)} - M_2^{(5)}\theta_2^{(1)} = 0 \qquad \text{(d)}$$

Dividing by $\theta_2^{(1)}$, relation (d) reduces to relation (c).

On the basis of the foregoing presentation, we may conclude that an equation of equilibrium corresponds to every collapse mechanism of a structure. However, all these equations of equilibrium are not independent. The choice of which of the equations of equilibrium are the independent and which are the dependent is arbitrary. The collapse mechanisms corresponding to the chosen independent equations of equilibrium are referred to as *elementary*. The other collapse mechanisms are referred to as *combined* and can be obtained by suitable superposition of elementary mechanisms. Thus, the choice of which of the collapse mechanisms are the elementary and which are the combined, is also arbitrary. For planar frames, it is usually convenient to choose the beam and sidesway mechanism as the elementary and the corresponding equations of equilibrium as the independent equations.

16.11 Theorems for Limit Analysis

In this section we present two theorems which are employed for the systematic plastic analysis of framed structures made from isotropic, linearly elastic–ideally plastic materials. For this purpose we define *as safe* a distribution of internal moments in a framed structure whose value at any cross section does not exceed the value of its fully plastic moment.

16.11.1 Lower Bound Theorem

Consider a beam or a frame, made from an isotropic, linearly elastic–ideally plastic material, subjected to external forces whose values depend on one parameter k. That is, these forces could include concentrated $kP^{(1)}$, $kP^{(2)}$, $kP^{(3)}$, ..., $kP^{(n)}$ and distributed kp forces. If for a set of values of these forces specified by $k = k_s$, one can find a statically admissible distribution of internal moments (see Section 13.3) which is also safe, then

$$k_c \geq k_s \qquad (16.57)$$

where k_c is the value of the load parameter which corresponds to the collapse load of the structure. This theorem is also known as the *static theorem*.

Proof

Consider a beam or a frame subjected to the loads described above with $k = k_c$. We denote by $M_2^{P(j)}(j = 1, 2, ..., m)$ the fully plastic moments acting at its cross sections $j = 1, 2, ..., m$ where plastic hinges have been formed. Moreover, we denote by $\Delta\theta_2^{(j)}$ the rotation of the plastic hinges during plastic collapse. Furthermore, we denote by Δu the additional displacement after the structure began to collapse plastically and by $\Delta u^{(i)}$ the value of this additional displacement at the point of application of the concentrated force $P^{(i)}$ in the direction of $P^{(i)}$. During this displacement the components of strain of the particles of the members of the structure vanish. Moreover, during that displacement, the plastic moments act in the direction opposite to that of $\Delta\theta_2^{(j)}$. Taking this into account choosing Δu with $\Delta\theta_2^{(j)}$ as the virtual displacement field, and applying the principle of virtual work (13.6), we have

$$\sum_{i=1}^{n} k_c P^{(i)}\Delta u^{(i)} + \iint_S k_c p\Delta u dS = \sum_{j=1}^{m} M_2^{P(j)}\Delta\theta_2^{(j)} \geq 0 \tag{16.58}$$

where S is the surface of the body and m is the number of plastic hinges formed in the structure at plastic collapse.

Consider the same structure subjected to the loads described above with $k = k_s$. We denote by $M_2^{S(j)}(j = 1, 2, ..., m)$ the moments, corresponding to $M_2^{P(j)}$, of a statically admissible and safe distribution of internal moments in the strucure under consideration. Choosing Δu with $\Delta\theta_2^{(j)}$ as the "virtual" displacement field, and applying the principle of virtual work (13.6), we have

$$\sum_{i=1}^{n} k_s P^{(i)}\Delta u^{(i)} + \iint_S k_s p\Delta u dS = \sum_{j=1}^{m} M_2^{s(j)}\Delta\theta_2^{(j)} \geq 0 \tag{16.59}$$

Substituting relation (16.59) from (16.58), we get

$$(k_c - k_s)\left[\sum_{i=1}^{n} P^{(i)}\Delta u^{(i)} + \iint_S p\Delta u dS\right] = \sum_{j=1}^{m}\left[M_2^{P(j)} - M_2^{s(j)}\right]\Delta\theta_2^{(j)} \geq 0 \tag{16.60}$$

Since $M_2^{S(j)}$ is less or equal to $M_2^{P(j)}$ the right side of relation (16.60) is either positive or zero. Moreover, the term in the bracket at the left side of relation (16.60) is positive since it is equal to the work performed by the external forces acting on the structure during plastic collapse of the structure. Consequently, from relation (16.60) we see that

$$k_c - k_s \geq 0$$

16.11.2 Upper Bound Theorem

Consider a framed structure, made from an isotropic linearly elastic–ideally plastic material, subjected to external forces whose values depend on one parameter k. That is, these forces could include concentrated $kP^{(1)}$, $kP^{(2)}$, $kP^{(3)}$, ..., $kP^{(n)}$ and distributed kp forces. If we denote by $\tilde{k}$ the value of the force parameter corresponding to a possible collapse mechanism of the structure not necessarily the actual one, we have

$$\tilde{k} \geq k_c \tag{16.61}$$

where k_c is the value of the load parameter which corresponds to the actual collapse load of the structure. This theorem is also known as *the kinematic theorem*.

Proof

Consider a structure subjected to the forces described above with $k = \tilde{k}$ which corresponds to the values of the external forces that render the structure a mechanism having plastic hinges at q cross sections. This mechanism may or may not be the actual collapse mechanism of the structure. We denote by $\tilde{M}_2^{P(j)}$ $(j = 1, 2, ..., q)$ the fully plastic moments of the structure subjected to the forces described above with $k = \tilde{k}$. Moreover, we denote by $\Delta\tilde{\theta}_2^{(j)}$ the rotations of the plastic hinges and by $\Delta\tilde{u}$ the additional displacement that the points of the structure would have undergone if the structure was collapsing as the mechanism under consideration. Furthermore, we denote by $\Delta\tilde{u}^{(i)}$ the additional displacement of the point of application of the force $P^{(i)}$ in the direction of $P^{(i)}$. During this displacement the components of strain of the particles of the members of the structure vanish. Taking this into account, choosing $\Delta\tilde{u}$ with $\Delta\tilde{\theta}_2^{(j)}$ as the virtual displacement field and applying the principle of virtual work (13.6), we have

$$\sum_{i=1}^{n} \tilde{k}P^{(i)}\Delta\tilde{u}^{(i)} + \iint_S \tilde{k}p\Delta\tilde{u}\, dS = \sum_{i=1}^{m} M^{P(j)}\Delta\tilde{\theta}_2^{(j)} \geq 0 \tag{16.62}$$

Consider the same structure subjected to the loads described above with $k = k_c$. We denote by $M_2^{(j)}(j = 1, 2, ..., q)$ the moment at the j^{th} cross section $(j = 1, 2, ..., q)$ of the structure subjected to the forces described above with $k = k_c$. The cross sections $(j = 1, 2, ..., q)$ are the ones at which plastic hinges are formed when $k = \tilde{k}$. Choosing $\Delta\tilde{u}$ with $\Delta\tilde{\theta}_2^{(j)}$ as the virtual displacement field, and applying the principle of virtual work (13.6) we have

$$\sum_{i=1}^{n} k_c P^{(i)}\Delta\tilde{u}^{(i)} + \iint_S k_c p\Delta\tilde{u}\, dS = \sum_{j=1}^{k} M_2^{(j)}\Delta\tilde{\theta}_2^{(j)} \geq 0 \tag{16.63}$$

Subtracting relation (16.62) from (16.63), we obtain

$$(k_c - \tilde{k})\left[\sum_{i=1}^{n} P^{(i)}\Delta\tilde{u}^{(i)} + \iint_S p\Delta\tilde{u}\, dS\right] = \sum_{j=1}^{k}\left(M_2^{(j)} - M_2^{P(j)}\right)\Delta\tilde{\theta}_2^{(j)} \leq 0 \tag{16.64}$$

The moment $M_2^{(j)}$ is less or equal the the fully plastic moment $M_2^{P(j)}$. Consequently, the right side of relation (16.64) is negative or zero. Moreover, the term in the bracket in the left side of relation (16.64) is always positive because it represents the work of the external forces during plastic collapse. Consequently, from relation (16.64), we see that

$$k_c - \tilde{k} \leq 0$$

16.12 Systematic Procedure for Plastic Analysis of Structures

A systematic procedure for plastic analysis of structures involves the following steps:

STEP 1 The elementary collapse mechanisms of a structure are listed and the upper bound to the collapse load corresponding to each elementrary collapse mechanism is established.

STEP 2 Some of the elementary collapse mechanisms having the lower upper bounds are combined and the upper bounds corresponding to the resulting combined mechanisms are established.

STEP 3 A lower bound to the collapse load is established. If the difference between the lowest upper bound established in step 2 and the lower bound is not within the desired accuracy, additional collapse mechanisms must be analyzed and a new upper bound, as well as a new lower bound must be established. The process is repeated until the forementioned difference is within the desired accuracy. Then the collapse load of the structure may be approximated to the lowest lower bound.

In what follows, we present an example.

Example 7 Compute the collapse load of the one bay frame loaded as shown in Fig. a. Notice that the fully plastic moment of the beam is twice that of the columns.

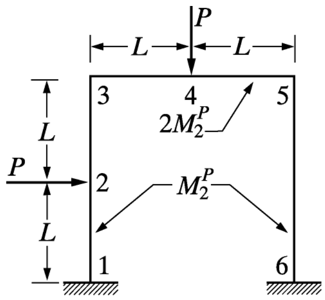
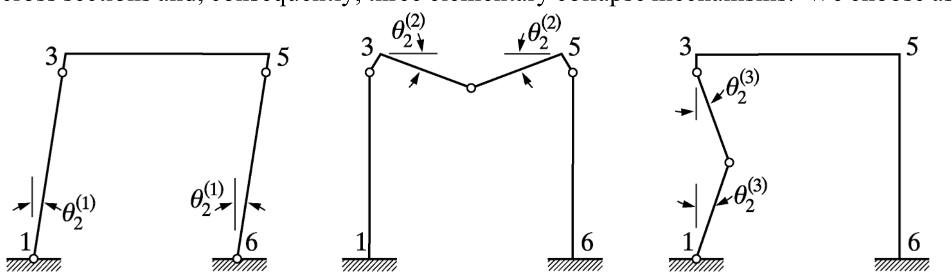


Figure a Geometry and loading of the frame.

Solution

STEP 1 Referring to Fig. a, we see that there are six cross sections in this structure where plastic hinges could possibly develop. These cross sections are at points 1, 2, 3, 4, 5, 6. Moreover, this is an indeterminate structure of the third degree. Therefore, there are 6-3 = 3, independent equations of equilibrium relating the moments at the forementioned cross sections and, consequently, three elementary collapse mechanisms. We choose as



(a) Sidesway collapse mechanism (b) Beam collapse mechanism (c) Column collapse mechanism

Figure b Elementary collapse mechanisms of the frame of Fig. a.

the elementary mechanisms the sidesway, the beam and the column collapse mechanisms shown in Fig. b. The equations of equilibrium corresponding to these mechanisms may be obtained by applying the principle of virtual work, as discussed in Section 16.10. Thus, referring to Fig. ba and denoting by $M_2^{(i)}(I = 1, 2, 3, ..., 6)$ the internal moments at points ($i = 1, 2, 3, ..., 6$) of the frame, the equilibrium equation corresponding to the sidesway collapse mechanism is

$$PL\theta_2^{(1)} = -M_2^{(1)}\theta_2^{(1)} + M_2^{(3)}\theta_2^{(1)} - M_2^{(5)}\theta_2^{(1)} + M_2^{(6)}\theta_2^{(1)} \quad \text{(a)}$$

or

$$PL = -M_2^{(1)} + M_2^{(3)} - M_2^{(5)} + M_2^{(6)} \quad \text{(b)}$$

In relation (a) positive moment produces tension on the inside fibers of the members of the frame (see Fig. c). The upper bound to the collapse load, corresponding to the sidesway collapse mechanism, may be obtained by setting the following values for the moments in equation (b):

$$M_2^{(1)} = -M_2^P \qquad M_2^{(3)} = M_2^P \qquad M_2^{(5)} = -M_2^P \qquad M_2^{(6)} = M_2^P \quad \text{(c)}$$

Thus, we get

$$P^C \leq P^{u.b.} = \frac{4M_2^P}{L} \quad \text{(d)}$$

Notice that the sign of the moments in relation (c) is obtained by nothing that the fully plastic moment acting at the ends of a segment of the structure extending between two plastic hinges has such a sense that it tends to oppose the rotation of the segment. For example, referring to Fig. ba we see that the rotation of ends 1 and 3 of member 13 is clockwise. Thus, the sense of the fully plastic moments acting at the ends of member 13 of the structure during sidesway plastic collapse is as shown in Fig. d, ($M_2^{(1)} = -M_2^P$ and $M_2^{(3)} = M_2^P$). Moreover, the sense of the fully plastic moments for a collapse mechanism may be established by noting that it must be such that all the terms on the right side of the corresponding equilibrium equation must be positive. Referring to Fig. bb, the equation of equilibrium corresponding to the beam collapse mechanism is

$$PL\theta_2^{(2)} = -M_2^{(3)}\theta_2^{(2)} + 2M_2^{(4)}\theta_2^{(2)} - M_2^{(5)}\theta_2^{(2)} \quad \text{(e)}$$

or

$$PL = -M_2^{(3)} + 2M_2^{(4)} - M_2^{(5)} \quad \text{(f)}$$

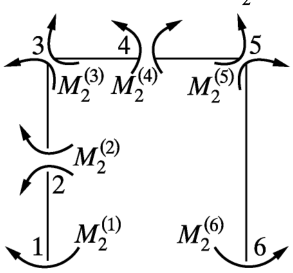


Figure c Positive internal moments acting on the members of the structure.

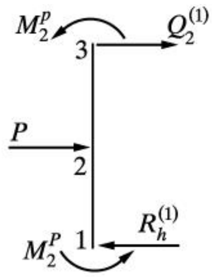


Figure d Free-body diagram of member 13 of the sidesway collapse mechanism of Fig. ba.

The upper bound to the collapse load corresponding to the beam collapse mechanism may be obtained by setting the following values for the moments in relation (f):

$$M_2^{(3)} = -M_2^P \qquad M_2^{(4)} = 2M_2^P \qquad M_2^{(5)} = -M_2^P \tag{g}$$

Thus, we get

$$P^C \leq P^{u.b.} = \frac{6M_2^P}{L} \tag{h}$$

Referring to Fig. bc, the equation of equilibrium corresponding to the column collapse mechanism is

$$PL\theta_2^{(3)} = -M_2^{(1)}\theta_2^{(3)} + 2M_2^{(2)}\theta_2^{(3)} - M_2^{(3)}\theta_2^{(3)} \tag{i}$$

or

$$PL = -M_2^{(1)} + 2M_2^{(2)} - M_2^{(3)} \tag{j}$$

The upper bound to the collapse load corresponding to the column collapse mechanism may be obtained by setting the following values for the moments in relation (j):

$$M_2^{(1)} = -M_2^P \qquad M_2^{(2)} = M_2^P \qquad M_2^{(3)} = -M_2^P \tag{k}$$

Thus, we get

$$P^C \leq P^{u.b.} = \frac{4M_2^P}{L} \tag{l}$$

STEP 2 Inasmuch as the two lowest of the upper bounds established in step 1 are those obtained from the sidesway and the column collapse mechanisms, it is possible that a lower upper bound could be obtained by combining these two mechanisms to yield the mechanism shown in Fig. ec. Applying the principle of virtual work to this mechanism, we obtain the following equation:

$$PL(\theta_2^{(1)} + \theta_2^{(3)}) = -M_2^{(1)}(\theta_2^{(1)} + \theta_2^{(3)}) + 2M_2^{(2)}\theta_2^{(3)} - M_2^{(3)}(\theta_2^{(3)} - \theta_2^{(1)}) - M_2^{(5)}\theta_2^{(1)} + M_2^{(6)}\theta_2^{(1)} \tag{m}$$

The same result may be obtained by adding equations (a) and (i). For any choice of the ratio $\theta_2^{(1)}/\theta_2^{(3)}$ equation (m) gives an equation of equilibrium. Assuming that $\theta_2^{(3)} > \theta_2^{(1)} > 0$ an upper bound to the collapse load corresponding to the combined mechanism of Fig. ec may be obtained by setting the following values for the moments in relation (m):

$$M_2^{(1)} = -M_2^P \qquad M_2^{(2)} = M_2^P \qquad M_2^{(3)} = -M_2^P \qquad M_2^{(5)} = -M_2^P \qquad M_2^{(6)} = M_2^P \tag{n}$$

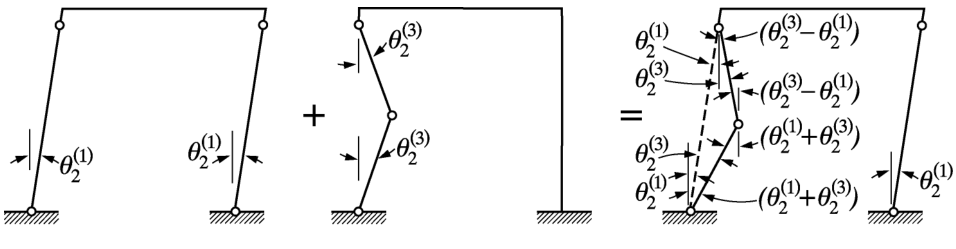


Figure e Superposition of the sidesway and column mechanisms.

Thus, relation (m) reduces to

$$\frac{P^{u.b.}L}{M_2^P} = \frac{2\theta_2^{(1)} + 4\theta_2^{(3)}}{\theta_2^{(1)} + \theta_2^{(3)}} = 2 + \frac{2}{\dfrac{\theta_2^{(1)}}{\theta_2^{(3)}} + 1} \tag{o}$$

From relation (o) we see that $P^{u.b.}$ depends on the value of the ratio $\theta_2^{(1)}/\theta_2^{(3)}$. Inasmuch as it has been assumed that $0 < \theta_2^{(1)}/\theta_2^{(3)} \leq 1$, the lowest value of $P^{u.b.}$ is obtained for $\theta_2^{(1)}/\theta_2^{(3)}$ equal to unity. This choice of the ratio $\theta_2^{(1)}/\theta_2^{(3)}$ cancels the hinge at point 3 and reduces the combined collapse mechanism of Fig. ec to that of Fig. f. Moreover, it reduces equation (m) to the following equation of equilibrium:

$$2PL = -2M_2^{(1)} + 2M_2^{(2)} - M_2^{(5)} + M_2^{(6)} \tag{p}$$

The same result may be obtained by combining equations (b) and (j) or by applying the principle of virtual work to the mechanism shown in Fig. f. The upper bound to the collapse load corresponding to the combined collapse mechanism shown in Fig. f is obtained by setting the following values for the moments in relation (p):

$$M_2^{(1)} = -M_2^P \qquad M_2^{(2)} = M_2^P \qquad M_2^{(5)} = -M_2^P \qquad M_2^{(6)} = M_2^P \tag{q}$$

Thus, we obtain

$$P^C \leq P^{u.b.} = \frac{3M_2^P}{L} \tag{r}$$

It is apparent that the upper bound to the collapse load corresponding to the combined (sidesway and column) collapse mechanism is the lowest of the ones corresponding to the mechanisms which we have thus far analyzed.

STEP 3 We compute a lower bound to the collapse load of the structure. In order to accomplish this, we note that for the frame under consideration, there are three independent equations of equilibrium. One of these equations has been used in obtaining $P^{u.b.}$ as the statically admissible load corresponding to the moment distribution given by relations (q). Thus, there are two independent equations of equilibrium available, for obtaining $M_2^{(3)}$ and $M_2^{(4)}$, the only moments not included in relations (q). We can use equations (b) and (f) as the independent equations. Thus, substituting relation (q) and (r) into (b) and (f), we get

$$M_2^{(3)} = 0 \qquad M_2^{(4)} = M_2^P \tag{s}$$

The moment distribution

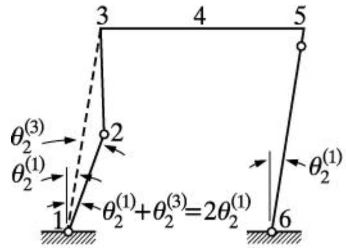


Figure f Combined collapse mechanism for $\theta_2^{(1)} = \theta_2^{(3)}$.

$$M_2^{(1)} = -M_2^P \quad M_2^{(2)} = M_2^P \quad M_2^{(3)} = 0 \quad M_2^{(4)} = M_2^P \quad M_2^{(5)} = -M_2^P \quad M_2^{(6)} = M_2^P$$

is statically admissible to the applied load $P = 3M^P/L$ and, moreover, is safe. Thus, on the basis of the lower bound theorem, this value of the load is a lower bound to the collapse load. That is,

$$P^{l.b.} = \frac{3M_2^P}{L} \leq P^C \tag{t}$$

Comparing relation (r) and (t), it is apparent that the collapse load is

$$P^C = \frac{3M_2^P}{L} \tag{u}$$

Remarks On the basis of the foregoing example, we note that

(a) The equation of equilibrium corresponding to a combined collapse mechanism may be obtained by combining the equations of equilibrium corresponding to each of the mechanisms which have been combined, to yield the combined collapse mechanism.

(b) We denote by $\theta_2^{(j)}$ the parameter that specifies the geometry of the i^{th} mechanism of a structure. When n mechanisms are superimposed, the number of hinges of the resulting combined mechanisms depends on the choice of the relative values of $\theta_2^{(i)}$ (i = 1, 2, ..., n). The combined mechanism which gives the lowest upper bound is obtained for relative values of $\theta_2^{(i)}$ which cancel as many hinges as possible.

(c) Only if one or more hinges are canceled during the combination of two or more mechanisms, the upper bound obtained from the resulting combined mechanism could be less than the lowest upper bound obtained from each of the elementary mechanisms which have been combined.

16.13 Problems

1. A shaft of length 1.2 m and solid circular cross sections of radius 40 mm is subjected to equal and opposite torsional moments at its ends. The shaft is made from an isotropic, linearly elastic–ideally plastic material with G = 75 GPa and $\tau_{1\theta}^{Y}$ = 160 MPa.

(a) Determine the magnitude of the torsional moments for which yielding begins at the particles of the lateral surface of the shaft. Ans. 16.085 kN·m

(b) Determine the magnitude of torsional moments which reduces the radius of the elastic core of the shaft to 20 mm. Ans. M_1 = 20.776 kN·m

(c) Determine the corresponding angle of twist of the one end of the shaft relative to the other. Ans. $\theta_1(L)$ = 7.334°

(d) Determine the distribution of the residual stresses on the cross sections of the shaft when the torsional moments determined in (b) are removed.

Ans. $\tau_{1\theta}(0.02)$ = 56.66 MPa $\tau_{1\theta}(0.004)$ = −46.8 MPa

(e) Determine the residual angle of twist. Ans. $\theta_1(L)$ = 2.597°

2. A shaft of length 1.5 m is made from a steel (G = 77 GPa) circular tube, whose cross section is shown in Fig. 16P2. Steel can be approximated by the isotropic, linearly elastic– ideally plastic model of material behavior having a yield stress in pure shear equal to $(\tau_{1\theta}^{Y})_s = 145$ MPa. The shaft is subjected to equal and opposite torsional moments at its ends.

(a) Establish M_1^Y and M_1^P. Ans. $M_1^Y = 48.58$ KN·m, $M_1^P = 63.16$ KN·m

(b) Suppose that the shaft is subjected at its ends to equal and opposite torsional moments of magnitude $(M_1^Y + M_1^P)/2$. Compute and show on a sketch the distribution of shearing stress on the cross section of the shaft.

Ans. $\rho = 49.09$ mm $\tau_{1\theta}^{*} = 2.953758r$ MPa for $20 \le r \le 49.09$

(c) Establish the distribution of the residual shearing stress when the shaft is unloaded.

Ans. at r = 20 mm $\tau_{1\theta} = 3.482$ at r = 49.09 mm $\tau_{1\theta} = 8.547$ MPa
at r = 60 mm $\tau_{1\theta} = -21.779$MPa

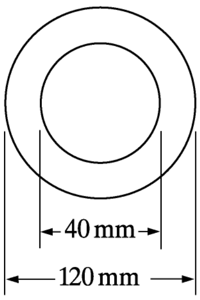


Figure 16P2

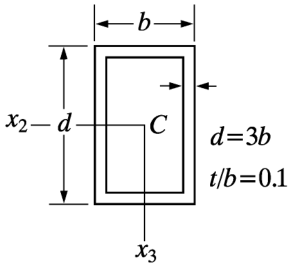


Figure 16P4

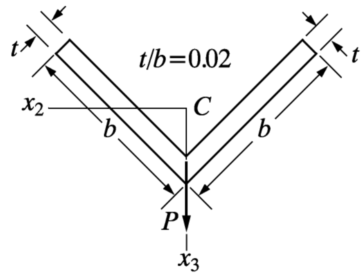


Figure 16P5

3. A prismatic shaft of circular cross section is fixed at its one end and is subjected to a distribution of traction on its other end which is statically equivalent to a concentrated axial centroidal force of 20 kN and a torsional moment of 0.6 kN·m. The shaft is made from an isotropic, linearly elastic material (E = 200 GPa, G = 75 GPa) obeying the Von Mises yield criterion with a yield stress in uniaxial tension or compression of $\tau_{11}^{Y} = 340$ MPa. Using a factor of safety of 2.5, determine the minimum required safe diameter D of the shaft. Assume that the shaft fails at the initiation of yielding. Ans. D = 42.7 mm

4. and **5**. Compute the elastic design moment M_2^Y, the fully plastic moment M_2^P and the shape factor of the beam whose cross section is shown in Fig. 16P4. Repeat with the beam whose cross section is shown in Fig. 16P5.

6. to **8**. Consider the 3 m long simply supported beam whose cross section is shown in Fig. 16P6. The beam is made from an isotropic linearly elastic–ideally plastic, material with modulus of elasticity E = 200GPa and yield stress in uniaxial tension τ_{11}^{Y} = **250 MPa**. The beam is subjected to a concentrated force of P_3 at the middle point of its span. Compute the elastic failure force P^Y and the plastic collapse force P^C.

(a) Establish the distribution of the normal and shearing components of stress acting on the cross section of the beam just to the left of the concentrated force

$P_3 = (P^Y + P^P)/2$.

(b) Establish the geometry of the plastic regions for $P_3 = (P^Y + P^P)/2$.

(c) Establish the distribution of the residual normal component of stress acting on the cross section just to the left of the concentrated force when the beam is unloaded after being subjected to $P_3 = (P^Y + P^P)/2$.

Repeat with the beams whose cross sections are shown in Figs. 16P7 and 16P8.

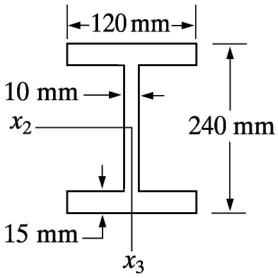


Figure 16P6

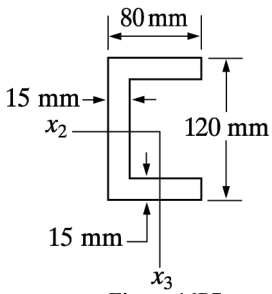


Figure 16P7

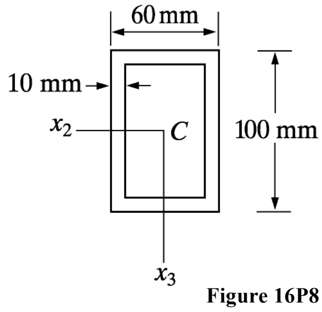


Figure 16P8

9. A machine part made from steel has the T cross section as shown in Fig. 16P9. The part is fixed at its one end and is subjected to a uniform load p_3. If the yield stress for steel in tension or in compression is 250 MPa, compute the maximum allowable value of the load p_3, using a factor of safety of 4 and (a) elastic design (b) plastic design.

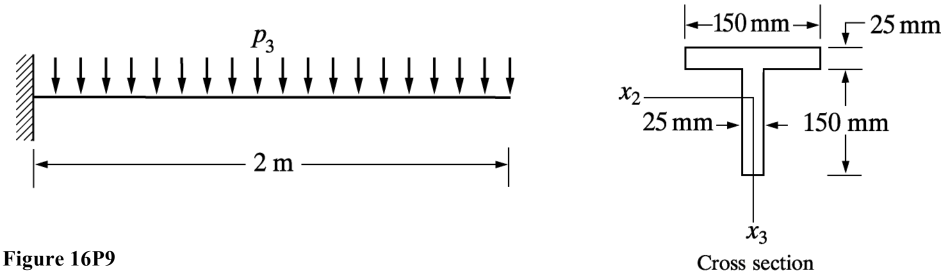


Figure 16P9

10. Consider the beam subjected to the loading shown in Fig. 16P10. The beam is made from an isotropic, linearly elastic material. Using a factor of safety of 4 and (a) elastic design and (b) plastic design, determine the maximum allowable value of the force P. The yield stress in uniaxial tension of the material from which the beam is made is $\tau_{11}^Y = 250$ MPa. Neglect the weight of the beam.

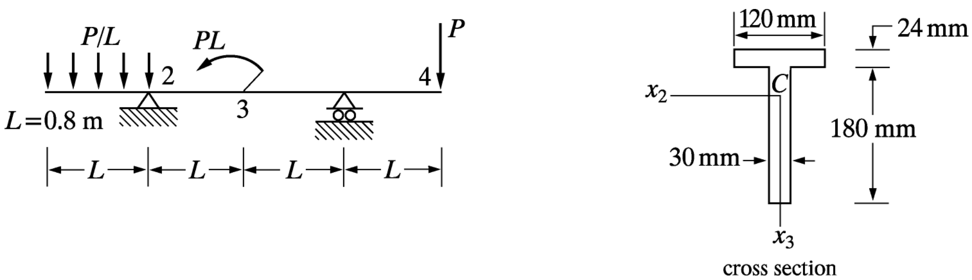


Figure 16P10

11. Using elastic design, find the maximum safe force P_3 which can be placed at the unsupported end of the tapered cantilever beam of rectangular cross section of constant width $b = 0.2$ m (see Fig. 16P11) in order to have a factor of safety of 2. The beam is made from an isotropic, linearly elastic-ideally plastic material ($E = 200$ GPa, $\nu = 0.3$, $\tau_{11}^{Y} = 240$ MPa) obeying the Von Mises yield criterion. (**Hint**: The shearing component of stress τ_{13} can be obtained from relation (b) of the example of Section 10.2.1.)

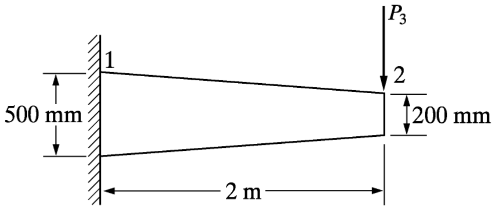


Figure 16P11

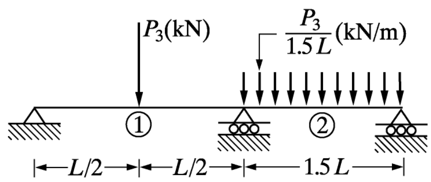


Figure 16P12

12. The two-span continuous beam of Fig. 16P12 is subjected to the loads shown. The left span has a fully plastic moment of M^P. Find the required value of the fully plastic moment of the right span so that the two spans collapse plastically simultaneously.

Ans. $0.693M^P$

13. Find the collapse load of the fixed at both ends beam of constant cross section of Fig. 16P13. The beam is made of one material ($E = 200$ GPa, $\nu = 0.3$, $\tau_{11}^{Y} = 200$ MPa).

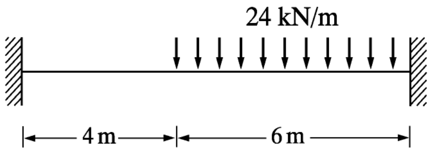


Figure 16P13

14. to **17**. Use the method of described in Section 16.12 to compute the collapse load P^C of the structure subjected to the loads shown in Fig. 16P14. The structure is made from an isotropic, linearly elastic–ideally plastic material. Repeat with the structures of Fig. 16P15 to 16P16.

Ans. 16. $P^C = 1.20M^P/L$
17. $P^C = 2.36M^P/L$

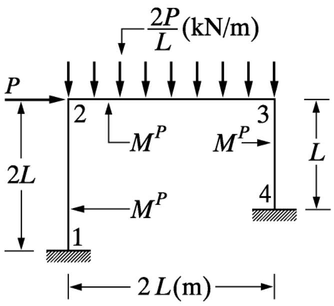


Figure 16P14

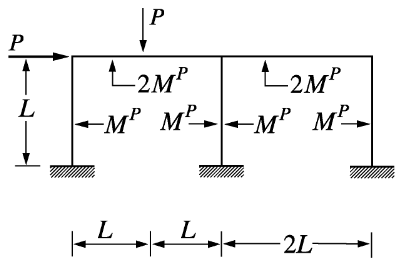


Figure 16P15

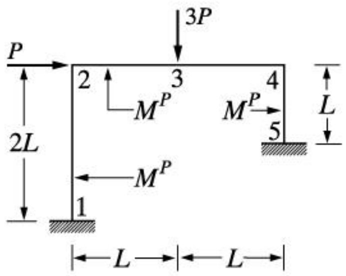


Figure 16P16

Chapter 17

Mechanics of Materials Theory for Thin Plates

17.1 Introduction

A plate is a body whose boundary consists of two plane surfaces, called its *faces*, located a small distance apart and one or more (if it has holes) prismatic lateral surfaces (see Fig. 17.1). The distance between the faces of a plate is called its thickness and is considerably smaller than its other two dimensions. The locus of the midpoints of the thickness of a plate is a plane known as its midplane. We call the closed line bounding the midplane of a plate its *edges*. We limit our attention to thin plates of constant thickness t. Plates are considered as thin when their thickness is less than about 1/20 of the smallest dimension of their midplane. As shown in Fig. 17.1, we choose the x_3 axis normal to the midplane of the plate and the x_1 and x_2 axes in its midplane.

We consider plates originally in their undeformed stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_0. Subsequently, the plates reach a second deformed state of mechanical, but not necessarily thermal equilibrium due to the application of specified boundary conditions on their lateral surfaces and to the application on their faces of one or more of the following loads:

1. Specified distributed $p_3(x_1, x_2)$ and concentrated $P_3^{(n)}(n = 1, 2, \ldots, n_3)$ transverse forces acting on their faces

2. Specified temperature distribution $T^{(+)}(x_1, x_2)$ and $T^{(-)}(x_1, x_2)$ at their faces $x_3 = t/2$ and $x_3 = -t/2$, respectively

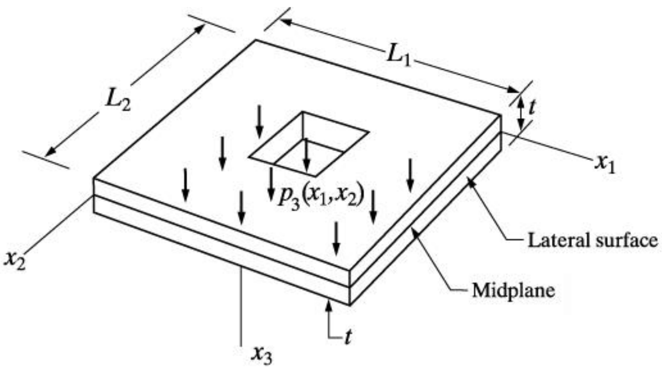
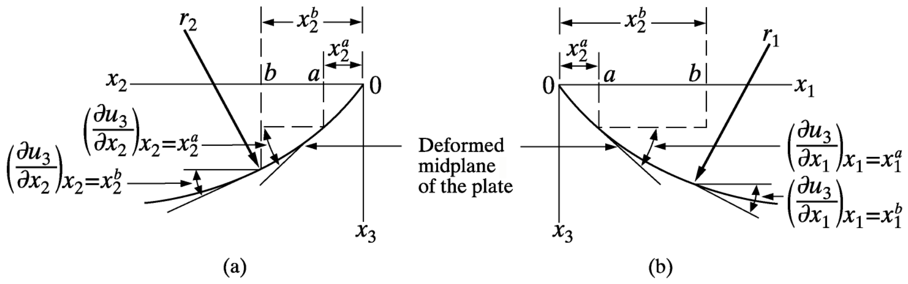
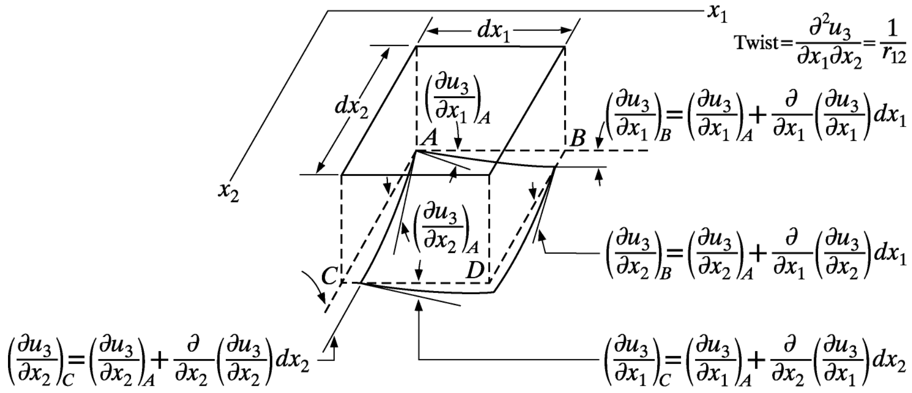


Figure 17.1 Thin plate.



(a) Positive radii of curvature



(b) Geometric interpretation of the twist of midplane of a plate

Figure 17.2 Radii of curvature and twist of the midplane of a plate in the plane normal to the x_2 and x_1 axes.

These loads deform the plate in a way that the particles of its midplane move only in the direction normal to it. That is

$$\hat{u}_1(x_1, x_2, 0) = u_1(x_1, x_2) = 0$$
$$\hat{u}_2(x_1, x_2, 0) = u_2(x_1, x_2) = 0 \tag{17.1}$$

We denote by r_1 and r_2 the radii of curvature of the deformed midplane of the plate in the planes normal to the x_2 and x_1 axis, respectively. In Fig. 17.2a we show the positive radii of curvature; referring to this figure we see that when the midplane of a plate has a positive curvature r_1 and r_2, the rate of change of its slopes $\partial u_3/\partial x_1$ and $\partial u_3/\partial x_2$ is negative. That is, the second derivative of u_3 with respect to x_1 or x_2, respectively, is negative. Taking this into account, referring to relations (9.25) and limiting our attention to plates whose deformation is within the range of validity of the assumption of small deformation, their radii of curvature may be approximated as

$$\frac{1}{r_1} = -\frac{\partial^2 u_3}{\partial x_1^2} \tag{17.2a}$$

$$\frac{1}{r_2} = -\frac{\partial^2 u_3}{\partial x_2^2} \tag{17.2b}$$

The midplane of a plate may also twist. That is, there may be a change of the slope $\partial u_3/\partial x_1$ or $\partial u_3/\partial x_2$ as one proceeds in the x_2 or x_1 direction, respectively. The measure of the twist is the rate of change of the slope $\partial u_3/\partial x_i (i = 1, 2)$ in the direction $x_j (j = 2, 1 \; j \neq i)$. That is,

$$\frac{1}{r_{12}} = \frac{\partial^2 u_3}{\partial x_1 \partial x_2} \tag{17.3}$$

The geometric interpretation of the twist is demonstrated in Fig. 17.2b.

17.2 Fundamental Assumptions of the Theories of Mechanics of Materials for Thin Plates

In the theories of mechanics of materials for thin plates the following assumptions are made:

Assumption 1. *The behavior of a plate may be approximated by that of the continuum model (see Section 2.1).*

Assumption 2. The effect of the deformation of the plates which we are considering on their temperature is negligible. On the basis of this assumption the temperature distribution of a plate can be computed independently of its deformation. In this book we assume that it has been computed and it is known.

Assumption 3. The normal component of stress τ_{33}, acting on the particles of plates is negligible compared to the other normal components of stress. That is,

$$\tau_{33} \approx 0 \tag{17.4}$$

On the basis of this assumption, referring to relations (3.95), the stress–strain relations for plates made from an isotropic linearly elastic material reduce to

$$\begin{aligned}
e_{11} &= \frac{1}{E}(\tau_{11} - v\tau_{22}) + \alpha(T - T_0) \\
e_{22} &= \frac{1}{E}(\tau_{22} - v\tau_{11}) + \alpha(T - T_0) \\
e_{33} &= -\frac{v}{E}(\tau_{11} + \tau_{22}) + \alpha(T - T_0) \\
e_{12} &= \frac{\tau_{12}}{2G} \qquad e_{13} = \frac{\tau_{13}}{2G} \qquad e_{23} = \frac{\tau_{23}}{2G}
\end{aligned} \tag{17.5}$$

or

$$\begin{aligned}
\tau_{11} &= \frac{E}{1-\nu^2}(e_{11}+\nu e_{22}) - \frac{E\alpha}{1-\nu}(T-T_0) \\
\tau_{22} &= \frac{E}{1-\nu^2}(e_{22}+\nu e_{11}) - \frac{E\alpha}{1-\nu}(T-T_0) \qquad (17.6) \\
\tau_{12} &= 2Ge_{12} \qquad \tau_{13} = 2Ge_{13} \qquad \tau_{23} = 2Ge_{23}
\end{aligned}$$

Assumption 4. The principle of Saint Venant can be applied to the lateral surfaces of plates. This principle indicates that the change of the stress field at points sufficiently removed from the lateral surfaces of a plate is negligible when the distribution of traction acting on its lateral surfaces is replaced by a statically equivalent one, that is, one which has the same resultant force and moment, per unit length of the edge of the plate. For this reason at the portion of the lateral surface of a plate at which the components of displacement are not specified it is sufficient to specify the resultant force and the resultant moment per unit length of its edge.

Assumption 5. When the theory of elasticity is employed to establish the distribution of the components of displacement and stress in a body in equilibrium under the influence of external forces (surface tractions and body forces) and/or change of temperature, it is required that the distribution of the components of stress satisfies the conditions for equilibrium for all the particles of the body.

In the theories of mechanics of materials for plates we do not ensure that every particle of a plate is in equilibrium. Instead we ensure that each segment of infinitesimal length and width, and finite thickness t of a plate is in equilibrium.

Assumption 6. Straight lines normal to the midplane of a plate before deformation remain straight after deformation. Consequently, the deformed configuration of a line which before deformation is normal to the midplane of a plate at point A is specified by the transverse component of translation $u_3(x_1, x_2)$ of its midpoint and by the two components of rotation $\theta_1(x_1, x_2)$ and $\theta_2(x_1, x_2)$ about the x_1 and x_2 axes, respectively. Referring to Fig. 17.3 consider the particles of the straight line $ABCD$ which before

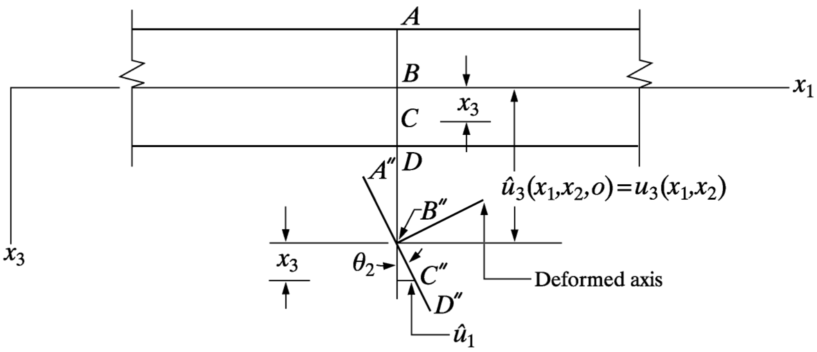


Figure 17.3 Deformed configuration of a material straight line.

deformation was normal to the midplane of the plate. On the basis of Assumption 4, due to the deformation of the plate, these particles move to another straight line $A'B'C'D'$ not shown in Fig. 17.3. Taking into account relation (17.1) the projection of line $A'B'C'D'$ on the x_1x_3 plane is the straight line $A''B''C''D''$ shown in Fig. 17.3. The movement of the particle located before deformation at point C is specified by its components of displacement $\hat{u}_1(x_1, x_2, x_3)$, $\hat{u}_2(x_1, x_2, x_3)$, and $\hat{u}_3(x_1, x_2, x_3)$. As can be seen from Fig. 17.3, we have

$$\hat{u}_1(x_1, x_2, x_3) = x_3\theta_2(x_1, x_2) \tag{17.7a}$$

Similarly, it can be shown that

$$\hat{u}_2(x_1, x_2, x_3) = -x_3\theta_1(x_1, x_2) \tag{17.7b}$$

Assumption 7. In view of the small thickness of a plate we assume that the component of translation of a particle in the direction normal to its midplane is approximately equal to that of its projection on the midplane. That is,

$$\hat{u}_3(x_1, x_2, x_3) \approx \hat{u}_3(x_1, x_2, 0) = u_3(x_1, x_2) \tag{17.8}$$

In order to be able to formulate the boundary value problem for computing the component of translation (deflection) $u_3(x_1, x_2)$ of a plate we need two relations relating it to the components of rotation $\theta_1(x_1, x_2)$ and $\theta_2(x_1, x_2)$. There are two theories available in the literature, for analyzing plates. They differ only in the relations of the component of translation $u_3(x_1, x_2)$ of the plate with the components of rotation $\theta_1(x_1, x_2)$ and $\theta_2(x_1, x_2)$. The one theory which we present in this text is called the *classical theory of plates*. It is based on the assumption that the geometry of the plate is such that the effect, of shear deformation [that is, of the shearing components of strain $e_{23}(x_1, x_2, x_3)$ and $e_{13}(x_1,x_2,x_3)$] of the particles of the plate, on its component of translation $u_3(x_1, x_2)$ is negligible. The other theory for analyzing plates is called the *Timoshenko theory of plates*. In this theory a portion of the effect of shear deformation of the particles of the plate on its component of translation $u_3(x_1, x_2)$ is retained. That is, following a reasoning similar to that adhered to for beams [see relations (9.27)] for the classical theory of plates, we have

$$\theta_2 = -\frac{\partial u_3}{\partial x_1} \qquad \theta_1 = \frac{\partial u_3}{\partial x_2} \tag{17.9}$$

Substituting relations (17.9) into (17.7), we get

$$\begin{aligned} \hat{u}_1(x_1, x_2, x_3) &= -x_3\frac{\partial u_3}{\partial x_1} \\ \hat{u}_2(x_1, x_2, x_3) &= -x_3\frac{\partial u_3}{\partial x_2} \end{aligned} \tag{17.10}$$

Substituting relations (17.10) into the strain–displacement relations (2.16) and referring to relations (17.2) and (17.3), we obtain

$$e_{11} = \frac{\partial \hat{u}_1}{\partial x_1} = -x_3\frac{\partial^2 u_3}{\partial x_1^2} = \frac{x_3}{r_1} \tag{17.11a}$$

$$e_{22} = \frac{\partial \hat{u}_2}{\partial x_2} = -x_3 \frac{\partial^2 u_3}{\partial x_2^2} = \frac{x_3}{r_2} \tag{17.11b}$$

$$e_{12} = -x_3\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right) = -\frac{2x_3}{r_{12}} \tag{17.11c}$$

where r_1 or r_2 is the radius of curvature of the deformed midplane of the plate in the plane normal to the x_2 or x_1 axis, respectively. In the classical theory of plates the components of strain e_{13} and e_{23} cannot be computed from the components of displacement on the basis of relations (2.16) because in obtaining relations (17.9) we have assumed that these components of strain are negligible and, consequently, substituting relations (17.10) into (2.16), we get that e_{13} and e_{23} are equal to zero. Substituting relations (17.11) into the stress–strain relation (17.6), we have

$$\tau_{11} = -\frac{Ex_3}{1-\nu^2}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \nu\frac{\partial^2 u_3}{\partial x_2^2}\right) - \frac{E\alpha}{1-\nu}(T - T_0) \tag{17.12a}$$

$$\tau_{22} = -\frac{Ex_3}{1-\nu^2}\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2}\right) - \frac{E\alpha}{1-\nu}(T - T_0) \tag{17.12b}$$

$$\tau_{12} = -2Gx_3\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right) \tag{17.12c}$$

17.3 Internal Action Intensities Acting on an Element of a Plate

Consider a plate originally in its undeformed stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_0. Subsequently the plate reaches a second state of mechanical but not necessary thermal equilibrium due to the application on it of one or more of the following loads and constraints:

1. Specified distributed $p_3(x_1, x_2)$ and concentrated $P_3^{(n)}(n = 1, 2, \ldots, n_3)$ transverse forces acting on its faces
2. Specified temperature distribution $T^{(+)}(x_1, x_2)$ and $T^{(-)}(x_1, x_2)$ at their faces $x_3 = t/2$ and $x_3 = -t/2$, respectively
3. Specified boundary conditions on its lateral surfaces (see Section 17.5)
Due to the application of these loads there will be stresses acting on the particles of the plate. We define the resultant force and moment intensities of the components of stress acting on a plane normal to the x_1 or x_2 axis, per unit length in the x_2 or x_1 direction, respectively, as

$$Q_1(x_1, x_2) = \int_{-t/2}^{t/2} \tau_{13}\, dx_3 \tag{17.13a}$$

$$Q_2(x_1, x_2) = \int_{-t/2}^{t/2} \tau_{23}\, dx_3 \tag{17.13b}$$

$$M_1(x_1, x_2) = \int_{-t/2}^{t/2} x_3 \tau_{11}\, dx_3 \tag{17.13c}$$

$$M_2(x_1, x_2) = \int_{-t/2}^{t/2} x_3 \tau_{22}\, dx_3 \tag{17.13d}$$

$$M_{21}(x_1, x_2) = \int_{-t/2}^{t/2} x_3 \tau_{12}\, dx_3 = M_{12}(x_1, x_2) \tag{17.13e}$$

$$N_1(x_1, x_2) = \int_{-t/2}^{t/2} \tau_{11}\, dx_3 = 0 \tag{17.13f}$$

$$N_2(x_1, x_2) = \int_{-t/2}^{t/2} \tau_{22}\, dx_3 = 0 \tag{17.13g}$$

Q_1 is the resultant shearing force acting on a plane perpendicular to the x_1 axis in the direction of the x_3 axis. It is taken per unit length in the x_2 direction. Q_2 is the resultant shearing force intensity acting on a plane perpendicular to the x_2 axis in the direction of the x_3 axis. It is taken per unit length in the x_1 direction. M_1 is the resultant bending moment intensity about the x_2 axis of the component of stress rotation τ_{11} acting on a plane perpendicular to the x_1 axis. It is taken per unit length in the x_2 direction. M_{12} is the resultant twisting moment intensity of the component of stress τ_{12} acting on a plane perpendicular to the x_1 axis. It is taken per unit length in the x_2 direction. M_2 is the resultant bending moment intensity about the x_1 axis of the component of stress τ_{22} acting on a plane perpendicular to the x_2 axis. It is taken per unit length in the x_1 direction. M_{21} is the resultant twisting moment of the component of stress τ_{21} acting on a plane perpendicular to the x_2 axis. It is taken per unit length in the x_1 direction. In Fig. 17.4 we show the resultant shearing force, bending moment and twisting moment intensities as they are assumed positive. N_i ($i = 1, 2$) is the resultant normal component of force intensity acting on the plane perpendicular to the x_i ($i = 1, 2$) axis. It is taken per unit length in the x_j ($j = 2, 1, j \neq i$) direction. The loads to which the plates under consideration are subjected do not produce such force intensities.

The quantities defined by relations (17.13) are also known as the *internal action intensities* of the plate

Substituting relations (17.12) into (17.13c) to (17.13e), we get

$$M_1 = -D\left(\frac{\partial^2 u_3}{\partial x_1^2} + \nu \frac{\partial^2 u_3}{\partial x_2^2}\right) - M^T = D\left(\frac{1}{r_1} + \frac{\nu}{r_2}\right) - M^T \tag{17.14a}$$

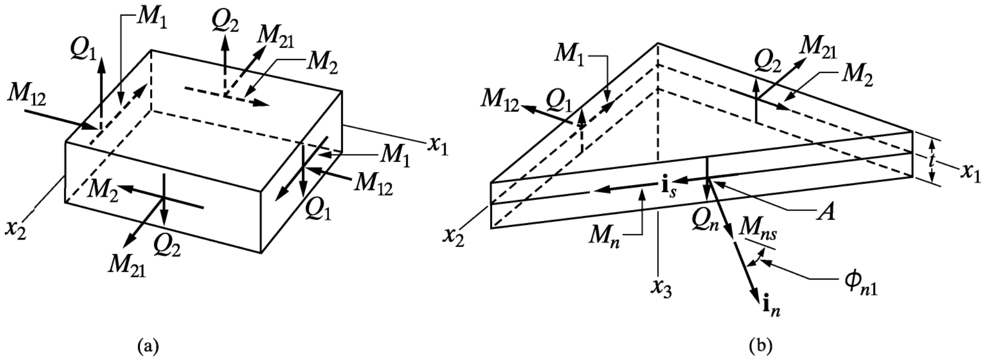


Figure 17.4 Positive internal actions.

$$M_2 = -D\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu \frac{\partial^2 u_3}{\partial x_1^2}\right) - M^T = D\left(\frac{1}{r_2} + \frac{\nu}{r_1}\right) - M^T \tag{17.14b}$$

$$M_{12} = -D(1-\nu)\frac{\partial^2 u_3}{\partial x_1 \partial x_2} = -\frac{D(1-\nu)}{r_{12}} \tag{17.14c}$$

where

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{Gt^3}{6(1-\nu)} \tag{17.15}$$

and

$$M^T = \frac{E\alpha}{1-\nu}\int_{-t/2}^{t/2} x_3(T - T_o)\,dx_3 \tag{17.16}$$

We assume that the variation of the temperature of the plate in the x_3 direction is linear. That is, denoting by T_c the temperature of the midplane of the plate, we have

$$T = Ax_3 + T_c \tag{17.17a}$$

Denoting by $T_3^{(+)}(x_1, x_2)$ and $T_3^{(-)}(x_1, x_2)$ the temperature of the faces $x_3 = t/2$ and $x_3 = -t/2$, respectively, of the plate, from relation (17.17a), we obtain

$$T^{(+)} = \frac{At}{2} + T_c \qquad T^{(-)} = -\frac{At}{2} + T_c \tag{17.17b}$$

or

$$A = \frac{T^{(+)} - T^{(-)}}{t} = \frac{\delta T_3}{t} \tag{17.17c}$$

We limit our attention to changes of *temperature which vanish at the midplane of the* *plate*. That is,

$$T_c = T_o \tag{17.17d}$$

Substituting relation (17.17c) into (17.17a) and using (17.17d), we get

$$T - T_o = T - T_c = \frac{x_3 \delta T_3}{t} \tag{17.17e}$$

Substituting relation (17.17c) into (17.16) and integrating, we obtain

$$M^T = \frac{E\alpha t^2 \delta T_3}{12(1 - \nu)} \tag{17.18}$$

Using relations (17.14) to eliminate u_3 from relations (17.12), we get

$$\tau_{11} = \frac{12(M_1 + M_T)x_3}{t^3} \tag{17.19a}$$

$$\tau_{22} = \frac{12(M_2 + M^T)x_3}{t^3} \tag*{(17.19a)}$$

$$\tau_{12} = \frac{12(1 - \nu)M_{12}x_3}{t^3} \tag{17.19c}$$

17.4 Internal Action Intensities Acting on Planes Which Are Inclined to the x_1 and x_2 Axes

Consider a plane normal to the midplane of a plate containing a point A of the midplane. As shown in Fig. 17.4b we denote by $\mathbf{i}_n$ and $\mathbf{i}_s$ the unit vectors normal and tangential to this plane at point A, respectively. The unit vectors $\mathbf{i}_n$, $\mathbf{i}_s$ and $\mathbf{i}_3$ specify a right-hand orthogonal system of directions. Referring to Fig. 17.4b we define the resultant shearing force intensity Q_n, bending moment intensity M_n and twisting moment M_{ns} intensity at point A acting on the plane normal to the unit vector $\mathbf{i}_n$, as

$$Q_n = \int_{-t/2}^{t/2} \tau_{n3}\, dx_3 \tag{17.20a}$$

$$M_n = \int_{-t/2}^{t/2} x_3\, \tau_{nn}\, dx_3 \tag{17.20b}$$

$$M_{ns} = \int_{-t/2}^{t/2} x_3\, \tau_{ns}\, dx_3 \tag{17.20c}$$

The components of stress τ_{nn} and τ_{ns} acting on the plane normal to the unit vector $\mathbf{i}_n$ are related to the components of stress τ_{11}, τ_{22} and τ_{12} acting on the planes normal to the unit vectors $\mathbf{i}_1$ and $\mathbf{i}_2$ by the transformation relations (1.116). That is,

$$\tau_{nn} = \frac{1}{2}(\tau_{11} + \tau_{22}) + \frac{1}{2}(\tau_{11} - \tau_{22}) \cos 2\,\phi_{11} + \tau_{12} \sin 2\,\phi_{11}$$
$$\tau_{ns} = -\frac{1}{2}(\tau_{11} - \tau_{22}) \sin 2\,\phi_{11} + \tau_{12} \cos 2\,\phi_{11} \tag{17.21}$$
$$\tau_{ss} = \frac{1}{2}(\tau_{11} + \tau_{22}) - \frac{1}{2}(\tau_{11} - \tau_{22}) \cos 2\,\phi_{11} - \tau_{12} \sin 2\,\phi_{11}$$

Substituting the relations (17.21) into (17.20b) and (17.20c) and taking into account relations (17.13c) to (17.13e), we obtain

$$M_n = \frac{1}{2}(M_1 + M_2) + \frac{1}{2}(M_1 - M_2) \cos 2\,\phi_{11} + M_{12} \sin 2\,\phi_{11}$$
$$M_{ns} = -\frac{1}{2}(M_1 - M_2) \sin 2\,\phi_{11} + M_{12} \cos 2\,\phi_{11} \tag{17.22}$$
$$M_s = \frac{1}{2}(M_1 + M_2) - \frac{1}{2}(M_1 - M_2) \cos 2\,\phi_{11} - M_{12} \sin 2\,\phi_{11}$$

Comparing relations (17.22) with (1.116), we see that the following matrix transforms as a planar symmetric tensor of the second rank:

$$\begin{bmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{bmatrix}$$

Consequently, we can use Mohr's circle to locate the planes of maximum and minimum bending and twisting moments as well as their values.

Referring to relations (17.14), and imaging that the x_1 and x_2 axes are oriented in the direction of unit vectors $\mathbf{i}_n$ and $\mathbf{i}_s$ shown in Fig. 17.4b $(x_1 \equiv x_n,\ x_2 \equiv x_s,$ subscripts $1 \rightarrow n$ and $2 \rightarrow s)$, we can deduce that

$$M_n = -D\left(\frac{\partial^2 u_3}{\partial x_n^{\,2}} + \nu \frac{\partial^2 u_3}{\partial x_s^{\,2}}\right) - M^T = D\left(\frac{1}{r_n} + \frac{\nu}{r_s}\right) - M^T \tag{17.23a}$$

$$M_s = -D\left(\frac{\partial^2 u_3}{\partial x_s^{\,2}} + \nu \frac{\partial^2 u_3}{\partial x_n^{\,2}}\right) - M^T = D\left(\frac{1}{r_s} + \frac{\nu}{r_n}\right) - M^T \tag{17.23b}$$

$$M_{ns} = -D(1 - \nu)\frac{\partial^2 u_3}{\partial x_n \partial x_s} = -\frac{D(1 - \nu)}{r_{ns}} \tag{17.23c}$$

17.5 Equations of Equilibrium for a Plate

Consider the free-body diagram of a segment of a plate, cut by two imaginary pairs of parallel planes $x_1 = x_1^*$, $x_1 = x_1^* + dx_1$ and $x_2 = x_2^*$, $x_2 = x_2^* + dx_2$, shown in Fig. 17.5. The segment does not include a concentrated force or any other discontinuity of the external forces acting on the faces of the plate. That is, it is only subjected to a transverse distributed forces $p_3(x_1, x_2)$ not shown in Fig. 17.5. From the equilibrium of this segment we have

$$\sum F_3 = 0 \quad -Q_1 dx_2 + \left(Q_1 + \frac{\partial Q_1}{\partial x_1} dx_1 \right) dx_2 - Q_2 dx_1 + \left(Q_2 + \frac{\partial Q_2}{\partial x_2} dx_2 \right) dx_1 + p_3 dx_1 dx_2 = 0 \tag{17.24a}$$

$$\begin{aligned} \sum M_2^{(A)} = 0 \quad & -M_1 dx_2 + \left(M_1 + \frac{\partial M_1}{\partial x_1} dx_1 \right) dx_2 - M_{21} dx_1 \\ & + \left(M_{21} + \frac{\partial M_{21}}{\partial x_2} dx_2 \right) dx_1 - Q_1 dx_2 dx_1 + p_3 dx_1 dx_2 \frac{dx_1}{2} = 0 \end{aligned} \tag{17.24b}$$

$$\begin{aligned} \sum M_1^{(A)} = 0 \quad & -M_2 dx_1 + \left(M_2 + \frac{\partial M_2}{\partial x_2} dx_2 \right) dx_1 - M_{12} dx_2 \\ & + \left(M_{12} + \frac{\partial M_{12}}{\partial x_1} dx_1 \right) dx_2 - Q_2 dx_1 dx_2 + p_3 dx_1 dx_2 \frac{dx_2}{2} = 0 \end{aligned} \tag{17.24c}$$

Disregarding infinitesimals of higher order and dividing relations (17.24) by $dx_1 dx_2$, we have

$$\frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} + p_3 = 0 \tag{17.25a}$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} = Q_1 \tag{17.25b}$$

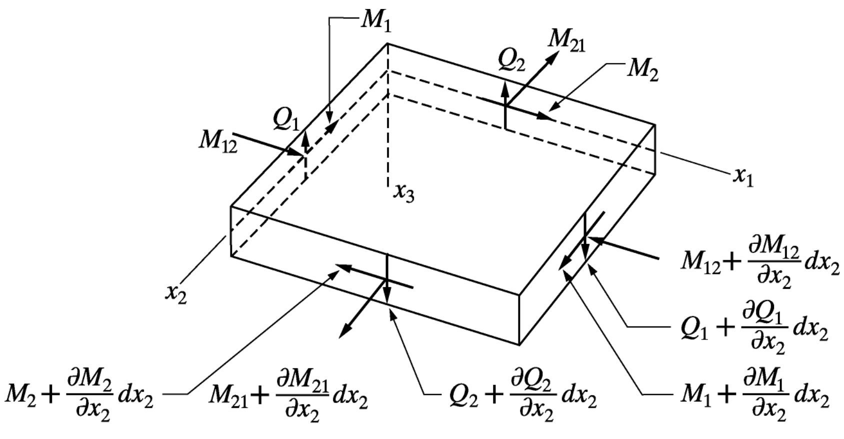


Figure 17.5 Resultant actions acting on a segment of infinitesimal length and width and thickness t of a plate.

$$\frac{\partial M_2}{\partial x_2} + \frac{\partial M_{21}}{\partial x_1} = Q_2 \tag{17.25c}$$

Substituting relations (17.14) into (17.25b) and (17.25c), we get

$$Q_1 = -D\frac{\partial}{\partial x_1}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) - \frac{\partial M^T}{\partial x_1} = -D\frac{\partial}{\partial x_1}(\nabla^2 u_3) - \frac{\partial M^T}{\partial x_1} \tag{17.26a}$$

$$Q_2 = -D\frac{\partial}{\partial x_2}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2}\right) - \frac{\partial M^T}{\partial x_2} = -D\frac{\partial}{\partial x_2}(\nabla^2 u_3) - \frac{\partial M^T}{\partial x_2} \tag{17.26b}$$

Moreover, referring to Fig. 17.4b, and imagining that the x_1 and x_2 axes are oriented in the direction of the unit vectors $\mathbf{i}_n$ and $\mathbf{i}_s$ from relations (17.26), we have

$$Q_n = -D\frac{\partial}{\partial x_n}(\nabla^2 u_3) - \frac{\partial M^T}{\partial x_n} \tag{17.26c}$$

$$Q_s = -D\frac{\partial}{\partial x_s}(\nabla^2 u_3) - \frac{\partial M^T}{\partial x_s} \tag{17.26d}$$

Differentiating relation (17.25b) with respect to x_1 and relation (17.25c) with respect to x_2, adding the resulting relations and using (17.25a) to eliminate Q_1 and Q_2, we obtain

$$\frac{\partial^2 M_1}{\partial x_1^2} + \frac{\partial^2 M_2}{\partial x_2^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + p_3 = 0 \tag{17.27}$$

Substituting relations (17.14) into (17.27), we get

$$\frac{\partial^4 u_3}{\partial x_1^4} + \frac{\partial^4 u_3}{\partial x_2^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} = \frac{p_3}{D} - \frac{1}{D}\left(\frac{\partial^2 M^T}{\partial x_1^2} + \frac{\partial^2 M^T}{\partial x_2^2}\right) \tag{17.28a}$$

or

$$\nabla^2(\nabla^2 u_3) = \nabla^4 u_3 = \frac{p_3}{D} - \frac{\nabla^2 M^T}{D} \tag{17.28b}$$

The operators ∇^2 and ∇^4 are known as the harmonic and the biharmonic operators, respectively, and are defined as

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_s^2} \tag{17.29a}$$

and

$$\nabla^4 = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4} \tag{17.29b}$$

The satisfaction of relation (17.28a) by the translation field $u_3(x_1, x_3)$ ensures that every segment of infinitesimal length and width and thickness t of a plate, which does not contain a point or a line of discontinuity of the external forces, is in equilibrium. The solution of equation (17.28a) involves two constants which are evaluated from appropriate boundary conditions, described in the next section.

On the basis of the foregoing presentation, the boundary value problem for computing the deflection of a plate may be stated as follows. Find a single-valued function $u_3(x_1, x_2)$ which has continuous first derivatives and satisfies the differential equations (17.28a) at every point of the midsurface of the plate and appropriate boundary conditions on its edges.

In the following section we present the boundary conditions that the deflection and internal actions of plates must satisfy at their edges.

17.6 Boundary Conditions for Plates

Consider a rectangular plate and choose the x_1 and x_2 axes parallel to its edges. In what follows we describe some of the most common boundary conditions for the edge $x_2 = L_2$ of the plate.

17.6.1 Build-In Edge $x_2 = L_2$

It is clear that the boundary conditions for such an edge are

$$u_3(x_1, L_2) = 0 \tag{17.30a}$$

$$\left.\frac{\partial u_3}{\partial x_2}\right|_{x_2 = L_2} = 0 \tag{17.30b}$$

17.6.2 Simply Supported Edge $x_2 = L_2$

A simply supported edge at $x_2 = L_2$ is restrained from moving in the x_3 direction but is free to rotate about the x_1 axis. Thus, its boundary conditions are

$$u_3(x_1, L_2) = 0 \tag{17.31a}$$

$$M_2(x_1, L_2) = -D\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2}\right)_{x_2 = L_2} - M^T = 0 \tag{17.31b}$$

Inasmuch as $u_3(x_1, L_2)$ does not change along $x_2 = L_2$, we have

$$\left.\frac{\partial^2 u_3}{\partial x_1^2}\right|_{x_2 = L_2} = 0 \tag{17.32}$$

Consequently, the boundary conditions (17.31) for the simply supported edge $x_2 = L_2$ of a plate reduce to

$$u_3(x_2, L_2) = 0 \tag{17.33a}$$

$$\left.\frac{\partial^2 u_3}{\partial x_2^2}\right|_{x_2 = L_2} + \frac{M^T(x_1, L_2)}{D} = 0 \tag{17.33b}$$

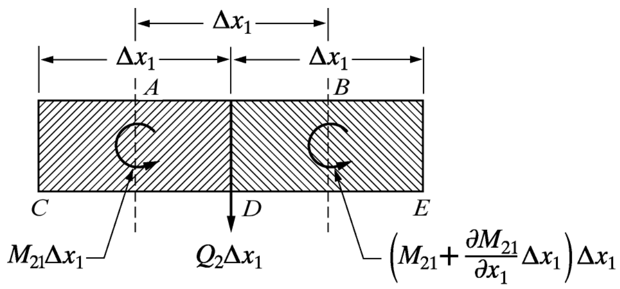
17.6.3 Unsupported Free Edge $x_2 = L_2$

If the edge $x_2 = L_2$ of a plate is unsupported and free, the bending moment, twisting moment or shearing force intensity along this edge must vanish. That is,

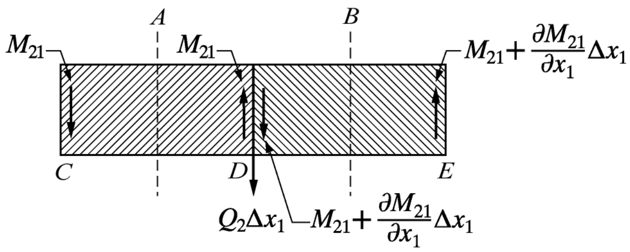
$$M_2(x_1, L_2) = 0 \qquad M_{21}(x_1, L_2) = 0 \qquad Q_2(x_1, L_2) = 0$$

However, only two boundary conditions are allowed for an equation of fourth order like equations (17.28). This paradox can be resolved by referring to Fig. 17.6, where we show a portion of the lateral surface $x_2 = L_2$ of a plate divided into two equal segments of length Δx_1 each. The twisting moment acting on segment CD is denoted as $M_{21}\Delta x_1$ while the twisting moment acting on segment DE is denoted by $[M_{21} + (\partial M_{21}/\partial x_1)\Delta x_1]\Delta x_1$. In Fig. 17.6a we also show a third segment AB of length Δx_1 and we denote the shearing force acting on this segment by $Q_2\Delta x_1$. In Fig.13.6b we replace the twisting moment acting on each of the segments CD and DE by a statically equivalent couple consisting of two equal and opposite shearing forces $Q_2^{CD}\Delta x_1$ and $Q_2^{DE}\Delta x_1$ of magnitude

$$Q_2^{CD} = \frac{M_{21}}{\Delta x_1} \tag{17.34}$$



(a) Twisting moment intensity acting on a portion of the boundary $x_2 = L_2$



(b) Effective shearing force intensity Q_2 for a portion of the boundary $x_2 = 0$

Figure 17.6 Portion of the lateral boundary $x_2 = 0$ of a plate.

$$Q_2^{DE} = \left(M_{21} + \frac{\partial M_{21}}{\partial x_1}\Delta x_1 \right) \frac{1}{\Delta x_1} \tag{17.35}$$

Referring to Fig. 17.6b, we see that the effective shearing force Q_2^e acting on segment AB is equal to

$$Q_2^e = Q_2 - Q_2^{CD} + Q_2^{DE} = Q_2 - \frac{M_{21}}{\Delta x_1} + \left(M_{21} + \frac{\partial M_{21}}{\partial x_1}\Delta x_1 \right) \frac{1}{\Delta x_1} \tag{17.36}$$

or

$$Q_2^e = Q_2 + \frac{\partial M_{21}}{\partial x_1} \tag{17.37}$$

Thus, the boundary conditions for an unsupported free edge ($x_2 = L_2$) of a plate are

1. The effective shearing force $Q_2^e(x_1, L_2)$ vanishes.

2. The moment $M_2(x_1, L_2)$ vanishes.

Substituting relations (17.14c) and (17.26b) into (17.37), we get

$$Q_2^e = -D\frac{\partial}{\partial x_2}\left[\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + (1-\nu)\frac{\partial^2 u_3}{\partial x_1^2}\right] - \frac{\partial M^T}{\partial x_2} = -D\frac{\partial}{\partial x_2}\left[\frac{\partial^2 u_3}{\partial x_2^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_1^2}\right] - \frac{\partial M^T}{\partial x_2} \tag{17.38a}$$

Similarly, we obtain

$$Q_1^e = -D\frac{\partial}{\partial x_1}\left[\frac{\partial^2 u_3}{\partial x_1^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_2^2}\right] - \frac{\partial M^T}{\partial x_n} \tag{17.38b}$$

For an edge which is normal to the unit vector $\mathbf{i}_n$ referring to Fig. 17.4b, we have

$$Q_n^e = -D\frac{\partial}{\partial x_n}\left[\frac{\partial^2 u_3}{\partial x_n^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_s^2}\right] - \frac{\partial M^T}{\partial x_n} \tag{17.38c}$$

On the basis of relations (17.14b) and (17.38a) the boundary conditions for an unsupported free edge ($x_2 = L_2$) of a plate are

$$Q_2^e = Q_2 + \frac{\partial M_{21}}{\partial x_1} = \left[-D\frac{\partial}{\partial x_2}\left[\frac{\partial^2 u_3}{\partial x_2^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_1^2}\right] - \frac{\partial M^T}{\partial x_2} \right]_{x_2=L_2} = 0 \tag{17.39a}$$

$$M_2 = \left[-D\left[\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2}\right] - M^T \right]_{x_2=L_2} = 0 \tag{17.39b}$$

17.6.4 Plate Rigidly Built into a Supporting Beam at $x_2 = L_2$

Consider a plate whose edge at $x_2 = L_2$ is rigidly connected to a supporting beam (see Fig. 17.7a). The deflection of the plate at $x_2 = L_2$ is equal to the deflection of the beam $u_3^B(x_1)$. That is,

$$u_3(x_1, L_2) = u_3^B(x_1) = u_3(x_1) \tag{17.40}$$

Moreover, the rotation per unit length $\theta_2(x_1, L_2)$ of the plate is equal to the angle of twist per unit length α of the beam. Referring to relation (6.53), the angle of twist per unit length α of the beam of Fig. 17.7a can be expressed as

$$\alpha(x_1) = \frac{d\theta_1^B}{dx_1} = \frac{M_1^B}{GR_C} \tag{17.41a}$$

where $M_1^B(x_1)$ is the internal torsional moment acting on the cross sections of the beam. $\theta_1^B(x_1)$ is the angle of twist of the cross sections of the beam. R_C is called the torsional constant and depends only on the geometry of the cross section of the beam. For a beam of rectangular cross section of width b and depth d it is given in Table 6.1, as

$$R_C = \frac{K_1 db^3}{3} \tag{17.41b}$$

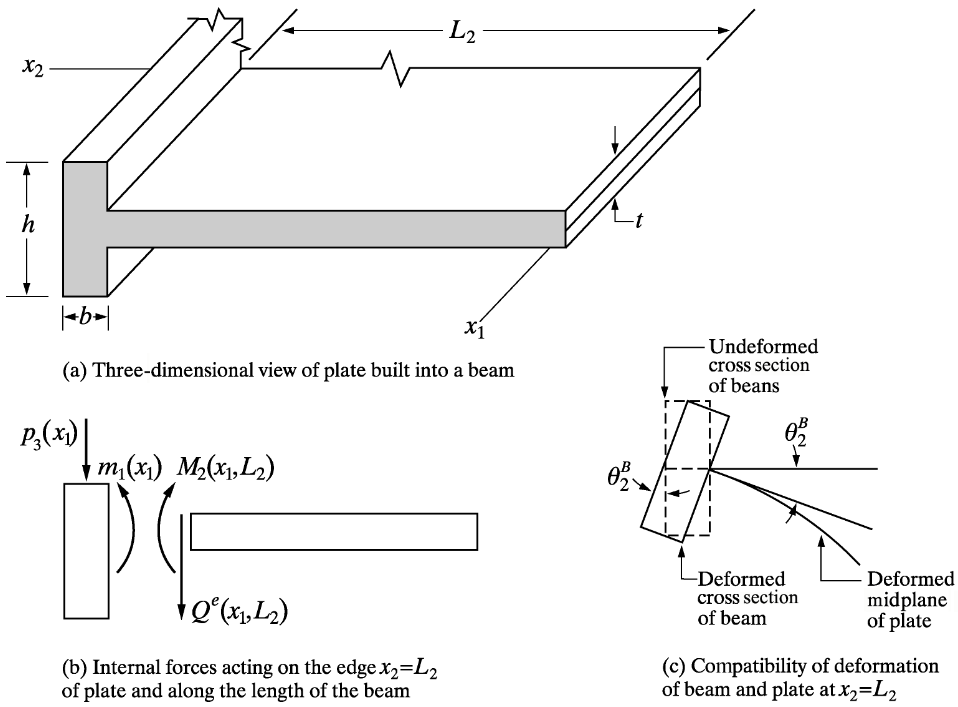


Figure 17.7 Plate rigidly built into an elastic beam.

The coefficient K_1 is given in Table 6.1 for various values of d/b. From the equilibrium of a segment of infinitesimal length of the beam referring to relation (8.19), we have

$$\frac{dM_1^B}{dx_1} = -m_1(x_1) \tag{17.42}$$

where m_1 is the external torsional moment per unit length acting on the beam. Substituting relation (17.41a) into (17.42), we get

$$\frac{d}{dx_1}\left(GR_C\frac{d\theta_1^B}{dx_1}\right) = -m_1(x_1) \tag{17.43}$$

Referring to Fig. 17.7b, we see that

$$M_2(x_1, L_2) = m_1(x_1) \tag{17.44}$$

Moreover, referring to Fig. 17.7c, we see that

$$\theta_1^B(x_1) = -\left(\frac{\partial u_3}{\partial x_2}\right)_{x_2=L_2} \tag{17.45}$$

Substituting relations (17.44) and (17.45) into (17.43), we obtain

$$\left[\frac{d}{dx_1}\left(GR_C\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)\right]_{x_2=L_2} = M_2(x_1, L_2)$$

Substituting relation (17.14b) into the above, we have

$$\left[\frac{d}{dx_1}\left(GR_C\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)\right]_{x_2=L_2} = -D\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial u_3}{\partial x_1^2} + \frac{M^T}{D}\right)_{x_2=L_2} \tag{17.46}$$

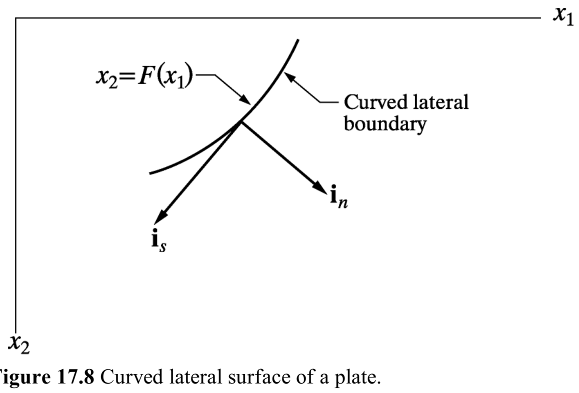


Figure 17.8 Curved lateral surface of a plate.

Finally, referring to Fig. 17.7b, we see that

$$Q_2^e(x_1, L_2) = -p_3(x_1) \tag{17.47}$$

Referring to relation (9.34b), the deflection of the beam must satisfy the following relation:

$$\frac{\partial^2}{\partial x_1^2}\left(EI_2\frac{\partial^2 u_3}{\partial x_1^2}\right)_{x_2=L_2} = p_3(x_1) \tag{17.48}$$

Substituting relation (17.38a) and (17.48) into (17.47), we obtain

$$\left[\frac{\partial^2}{\partial x_1^2}\left(EI_2\frac{\partial^2 u_3}{\partial x_1^2}\right)\right]_{x_1=L_2} = \left[D\frac{\partial}{\partial x_2}\left(\frac{\partial^2 u_3}{\partial x_2^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_1^2}\right) + \frac{\partial M^T}{\partial x_2}\right]_{x_2=L_2} \tag{17.49}$$

Thus, relations (17.46) and (17.49) are the two boundary conditions for the edge $x_2 = L_2$ of a plate which is rigidly connected to a beam. The boundary conditions for a straight edge $x_2 = L_2$ of a plate are tabulated in Table 17.1. The boundary conditions for plates with curved lateral surfaces can be obtained from the boundary conditions for straight lateral surfaces described above by referring the Fig. 17.8 and relations (17.23), (17.26c) and (17.26d). They are tabulated in Table 17.1.

17.6.5 Corner Forces of Rectangular Plates

As shown in Fig. 17.9 concentrated reacting forces of magnitude $2M_{12}$ are required at the corners of rectangular plates in order to keep them in equilibrium. Referring to relation (17.14c) these corner forces are equal to

$$R_{\text{corner}} = 2M_{12} = -2D(1-\nu)\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \tag{17.50}$$

In the above relation positive corner forces are acting in the direction shown in Fig. 17.9b.

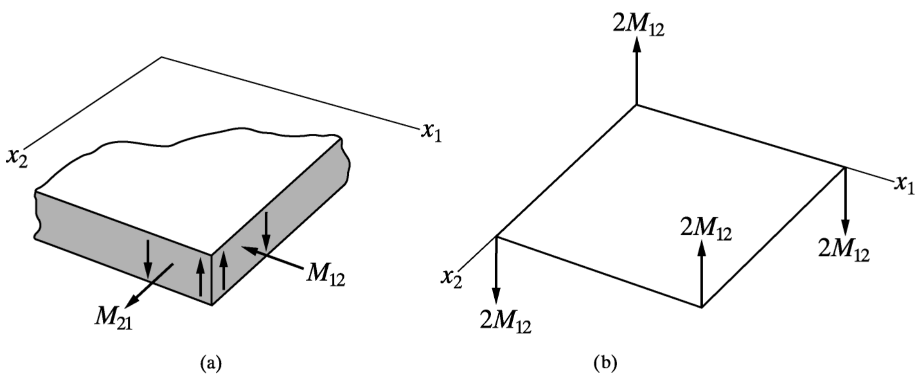


Figure 17.9 Concentrated reacting forces acting on the corners of a rectangular plate when M_{12} is assumed positive.

Table 17.1 Boundary conditions for plates.

Type of Boundary	Straight edge $x_2 = L_2$	Curved Edge $x_2 = f(x_1)$ with Unit Vectors Outward Normal $\mathbf{i}_n$ and Tangential $\mathbf{i}_s$
Build in edge	$u_3(x_1, L_2) = 0$ $\left.\frac{\partial u_3}{\partial x_2}\right\|_{x_2=L_2} = 0$	$\left.u_3\right\|_{x_2=f(x_1)} = 0$ $\left.\frac{\partial u_3}{\partial x_n}\right\|_{x_2=f(x_1)} = 0$
Simply supported edge	$u_3(x_1, L_2) = 0$ $\left.\frac{\partial^2 u_3}{\partial x_2^2}\right\|_{x_2=L_2} + \frac{M^T}{D} = 0$	$\left.u_3\right\|_{x_2=f(x_1)} = 0$ $\frac{M_n}{D} = \left(-\left(\frac{\partial^2 u_3}{\partial x_n^2} + \nu\frac{\partial^2 u_3}{\partial x_t^2} \right) - \frac{M^T}{D} \right)_{x_2=f(x_1)}$
Un-supported free edge	$\frac{Q_2^e}{D} = -\left[\frac{\partial}{\partial x_2}\left[\frac{\partial^2 u_3}{\partial x_2^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_1^2}\right] + \frac{M^T}{D}\right]_{x_2=L_2} = 0$ $\frac{M_2}{D} = -\left[\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2} + \frac{M^T}{D}\right]_{x_2=L_2} = 0$	$\frac{Q_n^e}{D} = \left[-\frac{\partial}{\partial x_n}\left[\frac{\partial^2 u_3}{\partial x_n^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_s^2}\right] - \frac{M^T}{D}\right]_{x_2=f(x_1)} = 0$ $\frac{M_n}{D} = -\left[\left[\frac{\partial^2 u_3}{\partial x_n^2} + \nu\frac{\partial^2 u_3}{\partial x_s^2}\right] + \frac{M^T}{D}\right]_{x_2=f(x_1)} = 0$
Edge build into a beam	$\left[\frac{d}{dx_1}\left(GR_C\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)\right]_{x_2=L_2}$ $= -\left[D\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial u_3}{\partial x_1^2}\right) + M^T\right]_{x_2=L_2}$ $\left[\frac{\partial^2}{\partial x_1^2}\left(EI_2\frac{\partial^2 u_3}{\partial x_1^2}\right)\right]_{x_2=L_2}$ $= \left[D\frac{\partial}{\partial x_2}\left(\frac{\partial^2 u_3}{\partial x_2^2} + (2-\nu)\frac{\partial^2 u_3}{\partial x_1^2}\right) + \frac{\partial M^T}{\partial x_2}\right]_{x_2=L_2}$	

17.7 Analysis of Simply Supported Rectangular Plates Subjected to a General Distribution of Transverse Forces

The deflection, the moments and the shearing forces of plates of any geometry supported in any way and subjected to any loads can be established using available market computer programs. In these programs the boundary value problems presented in the previous sections are solved approximately usually employing the finite element method (see Chapter 15).

In this section we present briefly the solution of boundary value problems for computing the deflection, the moments and the shearing forces of simply supported rectangular plates using double Fourier series.

Consider a simply supported rectangular plate in a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_0. Subsequently, the plate is subjected to a general distribution of transverse forces $p_3(x_1, x_2)$ and reaches a second state of mechanical and thermal equilibrium at the uniform temperature T_0. The deflection $u_3(x_1, x_2)$ of this plate must satisfy the differential equation (17.28) at every point of the midplane of the plate, and referring to relations (17.33) and to Fig. 17.10, the following boundary conditions:

$$\begin{aligned}
u_3(0, x_2) &= 0 & \left.\frac{d^2u_3}{dx_1^2}\right|_{x_1=0} &= 0 \\
u_3(L_1, x_2) &= 0 & \left.\frac{d^2u_3}{dx_1^2}\right|_{x_1=L_1} &= 0 \\
u_3(x_1, 0) &= 0 & \left.\frac{d^2u_3}{dx_2^2}\right|_{x_2=0} &= 0 \\
u_3(x_1, L_2) &= 0 & \left.\frac{d^2u_3}{dx_2^2}\right|_{x_2=L_2} &= 0
\end{aligned} \tag{17.51}$$

All these boundary conditions (17.51) are satisfied if we represent the deflection $u_3(x_1, x_2)$of the plate by a double Fourier series. That is,

$$u_3(x_1, x_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) \tag{17.52}$$

Substituting relation (17.52) into (17.28), we get

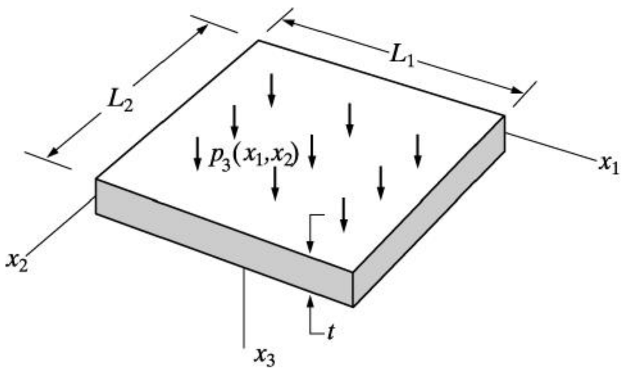


Figure 17.10 Geometry and loading of the plate.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\left(\frac{m\pi}{L_1}\right)^4 + 2\left(\frac{m\pi}{L_1}\right)^2\left(\frac{n\pi}{L_2}\right)^2 + \left(\frac{n\pi}{L_2}\right)^4\right]\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) = \frac{p_3(x_1, x_2)}{D} \tag{17.53}$$

From this equation we see that if $p_3(x_1, x_2)$ were represented by a double Fourier series it could be possible to compute A_{mn} by matching coefficients. Thus,

$$p_3(x_1, x_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{mn}\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) \tag{17.54}$$

Multiplying both sides of the above relation by $\sin(q_1\pi x_1/L_1)\sin(q_2\pi x_2/L_2)dx_1dx_2$ integrating the resulting relation and making use of the orthogonality property† of the sine functions, we get

$$p_{mn} = \frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} p_3(x_1, x_2)\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) dx_1 dx_2 \tag{17.55}$$

Substituting relation (17.54) into (17.53), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\left[\left(\frac{m\pi}{L_1}\right)^4 + 2\left(\frac{m\pi}{L_1}\right)^2\left(\frac{n\pi}{L_2}\right)^2 + \left(\frac{n\pi}{L_2}\right)^4\right] - \frac{p_{mn}}{D}\right]\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) = 0$$

Since the above relation must be valid for all values of x_1 and x_2, the coefficients of $\sin(m\pi x_1/L_1)\sin(n\pi x_2/L_2)$ must vanish. Hence,

$$A_{mn} = \frac{p_{mn}}{\pi^4 D\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2} \tag{17.57}$$

Substituting relation (17.57) into (17.52), we have

$$u_3(x_1, x_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_{mn}}{\pi^4 D\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) \tag{17.58}$$

Expressions for the moments and the shearing forces may be established by substituting relation (17.58) into (17.14) and (17.26), respectively.

The afore described method is straightforward. However, the double infinite series

† The orthogonality property of the sine functions is

$$\int_0^{L_i}\sin\frac{m\pi x_i}{L_i}\sin\left(\frac{q_1\pi x_i}{L_i}\right)dx_i = \begin{cases} 0 & \text{if } m \neq q_1 \\ \frac{L_i}{2} & \text{if } m = q_1 \end{cases} \tag{17.56}$$

that appear in the expression for the moments and the shearing forces of plates subjected to certain loads may not converge very fast.

In what follows we present two examples.

Example 1 Compute the deflection $u_3(x_1, x_2)$ and the normal components of stress of the simply supported rectangular plate whose geometry is shown in Fig. a, when subjected to a uniformly distributed transverse force p_3. The plate is made from an isotropic, linearly elastic material.

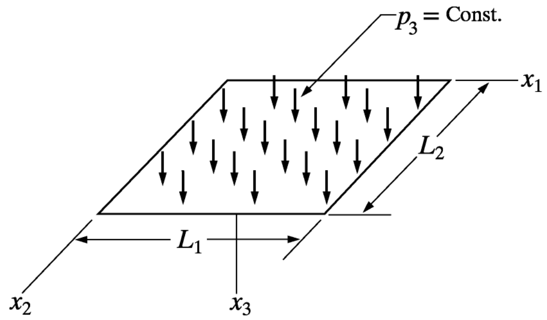


Figure a Geometry of the plate.

Solution Referring to relation (17.55), we obtain

$$p_{mn} = \frac{4p_3}{L_1 L_2}\int_0^{L_1}\int_0^{L_2} \sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right) dx_1 dx_2 = \frac{16p_3}{\pi^2 mn} \tag{a}$$

where m and n are odd integers. Substituting relation (a) into (17.58), we get

$$u_3(x_1, x_2) = \frac{16p_3}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right)}{mn\left[\left(\frac{m}{L_1}\right)^2+\left(\frac{n}{L_2}\right)^2\right]^2} \tag{b}$$

Substituting relation (b) into (17.14), we have

$$M_1(x_1, x_2) = \frac{16p_3}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{\left(\frac{m}{L_1}\right)^2+\nu\left(\frac{n}{L_2}\right)^2}{mn\left[\left(\frac{m}{L_1}\right)^2+\left(\frac{n}{L_2}\right)^2\right]^2}\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right)\right] \tag{c}$$

$$M_2(x_1, x_2) = \frac{16p_3}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{\left(\frac{n}{L_2}\right)^2 + \nu\left(\frac{m}{L_1}\right)^2}{mn\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right)\right] \tag{d}$$

$$M_{12}(x_1, x_2) = -\frac{16(1-\nu)p_3}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{1}{L_1L_2\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\cos\left(\frac{m\pi x_1}{L_1}\right)\cos\left(\frac{n\pi x_2}{L_2}\right)\right] \tag{e}$$

For any given simply supported rectangular plate the series involved in relation (b) to (e) may be evaluated with the help of an electronic computer.

Referring to relations (b), (c) and (d), we see that the maximum deflection and the maximum bending moments occur at $x_1 = L_1/2$ and $x_2 = L_2/2$ and are equal to

$$(u_3)_{\max} = \frac{16p_3}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{(1)^{\frac{m+n}{2}-1}}{mn\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2} \tag{f}$$

$$(M_1)_{\max} = \frac{16p_3}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{\left[\left(\frac{m}{L_1}\right)^2 + \nu\left(\frac{n}{L_2}\right)^2\right](1)^{\frac{m+n}{2}-1}}{mn\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2} \tag{g}$$

$$(M_2)_{\max} = \frac{16p_3}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{\left[\left(\frac{n}{L_2}\right)^2 + \nu\left(\frac{m}{L_1}\right)^2\right](1)^{\frac{m+n}{2}-1}}{mn\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2} \tag{h}$$

For $m = 1$ $n = 1$

$$(u_3)_{\max} = \frac{16p_3}{\pi^6 D}K_u \qquad (M_1)_{\max} = \frac{16p_3}{\pi^4}K_1 \qquad (M_2)_{\max} = \frac{16p_3}{\pi^4}K_2 \tag{i}$$

where

$$K_u = \frac{1}{\left(\frac{1}{L_1^2} + \frac{1}{L_2^2}\right)^2} \qquad K_1 = \left(\frac{1}{L_1^2} + \frac{\nu}{L_2^2}\right)K_u \qquad K_2 = \left(\frac{1}{L_2^2} + \frac{\nu}{L_1^2}\right)K_u \tag{j}$$

The contributions of the term with $m = 3$ $n = 3$ of the series in relations (f),(g) and (h) to the maximum deflection and moments on the plate are

$$\Delta(u_3)_{max} = \frac{16p_3K_u}{\pi^6 D(729)} \qquad \Delta(M_1)_{max} = \frac{16p_3K_1}{\pi^4(81)} \qquad (M_2)_{max} = \frac{16p_3K_2}{\pi^4(81)} \tag{k}$$

As can be seen from relation (k) the series in relations (f), (g) and (h) converge fast. In fact it can be shown that accurate results can be obtained for the deflection of a simply supported rectangular plate by retaining the first term of the series of relation (f). Moreover, accurate results can be obtained for the bending moments by retaining the first six terms [($m = n = 1$), ($m = 1$, $n = 3$), ($n = 1$, $m = 3$), ($m = 3$, $n = 3$), ($m = 1$, $n = 5$), ($n = 1$, $m = 5$)] of these series.

For a square plate ($L_1 = L_2 = L$), retaining the first four terms in the series of relations (f), (g) and (h), and using $\nu = 0.3$, we get

$$(u_3)_{max} = \frac{0.0443 p_3 L^4}{Et^3} \qquad (M_1)_{max} = (M_2)_{max} = 0.0472 p_3 L^2 \tag{l}$$

The maximum normal component of stress acting on the planes $x_1 = L_1/2$ or $x_2 = L_2/2$ occurs at $x_3 = \pm t/2$. Referring to relations (17.19) and (l) for a square plate it is equal to

$$(\tau_{11})_{max} = \frac{6(M_1)_{max}}{t^2} = \frac{0.2832 p_3 L^2}{t^2} = (\tau_{22})_{max} \tag{m}$$

Substituting relation (b) into (17.38a) and (17.38b), we obtain

$$\begin{aligned} Q_1^e(x_1, x_2) &= -D\left[\frac{\partial^3 u_3}{\partial x_1^3} + (2-\nu)\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2}\right] \\ &= \frac{16p_3}{\pi^3 L_1}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\left[\frac{\left(\frac{m}{L_1}\right)^2 + (2-\nu)\left(\frac{n}{L_2}\right)^2}{n\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\right]\cos\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right)\right] \end{aligned} \tag{n}$$

$$\begin{aligned} Q_2^e(x_1, x_2) &= -D\left[\frac{\partial^3 u_3}{\partial x_2^3} + (2-\nu)\frac{\partial^3 u_3}{\partial x_2 \partial x_1^2}\right] \\ &= \frac{16p_3}{\pi^3 L_2}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\left[\frac{\left(\frac{n}{L_2}\right)^2 + (2-\nu)\left(\frac{m}{L_1}\right)^2}{m\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\right]\sin\left(\frac{m\pi x_1}{L_1}\right)\cos\left(\frac{n\pi x_2}{L_2}\right)\right] \end{aligned} \tag{o}$$

We denote by R_1^0 and $R_1^{L_1}$ the resultants of the reacting forces exerted on the plate by its supports, at $x_1 = 0$ and $x_1 = L_1$, respectively. Moreover, we denote by R_2^0 and $R_2^{L_2}$ the resultants of the reacting forces exerted on the plate by its supports on its edges at $x_2 = 0$ and $x_2 = L_2$, respectively. We consider the resultants of the reacting forces positive when they act in the negative x_3 direction. Referring to relations (o) and (n), these forces are equal to

$$\begin{aligned}
R_1^0 &= \int_{x_2=0}^{x_2=L_2} Q_1^e(0, x_2)dx_2 = \frac{32p_3L_2}{\pi^4 L_1}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{\left(\frac{m}{L_1}\right)^2 + (2-\nu)\left(\frac{n}{L_2}\right)^2}{n^2\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\right] \\
R_1^{L_1} &= -\int_{x_2=0}^{x_2=L_2} Q_1^e(L_1, x_2)dx_2 = \frac{32p_3L_2}{\pi^4 L_1}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{\left(\frac{m}{L_1}\right)^2 + (2-\nu)\left(\frac{n}{L_2}\right)^2}{n^2\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\right] \\
R_2^0 &= \int_{x_1=0}^{x_1=L_1} Q_2^e(x_1, 0)dx_1 = \frac{32p_3L_1}{\pi^4 L_2}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{\left(\frac{n}{L_2}\right)^2 + (2-\nu)\left(\frac{m}{L_1}\right)^2}{m^2\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\right] \\
R_2^{L_2} &= -\int_{x_1=0}^{x_1=L_1} Q_2^e(x_1, L_2)dx_1 = \frac{32p_3L_1}{\pi^4 L_2}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left[\frac{\left(\frac{n}{L_2}\right)^2 + (2-\nu)\left(\frac{m}{L_1}\right)^2}{m^2\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]^2}\right]
\end{aligned} \tag{p}$$

Referring to Fig. 17.9a and to relations (17.50) and (e), and considering the reacting forces positive when they act in the positive x_3 direction, we have

$$\begin{aligned}
R_{\text{corner}}(0, 0) &= -2M_{12} = R_{\text{corner}}(L_1, L_2) = -R_{\text{corner}}(0, L_2) \\
&= \frac{32(1-\nu)p_3}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{L_1L_2\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]}
\end{aligned} \tag{q}$$

Referring to Fig. 17.9b and to relations (q) we see that all the corner forces are acting in the direction of the positive x_3 axis. From relation (p) and (q) we find that the resultant of all the reacting forces acting on the plate by its supports is equal to $p_2 L_1L_2$. That is, the sum of the reacting forces acting on the plate in the direction of the negative x_3 axis is equal, as it should be, to the resultant of the external forces ($p_3L_1L_2$) acting on the face of the plate in the direction of the positive x_3 axis.

Example 2 Compute the deflection $u_3(x_1, x_2)$ of a simply supported plate subjected to a concentrated force P_3 as shown in Fig. a. The plate is made from an isotropic, linearly elastic material.

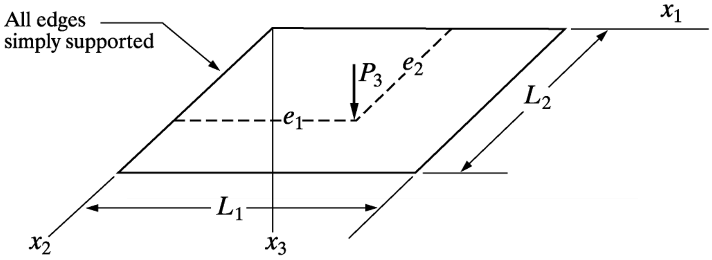


Figure a Geometry and loading of the plate.

Solution Referring to relation (9.37) and to Appendix G we can convert the concentrated force P_3 to mathematically equivalent distributed forces $p_3(x_1)$ as follows

$$p_3(x_1) = P_3\delta(x_1 - e_1)\delta(x_2 - e_2) \tag{a}$$

Substituting relation (a) into (17.55) and using relation (G.12), we obtain

$$\begin{aligned} p_{m,n} &= \frac{4P_3}{L_1L_2}\int_0^{L_2}\int_0^{L_1}\delta(x_1 - e_1)\delta(x_2 - e_2)\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right)dx_1dx_2 \\ &= \frac{4P_3}{L_1L_2}\sin\left(\frac{m\pi e_1}{L_1}\right)\sin\left(\frac{n\pi e_2}{L_2}\right) \end{aligned} \tag{b}$$

Substituting relation (b) into (17.58), we get

$$u_3(x_1, x_2) = \frac{4P_3}{L_1L_2}\sum_{m=1,2}^{\infty}\sum_{n=1,2}^{\infty}\frac{\sin\left(\frac{m\pi e_1}{L_1}\right)\sin\left(\frac{n\pi e_2}{L_2}\right)\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_1}{L_2}\right)}{\pi^4 D\left[\left(\frac{m}{L_1}\right)^2 + \left(\frac{n}{L_2}\right)^2\right]} \tag{c}$$

Expressions for the moments are obtained by substituting relation (c) into (17.14).

17.8 The Method of Levy for Computing the Deflection of Rectangular Plates Having a Simply Supported Pair of Parallel Edges

In this section we present the method of the French mathematician Levy which gives expressions for the deflection the bending and twisting moments, and the shears forces, which involve single infinite series. These series usually converge faster than the double series involved in the expression for the same quantities obtained using the method of double Fourier series presented in the previous section. Referring to Fig. 17.11 the method

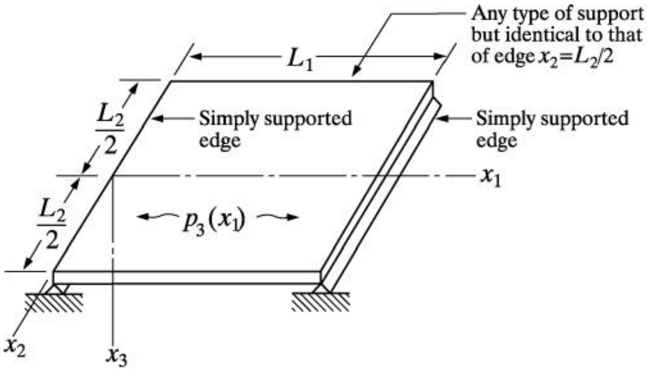


Figure 17.11 Rectangular plate subjected to a load $p_3(x_1)$.

of Levy can be employed to compute the deflection $u_3(x_1, x_2)$ of rectangular plates having the following attributes:

(a) They are simply supported along two opposite edges say $x_1 = 0$ and $x_1 = L_1$.
(b) They have identical boundary conditions along their other two edges ($x_2 = \pm L_2/2$).
(c) They are subjected to a distribution of transverse forces which does not vary in the x_2 direction.

In the method of Levy the deflection $u_3(x_1, x_2)$ of the plate is expressed as the sum of two parts

$$u_3(x_1, x_2) = u_3^{\text{aux}}(x_1) + u_3^{\text{cor}}(x_1, x_2) \tag{17.59}$$

$u_3^{\text{aux}}(x_1)$ is the deflection of an auxiliary plate simply supported along its edges $x_1 = 0$ and $x_1 = L_1$ while its other edges are unsupported and free. The auxiliary plate is subjected to the given forces and change of temperature if any acting on the actual plate. $u_3^{\text{cor}}(x_1)$ is the correction which must be added to $u_3^{\text{aux}}(x_1)$ in order to obtain the deflection of the actual plate. Referring to relation (17.28b) the function $u_3^{\text{aux}}(x_1)$ must satisfy the following ordinary deferential equation:

$$\nabla^4 u_3^{\text{aux}} = \frac{d^4 u_3^{\text{aux}}}{dx_1^4} = \frac{p_3}{D} \tag{17.60}$$

The solution of this equation involves four constants which are evaluated from the boundary conditions of the plate at $x_1 = 0$ and $x_1 = L_1$. That is, referring to relations (17.33) we have

$$\begin{aligned} u_3^{\text{aux}}(0) &= 0 \qquad & u_3^{\text{aux}}(L_1) &= 0 \\ \left.\frac{d^2 u_3^{\text{aux}}}{dx_1^2}\right|_{x_1=0} &= 0 \qquad & \left.\frac{d^2 u_3^{\text{aux}}}{dx_1^2}\right|_{x_1=L_1} &= 0 \end{aligned} \tag{17.61}$$

We shall seek the solution of the boundary value problem for computing $u_3^{\text{aux}}(x_1)$ using Fourier series. Every periodic function $f(x_1)$ can be expanded into such series. That is,

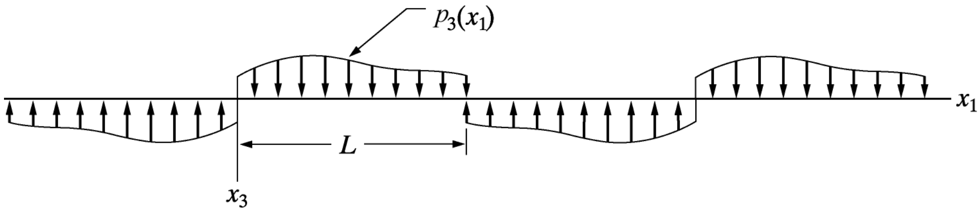


Figure 17.12 Periodic loading.

$$f(x_1) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx_1) + b_n \sin(nx_1)] \tag{17.62}$$

The load $p_3(x_1)$ can be considered as being periodic by imagining that it extends beyond the span of the plate as shown in Fig. 17.12. This extension of $p_3(x_1)$ deflects the beam in a way that is compatible with its boundary conditions. Thus, it can be expanded in the following Fourier series:

$$p_3(x_1) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x_1}{L_1}\right) \tag{17.63}$$

Multiplying both sides of relation (17.63) by $\sin(m\pi x_1/L_1)$ $(m = 1, 2, 3, ...)$ and integrating from $x_1 = 0$ to $x_1 = L_1$, we get

$$\int_0^{L_1} p_3(x_1)\sin\left(\frac{m\pi x_1}{L_1}\right) dx_1 = \sum_{n=1}^{\infty} b_n \int_0^{L_1} \sin\left(\frac{n\pi x_1}{L_1}\right)\sin\left(\frac{m\pi x_1}{L_1}\right) dx_1 \tag{17.64}$$

Using the orthogonality property (17.56) of the functions $\sin(m\pi x_1/L_1)$ (see footnote p. 827) relation (17.64) reduces to

$$\int_0^{L_1} p_3(x_1)\sin\left(\frac{n\pi x_1}{L_1}\right) dx_1 = b_n \int_0^{L_1} \sin^2\left(\frac{n\pi x_1}{L_1}\right) dx_1 = \frac{b_n L_1}{2} \tag{17.65}$$

or

$$b_n = \frac{2}{L_1}\int_0^{L_1} p_3(x_1)\sin\left(\frac{n\pi x_1}{L_1}\right) \tag{17.66}$$

Referring to relation (17.62) and taking into account the boundary conditions (17.61) of the auxiliary plate, we expand its deflection $u_3^{\text{aux}}(x_1)$ into the following Fourier series:

$$u_3^{\text{aux}} = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x_1}{L_1}\right) \tag{17.67}$$

Substituting relations (17.63) and (17.67) into (17.60), we obtain

$$\sum_{n=1}^{\infty}\left[c_n\left(\frac{n\pi}{L_1}\right)^4 - \frac{b_n}{D}\right]\sin\left(\frac{n\pi x_1}{L_1}\right) = 0 \tag{17.68}$$

Inasmuch as the above relation must be valid for all values of x_1, the coefficients of $\sin(n\pi x_1/a)$ must vanish. That is, using relation (17.66), we get

$$c_n = \frac{b_n L_1^4}{D(n\pi)^4} = \frac{2L_1^3}{D(n\pi)^4}\int_0^{L_1} p_3(x_1)\sin\left(\frac{n\pi x_1}{L_1}\right) dx_1 \tag{17.69}$$

Substituting relation (17.59) into (17.28), we have

$$\nabla^4 u_3^{\text{aux}} + \nabla^4 u_3^{\text{cor}} = \frac{p_3}{D} \tag{17.70}$$

Using relation (17.60), relation (17.70) reduces to

$$\nabla^4 u_3^{\text{cor}} = 0 \tag{17.71}$$

Since both $u_3(x_1, x_2)$ and $u_3^{\text{aux}}(x_1)$ satisfy identical boundary conditions [see relation (17.61)] at $x_1 = 0$ and $x_1 = L_1$ the correction $u_3^{\text{cor}}(x_1, x_2)$ must satisfy the following boundary conditions at $x_1 = 0$ and $x_1 = L_1$:

$$\begin{aligned} &u_3^{\text{cor}}(0, x_2) = 0 && u_3^{\text{cor}}(L_1, x_2) = 0 \\ &\left.\frac{d^2 u_3^{\text{cor}}}{dx_1^2}\right|_{x_1=0} = 0 && \left.\frac{d^2 u_3^{\text{cor}}}{dx_1^2}\right|_{x_1=L_1} = 0 \end{aligned} \tag{17.72}$$

We assume a solution of equation (17.71) of the following form:

$$u_3^{\text{cor}}(x_1, x_2) = \sum_{n=1}^{\infty} Y_n(x_2)\sin\left(\frac{n\pi x_1}{L_1}\right) \tag{17.73}$$

This solution satisfies automatically the boundary conditions at $x_1 = 0$ and $x_1 = L_1$. Substituting relation (17.73) into the differential equation (17.71), we get

$$\sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{L_1}\right)^4 Y_n - 2\left(\frac{n\pi}{L_1}\right)^2 Y_n^{//} + Y_n^{////}\right]\sin\left(\frac{n\pi x_1}{L_1}\right) = 0 \tag{17.74}$$

where $Y_n^{//}$ and $Y_n^{////}$ are the second and fourth derivatives, respectively, of the function $Y_n(x_2)$. Inasmuch as the functions $\sin(n\pi x_1/L_1)(n = 1, 2, \ldots)$ are linearly independent the coefficient of each one of them must vanish. Thus,

$$Y_n^{////} - 2\left(\frac{n\pi}{L_1}\right)^2 Y_n^{//} + \left(\frac{n\pi}{L_1}\right)^4 Y_n = 0 \qquad n = 1, 2, 3, \ldots \tag{17.75}$$

The solution of this equation is

$$Y_n(x_2) = (C_n + E_n x_2)\sinh\left(\frac{n\pi x_2}{L_1}\right) + (B_n + D_n x_2)\cosh\left(\frac{n\pi x_2}{L_1}\right) \tag{17.76}$$

The constants are evaluated by requiring that the solution (17.73) satisfies the boundary conditions at $x_2 = \pm L_2/2$.

Notice that the loading is not a function of x_2. Moreover, since the support of the edge of the plate at $x_2 = -L_2/2$ is identical to that of its edge at $x_2 = L_2/2$ the deflected surface of the plate will be symmetric with respect to the axis x_1. Therefore $u_3^{\text{cor}}(x_1, x_2)$ and $Y_n(x_2)$ must be even functions of x_2. Consequently, C_n and D_n must vanish. Thus, for such plates introducing the constant $A_n = E_n\,\alpha/n\pi$, relation (17.76) reduces to

$$Y_n(x_2) = A_n\left(\frac{n\pi x_2}{a}\right)\sinh\left(\frac{n\pi x_2}{a}\right) + B_n\cosh\left(\frac{n\pi x_2}{a}\right) \tag{17.77}$$

Substituting relation (17.77) into (17.73), we have

$$u_3^{\text{cor}}(x_1, x_2) = \sum_{n=1}^{\infty}\left[A_n\left(\frac{n\pi x_2}{L_1}\right)\sinh\left(\frac{n\pi x_2}{L_1}\right) + B_n\cosh\left(\frac{n\pi x_2}{L_1}\right)\right]\sin\left(\frac{n\pi x_2}{L_1}\right) \tag{17.78}$$

Substituting relations (17.78) and (17.67) into (17.59), we get

$$u_3(x_1, x_2) = \sum_{n=1}^{\infty}\left[c_n + A_n\left(\frac{n\pi x_2}{L_1}\right)\sinh\left(\frac{n\pi x_2}{L_1}\right) + B_n\cosh\left(\frac{n\pi x_2}{L_1}\right)\right]\sin\left(\frac{n\pi x_1}{L_1}\right) \tag{17.79}$$

The moments can be obtained by substituting relation (17.79) into relations (17.14).

For a given distribution of the external forces $p_3(x_1)$ which varies only in one direction the coefficients $c_n (n = 1, 2, ...)$ are established using relation (17.69). The constants A_n and B_n are evaluated by requiring that the solution (17.79) satisfies the boundary conditions at the edges $x_2 = \pm L_2/2$ of the plate.

In what follows we present an example.

Example 3 Consider a rectangular plate simply supported at its edges $x_1 = 0$ and $x_1 = L_1$

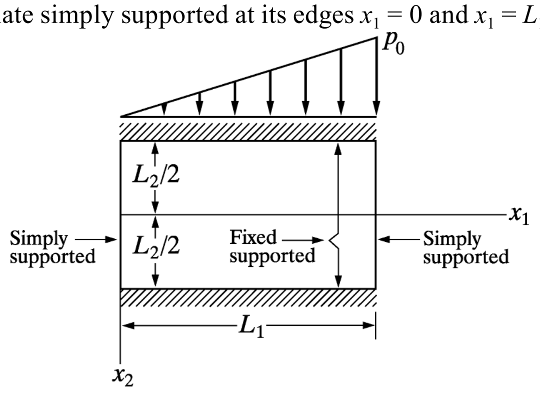


Figure a Geometry and loading of the plate.

and fixed at its edges $x_2 = \pm L_2/2$. The plate is made from an isotropic, linearly elastic material and is subjected to hydrostatic pressure $p_3(x_1)$ as shown in Fig. a. Compute the deflection $u_3(x_1, x_2)$ of the plate and the internal actions acting on its cross sections.

Solution The external forces acting on the plate are

$$p_3(x_1) = \frac{p_0 x_1}{L_1} \tag{a}$$

The solution of this problem is given by relation (17.79), where the coefficients c_n are obtained by substituting relation (a) into (17.69) and integrating the resulting relation. That is,

$$c_n = \frac{2L_1^2 p_0}{D(n\pi)^4}\int_0^{L_1} x_1 \sin\left(\frac{n\pi x_1}{L_1}\right) dx_1 = \frac{2L_1^4 p_0}{D(n\pi)^6}\left[\sin\left(\frac{n\pi x_1}{L_1}\right) - \left(\frac{n\pi x_1}{L_1}\right)\cos\left(\frac{n\pi x_1}{L_1}\right)\right]_0^{L_1}$$

$$= -\frac{2L_1^4 p_0 (-1)^n}{D(n\pi)^5} \tag{b}$$

Moreover, the constants A_n and B_n in relation (17.79) are evaluated from the boundary conditions at $x_2 = L_2/2$ or $x_2 = -L_2/2$. That is, referring to relations (17.30), we have

$$u_3\left(x_1, \pm\frac{L_2}{2}\right) = 0 \qquad \left.\frac{du_3}{dx_2}\right|_{x_2 = \pm\frac{L_2}{2}} = 0 \tag{c}$$

Substituting relation (17.79) with (a) into (c), we obtain

$$\sum_{n=1}^{\infty}\left[c_n + A_n g_n \sin h(g_n) + B_n \cos h(g_n)\right]\sin\left(\frac{n\pi x_1}{L_1}\right) = 0 \tag{d}$$

$$\sum_{n=1}^{\infty}\left[\left(\frac{2A_n g_n}{L_2} + \frac{2B_n g_n}{L_2}\right)\sin h(g_n) + \frac{2A_n g_n^2}{L_2}\cos h(g_n)\right]\sin\left(\frac{n\pi x_1}{L_1}\right) = 0 \tag{e}$$

where

$$g_n = \pm\frac{n\pi L_2}{2L_1} \tag{f}$$

Inasmuch as relations (d) and (e) must be valid for all values of x_1 the coefficients of $\sin(n\pi x_1/L_1)$ must vanish. That is,

$$\begin{aligned} c_n + A_n g_n \sin h(g_n) + B_n \cos h(g_n) &= 0 \\ (A_n + B_n)\sin h(g_n) + A_n g_n \cos h(g_n) &= 0 \end{aligned} \tag{g}$$

Solving relations (g), we get

$$
\begin{aligned}
A_n &= \frac{c_n \sinh(g_n)}{g_n + \sinh(g_n)\cosh(g_n)} \\
B_n &= -\frac{c_n\left[\sinh(g_n) + g_n \cosh(g_n)\right]}{g_n + \sinh(g_n)\cosh(g_n)}
\end{aligned}
\tag{h}
$$

Substituting relations (h) and (b) into (17.79), we obtain

$$
\begin{aligned}
u_3(x_1, x_2) = -\sum_{n=1}^{\infty}\Bigg[\frac{2L_1^4 p_0(-1)^n}{Dn^5\pi^5\left[g_n + \sinh(g_n)\cosh(g_n)\right]}\Bigg[g_n + \sinh(g_n)\cosh(g_n) \\
+ \sinh(g_n)\left(\frac{n\pi x_2}{L_1}\right)\sinh\left(\frac{n\pi x_2}{L_1}\right) - \left[\sinh(g_n) + g_n\cosh(g_n)\right]\cosh\left(\frac{n\pi x_2}{L_1}\right)\Bigg]\sin\left(\frac{n\pi x_1}{L_1}\right)\Bigg]
\end{aligned}
\tag{i}
$$

Substituting relation (i) into (17.14b), we have

$$
\begin{aligned}
M_2(x_1, x_2) &= -D\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2}\right) \\
&= \sum_{n=1}^{\infty}\left[\frac{2L_1^2 p_0(-1)^n \sin\left(\dfrac{n\pi x_1}{L_1}\right)}{n^3\pi^3[g_n + \sinh(g_n)\cosh(g_n)]}\right[-\nu g_n - \nu\sinh(g_n)\cosh(g_n) \\
&\quad + (1-\nu)\left(\frac{n\pi x_2}{L_1}\right)\sinh(g_n)\sinh\left(\frac{n\pi x_2}{L_1}\right) - (1-\nu)[\sinh(g_n) \\
&\quad + g_n\cosh(g_n)]\cosh\left(\frac{n\pi x_2}{L_1}\right) + 2\sinh(g_n)\cosh\left(\frac{n\pi x_2}{L_1}\right)\Bigg]\Bigg]
\end{aligned}
\tag{j}
$$

The moment at $x_2 = \pm L_2/2$ is equal to

$$
M_2(x_1, \pm L_2/2) = \sum_{n=1}^{\infty}\left[\frac{2L_1^2 p_0(-1)^n[-g_n + \sinh(g_n)\cosh(g_n)]\sin\left(\dfrac{n\pi x_1}{L_1}\right)}{n^3\pi^3[g_n + \sinh(g_n)\cosh(g_n)]}\right]
\tag{k}
$$

The same result has been obtained by Timoshenko and Woinowsky-Kreiger[†] by superimposing the solution for the simply supported plate subjected to the triangular load ($p_0 x_1 / L_1$) with the solution to the same plate subjected to an unknown distribution of moment $M_2^*(x_1)$ along its fixed edges $x_2 = \pm L_2/2$. The moment $M^*(x_1)$ is established

† Timoshenko, S. and Woinowsky-Kreiger, S., *Theory of Plates and Shells,* McGraw-Hill, New York, 1959, p. 190.

as the one which renders $\partial u_3/\partial x_2$ equal to zero.

The series in relations (j) and (k) converge fast and the value of the moment M_2 can be readily established. The moment $M_1(x_1, x_2)$ and the twisting moment $M_{12}(x_1, x_2)$ are obtained by substituting relation (i) into (17.14a) and (17.14c).

Numerical values of the bending moment divided by $p_3 L_2^2$ are presented on p. 190 of the book by Timoshenko and Woinowsky-Kreiger†.

17.9 Bending of Circular Plates

For circular plates we will use polar coordinates. That is, referring to Fig. 17.13, we have

$$\begin{aligned} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta \end{aligned} \tag{17.80a}$$

and

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \theta &= \arctan\left(\frac{x_2}{x_1}\right) \end{aligned} \tag{17.80b}$$

From relation (17.80b), we get

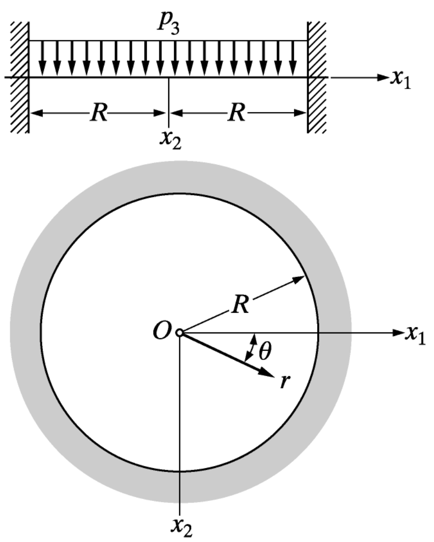


Figure 17.13 Circular plate fixed at $r = R$ and polar coordinates.

† Timoshenko, S. and Woinowsky-Kreiger, S., *Theory of Plates and Shells,* McGraw-Hill, New York, 1959, p. 190.

$$\frac{\partial r}{\partial x_1} = x_1(x_1^2 + x_2^2)^{-1/2} = \frac{x_1}{r} = \cos\theta \tag{17.81a}$$

$$\frac{\partial r}{\partial x_2} = \frac{x_2}{r} = \sin\theta \tag{17.81b}$$

$$\frac{\partial\theta}{\partial x_1} = -\frac{\dfrac{x_2}{x_1^2}}{1 + \left(\dfrac{x_2}{x_1}\right)^2} = -\frac{x_2}{x_1^2 + x_2^2} = -\frac{x_2}{r^2} = -\frac{\sin\theta}{r} \tag{17.81c}$$

$$\frac{\partial\theta}{\partial x_2} = \frac{\dfrac{1}{x_1}}{1 + \left(\dfrac{x_2}{x_1}\right)^2} = \frac{x_1}{x_1^2 + x_2^2} = \frac{x_1}{r^2} = \frac{\cos\theta}{r} \tag{17.81d}$$

Using relations (17.81), we have

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{17.82b}$$

$$\begin{aligned}\frac{\partial^2}{\partial x_1^2} &= \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\ &= \cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial^2}{\partial\theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}\end{aligned} \tag{17.83a}$$

Expressions for $\partial^2/\partial x_2^2$ and $\partial^2/\partial x_1\partial x_2$ can be obtained by following the procedure adhered to in obtaining relation (17.83a). That is,

$$\begin{aligned}\frac{\partial^2}{\partial x_2^2} &= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) = \sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\ &\quad + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x_1\partial x_2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) \\ &= \cos\theta\sin\theta\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta - \sin^2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \\ &\quad - \frac{\cos^2\theta - \sin^2\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\end{aligned} \tag{17.83b}$$

Adding relations (17.83a) and (17.83b), we have

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{17.84}$$

Using relation (17.84), relation (17.28b) can be written as

$$\begin{aligned}\nabla^4 u_3 &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 u_3}{\partial r} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}\right)\\ &= \frac{p_3}{D} - \frac{1}{D}\left(\frac{\partial^2 M^T}{\partial r^2} + \frac{1}{r}\frac{\partial M^T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 M^T}{\partial \theta^2}\right)\end{aligned} \tag{17.85}$$

In what follows we convert to polar coordinates relations (17.23) and (17.38c). In order to accomplish this we take into account referring to Fig. 17.13 that if the x_1 and x_2 axes were chosen to coincide with the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\theta$ the angle θ would be zero. Thus, referring to relations (17.83), we have

$$\begin{aligned}\frac{\partial^2}{\partial x_n^2} &= \left.\frac{\partial^2}{\partial x_1^2}\right|_{\theta=0} = \frac{\partial^2}{\partial r^2}\\ \frac{\partial^2}{\partial x_s^2} &= \left.\frac{\partial^2}{\partial x_2^2}\right|_{\theta=0} = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}\\ \frac{\partial^2}{\partial x_n \partial x_s} &= \left.\frac{\partial^2}{\partial x_1 \partial x_2}\right|_{\theta=0} = \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\end{aligned} \tag{17.86}$$

Using relations (17.86) relations (17.23), (17.26c) and (17.26d) can be written as

$$M_r = M_n = -D\left(\frac{\partial^2 u_3}{\partial x_n^2} + \nu\frac{\partial^2 u_3}{\partial x_s^2}\right) - M^T = -D\left[\frac{\partial^2 u_3}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial u_3}{\partial \theta}\right)\right] - M^T \tag{17.87a}$$

$$M_\theta = M_s = -D\left(\frac{\partial^2 u_3}{\partial x_s^2} + \nu\frac{\partial^2 u_3}{\partial x_n^2}\right) - M^T = -D\left(\frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial u_3}{\partial \theta} + \nu\frac{\partial u_3}{\partial r}\right) - M^T \tag{17.87b}$$

$$M_{\theta r} = M_{ns} = -D(1-\nu)\frac{\partial^2 u_3}{\partial x_n \partial x_s} = -D(1-\nu)\left(\frac{1}{r}\frac{\partial^2 u_3}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}\right) \tag{17.87c}$$

$$Q_r = Q_n = -D\frac{\partial}{\partial x_n}\left[\frac{\partial^2 u_3}{\partial x_n^2} + \frac{\partial^2 u_3}{\partial x_s^2}\right] - M^T = -D\frac{\partial}{\partial r}\left[\frac{\partial^2 u_3}{\partial r^2} + \left(\frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2} + \frac{1}{r}\frac{\partial u_3}{\partial r}\right)\right] - M^T \tag{17.87d}$$

$$Q_\theta = Q_s = -D\frac{\partial}{\partial x_s}\left[\frac{\partial^2 u_3}{\partial x_s^2} + \frac{\partial^2 u_3}{\partial x_n^2}\right] - M^T = -\frac{D}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{\partial^2 u_3}{\partial r^2}\right] - M^T \tag{17.87e}$$

Referring to Table 17.1, the boundary conditions for a circular plate of radius R without holes are

Simply supported circular plate

$$u_3\Big|_{r=R} = 0$$

$$M_r\Big|_{r=R} = \left(\frac{\partial^2 u_3}{\partial r^2} + \frac{\nu}{r}\frac{\partial u_3}{\partial r} + \frac{\nu}{r^2}\frac{\partial u_3}{\partial \theta}\right)_{r=R} = 0 \tag{17.88}$$

Build in circular plate

$$u_3\Big|_{r=R} = 0 \qquad \frac{\partial u_3}{\partial r}\Big|_{r=R} = 0 \tag{17.89}$$

17.9.1 Circular Plates Whose Geometry and Loading Do Not Vary With θ.

If the geometry of circular plates and their loading does not vary with θ, relation (17.85) reduces to

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)\right]\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_3}{dr}\right)\right] = \frac{p_3}{D}$$

or

$$\frac{d}{dr}\left(r\frac{d}{dr}\right)\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_3}{dr}\right)\right] = \frac{p_3 r}{D} \tag{17.90}$$

Integrating relation (17.90), we get

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{du_3}{dr}\right)\right] = \frac{p_3 r}{2D} + \frac{A}{r} \tag{17.91a}$$

$$\frac{d}{dr}\left(r\frac{du_3}{dr}\right) = \frac{p_3 r^3}{4D} + rA\ln(r) + rB' \tag{17.91b}$$

$$\begin{aligned}\frac{du_3}{dr} &= \frac{p_3 r^3}{16D} + \frac{Ar\ln(r)}{2} - \frac{Ar}{4} + \frac{B'r}{2} + \frac{C}{r} \\ &= \frac{p_3 r^3}{16D} + \frac{Ar\ln(r)}{2} + \frac{Br}{2} + \frac{C}{r}\end{aligned} \tag{17.91c}$$

$$u_3(r) = \frac{p_3 r^4}{64D} + \frac{Ar^2\ln(r)}{4} + \left(\frac{2B - A}{8}\right)r^2 + C\ln(r) + E \tag{17.91d}$$

For plates without a hole the deflection must be finite for any value of r. Therefore, the constants A and C must vanish since $(\ln r) \to \infty$ as $r \to 0$. Thus, for such plates relation (17.91d) reduces to

$$u_3(r) = \frac{p_3 r^4}{64D} + \frac{Br^2}{4} + E \tag{17.92}$$

The constants B and E are obtained by requiring that relation (17.92) satisfies the boundary conditions of the plate. Substituting relation (17.92) into (17.87), we obtain

$$\begin{aligned} M_r &= -\frac{(3+\nu)p_3 r^2}{16} - \frac{BD}{2}(1+\nu) \qquad M_\theta = -\frac{(3\nu+1)p_3 r^2}{16} - \frac{BD}{2}(1+\nu) \\ M_{\theta r} &= 0 \qquad Q_r = -\frac{p_3 r}{2} \qquad Q_\theta = 0 \end{aligned} \tag{17.93}$$

In what follows we present two examples.

Example 4 Establish the deflection and the internal actions of a circular plate of radius R without holes build in at $r = R$ and subjected to uniformly distributed transverse forces.

Solution Substituting relation (17.92) into the boundary conditions (17.89), we get

$$\begin{aligned} &\frac{p_3 R^4}{64D} + \frac{BR^2}{4} + E = 0 \\ &\left.\frac{\partial u_3}{\partial r}\right|_{r=R} = \frac{p_3 R^3}{16D} + \frac{BR}{2} = 0 \end{aligned} \tag{a}$$

From relations (a), we obtain

$$B = -\frac{p_3 R^2}{8D} \qquad E = \frac{p_3 R^4}{64D} \tag{b}$$

Substituting the values of the constants (b) into relations (17.92) and (17.93), we have

$$u_3(r) = \frac{p_3}{64D}\left(r^4 - 2r^2R^2 + R^4\right) = \frac{p_3}{64D}\left(R^2 - r^2\right)^2 \tag{c}$$

$$M_r(r) = -\frac{p_3}{16}\left[(3 + \nu)r^2 - R^2(1 + \nu)\right] \quad M_\theta(r) = -\frac{p_3}{16}\left[(3\nu + 1)r^2 - R^2(1 + \nu)\right] \tag{d}$$

$$M_{r\theta} = 0 \qquad Q_r(r) = -\frac{p_3 r}{2} \tag{e}$$

Referring to relation (c) we see that the maximum deflection occurs at $r = 0$ and it is equal to

$$(u_3)_{\max} = u_3(0) = \frac{p_3R^4}{64D} \tag{f}$$

Moreover, referring to relation (d) and (e) we see that the maximum negative bending moments occurs at $r = R$ and are equal to

$$M_r = -\frac{p_3R^2}{8} \qquad M_\theta = -\frac{\nu p_3R^2}{8} \tag{g}$$

while the maximum positive bending moments occur at r=0 and are equal to

$$M_r = M_\theta = \frac{p_3(1 - \nu)R^2}{16} \tag{h}$$

From relations (g) and (h) we see that the maximum value of the radial component τ_{rr} of stress occurs at $r = R$. Thus, substituting relation (g) into (17.19a), we have

$$(\tau_{rr})_{\max} = \tau_{rr}(R, \theta, t/2) = \frac{6M_r}{t} = -\frac{3p_3R^2}{4t^2} \tag{i}$$

Example 5 Establish the deflection and internal actions of a simply supported circular plate of radius R, without holes subjected to uniformly distributed transverse forces.

Solution Substituting relation (17.92) and (17.93) into the boundary conditions (17.88), we get

$$\frac{p_3R^4}{64D} + \frac{BR^2}{4} + E = 0 \qquad -\frac{(3 + \nu)p_3R^2}{16} - \frac{BD}{2}(1 + \nu) = 0 \tag{a}$$

From relation (a), we obtain

$$B = -\frac{(3 + \nu)p_3R^2}{8(1 + \nu)D} \qquad E = \left(\frac{5 + \nu}{1 + \nu}\right)\frac{p_3R^4}{64D} \tag{b}$$

Substituting the values of the constants (b) into relations (17.93), we have

$$u_3(r) = \frac{p_3}{64D(1+\nu)}\left[(1+\nu)r^4 - 2(3+\nu)R^2r^2 + (5+\nu)R^4\right] \tag{c}$$

$$M_r(r) = \frac{p_3(3+\nu)}{16}\left(R^2 - r^2\right) \qquad M_\theta(r) = \frac{p_3}{16}\left[R^2(3+\nu) - r^2(1+3\nu)\right] \tag{d}$$

$$M_{r\theta} = 0 \qquad Q_r(r) = -\frac{p_3 r}{2} \tag{e}$$

Referring to relation (c), (d) and (e) we see that the maximum deflection and bending moments occurs at $r = 0$ and are equal to

$$(u_3)_{\max} = u_3(0) = \frac{p_3R^4}{64D}\left(\frac{5+\nu}{1+\nu}\right) \tag{f}$$

$$(M_r)_{\max} = (M_\theta)_{\max} = M_r(0) = M_\theta(0) = \frac{(3+\nu)p_3R^2}{16} \tag{g}$$

The maximum value of the radial component τ_{rr} of stress occurs at $r = 0$. Referring to relation (17.19a) and (g), it is equal to

$$(\tau_{rr})_{\max} = \tau_{rr}(0,\, \theta,\, t/2) = \frac{6M_r}{t^2} = \frac{3(3+\nu)p_3R^2}{8t^2}$$

17.10 Use of the Weighted Residual Methods to Construct Approximate Expressions for the Deflection of Plates

Many boundary value problems of practical interest involving the determination of the deflection and the internal actions of plates cannot be solved analytically. For such problems it is necessary to resort to approximate solutions. In this section we use the weighted residual methods to construct approximate solutions of the following form for the boundary value problem for computing the deflection field of a plate:

$$\begin{aligned}\tilde{u}_3(x_1, x_2) &= \phi_0 + \sum_{s=1}^{S} \tilde{c}_s \phi_s(x_1, x_2) \\ &= \phi_0 + \{\phi\}^T\{\bar{c}\}\end{aligned} \tag{17.94a}$$

where

$$\begin{aligned}\{\phi\}^T &= [\phi_1\, \phi_2,\, \ldots,\, \phi_S] \\ \{\bar{c}\}^T &= [\bar{c}_1\, \bar{c}_2,\, \ldots,\, \bar{c}_S]\end{aligned} \tag{17.94b}$$

$\phi_0(x_1, x_2)(s = 1, 2, \ldots, S)$ = continuous functions of x_1 and x_2 chosen to satisfy at least the essential boundary conditions of the plate.

$\bar{c}_s(s = 1, 2, \ldots, S)$ = undeterminate coefficients also known as degree of freedom.

$\phi_s(x_1, x_2)(s = 1, 2, \ldots, S)$ = linearly independent functions of x_1 and x_2 chosen to satisfy at least the homogeneous part of the boundary conditions of the plate which were satisfied by $\phi_0(x_1, x_2)$. They are known as *interpolation or trial functions*.

The selection of the functions $\phi_s(x_1, x_2)(s = 1, 2, \ldots, S)$ affects the accuracy of the approximate solution of a boundary value problem. In order to ensure that as s increases the approximate solution (17.94a) converges to the actual solution of the boundary value problem, the trial functions must be a sequence of functions from a complete set of functions starting from the lowest order up to the order S without missing an intermediate term. Moreover, the approximate solution (17.94a) should not vanish at points where the actual solution does not vanish. Furthermore, the speed of convergence of the approximate solution improves if the trial functions $\phi_s(x_1, x_2)(s = 1, 2, \ldots, S)$ satisfy the symmetry conditions of the problem, if any.

17.10.1 The Classical Weighted Residual Equation for the Deflection of Rectangular Plates

We consider a rectangular plate in the x_1x_2 plane of dimensions L_1 and L_2, in the x_1 and x_2 directions, respectively. We assume that relation (17.94a) satisfies at least its essential boundary conditions. Thus, when the approximate solution (17.94a) is substituted into the differential equation (17.28a), there could be a residual R_d. That is,

$$R_d = D\left(\frac{\partial^4 u_3}{\partial x_1^4} + \frac{\partial^4 u_3}{\partial x_2^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2}\right) - p_3(x_1 x_2) + \frac{\partial^2 M^T}{\partial x_1^2} + \frac{\partial^2 M^T}{\partial x_2^2} \tag{17.95}$$

Moreover, when the approximate solution (17.94a) is substituted into the natural boundary conditions there could be residuals. That is,

$$\begin{aligned}
R_{b1} &= M_1(0, x_2) - M_1^0(x_2) & R_{b2} &= Q_1^e(0, x_2) - Q_1^{e0}(x_2) \\
R_{b3} &= M_1(L_1, x_2) - M_1^{L_1}(x_2) & R_{b4} &= Q_1^e(L_1, x_2) - Q_1^{eL_1}(x_2) \\
R_{b5} &= M_2(x_1, 0) - M_2^0(x_1) & R_{b6} &= Q_2^e(x_1, 0) - Q_2^{e0}(x_2) \\
R_{b7} &= M_2(x_1, L_2) - M_2^{L_2}(x_1) & R_{b8} &= Q_2^e(0, x_2) - Q_2^{eL_2}(x_2)
\end{aligned} \tag{17.96}$$

where $M_1^0(x_2)$, $M_1^{L_1}(x_2)$, $M_2^0(x_1)$, $M_2^{L_2}(x_1)$, $Q_1^{e0}(x_2)$, $Q_1^{eL_1}(x_2)$, $Q_2^{e0}(x_1)$, $Q_2^{eL_2}(x_1)$ are given values of the applied moments and of the equivalent shears [see relation (17.37)] on the lateral boundary of the plate.

We are interested to establish a set of values of the parameters $c_s(s = 1, 2, ..., S)$ of relation (17.94a) which reduce in a uniform way throughout the area of the plate the value of the residual R_d and on the lateral boundary of the plate the values of the non-vanishing residuals $R_{bi}(i = 1, 2, ..., 8)$ if there are any. That is, using relations (17.95) and (17.96), we get

$$
\begin{aligned}
&\epsilon_1 \int_0^{L_2} W_r^{b1}\left[M_1(0, x_2) - M_1^0(x_2)\right]dx_2 + \epsilon_2 \int_0^{L_2} W_r^{b2}\left[Q_1^e(0, x_2) - Q_1^{e0}(x_2)\right]dx_2 \\
&+ \epsilon_3 \int_0^{L_2} W_r^{b3}\left[M_1(L_1, x_2) - M_1^{L_1}(x_2)\right]dx_2 + \epsilon_4 \int_0^{L_2} W_r^{b4}\left[Q_1^e(L_1, x_2) - Q_1^{eL_1}(x_2)\right]dx_2 \\
&+ \epsilon_5 \int_0^{L_1} W_r^{b5}\left[M_2(x_1, 0) - M_2^0(x_1)\right]dx_1 + \epsilon_6 \int_0^{L_1} W_r^{b6}\left[Q_2^e(x_1, 0) - Q_2^{e0}(x_2)\right]dx_1 \\
&+ \epsilon_7 \int_0^{L_1} W_r^{b7}\left[M_2(x_1, L_2) - M_2^{L_2}(x_1)\right]dx_1 + \epsilon_8 \int_0^{L_1} W_r^{b8}\left[Q_2^e(x_1, L_2) - Q_2^{eL_2}(x_2)\right]dx_1 \\
&+ \iint_\Omega W_r R_d \, d\Omega = 0
\end{aligned}
\tag{17.97}
$$

where
W_r, W_r^{bi} $(i = 1, 2, ..., 8)$ = chosen functions of x_1, and x_2 known as weighting functions.

$$
\epsilon_i \quad (i = 1, 2, \ldots, 8) = \begin{cases} 0 & \text{when the approximate solution (17.94a) satisfies the } i^{\text{th}} \text{ natural boundary condition (17.96).} \\ 1 & \text{when the approximate solution (17.94a) does not satisfy the } i^{\text{th}} \text{ natural boundary condition (17.96).} \end{cases}
\tag{17.98}
$$

Relation (17.97) is the weighted residual equation for the boundary value for computing the deflection of rectangular plates.

17.10.2 The Modified Weighted Residual Equation for the Deflection of a Plate

We can obtain the modified weighted residual (weak) form of the boundary value problem for computing the deflection of plates by adhering to the following steps:

STEP 1 We write the weighted residual equation (17.97).

STEP 2 We rewrite the last term of equation (17.97) as follows:

$$
\begin{aligned}
\iint_\Omega W_r R_d d\Omega = D \iint_\Omega W_r \left(\frac{\partial^4 u_3}{\partial x_1^4} + \frac{\partial^4 u_3}{\partial x_2^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} \right) d\Omega - \iint_\Omega W_r p_3(x_1 x_2) d\Omega \\
+ \iint_\Omega \left(\frac{\partial^2 M^T}{\partial x_1^2} + \frac{\partial^2 M^T}{\partial x_2^2} \right) d\Omega
\end{aligned}
$$

$$= D\iint_{\Omega}\left[W_r\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^3 u_3}{\partial x_1^3} + \nu\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2} + \frac{\partial M^T}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial^3 u_3}{\partial x_2^3} + \nu\frac{\partial^2 u_3}{\partial x_1^2 \partial x_2} + \frac{\partial M^T}{\partial x_2}\right)\right.\right.$$

$$\left.\left. + 2(1-\nu)\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2}\right]\right] d\Omega - \iint_{\Omega} W_r p_3(x_1 x_2)\, d\Omega$$

$$= D\iint_{\Omega}\left[\frac{\partial}{\partial x_1}\left[W_r\left(\frac{\partial^3 u_3}{\partial x_1^3} + \nu\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2} + \frac{\partial M^T}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_2}\left[W_r\left(\frac{\partial^3 u_3}{\partial x_2^3} + \nu\frac{\partial^3 u_3}{\partial x_1^2 \partial x_2}\right.\right.\right.$$

$$\left.\left.\left. + \frac{\partial M^T}{\partial x_2}\right)\right]\right] d\Omega + D\left[\iint_{\Omega}\left[(1-\nu)\frac{\partial}{\partial x_1}\left(W_r\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2}\right) + (1-\nu)\frac{\partial}{\partial x_2}\left(W_r\frac{\partial^3 u_3}{\partial x_1^2 \partial x_2}\right)\right]\right] d\Omega$$

$$- D\iint_{\Omega}\left[\frac{\partial W_r}{\partial x_1}\left(\frac{\partial^3 u_3}{\partial x_1^3} + \nu\frac{\partial^2 u_3}{\partial x_1 \partial x_2^2} + \frac{\partial M^T}{\partial x_1}\right) + \frac{\partial W_r}{\partial x_2}\left(\frac{\partial^3 u_3}{\partial x_2^3} + \frac{\partial^3 u_3}{\partial x_1^2 \partial x_2} + \frac{\partial M^T}{\partial x_2}\right)\right] d\Omega$$

$$- D\iint_{\Omega}\left[(1-\nu)\frac{\partial W_r}{\partial x_1}\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2} + (1-\nu)\frac{\partial W_r}{\partial x_2}\frac{\partial^3 u_3}{\partial x_1^2 \partial x_2}\right] d\Omega - \iint_{\Omega} W_r p_3(x_1 x_2)\, d\Omega \tag{17.99}$$

We apply Green's theorem (6.19a) to the first and second integrals of relation (17.99) and we rewrite the remaining integrals as follows:

$$\iint_{\Omega} W_r R_d\, dA = D\oint\left[W_r\left(\frac{\partial^3 u_3}{\partial x_1^3} + \nu\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2} + \frac{\partial M^T}{\partial x_1}\right) dx_2 - W_r\left(\frac{\partial^3 u_3}{\partial x_2^3} + \nu\frac{\partial^3 u_3}{\partial x_1^2 \partial x_2}\right.\right.$$

$$\left.\left. + \frac{\partial M^T}{\partial x_2}\right) dx_1 + D(1-\nu)\oint\left[W_r\frac{\partial^3 u_3}{\partial x_1 \partial x_2^2} dx_2 - W_r\frac{\partial^3 u_3}{\partial x_1^2 \partial x_2} dx_1\right]\right.$$

$$- D\iint_{\Omega}\left[\frac{\partial}{\partial x_1}\left[\frac{\partial W_r}{\partial x_1}\left(\frac{\partial^2 u_3}{\partial x_1^2} + \nu\frac{\partial^2 u_3}{\partial x_{22}} + M^T\right)\right] + \frac{\partial}{\partial x_2}\left[\frac{\partial W_r}{\partial x_2}\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2} + M^T\right)\right]\right] d\Omega$$

$$+ D\iint_{\Omega}\left[\frac{\partial^2 W_r}{\partial x_1^2}\left(\frac{\partial^2 u}{\partial x_1^2} + \nu\frac{\partial^2 u_3}{\partial x_2^2} + M^T\right) + \frac{\partial^2 W_r}{\partial x_2^2}\left(\frac{\partial^2 u_3}{\partial x_2^2} + \nu\frac{\partial^2 u_3}{\partial x_1^2} + M^T\right)\right] d\Omega$$

$$- D(1-\nu)\iint_{\Omega}\left[\frac{\partial}{\partial x_2}\left(\frac{\partial W_r}{\partial x_1}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{\partial W_r}{\partial x_2}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)\right] d\Omega$$

$$+ \ 2D(1-\nu)\iint_{\Omega}\frac{\partial^2 W_r}{\partial x_1 \partial x_2}\left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2}\right)d\Omega - \iint_{\Omega} W_r p_3(x_1, x_2) d\Omega \tag{17.100}$$

The line integrals are taken in the counterclockwise direction.

We combine the line integrals of relation (17.100) and we use relations (17.26). Moreover, we apply Green's theorem (6.19a) to the first and third surface integrals of relation (17.100) and we use relations (17.14). Thus, we obtain

$$\begin{aligned}\iint_{\Omega} W_r R_d d\Omega = &\iint_{\Omega}\left[\frac{\partial^2 W_r}{\partial x_1^2}M_1 + \frac{\partial^2 W_r}{\partial x_1^2}M_2 + 2\frac{\partial^2 W_r}{\partial x_1 \partial x_2}M_{12}\right]d\Omega - \iint_{\Omega} W_r p_3 d\Omega \\ &- \oint W_r Q_1 dx_2 + \oint W_r Q_2 dx_1 + \oint \frac{\partial W_r}{\partial x_1}M_1 dx_2 \\ &- \oint \frac{\partial W_r}{\partial x_2}M_2 dx_1 + \oint \frac{\partial W_r}{\partial x_2}M_{12}dx_2 - \oint \frac{\partial W_r}{\partial x_1}M_{12}dx_1\end{aligned} \tag{17.101}$$

We integrate by parts the fifth and sixth line integrals of relation (17.101), to obtain

$$\begin{aligned}\oint_{\Omega}\frac{\partial W_r}{\partial x_2}M_{12}dx_2 - \oint \frac{\partial W}{\partial x_1}M_{12}dx_1 &= -\oint W_r \frac{\partial M_{12}}{\partial x_2}dx_2 + W_r M_{12} + \oint W_r \frac{\partial M_{12}}{\partial x_1}dx_1 - W_r M_{12} \\ &= -\oint W_r \frac{\partial M_{12}}{\partial x_2}dx_2 + \oint W_r \frac{\partial M_{12}}{\partial x_1}dx_1\end{aligned} \tag{17.102}$$

We substitute relation (17.102) into (17.101) and we use relations (17.37), to get

$$\begin{aligned}\iint_{\Omega} W_r R_d d\Omega = &\iint_{\Omega}\left[\frac{\partial^2 W_r}{\partial x_1^2}M_1 + \frac{\partial^2 W_r}{\partial x_2^2}M_2 + 2\frac{\partial^2 W_r}{\partial x_1 \partial x_2}M_{12}\right]d\Omega - \iint_{\Omega} W_r p_3 d\Omega \\ &- \oint W_r Q_1^e dx_2 + \oint W_r Q_2^e dx_1 + \oint \frac{\partial W_r}{\partial x_1}M_1 dx_2 - \oint \frac{\partial W_r}{\partial x_2}M_2 dx_1\end{aligned} \tag{17.103}$$

where the line integrals are taken in the counterclockwise direction. For example

$$\begin{aligned}-\oint W_r Q_1^e dx_2 &= -\int_0^{L_2} W_r(0, x_2)Q_1^e(0, x_2)dx_2 - \int_{L_2}^{0} W_r(L_1, x_2)Q_1^e(L_1, x_2)dx_2 \\ &= -\int_0^{L_2} W_r(0, x_2)Q_1^e(0, x_2)dx_2 + \int_0^{L_2} W_r(L_1, x_2)Q_1^e(L_1, x_2)dx_2 \\ \oint W_r Q_2^e dx_1 &= \int_0^{L_1} W_r(x_1, L_2)Q_2^e(x_1, L_2)dx_1 + \int_{L_1}^{0} W_r(x_1, 0)Q_2^e(x_1, 0)dx_1\end{aligned}$$

$$= -\int_0^{L_1} W_r(x_1, L_2) Q_2^e(x_1, L_2)\, dx_1 - \int_0^{L_1} W_r(x_1, 0) Q_2^e(x_1, 0)\, dx_1$$

$$\oint \frac{\partial W_r}{\partial x_1} M_1\, dx_2 = \int_0^{L_2} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=0} M_1(0, x_2)\, dx_2 - \int_0^{L_2} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=L_2} M_1(L_2, x_1)\, dx_2 \tag{17.104}$$

$$-\oint \frac{\partial W_r}{\partial x_2} M_2\, dx_1 = -\int_0^{L_1} \left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=L_2} M_2(x_1, L_2)\, dx_1 + \int_0^{L_1} \left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=0} M_2(x_1, 0)\, dx_1$$

We take into account relation (17.104) and we rewrite relation (17.103) as follows:

$$\begin{aligned}
\iint_\Omega W_r R_d\, d\Omega &= \iint_\Omega \left[\frac{\partial^2 W_r}{\partial x_1^2} M_1 + \frac{\partial^2 W_r}{\partial x_2^2} M_2 + 2\frac{\partial^2 W_r}{\partial x_1 \partial x_2} M_{12}\right] d\Omega - \iint_\Omega W_r p_3\, d\Omega \\
&= -\int_0^{L_2} W_r(0, x_2) Q_1^e(0, x_2)\, dx_2 + \int_0^{L_2} W_r(L_1, x_2) Q_1^e(L_1, x_2)\, dx_2 \\
&\quad - \int_0^{L_1} W_r(x_1, 0) Q_2^e(x_1, 0)\, dx_1 + \int_0^{L_1} W_r(x_1, L_2) Q_2^e(x_1, L_2)\, dx_1 \\
&\quad + \int_0^{L_2} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=0} M_1(0, x_2)\, dx_2 - \int_0^{L_2} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=L_1} M_1(L_1, x_2)\, dx_2 \\
&\quad + \int_0^{L_1} \left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=0} M_2(x_1, 0)\, dx_1 - \int_0^{L_1} \left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=L_2} M_2(x_1, L_2)\, dx_1
\end{aligned} \tag{17.105}$$

STEP 3 We simplify relation (17.97) by limiting the choice of the weighting functions W_r^{bi} ($i = 1, 2, ..., 8$) as follows:

$$\begin{aligned}
&W_r^{b1} = -\left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=0} \quad W_r^{b2} = W_r \quad W_r^{b3} = -\left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=L_1} \quad W_r^{b4} = -W_r \\
&W_r^{b5} = -\left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=0} \quad W_r^{b6} = W_r \quad W_r^{b7} = -\left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=L_2} \quad W_r^{b8} = -W_r
\end{aligned} \tag{17.106}$$

We substitute relations (17.106) into relation (17.105) and the resulting relations into (17.97), to obtain

$$\iint_\Omega \left[\frac{\partial^2 W_r}{\partial x_1^2} M_1 + \frac{\partial^2 W_r}{\partial x_2^2} M_2 + 2\frac{\partial^2 W_r}{\partial x_1 \partial x_2} M_{12}\right] - \iint_\Omega W_r p_3\, d\Omega$$

$$+ \epsilon_1 \int_0^{L_2} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=0} M_1^0(x_2)\, dx_2 - \epsilon_2 \int_0^{L_2} W_r Q_1^{e0}(x_2)\, dx_2$$

$$
\begin{aligned}
&- \epsilon_3 \int_0^{L_2} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=L_1} M_1^{L_1}(x_2)\,dx_2 + \epsilon_4 \int_0^{L_2} W_r Q_1^{eL_1}(x_2)\,dx_2 \\
&+ \epsilon_5 \int_0^{L_1} \left.\frac{\partial W_r}{\partial x_2}\right|_{x_2=0} M_2^{0}(x_1)\,dx_1 - \epsilon_6 \int_0^{L_1} W_r Q_1^{e0}(x_1)\,dx_1 \\
&- \epsilon_7 \int_0^{L_1} \left.\frac{\partial W_r}{\partial x_1}\right|_{x_1=L_2} M_2^{L_2}(x_1)\,dx_1 + \epsilon_8 \int_0^{L_1} W_r Q_2^{eL_2}(x_1)\,dx_1 = 0
\end{aligned} \tag{17.107}
$$

where the parameters $\epsilon_i (i = 1, 2, ..., 8)$ are defined by relation (17.98). Equation (17.109) is the modified weighted residual equation for computing the deflection of plates.

17.10.2 Discretization of the Boundary Value Problem for Computing the Deflection of Rectangular Plates

In this section we use the modified weighted residual equation (17.107) to obtain approximate expressions for the deflection of rectangular plates, using the Gallerkin assumption. That is,

$$W_r = \phi_r(x_1, x_2) \tag{17.108}$$

We assume a solution of the form (17.94a) and we substitute it into relations (17.14) to get

$$
\begin{aligned}
\tilde{M}_1 &= -D\left[\frac{\partial^2 \phi_0}{\partial x_1^2} + \nu\frac{\partial^2 \phi_0}{\partial x_2^2} + \sum_{s=1}^{S}\left(\frac{\partial^2 \phi_s}{\partial x_1^2} + \nu\frac{\partial^2 \phi_s}{\partial x_2^2}\right)\tilde{c}_s\right] - M^T \\
\tilde{M}_2 &= -D\left[\frac{\partial^2 \phi_0}{\partial x_2^2} + \nu\frac{\partial \phi_0}{\partial x_1^2} + \sum_{s=1}^{S}\left(\frac{\partial^2 \phi_s}{\partial x_2^2} + \nu\frac{\partial^2 \phi_s}{\partial x_1^2}\right)\tilde{c}_s\right] - M^T \\
\tilde{M}_{12} &= -D(1-\nu)\left[\frac{\partial^2 \phi_0}{\partial x_1 \partial x_2} + \sum_{s=1}^{S}\left(\frac{\partial^2 \phi_s}{\partial x_1 \partial x_2}\tilde{c}_s\right)\right]
\end{aligned} \tag{17.109}
$$

We substitute relations (17.109) into the relation (17.107) with (17.108) to obtain

$$
\begin{aligned}
-D\int\int_{\Omega} \Bigg\{ &\frac{\partial^2 \phi_r}{\partial x_1^2}\left[\frac{\partial^2 \phi_0}{\partial x_1^2} + \nu\frac{\partial^2 \phi_0}{\partial x_2^2} + \sum_{s=1}^{\infty}\left(\frac{\partial^2 \phi_s}{\partial x_1^2} + \nu\frac{\partial^2 \phi_s}{\partial x_2^2}\right)\tilde{c}_s - M^T\right] \\
&+ \frac{\partial^2 \phi_r}{\partial x_2^2}\left[\frac{\partial^2 \phi_0}{\partial x_2^2} + \nu\frac{\partial^2 \phi_0}{\partial x_1^2} + \sum_{s=1}^{\infty}\left(\frac{\partial^2 \phi_s}{\partial x_2^2} + \nu\frac{\partial^2 \phi_s}{\partial x_1^2}\right)\tilde{c}_s - M^T\right]
\end{aligned}
$$

$$+2(1-\nu)\frac{\partial^2\phi_r}{\partial x_1\partial x_2}\left(\frac{\partial^2\phi_0}{\partial x_1\partial x_2}+\sum_{s=1}^{\infty}\frac{\partial^2\phi_s}{\partial x_1\partial x_2}\tilde{c}_s\right)\Bigg\}d\Omega-\iint_\Omega W_r p_3 d\Omega$$

$$-\epsilon_3\int_0^{L_2}\frac{\partial W_r}{\partial x_1}\bigg|_{x_1=L_1}M_1^{L_1}(x_2)dx_2+\epsilon_4\int_0^{L_2}W_r Q_1^{eL_1}(x_2)dx_2$$

$$+\epsilon_1\int_0^{L_2}\frac{\partial W_r}{\partial x_1}\bigg|_{x_1=0}M_1^{0}(x_2)dx_2-\epsilon_2\int_0^{L_2}W_r Q_1^{e0}(x_2)dx_2 \tag{17.110}$$

$$+\epsilon_5\int_0^{L_1}\frac{\partial W_r}{\partial x_2}\bigg|_{x_2=0}M_2^{0}(x_1)dx_1-\epsilon_6\int_0^{L_1}W_r Q_1^{e0}(x_1)dx_1$$

$$-\epsilon_7\int_0^{L_1}\frac{\partial W_r}{\partial x_1}\bigg|_{x_1=L_2}M_2^{L_2}(x_1)dx_1+\epsilon_8\int_0^{L_1}W_r Q_2^{eL_2}(x_1)dx_1=0 \qquad r,s=1,2,\ldots,S$$

Relation (17.110) can be rewritten as

$$\{F\} = [S]\{\tilde{c}\} \tag{17.111}$$

where $[S]$ is a square matrix known as the *stiffness matrix* of the problem whose terms S_{rs} (r, s = 1, 2, ..., S) are known as the stiffness coefficients and are given as

$$S_{rs} = D\iint_A\Bigg[\left(\frac{\partial^2\phi_s}{\partial x_1^2}+\nu\frac{\partial^2\phi_s}{\partial x_2^2}\right)\frac{\partial^2\phi_r}{\partial x_1^2}+\left(\frac{\partial^2\phi_s}{\partial x_2^2}+\nu\frac{\partial^2\phi_s}{\partial x_1^2}\right)\frac{\partial^2\phi_r}{\partial x_2^2}$$

$$+2(1-\nu)\left(\frac{\partial^2\phi_s}{\partial x_1\partial x_2}\right)\frac{\partial^2\phi_r}{\partial x_1\partial x_2}\Bigg]dA \tag{17.112}$$

$$r,s = 1,2,\ldots,S$$

The matrix $[F]$ is called the *load vector*. Its terms F_r (r = 1, 2, ...) are equal to

$$F_r = \iint_\Omega p_3\phi_r d\Omega + \iint_\Omega\Bigg[\frac{\partial^2\phi_r}{\partial x_1^2}\left(\frac{\partial^2\phi_0}{\partial x_1^2}+\nu\frac{\partial^2\phi_0}{\partial x_2^2}\right)+\frac{\partial^2\phi_r}{\partial x_2^2}\left(\frac{\partial^2\phi_0}{\partial x_2^2}+\nu\frac{\partial^2\phi_0}{\partial x_1^2}\right)$$

$$+2(1-\nu)\frac{\partial^2\phi_r}{\partial x_1\partial x_2}\left(\frac{\partial^2\phi_0}{\partial x_1\partial x_2}\right)\Bigg]d\Omega-\iint_\Omega M^T\left(\frac{\partial^2\phi_r}{\partial x_1^2}+\frac{\partial^2\phi_r}{\partial x_2^2}\right)d\Omega \tag{17.113}$$

On the basis of the foregoing presentation in order to construct approximate solutions for the boundary value problem for computing the deflection field of rectangular plates using the modified weighted residual equation (17.107) with the Gallerkin assumption (17.108), we adhere to the following steps:

STEP 1 We choose the set of trial functions $\phi_s(x_1, x_2)(s = 1, 2, 3, \ldots, S)$ and $\phi_0(x_1, x_2)$.

STEP 2 We substitute the trial functions $\phi_s(x_1, x_2)(s = 1, 2, 3, \ldots, S)$ and $\phi_0(x_1, x_2)$ into relations (17.112) and (17.113) to obtain the values of $S_{rs}(r,s = 1, 2, \ldots, S)$ and $F_r(r = 1, 2, \ldots, S)$.

STEP 3 We substitute the values of S_{rs} and F_r obtained in step 2 into relation (17.111) and solve the resulting relations to find the matrix $\{\tilde{c}\}$.

STEP 4 We substitute the matrix $\{\tilde{c}\}$ obtained in step 3 into relation (17.94a) to obtain an approximation to the deflection field $u_3(x_1, x_2)$ of the plate.

STEP 5 We substitute the approximation for the deflection of the plate obtained in step 4 in relations (17.14) and (17.26) to obtain approximations to the internal actions of the plate.

In what follows we present an example.

Example 6 Establish an approximation for the deflection field and the internal actions of the rectangular plate whose geometry is shown in Fig. a. The plate is made from an isotropic, linearly elastic material, is fixed on all edges and is subjected to uniformly distributed transverse forces p_3.

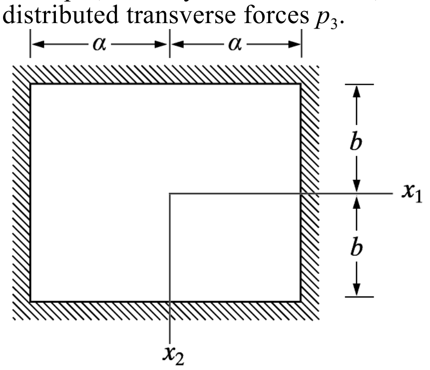


Figure a Geometry of the plate.

Solution

STEP 1 We choose the following trial functions:

$$\begin{aligned}
\phi_1 &= (x_1^2 - a^2)^2(x_2^2 - b^2)^2 \\
\phi_2 &= x_1^2(x_1^2 - a^2)^2(x_2^2 - b^2)^2 \\
\phi_3 &= x_2^2(x_1^2 - a^2)^2(x_2^2 - b^2)^2
\end{aligned} \tag{a}$$

These functions satisfy all the boundary conditions of the plate which are essential. That is

$$\tilde{u}_3(\pm a, x_2) = 0 \qquad \left.\frac{d\tilde{u}_3}{dx_1}\right|_{x_1=\pm a} = 0 \qquad \tilde{u}_3(x_1, \pm b) = 0 \qquad \left.\frac{d\tilde{u}_3}{dx_2}\right|_{x_2=\pm b} = 0 \tag{b}$$

STEP 2 We compute the values of S_{rs} and F_r. From relations (a) we obtain

$$\frac{\partial^2 \phi_1}{\partial x_1^2} = 4(3x_1^2 - a^2)(x_2^2 - b^2)^2$$

$$\frac{\partial^2 \phi_1}{\partial x_1 \partial x_2} = 16x_1 x_2(x_1^2 - a^2)(x_2^2 - b^2) \tag{c}$$

$$\frac{\partial^2 \phi_1}{\partial x_2^2} = 4(x_1^2 - a^2)^2(3x_2^2 - b^2)$$

$$\frac{\partial^2 \phi_2}{\partial x_1^2} = 2(x_2^2 - b^2)^2(15x_1^4 - 12a^2x_1^2 + a^4)$$

$$\frac{\partial^2 \phi_2}{\partial x_2^2} = 4x_1^2(x_1^2 - a^2)^2(3x_2^2 - b^2) \tag{d}$$

$$\frac{\partial^2 \phi_2}{\partial x_1 \partial x_2} = 8x_1 x_2(3x_1^2 - a^2)(x_1^2 - a^2)(x_2^2 - b^2)$$

$$\frac{\partial^2 \phi_3}{\partial x_1^2} = 4x_2^2(x_2^2 - b^2)^2(3x_1^2 - a^2)$$

$$\frac{\partial^2 \phi_3}{\partial x_2^2} = 2(x_1^2 - a^2)^2(15x_2^4 - 12b^2x_2^2 + b^4) \tag{e}$$

$$\frac{\partial^2 \phi_3}{\partial x_1 \partial x_2} = 8x_1 x_2(x_2^2 - b)^2(x_1^2 - a^2)(3x_2^2 - b^2)$$

Substituting relation (c), (d) and (e) into (17.108) and then integrating[†] and evaluating the resulting relations for $a = 1.5$ m, $b = 1.0$ m, and $\nu = 0.3$, we get

$$\begin{aligned} S_{11} = D \int_{x_2=-1}^{x_1=1} \int_{x_1=-1.5}^{x_1=1.5} \Big[& \left[4(3x_1^2 - a^2)(x_2^2 - b^2)^2 + 4\nu(x_1^2 - a^2)^2(3x_2^2 - b^2)\right] 4(3x_1^2 - a^2)(x_2^2 - b^2)^2 \\ & + \left[4(x_1^2 - a^2)^2(3x_2^2 - b^2) + 4\nu(3x_1^2 - a^2)(x_2^2 - b^2)^2\right] 4(x_1^2 - a^2)(3x_2^2 - b^2) \\ & + 2(1-\nu)(16)^2 x_1^2 x_2^2 (x_1^2 - a^2)^2 (x_2^2 - b^2)^2 \Big] dx_1\, dx_2 \\ = {} & 1160.934D \end{aligned}$$

[†] The integration has been performed by the computer using an integration subroutine.

$$\begin{aligned} S_{12} &= 214.381D \\ S_{13} &= 128.622D \\ S_{22} &= 521.21D \\ S_{23} &= 27.988D \\ S_{33} &= 364.559D \end{aligned} \tag{f}$$

and

$$\begin{aligned} F_1 &= p_3\iint_A (x_1^2 - a^2)^2(x_2^2 - b^2)^2\, dA = 8.64\, p_3 \\ F_2 &= 2.777\, p_3 \\ F_3 &= 1.234\, p_3 \end{aligned} \tag{g}$$

STEP 3 We substitute relations (f) and (g) into (17.109) and solve it for $\{c\}$. That is,

$$\{c\} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1160.934 & 214.391 & 128.622 \\ 214.381 & 521.210 & 27.988 \\ 128.622 & 27.988 & 364.559 \end{bmatrix}^{-1} \frac{p_3}{D}\begin{Bmatrix} 8.640 \\ 2.777 \\ 1.234 \end{Bmatrix} = \frac{p_3(10^{-3})}{D}\begin{Bmatrix} 6.90611 \\ 2.4468 \\ 0.76126 \end{Bmatrix} \tag{h}$$

STEP 4 We substitute the values of the constants (h) and the expressions for the trial functions (a) into relation (17.94a) to obtain

$$\begin{aligned} \tilde{u}_3 &= c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3 \\ &= 10^{-3}(x^2 - a^2)^2(x_1^2 - b^2)^2(6.90611 + 2.4468\, x_1^2 + 0.76126 x_2^2)\frac{p_3}{D} \end{aligned} \tag{i}$$

The maximum deflection of the plate occurs at $x_1 = 0$ and $x_2 = 0$. That is,

$$(\tilde{u}_3)_{\max} = 0.00690622\,\frac{p_3 b^4 a^4}{D} = 0.034965\,\frac{p_3}{D} \tag{j}$$

Substituting relation (a) into (17.109), we have

$$\begin{aligned} \tilde{M}_1 &= -D\left[c_1\frac{\partial^2\phi_1}{\partial x_1^2} + c_2\frac{\partial^2\phi_2}{\partial x_1^2} + c_3\frac{\partial^2\phi_3}{\partial x_1^2} + \nu\left(c_1\frac{\partial^2\phi_1}{\partial x_2^2} + c_2\frac{\partial^2\phi_2}{\partial x_2^2} + c_3\frac{\partial^2\phi_3}{\partial x_2^2}\right)\right] \\ \tilde{M}_2 &= -D\left[c_1\frac{\partial^2\phi_1}{\partial x_2^2} + c_2\frac{\partial^2\phi_2}{\partial x_2^2} + c_3\frac{\partial^2\phi_3}{\partial x_2^2} + \nu\left(c_1\frac{\partial^2\phi_1}{\partial x_1^2} + c_2\frac{\partial^2\phi_2}{\partial x_1^2} + c_3\frac{\partial^2\phi_3}{\partial x_1^2}\right)\right] \\ \tilde{M}_{12} &= -D(1-\nu)\left(c_1\frac{\partial^2\phi_1}{\partial x_1\partial x_2} + c_2\frac{\partial^2\phi_2}{\partial x_1\partial x_2} + c_3\frac{\partial^2\phi_3}{\partial x_1\partial x_2}\right) \end{aligned} \tag{k}$$

where the derivatives of $\phi_i (i = 1, 2, 3)$ are given by relations (c) to (e) and the constant c_s $(s = 1, 2, 3)$ are given by relation (h). Substituting these relations into (k) and evaluating the resulting expressions at $x_1 = 0$ and $x_2 = 0$ we obtain the following approximations for the values of the bending moments at the middle point of the plate:

$$\tilde{M}_1(0,0) = 0.0783p_3$$
$$\tilde{M}_2(0,0) = 0.14336p_3 \tag{l}$$

These results differ by less than 5% from those given $(\tilde{M}_1 = 0.0812p_3 \quad \tilde{M}_2 = 0.1472p_3)$ by Timoshenko and Woinowsky-Krieger†.

17.11 The Theorem of Total Stationary Potential Energy for Plates

The theorem of total stationary potential energy for an elastic deformable body has been proved in Section 13.12 and it can be stated as follows.
Consider a body made from an elastic material subjected to external forces in an environment of constant temperature. The actual displacement field of the body makes its total potential stationary. Moreover, if a geometrically admissible displacement field renders the total potential energy of a body stationary is the actual displacement field of the body. Referring to relation (13.121), the total potential energy of an elastic body is defined as

$$\Pi_s = \iiint_V U_s \, dV - \iiint_V \sum_{i=1}^{3} B_i \hat{u}_i \, dX - \iint_{S_t} \sum_{i=1}^{3} \overset{n}{T}{}_i^s \hat{u}_i \, dS \tag{17.114}$$

S_t is the portion of the surface of the body where the components of traction $\overset{n}{T}{}_i^s (i = 1, 2, 3)$ are specified. U_s is the strain energy density defined by relation (3.55). Referring to relation (3.83) the strain energy density for a linearly elastic body is equal to

$$U_s = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij} e_{ij} \tag{17.115}$$

For a plate in the x_1x_2 plane, made from a linearly elastic material, disregarding the effect of shear deformation and recalling that τ_{33} is small and can be neglected, relation (17.115) reduces to

$$U_s = \frac{1}{2}(\tau_{11}e_{11} + \tau_{22}e_{22} + 2\tau_{12}e_{12}) \tag{17.116}$$

Substituting relation (17.11) into (17.116), we get

$$U_s = -\frac{x_3}{2}\left(\tau_{11}\frac{\partial^2 u_3}{\partial x_1^2} + \tau_{22}\frac{\partial^2 u_3}{\partial x_2^2} + 2\tau_{12}\frac{\partial^2 u_3}{\partial x_1 \partial x_2} \right) \tag{17.117}$$

The last two terms in relation (17.114) represent the work that the external forces (body forces and surface tractions) will perform in going from their position in the deformed configuration to their position in the undeformed configuration of the body. For a plate this work is equal to

$$-\iiint_Y \left(\sum_{i=1}^{3} B_i u_i \right) dV - \iint_{S_t} \sum_{i=1}^{3} \overset{n}{T}{}_i^s u_i dS = -\iint_A p_3 u_3 dA \tag{17.118}$$

† Timoshenko, S. and Woinowsky-Kreiger, S., *Theory of Plates and Shells,* 2nd edition, McGraw-Hill, New York, 1959, Table 35 p. 202.

Substituting relations (17.117) and (17.118) into (17.114) and using relations (17.13c), (17.13d) and (17.13e), we obtain the following expression for the total potential energy of a plate:

$$\begin{aligned}\Pi_s &= -\frac{1}{2}\iint_A\left(\frac{\partial^2 u_3}{\partial x_1^2}\int_{-t/2}^{t/2} x_3\,\tau_{11}\,dx_3 + \frac{\partial^2 u_3}{\partial x_2^2}\int_{-t/2}^{t/2} x_3\,\tau_{22}\,dx_3 + 2\frac{\partial^2 u_3}{\partial x_1\partial_2}\int_{-t/2}^{t/2} x_3\,\tau_{12}\,dx_3\right) dA \\ &\qquad - \iint_A p_3 u_3\, dA \\ &= -\frac{1}{2}\iint_A\left(M_1\frac{\partial^2 u_3}{\partial x_1^2} + M_2\frac{\partial^2 u_3}{\partial x_2^2} + 2M_{12}\frac{\partial^2 u_3}{\partial x_1\partial_2}\right) dA - \iint_A p_3 u_3\, dA\end{aligned} \tag{17.119}$$

Substituting relations (17.14) into (17.119), we get

$$\Pi_s = \frac{D}{2}\iint_A\left[(\nabla^2 u_3)^2 + 2(1-\nu)\left[\left(\frac{\partial^2 u_3}{\partial x_1\partial x_2}\right)^2 - \frac{\partial^2 u_3}{\partial x_1^2}\frac{\partial^2 u_3}{\partial x_2^2}\right] - \frac{2p_3 u_3}{D}\right] dA \tag{17.120}$$

where the harmonic operator ∇^2 is defined by relation (7.29a).

17.11.1 Use of the Theorem of Stationary Total Potential Energy to Construct Approximate Expressions for the Deflection of a Plate

The theorem of stationary total potential energy can be used in conjunction with the Ritz method (see Section 13.17.2) to obtain approximate expressions for the deflection field of plates. In this method an approximation to the deflection is chosen of the form (17.94a) with $\phi_0 = 0$ and is substituted into relation (17.120) to give the following approximate expression for Π_s:

$$\begin{aligned}\tilde{\Pi}_s = \frac{D}{2}\iint_A\Bigg[&\left(\sum_{s=1}^{S}\tilde{c}_s\nabla^2\tilde{\phi}_s\right)^2 \\ &+ 2(1-\nu)\left[\left[\sum_{s=1}^{S}\tilde{c}_s\left(\frac{\partial^2\phi_s}{\partial x_1\partial x_2}\right)\right]^2 - \left[\sum_{s=1}^{S}\tilde{c}_s\left(\frac{\partial^2\phi_s}{\partial x_1^2}\right)\right]\left[\sum_{s=1}^{S}\tilde{c}_s\left(\frac{\partial^2\phi_s}{\partial x_2^2}\right)\right]\right] - \frac{2p_3}{D}\sum_{s=1}^{S}\tilde{c}_s\phi_s\Bigg] dA\end{aligned} \tag{17.121}$$

The values of the coefficients $\tilde{c}_s(s = 1, 2, \ldots, S)$ which render the total potential energy of the plate stationary are established by setting

$$\frac{\partial\tilde{\Pi}_s}{\partial\tilde{c}_r} = 0 \qquad r = 1, 2, \ldots, S \tag{17.122}$$

Substituting relation (17.121) into (17.122), we obtain

$$\frac{\partial\tilde{\Pi}_s}{\partial\tilde{c}_r} = \frac{D}{2}\iint_A\left[2\left(\sum_{s=1}^{S}\tilde{c}_s\nabla^2\phi_s\right)\nabla^2\phi_r + 2(1-\nu)\left[2\left(\sum_{s=1}^{S}\tilde{c}_s\frac{\partial^2\phi_s}{\partial x_1\partial x_2}\right)\frac{\partial^2\phi_r}{\partial x_1\partial x_2}\right.\right.$$

$$-\left(\sum_{s=1}^{S}\tilde{c}_s\frac{\partial^2\phi_s}{\partial x_2^2}\right)\frac{\partial^2\phi_r}{\partial x_1^2} - \left(\sum_{s=1}^{S}\tilde{c}_s\frac{\partial^2\phi_s}{\partial x_1^2}\right)\frac{d^2\phi_r}{\partial x_2^2}\Bigg] - \frac{2p_3}{D}\phi_r\Bigg]dA = 0$$

$$r = 1, 2, \ldots, S \tag{17.123}$$

This relation can be written as

$$[S]\{\tilde{c}\} = \{F\} \tag{17.124}$$

where the terms of the matrices $[S]$ and $\{F\}$ are

$$S_{rs} = D\iint_A\left[\nabla^2\phi_s\nabla^2\phi_r + (1-\nu)\left(2\frac{\partial^2\phi_s}{\partial x_1\partial x_2}\frac{\partial^2\phi_r}{\partial x_1\partial x_2} - \frac{\partial^2\phi_s}{\partial x_2^2}\frac{\partial^2\phi_r}{\partial x_1^2} - \frac{\partial^2\phi_s}{\partial x_1^2}\frac{\partial^2\phi_r}{\partial x_2^2}\right)\right]dA \tag{17.125}$$

$$F_r = \iint_A p_3\phi_r dA \tag{17.126}$$

Comparing relation (17.125) with (17.112) and (17.126) with relation (17.113), we see that the results obtained using the theorem of total potential energy in conjunction with the Ritz method are identical with those obtained using the Gallerkin method.

17.12 Problems

1. Consider the simply, supported rectangular plate of constant thickness t whose geometry is shown in Fig. 17P1. The plate is made from an isotropic, linearly elastic material of modulus of elasticity E and Poisson's ratio ν and it is subjected to the following distribution of lateral forces:

$$p_3(x_1, x_2) = \frac{p_0(L_1 - x_1)(L_2 - x_2)}{L_1L_2} \quad \left(\frac{\text{kN}}{\text{m}^2}\right)$$

Using the method described in Section 17.7, derive a formula for the deflection of the plate in terms of p_0, L_1, L_2, t, E and ν. Compute the deflection at the center of the plate and the concentrated reactions acting at the corners of the plate when $p_0 = 20$ kN/m^2, $L_1 = 2$ m, $L_2 = 3$ m, $t = 10$ mm, $E = 200$ GPa, and $\nu = 0.3$.

$$\text{Ans. } u_3(x_1, x_2) = \frac{4p_0L_1^4L_2^4}{D\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn(L_2^2m^2 + 4L_1n^2)^2}\sin\left(\frac{m\pi x_1}{L_1}\right)\sin\left(\frac{n\pi x_2}{L_2}\right)$$

2. Consider the simply, supported rectangular plate of constant thickness t subjected to uniform distribution of moments along its two edges as shown in Fig. 17P2. The plate is made from an isotropic, linearly elastic material. Using the method of Levi establish formulas for the deflection $u_3(x_1, x_2)$ of the plate.

$$\text{Ans. } u_3\left(\frac{L}{2},0\right) = \sum_{n=1,3,5}^{\infty} - \frac{M_2^*L_2^2(-1)^{\frac{n+1}{2}}g_n\tan hg_n}{2D(n\pi)g_n^2\cos hg_n}, \text{ where } g_n = \frac{\pm n\pi^2L_2}{2L_1}$$

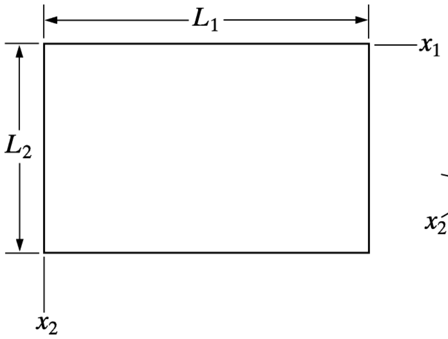


Figure 17P1

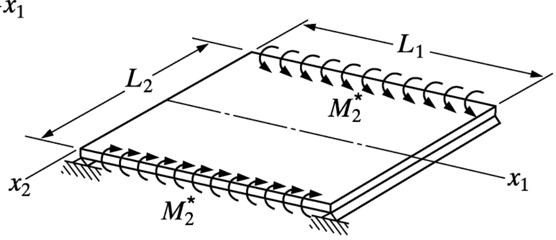


Figure 17P2

3. Consider a simply, supported circular plate of radius R and constant thickness t subjected to moments M_r^* uniformly distributed along its edge. The plate is made from an isotropic, linearly elastic material. Establish a formula for the deflection and the radial τ_{rr} and the tangential $\tau_{\theta\theta}$ components of stress of the plate.

$$\text{Ans. } u_3(r) = \frac{M_r^*}{2(1+\nu)D}(R^2 - r^2) \quad (\tau_{rr})_{\max} = (\tau_{\theta\theta})_{\max} = \frac{6M_r^*}{t^2}$$

4. Consider the simply supported circular plate with a concentric circular hole shown in Fig. 17P4 subjected to a moment M_r^* uniformly distributed along its inner edge. The plate is made from an isotropic, linearly elastic material. Establish a formula for the deflection of the plate and the radial and tangential components of stress.

$$\text{Ans. } u_3(r) = \frac{M_r^* b^2}{2(1+\nu)D(R^2-b^2)}\left(r^2+2R^2\left(\frac{1+\nu}{1-\nu}\right)\ln\frac{r}{R}-R^2\right)$$

5. The circular plate of Fig. 17P5 is simply supported at its edge and point supported at its center. The plate is made from an isotropic, linearly, elastic material. Compute the reaction at the center of the plate when its is subjected to uniformly distributed transverse forces p_3. The deflection of the center of a simply supported circular plate subjected at its center to a concentrated force P_3 is equal to

$$(u_3)_{\text{center}} = \frac{P_3 R^2}{16\pi D} \qquad \text{Ans. } R_c = \frac{p_3(5+\nu)R^2\pi}{4(1+\nu)}$$

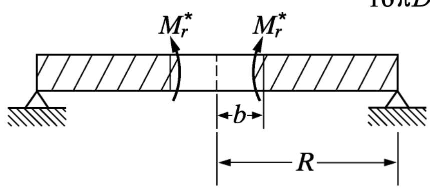


Figure 17P4

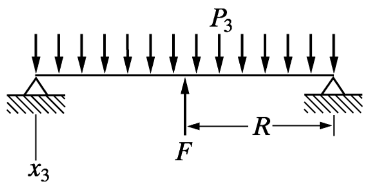


Figure 17P5

6. Consider a simply supported circular plate of constant thickness t and radius R subjected to uniformly distributed transverse forces p_3 kN/m^2. The plate is made from an isotropic, linearly elastic material. The deflection of this plate is given by the following formula:

$$u_3(r) = \frac{p_3}{64(1+\nu)D}\left[(1+\nu)r^4 - 2(3+\nu)r^2R^2 + (5+\nu)R^4\right] \tag{a}$$

Consider the same plate subjected to a moment M_r^R uniformly distributed along its edge. The deflection of the plate under this loading is given by the following formula:

$$u_3(r) = \frac{M_r^R}{2(1+\nu)D}\left(R^2 - r^2\right) \tag{b}$$

Using formulas (a) and (b) establish a formula for the deflection of a circular plate of radius R built in along its edge and subjected to uniformly distributed transverse forces p_3 (kN/m^2). Ans. $u_3(r) = \dfrac{p_3}{64D}\left(R^2 - r^2\right)^2$

7. A simply supported circular plate of radius 320 mm and thickness t = 30 mm is subjected to uniformly distributed transverse forces p_3 = 1.5 MPa. The plate is made from an isotropic, linearly elastic material (E = 200 GPa, ν = 0.30, and τ^Y = 280 MPa).

(a) Determine the maximum bending stress and the maximum deflection of the plate.

(b) Determine the value p_3^Y of the external transverse forces which are required to initiate yielding in the plate.

(c) Determine the factor of safety against initiation of yielding in the plate.

8. Assume a solution of the form (17.94a) with S = 6 and use the Gallekin method to compute an approximate value for the maximum deflection and maximum moments of the simply supported rectangular plate subjected to the transverse forces shown in Fig. 17P8. The plate is made from an isotropic, linearly elastic material with E = 200 GPa and ν = 0.3. Ans. $u_3(1,2) = (u_3)_{max} = \dfrac{2.169}{D}$

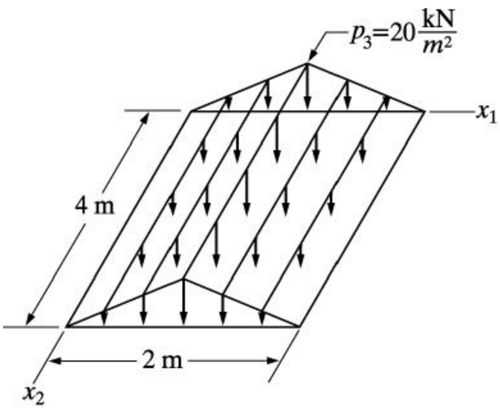


Figure 17P8

References: For a more detailed presentation and many solved examples, see:
1. Timoshenko, S. and Woinowsky-Krieger, S., *Theory of Plates and Shells,* 2nd edition, McGraw-Hill, New York, 1959.
2. Szilard, R., *Theory and Analysis of Plates,* Prentice-Hall, Englewood Cliffs, NJ, 1974.

Chapter 18

Instability of Elastic Structures

18.1 States of Unstable Equilibrium of Structures

The availability of electronic computers permits the use of more accurate methods in analyzing structures. Moreover, recent advances in manufacturing resulted in stronger engineering materials whose actual strength varies little from that specified by the manufacturer. These developments led to the reduction of the factors of safety used in the design of structures and the concomitant decrease of the area of the cross sections of their members. However, members of thin cross sections may reach a state of unstable equilibrium and fail by buckling at loads less than those which initiate yielding of one or more of their particles.

In this chapter we describe the phenomenon of buckling of certain structures and we present methods for computing the load (critical load) which produces buckling and for investigating the postbuckling behavior of structures. The magnitude of the critical load of certain structures depends on the type of forces to which they are subjected. We limit our attention to structures subjected to static, conservative, compressive forces, that is, forces whose work in moving from one position to another depends only on their initial and final position and not on the path which they follow. An example of a non-conservative force is a centroidal force of magnitude P, acting on the unsupported end of a cantilever column, which is axial when the column is straight but remains tangent to its elastic curve as the column deforms (see Fig. 18.1). A column subjected to such a force

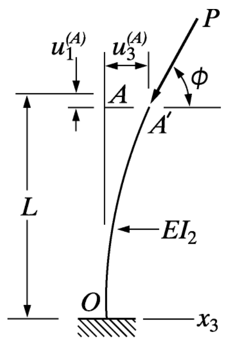


Figure 18.1 Beck's column.

is known as *Beck's column.* The work of such a force depends on the way the unsupported end of the column reaches its deformed position. If the unsupported end A of the column moves to its deformed position A' first by translating and then by rotating referring to Fig. 18.1 the work of the force is $Pu_1^{(A)}$. If the unsupported end of the column moves to its deformed position by first rotating and then translating, referring to Fig. 18.1 the work of the force is equal to $Pu_3^{(A)} \cos \phi + Pu_1^{(A)} \sin \phi$.

Consider a body made from a linearly elastic material originally at a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_0. Subsequently, the body is subjected to loads (body forces, surface tractions, change of temperature) and reaches a second state of mechanical equilibrium at a temperature $T(x_1, x_2, x_3)$. In the previous chapters we limit our attention to bodies made from a linearly elastic material and subjected to external loads of such magnitude that the unit elongations or shrinkages, the unit shears and the components of rotation of their particles are very small compared to unity; and, moreover, the components of rotation are not of a higher order of magnitude than the unit elongations or shrinkages and the unit shears. For this reason, we can approximate the unit elongations or shrinkages with the corresponding normal components of strain and the unit shears with twice the corresponding shearing components of strain (see Sections 2.3 and 2.4). Moreover, we can disregard the effect of the change of the geometry of a body, due to its deformation, on the components of stress and displacement of its particles; and as long as we take into account appropriate boundary conditions, we obtain unique solutions of the boundary value problems which we consider. Moreover, the relations between cause (external loads) and effect (deformation and stress) can be considered linear. However, bodies of certain geometries when they are subjected to certain values of certain types of loads, can assume more than one equilibrium configuration all of which are not stable. These equilibrium configurations cannot all be established using a theory based on the assumption of small deformation.

In investigating problems involving multiple equilibrium configurations it is necessary to take into account at least the effect of rotation of the particles of the body. A theory which assumes that the unit elongations or shinkages and the unit shears are small and can be disregarded compared to unity but the components of rotation are of a higher order of magnitude than the unit elongations or shrinkages and the unit shears, is known as the *theory of moderate rotations* and can be used to compute the critical load at buckling and to investigate the postbuckling behavior of bodies. The theory of moderate rotations is non-linear. That is, the relations between the cause (external forces) and the effect (components of displacement) is not linear.

In what follows we describe six examples of structures which reach a state of instability.

18.1.1 Perfectly Straight Column Subjected to Equal and Opposite Perfectly Axial Perfectly Centroidal Compressive Forces at Its Ends

Consider a long, perfectly straight column, made from a homogeneous, isotropic, linearly elastic material. We choose the principal centroidal axes of the cross sections of the column as the x_2 and x_3 axes such that $I_2 < I_3$. The one end of the column cannot translate and cannot rotate about the x_1 axis ($u_1^{\circ} = u_2^{\circ} = u_3^{\circ} = \theta_1^{o} = 0$), while its other end is freed to translate in the direction of its axis ($u_1 \neq 0$). For example, one end of the

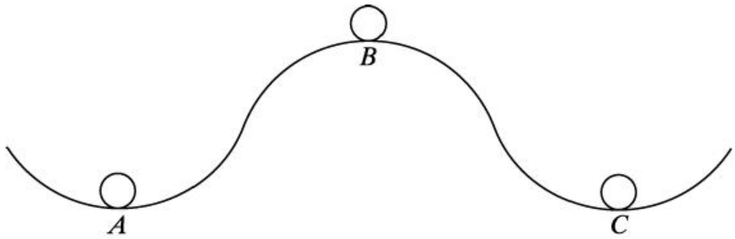


Figure 18.2 Positions of stable and unstable equilibrium of a ball.

column could be pinned to a fixed support ($u_1^o = u_2^o = u_3^o = \theta_1^o = 0$), while its other end is connected with a ball and sucket to a support which is free to translate only in the direction of the axis of the column ($u_2^{(A)} = u_3^{(A)} = 0$). As another example, one end of the column could be fixed ($u_1^o = u_2^o = u_3^o = \theta_1^o = \theta_2^o = \theta_3^o = 0$) while its other end is unsupported. The column is subjected to a perfectly centroidal, compressive, axial force P at its translating end, whose direction does not change during deformation. For values of the force P less than a critical value, which we denote by P_{cr}, the column remains straight and its cross sections do not twist ($u_2 = u_3 = 0, = \theta_1 = \theta_2 = \theta_3 = 0$). Moreover, when the column is subjected to a small transverse force F_2 or to a small torsional moment, it undergoes a small deflection u_2 or a small twisting rotation θ_1, respectively, which, however, vanishes after the transverse force or the torsional moment is removed. Furthermore, it can be shown that for any value of the force $P = P^* < P_{cr}$ the total potential energy of the column $\Pi_s(P^*, u_3^{(A)})$ or $\Pi_s(P^*, \theta_1^{(A)})$ (see Section 13.17) is a relative minimum when $u_3^{(A)}$ or $\theta_1^{(A)}$, respectively, vanish. $u_3^{(A)}$ and $\theta_1^{(A)}$ are the transverse component of translation and the twisting component of rotation, respectively, of a cross section of the column which specify a possible deflected equilibrium configuration. For values of the force in the range from $P = 0$ to $P = P_{cr}$ the column, in its perfectly straight configuration, is in a state of stable equilibrium analogous to that of a ball in equilibrium at position A of Fig. 18.2. If the ball is displaced slightly from this equilibrium position, it will return to it after the displacing force is removed. Moreover, the potential energy of the ball at position A is a relative minimum. For values of the compressive force higher than its critical value (P_{cr}) the column may stay in the straight configuration or it may deflect (flexural buckling) (see Fig. 18.3a) or in some cases it may twist (torsional buckling) (see Fig. 18.3b) or it may deflect and twist (flexural–torsional buckling) (see Fig. 18.3c). In general only columns of very thin-walled, open cross sections could exhibit torsional or flexural–torsional buckling when subjected to compressive axial forces. Columns of doubly symmetric, thin-walled, open cross sections buckle either in the flexural or in the torsional mode depending on whether the buckling load

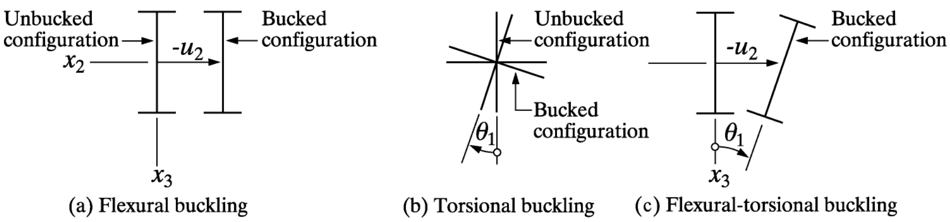


Figure 18.3 Modes of buckling of perfectly straight columns subjected to perfectly centroidal axial compressive forces undergoing flexural or torsional or flexural–torsional buckling.

corresponding to the flexural or to the torsional mode is the lowest. However, columns of not doubly symmetric, thin-walled, open cross sections, as, for example, a T-beam, may buckle either in the flexural mode or in a combined flexural–torsional mode (see Fig. 18.3c).

For values of the compressive force higher than its critical value ($P = P^* > P_{cr}$) the column will be in a state of unstable equilibrium in its straight configuration, ready to move to its deflected or twisted or deflected and twisted stable equilibrium configuration when subjected to a very small transverse force or torsional moment, which is immediately removed. We say that the column has *buckled* under the compressive force P^*. It can be shown that for $P = P^* > P_{cr}$, the total potential energy of the column in its buckled configuration is a minimum, while in its straight configuration it is a maximum. Therefore, when a column is in its perfectly straight configuration under a perfectly centroidal axial force greater than its critical value, it is in a state analogous to that of a ball at position B of Fig.18.2. When the ball is at position B and it is subjected to a very small force, which is immediately removed, it will move to the stable equilibrium position A or C depending on the direction of the applied force. Moreover, the potential energy of the ball at position B is a relative maximum while at position A or C is a relative minimum.

In this text, we consider only columns which buckle in the flexural mode. Consider a perfectly straight such column subjected to a perfectly centroidal axial force P. In Fig. 18.4b, we plot the force P versus the transverse component of translation $u_3^{(A)}$ diagram for this column. When the column is subjected to $P = P^* > P_{cr}$, it could stay in the unstable equilibrium configuration specified in Fig. 18.4b by point C but when a very small transverse force is applied to it and is immediately removed, it will move to the stable equilibrium configuration specified in Fig. 18.4b by point D. It can be shown that the total potential energy of the column (see Section 13.17) at position C is a maximum while at position D is a minimum. Moreover, when the column is subjected to $P = P^* > P_{cr}$ and is displaced from its stable equilibrium configuration D by the application of a very small transverse force, it will return to that configuration when the transverse force is removed. Referring to Fig. 18.4b, point A is called *bifurcation* or *branching point.* For values of the compressive force higher than P_{cr} the straight line AC represents an *unstable equilibrium path* while the curve AD is the *stable equilibrium path* known as the *post-buckling path*. The straight line AC can be established using both a linear and a non-

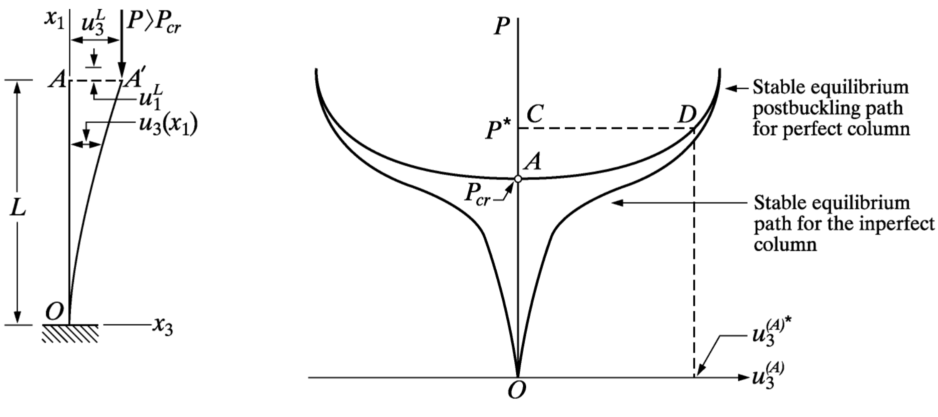


Figure 18.4 Load–deflection diagram of a perfectly straight column fixed at its one end and subjected to a perfectly centroidal, compressive, axial force at its other end, undergoing flexural buckling.

linear theory while the curve AD can be established only using a non-linear theory like the theory of moderate rotations.

Suppose that the column under consideration and/or its loading had an initial imperfection. For example, the imperfect column could have a small initial curvature and/or the compressive axial force could have been applied a small distance from the centroid of the cross section of the column. In such a case, for every value of the compressive axial force, no matter how small, there corresponds a unique bent configuration of stable equilibrium at which the total potential energy of the column is a minimum. However, for values of the compressive force close to the critical value of a perfectly centroidal perfectly axial compressive force acting on the corresponding perfectly straight column, the deflection of the imperfect column increases rapidly and as shown in Fig. 18.4b the equilibrium path of the imperfect column approaches the post-buckling equilibrium path of the perfect column. Thus, in this example, the presence of initial imperfections does not change significantly the ability of the column to resist the compressive force. However, it does change the response of the column. That is, its load-deflection diagram does not have a point of bifurcation.

On the basis of the foregoing presentation we may arrive at the following conclusions for a column undergoing flexural buckling:

1. When a perfectly straight column is subjected to a perfectly centroidal, compressive, axial force (perfect column), higher than its critical value, it can assume one of two possible equilibrium configurations: the straight line which is unstable or the bent which is stable. Consequently, we cannot use a theory based on the assumption of small deformation, to establish the two equilibrium configurations of a perfect column because such a theory gives a unique solution. For values of the applied compressive, axial force higher than its critical value, a theory based on the assumption of small deformation gives only the straight-line unstable equilibrium path of the column. The straight line and the bent configurations of a perfect column can be established using a theory which is more accurate than a theory based on the assumption of small deformation.

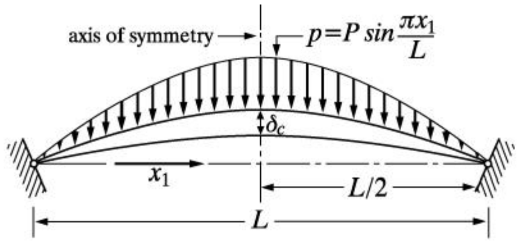
2. When an imperfect column is subjected to a compressive axial force, for every value of the force, it assumes a bent configuration of stable equilibrium. However, for values of the compressive force, close to the critical force of the corresponding perfect column, the deflection of the imperfect column is large and in order to compute it we must use a more accurate theory than the one based on the assumption of small deformation.

3. Usually the deflection of a column subjected to an axial compressive force of magnitude higher than its critical value is very large, and for this reason the column can not perform properly the task which it has been designed to perform. Thus, *for design purposes, we consider that the column fails when the compressive force acting on it reaches its critical value*.

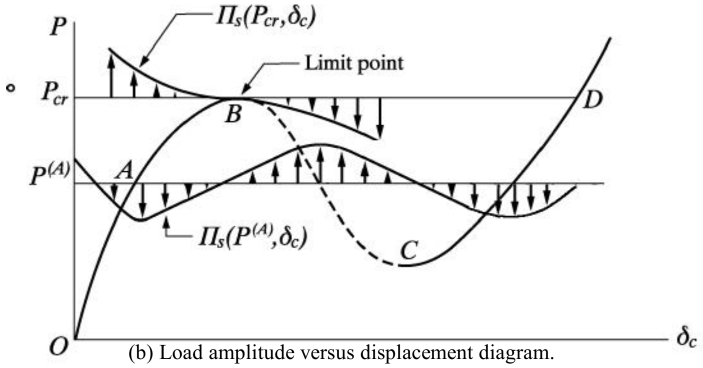
Notice that the critical value of the centroidal, compressive, axial force acting on slender straight columns can be considerably smaller than the smallest value of the centroidal, compressive, axial force required to produce yielding of one or more particles of this column. Consequently, the critical value of the centroidal, compressive, axial force of slender columns could be their important design parameter.

18.1.2 Shallow Sinusoidal Arch, Pinned to Rigid Abutments, Subjected to Sinusoidally Varying, Distributed Forces

Consider a symmetric, shallow sinusoidal arch made from a homogeneous, isotropic,



(a) Geometry and loading of the sinusoidal arch

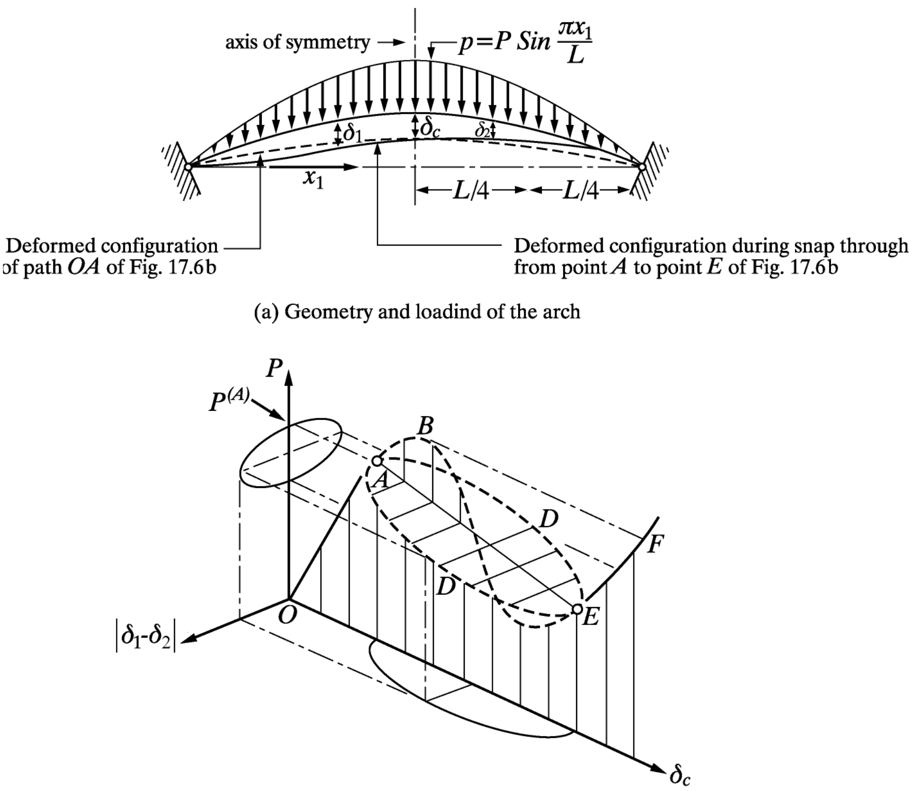


(b) Load amplitude versus displacement diagram.

Figure 18.5 Load amplitude versus displacement and total potential energy versus displacement diagrams of a shallow, sinusoidal arch subjected to sinusoidally distributed forces.

linearly elastic material and pinned to rigid abutments. As shown in Fig. 18.5a the arch is subjected to sinusoidally varying, distributed forces of amplitude P. We take as measure of the magnitude of the deformation of the arch the deflection δ_c of its middle point (see Fig. 18.5a). In Fig. 18.5b we plot the amplitude P of the external forces acting on the arch versus its deflection δ_c . As P increases from zero, the arch deforms to symmetric about its axis of symmetry, stable, equilibrium configurations. In Fig. 18.5b these configurations are represented by the path from point O to a point just to the left of point B, which is a local maximum of the load versus the deflection diagram and it is known as the *limit point*. The value of the amplitude of the external forces corresponding to point B is called the *critical load* of the arch and we denote it by P_{cr}. When the amplitude of the external forces reaches its critical value, in the presence of any disturbance no matter how small, the arch is inverted dynamically and it vibrates about its stable equilibrium configuration represented in Fig. 18.5b by point D. Finally, the vibrations are damped out and the arch comes to rest at the stable equilibrium configuration D. This phenomenon is known as *snap through buckling*.

For any value $P^{(A)}$ of the amplitude of the external forces less than that corresponding to the limit point B, as shown in Fig. 18.5b, the arch could assume one symmetric unstable equilibrium configuration, on the path from point B to point C, as well as two symmetric stable equilibrium configurations: one on the path from point O to a point just to the left of point B and the other on the path from point C to point D. In the two stable configurations the total potential energy $\Pi_s(P^{(A)}, \delta_c)$ versus δ_c diagram of the arch, as shown in Fig. 18.5b, has a relative minimum, while in the unstable equilibrium configuration it has a local maximum. Moreover, when the amplitude of the external



(a) Geometry and loadind of the arch

(b) Load amplitude versus displacement† diagram

Figure 18.6 Load amplitude versus displacement of a moderately shallow, sinusoidal arch subjected to sinusoidally distributed forces.

forces reaches its critical value ($P = P_{cr}$), as shown in Fig. 18.5b, the total potential energy $\Pi_s(P_{cr.}, \delta_c)$ versus δ_c diagram of the arch in its symmetric unstable configuration at the limit point B has a horizontal point of inflection.

In its buckled configuration the arch cannot perform the function for which it has been designed. Thus, for design purposes, we consider that the arch fails when the amplitude of the external forces reaches its critical value. The unstable equilibrium path BC is not followed in a normal loading sequence. However, the equilibrium configurations represented by this path can be observed experimentally by controlling the deformation of the arch.

18.1.3 Moderately Shallow Sinusoidal Arch Pinned to Rigid Abutments Subjected to Sinusoidally Varying Distributed Forces

Consider a symmetric, moderately shallow, sinusoidal arch made from a homogeneous isotropic, linearly elastic material. As shown in Fig. 18.6a the arch is subjected to a

† Taken from Thompson, J. M.T. amd Hunt, G.W., *A General Theory of Elastic Stability,* John Wiley and Sons, New York, 1973.

sinusoidally varying force of amplitude P. We take as measure of the deformation of the arch the deflections δ_c of its middle ($x_1 = L/2$) and δ_1 and δ_2 of its quarter ($x_1 = L/4$ and $x_1 = 3L/4$) points. In Fig. 18.6b we plot the amplitude P of the external forces versus the deflections δ_c and ($|\delta_1 - \delta_2|$). As P increases from zero the arch deforms to symmetric about its axis of symmetry stable equilibrium configurations ($|\delta_1 - \delta_2|$), until the amplitude P of the external forces reaches a critical value ($P_{cr} = P^{(A)}$) less than that corresponding to the limit point B of Fig. 18.6b. At this value of the amplitude P of the external forces, in the presence of any disturbance, no matter how small, the arch is inverted dynamically (snaps through) and it vibrates about its stable equilibrium configuration represented in Fig. 18.6b by point E. Finally, the vibrations are dumped out and the arch comes to rest at the symmetric stable equilibrium configuration E ($|\delta_1 - \delta_2|$). However, during its dynamic inversion the deformation of a moderately shallow arch is not symmetric. Referring to Fig. 18.6b, we see that A is a point of bifurcation from which emanate two symmetric paths (ADE), with respect to the plane P, δ_c of unsymmetric ($|\delta_1 - \delta_2| \neq 0$) unstable equilibrium configurations. If the unsymmetric movement of the arch is prevented by controlling its deformation, the arch will reach a state of instability when the amplitude of the external forces assumes the value corresponding to the limit point B of Fig. 18.6b. Under this loading, in the presence of any disturbance, no matter how small, the arch will be inverted dynamically to the symmetric configuration specified by point F in Fig. 18.6b.

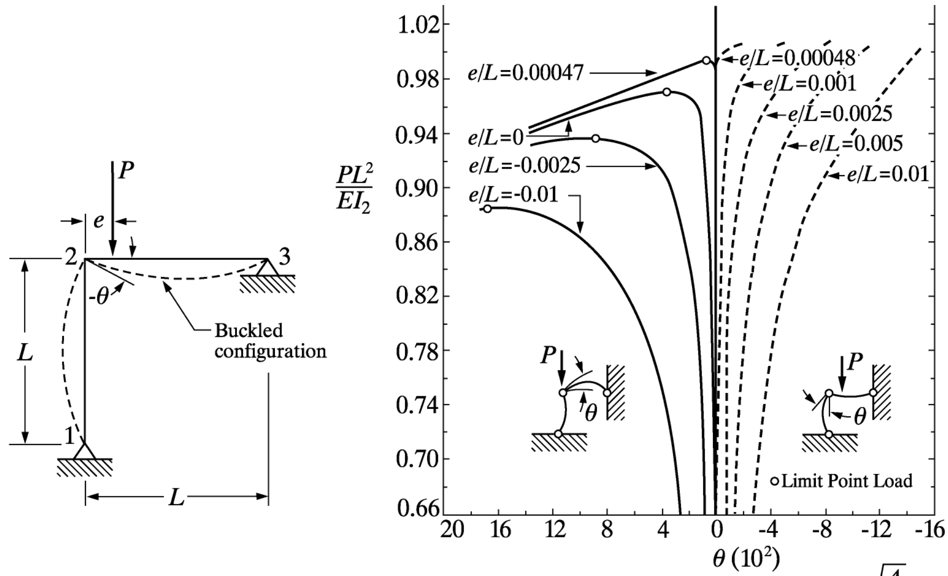
Notice that if the loading of the arch and/or its geometry were slightly non-symmetric the critical load at buckling would be less than that of the perfect arch. We say that the *arch is imperfection sensitive*.

Notice that the critical value of the load for a shallow or moderately shallow arch could be considerably smaller than that which produces yielding at one or more of its particles. Consequently, in such cases the critical value of the load for buckling is the important design parameter.

18.1.4 Planar Frame Consisting of a Horizontal and a Vertical Member Subjected to a Vertical Force on Its Horizontal Member

Consider a planar frame consisting of two perfectly straight, perfectly prismatic members made from an isotropic, linearly elastic material whose cross sections have an axis of symmetry which lies in the plane of the frame. The two members have the same length L and cross sectional area A. As shown in Fig. 18.7a the two members are rigidly connected at point 2 and pinned to a rigid support at points 1 and 3. Moreover, the frame is subjected to a vertical force P acting on its horizontal member at a point located e distance from the centroid of the cross sections of the vertical member. The line of action of the force P lies in the plane specified by the axis of the members of the frame. We take as measure of the deformation of the frame the angle of rotation θ of joint 2. It is apparent that as the load increases from zero with or without eccentricity both members are subjected to bending. It can be shown† that the force–rotation diagram of joint 2 of the frame exhibits a limit point for all values of e less than a certain small positive eccentricity e_0. In Fig. 18.7b $e_0/L = 0.000473$. Moreover, as e increases from minus infinity to a value less than e_0 the force corresponding to the limit point (critical force) increases. For values of the eccentricity e of the force, greater than e_0 as shown in Fig.

† See Kounadis, A.N., Giri, J. and Simitses, G.J., Non-linear stability analysis of an eccentrically loaded two-bar frame, *Journal of Applied Mechanics,* Dec. 1977/70, p. 39.



(a) Geometry and loading of the frame

(b) Load versus rotation of joint 2 for a frame with $L\sqrt{\frac{A}{I_2}}=80$

Figure 18.7 Response of a two members frame.

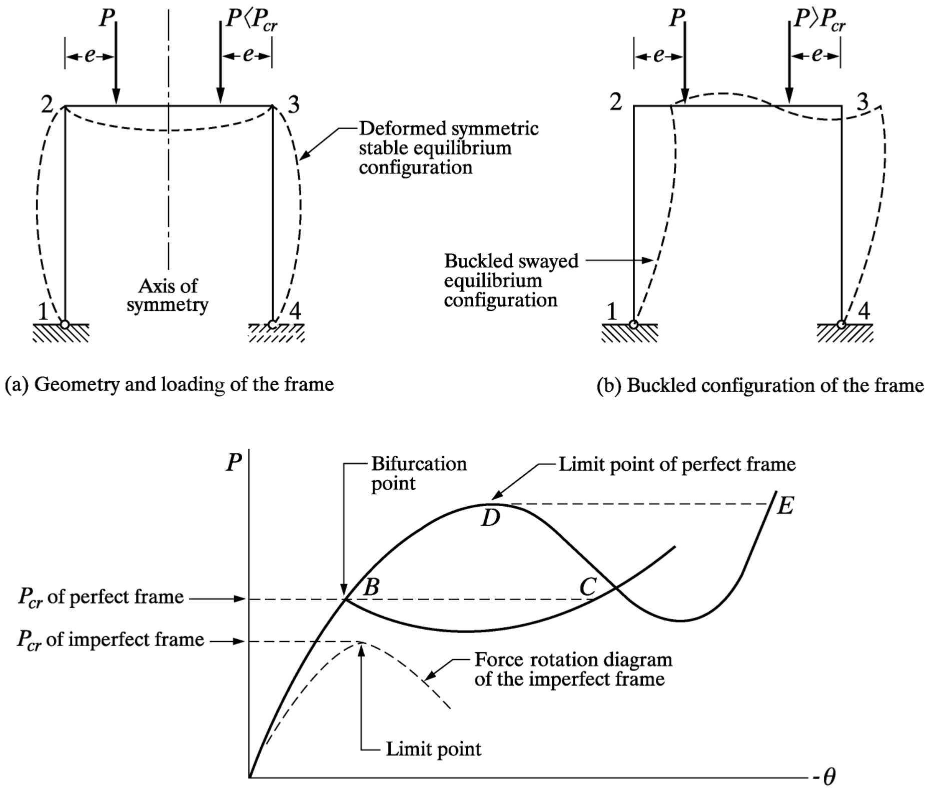
18.7b, the force–rotation diagram does not exhibit a limit point but a continuously rising path. That is, the frame does not loose its stability. For a certain value of the eccentricity slightly to the left of the axis of the vertical member, the limit point degenerates into a bifurcation point. For this value of e the angle θ in the initial stable equilibrium path ($0 < P < P_{cr}$) is equal to zero.

18.1.5 Symmetric Frame Consisting of Two Vertical and One Horizontal Members Subjected to Two Symmetric Forces on Its Horizontal Member

Consider the symmetric frame shown in Fig. 18.8a consisting of three perfectly straight, prismatic members made from an isotropic, linearly elastic material. As shown in Fig. 18.8a the frame is subjected to two equal forces which are acting on its horizontal member at points located at equal distances from the axis of symmetry of the frame. As the forces increase proportionally from zero the frame assumes symmetric about its axis of symmetry stable equilibrium configurations (see Fig. 18.8a) until for a certain critical value of the forces, as shown in Fig. 18.8b, the frame buckles (snaps through) to an asymmetric configuration, involving sidesway, denoted by point *C* in Fig. 18.8c. The line from point *O* just to the left of point *B* represents the stable equilibrium path. Point *B* is an asymmetric unstable bifurcation point. That is, the point at which the stable equilibrium path *OB* meets the equilibrium path *BC*. The value of the force corresponding to the bifurcation point is the critical load of the frame.

If the frame is prevented from swaying, it snaps through to a symmetric configuration denoted by *E* in Fig. 18.8c when the value of applied forces becomes equal to that corresponding to the limit point *D* of Fig. 18.8c.

Initial imperfections such as asymmetric forces or initially bent members change the response of the frame. Its force–rotation diagram exhibits a limit point with a critical



(a) Geometry and loading of the frame

(b) Buckled configuration of the frame

(c) External forces versus rotation of joint 2 diagram

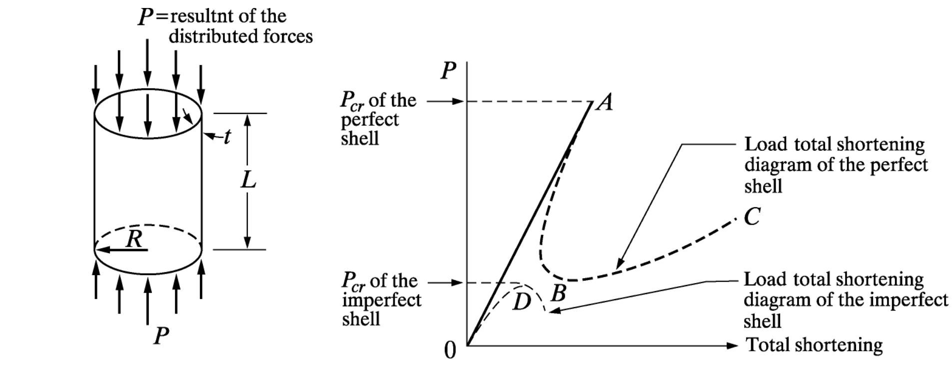
Figure 18.8 Response of a three-member symmetric frame subjected to symmetric forces.

force considerably smaller than that of the perfect frame (see Fig. 18.8c).

18.1.6 Thin Circular Cylindrical Shell Subjected to a Uniform Distribution of Axial Compressive Forces on Its End Surfaces

Consider a thin, perfectly cylindrical shell of perfectly circular cross section of radius R and constant thickness t. Each of the end surfaces of the shell is subjected to a perfectly uniform distribution of axial compressive forces whose resultant we denote by P (see Fig. 18.9a). The ratio R/t of the shell is large and its end surfaces are supported in such a way that their particles are free to move in any direction normal to its axis. The state of stress of the particles of the shell is specified with respect to a cylindrical system of axes x_1, r, θ by the following matrix:

$$[\tau] = \begin{bmatrix} \tau_{11} = \dfrac{P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18.1}$$



(a) Geometry and loading of the shell (b) Load total shortening diagram

Figure 18.9 Response of a circular cylindrical shell subjected to forces at its ends.

where A is the area of the cross section of the shell.

In Fig. 18.9b we plot the force P versus the total axial shortening δ of the shell. As P increases from zero the shell deforms to stable equilibrium configurations with its walls straight. In Fig. 18.9b these configurations are represented by the equilibrium path from point O to a point just to the left of point A. The latter is a branching or bifurcation point. As can be seen from Fig. 18.9b at the intersection of the fundamental path OA of stable equilibrium and the postbuckling path ABC the latter drops precipitously. The value of the force corresponding to point A is the critical load of the shell and we denote it by P_{cr}. It can be shown that it is equal to

$$P_{cr} = \frac{2\pi E t^2}{\sqrt{3(1 - \nu^2)}} \tag{18.2}$$

However, many experiments† have placed the actual buckling load of the shell as low as $0.1P_{cr}$. The reason for this discrepancy is that small imperfections change the response of the shell significantly and it looses its stability at a limit point [see point D of Fig. 18.9b)] at a considerably reduced value of the computed critical load (18.2) and snaps through to a stable equilibrium large amplitude diamond pattern configuration.

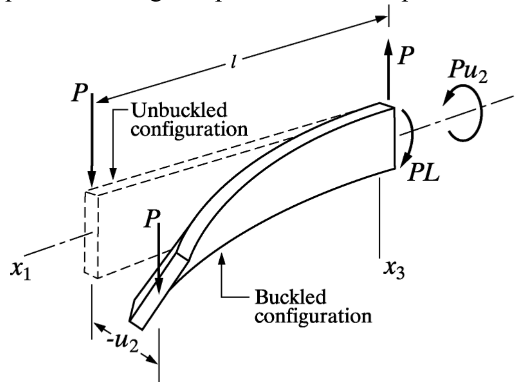


Figure 18.10 Free-body diagram of a cantilever beams of rectangular cross section undergoing flexural–torsional buckling.

† See for example, Almroth, B.O., Holmes A.M.C. and Brush, D.O., An experimental study of the buckling of cylinders under axial compression, *Journal of Experimental Mechanics,* 1964.

18.1.7 Beam of Thin-Walled Open Cross Sections Subjected to Bending About Its Principal Centroidal Axis of Maximum Moment of Inertia

A beam of thin-walled, open cross sections when subjected to bending about its principal centroidal axis of maximum moment of inertia may reach a state of instability and buckle by deflecting in the direction of the axis of maximum moment of inertia and twisting (see Fig. 18.10). This form of buckling may occur in beams whose torsional stiffness is relatively small and their bending stiffness about the centroidal axis of minimum principal moment of inertia is small compared to its stiffness about the centroidal axis of maximum principal moment of inertia. For example, an I-beam loaded in the plane of its web may reach a state of instability and buckle by twisting and deflecting in the direction normal to its web. This buckling phenomeon is known as *flexural–torsional buckling of beams*.

18.1.8 Comments

From the above examples we can make the following observations:

1. Bodies of certain geometry when they are subjected to certain values of a certain type of conservative forces, specified by one parameter P, assume equilibrium configurations which cannot be established using a theory based on the assumption of small deformation (see Section 2.4). The deformed configuration of such bodies can be established using theories based on less restrictive assumptions than the assumption of small deformation.

2. If we denote by δ a parameter which gives an indication of the magnitude of the deformation of such a body, its P versus δ diagram has either a limit point (see Fig. 18.5b) or a bifurcation point (see Figs. 18.4b, 18.6b, 18.8c and 18.9b); or for values of P higher than a certain value, δ grows rapidly as the load increases (see Figs. 18.4b and 18.7b). A limit point is a local maximum of the P versus δ diagram (see Fig. 18.5b). A bifurcation point is the point of intersection of the initial stable equilibrium path with another equilibrium path (postbukling path). The pattern of the deformed configuration of the body in the postbukling path is different than that in the initial stable equilibrium path. We distinguish three types of bifurcation points — stable-symmetric (see point A of Fig. 18.4b) unstable-symmetric (see point A of Fig. 18.6b) and unstable asymmetric (see point A of Fig. 18.9b). The value of the load parameter P corresponding to a limit point or to a bifurcation point is called the critical load at buckling of the structure.

3. The total potential energy of a body at a stable equilibrium configuration is a local minimum, while at an unstable equilibrium configuration is a local maximum .

4. Initial imperfections affect the pattern of deformation of bodies whose load deflection diagram has a bifurcation point. Moreover, initial imperfections reduce the value of the buckling load of bodies whose load–deflection diagram has either a limit point or an unstable bifurcation point.

5. The critical load at buckling is considered an upperbound to the load a structure can carry before loosing its ability to perform the task for which it has been designed, that is, before failing.

18.2 The Non-Linear Theory of Elasticity and the Theory of Moderate Rotations

Consider a particle of a body, which in the undeformed state of mechanical and thermal equilibrium at the uniform temperature T_o, is an orthogonal parallelepiped

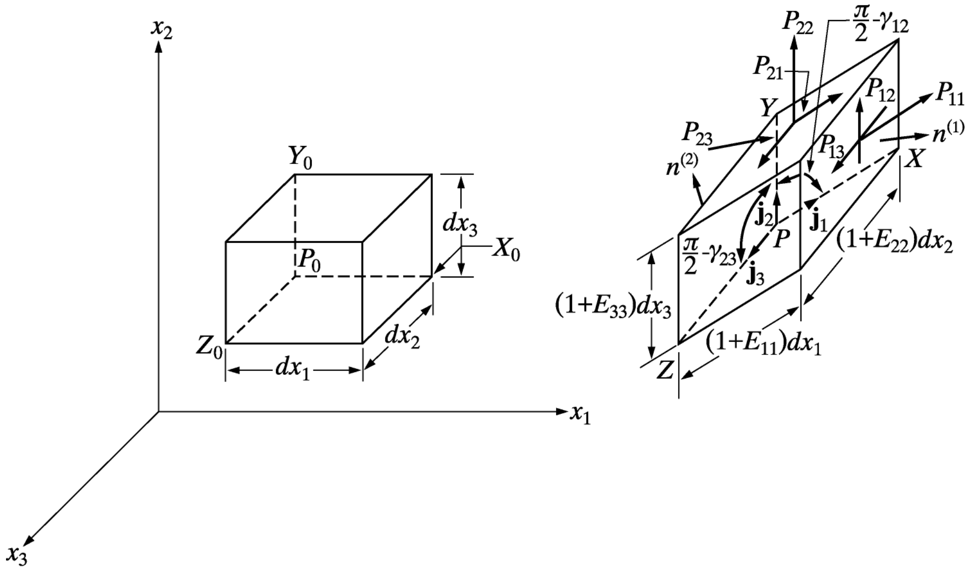


Figure 18.11 Deformed configuration of an infinitesimal orthogonal parallelepiped and components of traction P_{ij}.

$P_oX_oY_oZ_o$ with edges dx_1, dx_2, dx_3 (see Fig. 18.11). In general as the body goes from its undeformed to its deformed state its particles translate, rotate and deform (elongate or shrink and distort). As shown in Fig. 18.11 the particle under consideration in the deformed state is a non-orthogonal parallelepiped $PXYZ$ whose edges have lengths $(1 + E_{11})dx_1$, $(1 + E_{22})dx_2$, $(1 + E_{33})dx_3$ where E_{11}, E_{22} or E_{33} are the unit elongations or shinkages in the direction of the x_1, x_2 or x_3 axis, respectively, defined by relations (2.5). Moreover, as shown in Fig. 18.11 the before deformation right angles $\triangle X_oP_oY_o$, $\triangle X_oP_oZ_o$ and $\triangle Y_oP_oZ_o$ after deformation become equal to $\pi/2 - \gamma_{12}$, $\pi/2 - \gamma_{13}$ and $\pi/2 - \gamma_{23}$, respectively, where γ_{12}, γ_{13} and γ_{23} are the unit shears in the directions x_1x_2, x_1x_3 and x_2x_3, respectively, defined by relations (2.7). We denote by $\overset{n_i}{\mathbf{T}}$ $(i = 1, 2, 3)$ the tractions per unit undeformed area acting on the planes of the particle under consideration which before deformation were normal to the $x_i (i = 1, 2, 3)$ axes. We decompose each of these tractions into three components along the three non-orthogonal directions $\mathbf{j}_1$, $\mathbf{j}_2$ and $\mathbf{j}_3$, which are parallel to the edges of the deformed parallelepiped (see Fig. 18.11). That is,

$$
\begin{aligned}
\overset{n_1}{\mathbf{T}} &= P_{11}\mathbf{j}_1 + P_{12}\mathbf{j}_2 + P_{13}\mathbf{j}_3 \\
\overset{n_2}{\mathbf{T}} &= P_{21}\mathbf{j}_1 + P_{22}\mathbf{j}_2 + P_{23}\mathbf{j}_3 \qquad (18.3) \\
\overset{n_3}{\mathbf{T}} &= P_{31}\mathbf{j}_1 + P_{32}\mathbf{j}_2 + P_{33}\mathbf{j}_3
\end{aligned}
$$

It can be shown that the quantities $P_{ij} (i, j = 1, 2, 3)$ do not transform as components of a symmetric tensor of the second rank upon rotation of the system of axes $x_i (i = 1, 2, 3)$. For this reason we define another set of nine quantities known as the components of the *Lagrangian or material stress tensor* which can be shown that transform as components of a symmetric tensor of the second rank and are defined as

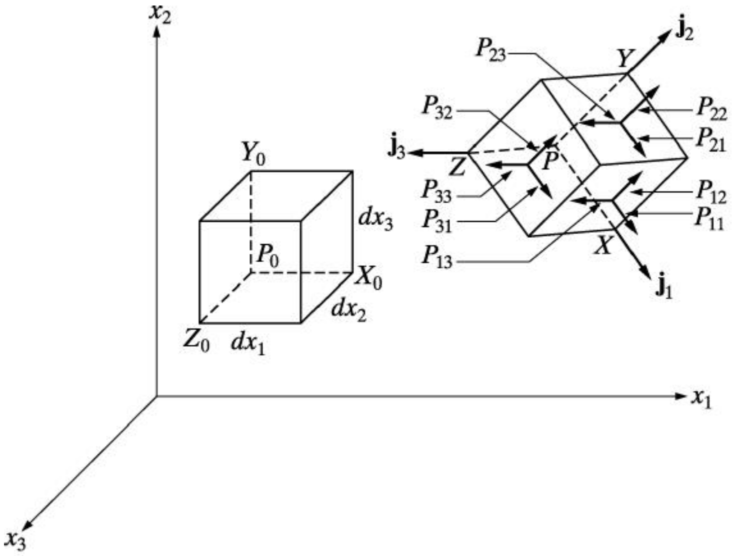


Figure 18.12 Approximation of the deformed configuration of an infinitesimal material parallelepiped in the theory of moderate rotations.

$$\sigma_{ij} = \frac{P_{ij}}{1 + E_{ij}} \qquad (i,j = 1,2,3 \quad E_{ij} = 0 \text{ if } i \neq j) \tag{18.4}$$

In the theory of moderate rotations it is assumed that the unit elongations or shinkages and the unit shears are negligible compared to unity. Consequently, as shown in Section 2.4 the unit elongations or shrinkages may be approximated by the normal components of the Lagrangian strain and the unit shears by twice the shearing components of the Lagrangian strain [see relations (2.14)]. That is,

$$E_{ii} = \epsilon_{ii} \qquad \gamma_{ij} = 2\epsilon_{ij} \qquad (i,j = 1,2,3 \;\; i \neq j) \tag{18.5}$$

Moreover, an infinitesimal material orthogonal parallelepiped before deformation may be considered as being approximately orthogonal after deformation (see Fig. 18.12) and the length of its deformed edges may be approximated by the length of its corresponding undeformed edges. However, the rotation of an infinitesimal material parallelepiped, due to the deformation of the body, cannot be disregarded (see Fig. 18.12). Furthermore, referring to relation (18.4) we see that the components of the Lagrangian stress may be approximated by the components P_{ij} $(i, j = 1, 2, 3)$ of traction acting on the surfaces of an infinitesimal material parallelepiped which before deformation was orthogonal with edges parallel to the x_1, x_2, x_3 axes. That is,

$$\sigma_{ij} \approx P_{ij} \qquad (i, j = 1, 2, 3) \tag{18.6}$$

where P_{ij} $(i, j = 1, 2, 3)$ are the components of traction in the directions $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$ which, in the theory of moderate rotations, may be considered as being approximately orthogonal.

In the non-linear (large deformation) theory of elasticity and in the theory of moderate rotations it is assumed that when a body is subjected to external forces in an environment of constant temperature, the components of the Lagrangian stress of a particle are related to its components of the Lagrangian strain defined by relations (2.13) by linear relations analogous to (3.47) and (3.48). Moreover, it can be shown that the theorem of stationary total potential energy proved in Section 13.17 for the linear theory of elasticity is valid *for the non-linear theory of elasticity and for the theory of moderate rotations.* Referring to relation (13.121) the total potential energy of a body, in an environment of constant temperature, made from an elastic material is

$$\mathit{\Pi}_s = \iiint_V U_s dV + \begin{array}{l}\text{work of the external actions in } \textit{going} \text{ from} \\ \text{their position in the deformed configuration} \\ \text{of the body to their position in the} \\ \text{undeformed configuration of the body}\end{array} \tag{18.7}$$

In the non-linear theory of elasticity and in the theory of moderate rotations the strain energy density U_s is considered a function of the nine components of the Lagrangian strain ϵ_{ij} $(i, j = 1, 2, 3)$ [see relations (2.13)] and it is defined as

$$\sigma_{ij} = \frac{\partial U_s}{\partial \epsilon_{ij}} \tag{18.8}$$

where σ_{ij} are the Lagrangian components of stress. It can be shown that the strain energy density of a particle of a body made from an isotropic, linearly elastic material is equal to

$$U_s = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\epsilon_{ij} \tag{18.9}$$

18.3 Criterion for the Stability or Instability of an Equilibrium Configuration of Structures

An effective criterion for the stability or instability of an equilibrium configuration of a body subjected to conservative forces is the following axiom which it is known as the *stability axiom*:

A necessary and sufficient condition for an equilibrium configuration of a body subjected to conservative forces to be stable is that its total potential energy is a local minimum. That is, the second variation of the total potential energy is positive (see Section E.4 of Appendix E)

$$\delta^{(2)} \mathit{\Pi}_s > 0 \tag{18.10}$$

Moreover, a necessary and sufficient condition for an equilibrium configuration of a body subjected to conservative forces to be unstable is that its total potential energy is a local maximum. That is, the second variation of its total potential energy is negative. Furthermore, if the second variation of the total potential energy of a body subjected to conservative forces is zero, a necessary and sufficient condition for an equilibrium configuration of the body to be stable is that the third variation of its total potential energy is positive.

18.4 Investigation of the Beginning of Buckling

In most cases of practical interest, we are only interested in establishing the critical load at buckling. For structures which exhibit a bifurcation point, when they are assumed perfect, this may be accomplished by investigating only the beginning of buckling, wherein the unit elongations or shrinkages, the unit shears and the rotations are very small and can be disregarded compared to unity. Thus, the Lagrangian components of strain ϵ_{ij}

($i, j = 1, 2, 3$) [see relations (2.13)] can be approximated by the components of strain e_{ij} ($i, j = 1, 2, 3$) [see relations (2.15)]. However, in drawing the free-body diagrams of the elements of a structure their rotation must be taken into account. The resulting approach yields the value of the critical load at buckling, as well as the shape of the buckled configuration of the structure at the beginning of buckling. It does not give, however, the postbuckled configuration of the structure.

18.5 Buckling of Structures Having One Degree of Freedom

Consider a structure subjected to conservative external forces whose magnitude is specified by one parameter P while its buckled configuration is specified by one parameter δ. For example, for the column of Fig. 18.4a δ could be the transverse component of translation $u_3^{(A)}$ of the end A of the column. For the arch of Fig. 18.5a, δ could be the deflection δ_c of its middle point. The critical load at buckling and the post-buckling behavior of such a structure can be established using the *theory of moderate rotations* and adhering to the following steps:

STEP 1 The equation for equilibrium for the structure is established. This is a non-linear relation between the force parameter P and the deformation parameter δ and can be established using one of the following methods:

1. *The direct equilibrium method*
2. *The method of stationary total potential energy*

STEP 2 The equation of equilibrium is solved and the P versus δ diagram is plotted. For small values of P the equation of equilibrium has an unique solution, which represents a stable equilibrium configuration of the structure. However, for higher values of P the equation of equilibrium may have more than one solution, each of which represents an equilibrium configuration which could be stable or unstable. The P versus δ diagram could consist of one continuous curve or of two intersecting curves known as *equilibrium paths*. The point of intersection of the two curves is known as the bifurcation point. For a structure whose load–deflection diagram has a bifurcation point, the extension of the original stable equilibrium path is unstable. The postbuckling equilibrium path is stable if $|\delta|$ increases when P increases.

In this section we establish the critical load at buckling and the postbuckling behavior of two simple structures having one degree of freedom. The load–deformation parameter diagram of the first structure when its loading is perfectly centroidal has a bifurcation point while that of the second structure has a limit point.

Example 1 Consider the structure shown in Fig. a consisting of two rigid prismatic bars AB and BC of length L joined by a frictionless pin. The bars are constrained from rotating relative to each other about the axis of the pin by a linear rotational spring of stiffness k_r. One end of the structure is pinned to a rigid support while its other end is pinned to a support which can move only in the direction of the axis of the structure. Moreover, the translation of the middle point of the structure is constrained by a linear extensional spring of stiffness k_e. Both springs are unstretched when the structure is in its straight-line configuration. The structure is subjected to an axial centroidal compressive force P.

Establish the critical value of P for which the structure can assume a stable equilibrium configuration (buckled) other than the straight. Moreover, determine the post-bucking behavior of the structure.

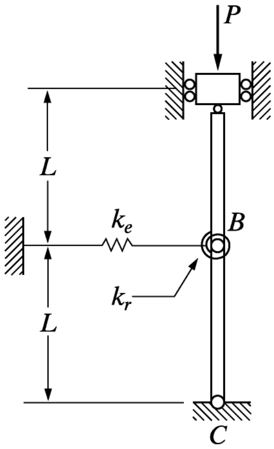


Figure a Geometry and loading of the structure.

Solution The structure of Fig. a has one degree of freedom. That is, as shown in Fig. b, its deformed configuration can be specified by one variable the angle of rotation θ.

Determination of the equation of equilibrium using the direct equilibrium method

In Fig. c we show the free-body diagram of bar AB in its deformed configuration. Referring to this diagram we have

$$\sum M_2^{(B')} = 0 \qquad P(L\sin\theta) - \frac{k_e}{2}L\sin\theta(L\cos\theta) - k_r(2\theta) = 0 \tag{a}$$

This is the equation of equilibrium for the structure of Fig. a. It has the following two solutions:

$$\theta = 0 \quad \text{For all choices of } k_e,\ k_r,\ P \text{ and } L \tag{b}$$

$$P = \frac{k_e L\cos\theta}{2} + \frac{2k_r\theta}{L\sin\theta} \tag{c}$$

Therefore, there are two equilibrium paths in the P versus θ diagram which intersect at a bifurcation point. The critical value of the load is obtained from equation (c) in the limit as θ approaches zero and noting that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$. That is,

$$P_{cr} = \frac{k_e L}{2} + \frac{2k_r}{L} \tag{d}$$

Determination of equation of equilibrium using the method of stationary total potential energy

The equation of equilibrium (a) can be established by applying the theorem of stationary total potential energy presented in Section 18.2. Recall that the total strain

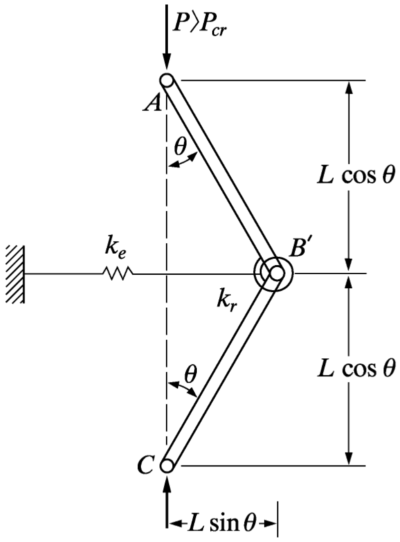


Figure b Deformed configuration of the structure.

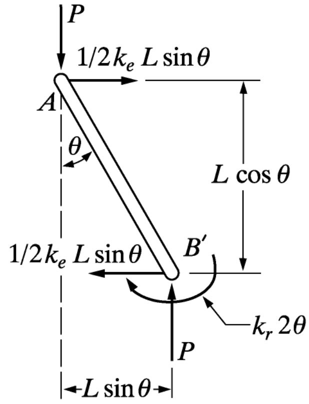


Figure c Free-body diagram of bar *AB* in its deformed configuration.

energy stored in a linear extensional spring is equal to the work of the applied force. That is, denoting by δ the elongation of the spring, we have

$$U_{\text{exten spr}} = \frac{k_e\delta^2}{2} \tag{e}$$

Moreover, the total strain energy stored in a linear rotational spring is equal to the work of the applied moment. That is, referring to Fig. b, we get

$$U_{\text{rotat spr}} = \frac{k_r(2\theta)^2}{2} \tag{f}$$

Thus, the total strain energy U_s stored in the two springs of the structure of Fig. a, when it assumes the equilibrium configuration shown in Fig. b, is equal to

$$U_s = \frac{k_e\delta^2}{2} + \frac{k_r(2\theta)^2}{2} = \frac{1}{2}(k_e L^2\sin^2\theta + 4k_r\theta^2) \tag{g}$$

Referring to Fig. b, the work of the external forces in moving from their position in the deform configuration of the structure to their position in its undeformed configuration, is given as

$$W_{\text{ext forces}} = -2PL(1 - \cos\theta) \tag{h}$$

Substituting relations (g) and (h) into relation (18.7), we obtain the following expression for the total potential energy of the structure:

$$\Pi_s(P,\theta) = \frac{1}{2}(k_e L^2\sin^2\theta + 4k_r\theta^2) - 2PL(1 - \cos\theta) \tag{i}$$

The theorem of stationary total potential energy of the non-linear theory of elasticity and the theory of moderate rotations, states that a function $\theta(P)$ which renders the total potential energy of a structure stationary specifies a configuration of equilibrium of this structure. This implies that the function $\theta(P)$ which renders the first variation of the total potential energy of the structure equal to zero specifies a position of equilibrium of this structure (see Section E.4 of Appendix E). Thus,

$$\delta^{(1)}\Pi_s = \frac{\partial \Pi_s}{\partial \theta}\delta\theta = (k_e L^2 \sin\theta\cos\theta + 4k_r\theta - 2PL\sin\theta)\,\delta\theta = 0 \tag{j}$$

Relation (j) is valid for any variation $\delta\theta$. Consequently, the term in parentheses in relation (j) must vanish. That is,

$$k_e L^2 \sin\theta\cos\theta + 4k_r\theta - 2PL\sin\theta = 0$$

This equation is the equation of equilibrium for the structure of Fig. a. As expected it is identical to equation (a) and it is satisfied if $\theta = 0$ or if P and θ satisfy relation (c).

Investigation of the stability or instability of the equilibrium paths of the column of Fig. a using the stability axiom presented in Section 18.3.

Recall that the stability action states that a necessary and sufficient condition for a path to be stable (unstable) is that the total potential energy of the structure is a local minimum (maximum). This implies that a structure is stable if the second variation of its total potential energy is positive and unstable if it is negative. Referring to relation (E.9) of Appendix E and using relation (i) the second variation of Π_s is

$$\delta^{(2)}\Pi_s = \frac{\partial^2 \Pi_s}{\partial \theta^2}\delta^2\theta = [k_e L^2(\cos^2\theta - \sin^2\theta) + 4k_r - 2PL\cos\theta]\,\delta^2\theta \tag{k}$$

For the path $\theta = 0$ relation (j) reduces to

$$\delta^{(2)}\Pi_s = \left(\frac{k_e L}{2} + \frac{2k_r}{L} - P\right) 2L\delta^2\theta \tag{l}$$

Thus, using relation (d), we see that the second variation $\delta^{(2)}\Pi_s$ of Π_s is positive for

$$P < \frac{k_e L}{2} + \frac{2k_r}{L} = P_{cr} \tag{m}$$

and negative for

$$P > \frac{k_e L}{2} + \frac{2k_r}{L} = P_{cr} \tag{n}$$

Therefore, as expected, for any set of values of the parameters $k_e L^2$ and k_r the path $\theta = 0$ is stable for $P < P_{cr}$ and unstable for $P > P_{cr}$.

For the path specified by relation (c), relation (j) reduces to

$$\delta^{(2)}\Pi_s = \frac{\partial^2 \Pi_s}{\partial \theta^2}\delta^2\theta = \left[-k_e L^2 \sin^2\theta + 4k_r - 4k_r\theta\frac{\cos\theta}{\sin\theta}\right]\delta^2\theta \tag{o}$$

From relation (o) we see that the sign of $\delta^{(2)}\Pi_s$ and, consequently, the stability or instability of the postbuckling path, depends on the values of the parameters $k_e L^2$ and k_r.

Moreover, from relation (o) we find that $\delta^{(2)}\Pi_s$ vanishes at the bifurcation point $\theta = 0$. Thus, in order to establish whether Π_s is a maximum or a minimum at the bifurcation point we must examine the sign of the third or higher variations of Π_s. That is, expanding $\Pi_s(P_{cr}, \theta)$ into a Taylor series about $\theta = 0$, we have

$$\delta \Pi_s = \Pi_s(P_{cr}, \theta) - \Pi_s(P_{cr}, 0) = \left(\frac{\partial \Pi_S}{\partial \theta}\right)_{\theta=0} \delta\theta + \frac{1}{2}\left(\frac{\partial^2 \Pi_s}{\partial \theta^2}\right)_{\theta=0} \delta^{(2)}\theta + \frac{1}{3!}\left(\frac{\partial^3 \Pi_s}{\partial \theta^3}\right)_{\theta=0} \delta^{(3)}\theta + \frac{1}{4!}\left(\frac{\partial^4 \Pi_s}{\partial \theta^4}\right)_{\theta=0} \delta^{(4)}\theta + \cdots \tag{p}$$

At the bifurcation point the first and second terms on the right side of relation (p) vanish, while the third and fourth terms may be obtained from relation (k) as

$$\delta^3 \Pi_s = \frac{1}{3!}\left(\frac{\partial^3 \Pi}{\partial \theta^3}\right)_{\theta=0} \delta^3\theta = \Big[-4\, k_e L^2 (\sin\theta \cos\theta + 2\, PL \sin\theta)\Big]_{\theta=0} \delta^3\theta = 0$$

$$\delta^4 \Pi_s = \frac{1}{4!}\left(\frac{\partial^4 \Pi}{\partial \theta^4}\right)_{\theta=0} \delta^4\theta = \frac{1}{4!}\Big[4 k_e L^2 (\sin^2\theta - \cos^2\theta) + 2PL\cos\theta\Big]_{\theta=0} \delta^4\theta = (-4 k_e L^2 + 2PL)\delta^4\theta \tag{q}$$

Thus, the third variation of Π_s vanishes at the biforcation point and in order to establish if Π_s is a minimum or a maximun at the biforcation point, we must examine the sign of the fourth variation of Π_s.

Substituting relation (d) into (p) at $P = P_{cr}$, we get

$$\delta^4 \Pi_s = \frac{1}{4!}(-3 k_e L^2 + 4 k_r)\delta^4\theta \tag{r}$$

This indicates that, at the bifurcation point, $\delta^4 \Pi_s$ is positive if

$$k_r > \frac{3}{4} k_e L^2 \tag{s}$$

Therefore, if the values of the parameters $k_e L^2$ and k_r satisfy relation (r), Π_s is a minimum at the bifurcation point and, consequently, when the structure under consideration is subjected to the value of the force $P = P_{cr}$, it is in a state of stable equilibrium in the straight configuration.

For $k_e L^2 = 0$ we obtain the structure of Fig. d and relations (c), (d) and (i) reduce to

$$P = \frac{2 k_r \theta}{L \sin\theta} \tag{t}$$

$$P_{cr} = \frac{2 k_r}{L} \tag{u}$$

$$\Pi_s = 2 k_r \theta^2 - 2\, PL(1 - \cos\theta) \tag{v}$$

In Fig. e we plot for the structure of Fig. d, the P/P_{cr} versus θ diagram [relation (t)] and the $\Pi_s/P_{cr}L$ versus θ diagrams [using relation (u)] for $P = P_{cr}$ and $P = 1.1P_{cr}$;

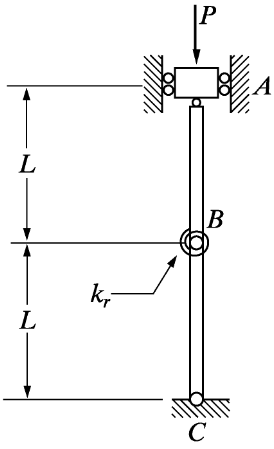


Figure d Geometry and loading of the structure with $k_eL = 0$.

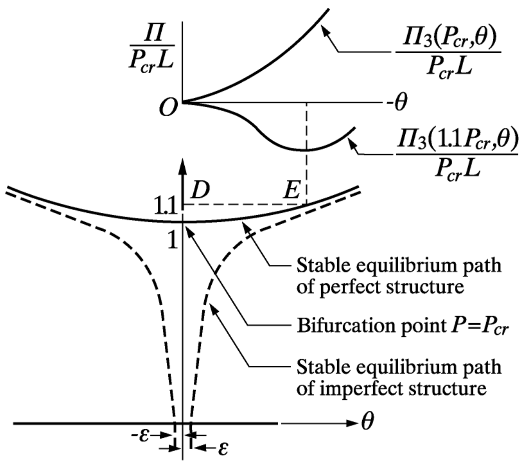


Figure e Load deflection and total potential energy deflection diagrams for the structure of Fig. d.

referring to this figure, we see that $\delta P/\delta\theta = 2k_r\theta/L\sin\theta\left(1 - \frac{\theta\cos\theta}{\sin\theta}\right) \approx$ $2k_r/L(1-\cos\theta)$ is greater than zero on the postbuckling path. Therefore the postbuckling path is a stable equilibrium path. Moreover, from relation (r) we see that at the bifurcation point $\delta\Pi_s$ is positive. This indicates that as shown in Fig. e the total potential energy of the structure at the bifurcation point is a relative minimum and therefore, at this point the equilibrium is stable. We say that the structure of Fig. d has *a stable symmetric bifurcation point*. Referring to Fig. e, we see that for $P = 1.1P_{cr}$ the total potential energy at point D ($\theta = 0$) is a relative maximum while at point E is a relative minimum. That is, when $P = 1.1P_{cr}$, the straight-line equilibrium configuration of the structure is unstable while the bent equilibrium configuration, specified by point E, is stable.

For $k_r = 0$ we obtain the structure of Fig. f and relations (c), (d) and (i) reduce to

$$P = \frac{k_eL\cos\theta}{2} \tag{w}$$

$$P_{cr} = \frac{k_eL}{2} \tag{x}$$

$$\Pi_s = \frac{1}{2}k_eL^2\sin^2\theta - 2PL(1 - \cos\theta) \tag{y}$$

In Fig. g we plot the P/P_{cr} versus θ diagram for the structure of Fig. f; referring to this figure, we see that the postbuckling path is an unstable equilibrium path. Therefore as soon as the load reaches its critical value in the presence of any disturbance, no matter how small, the structure will collapse. We say that the structure of Fig. f *has an unstable symmetric bifurcation point*.

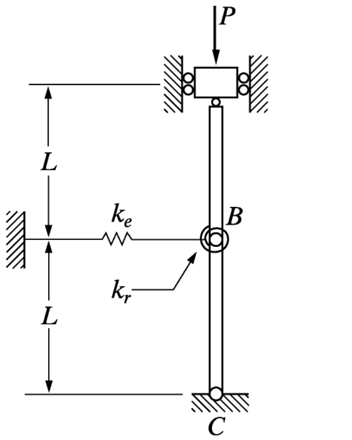


Figure f Geometry and loading of the structure with $k_r = 0$.

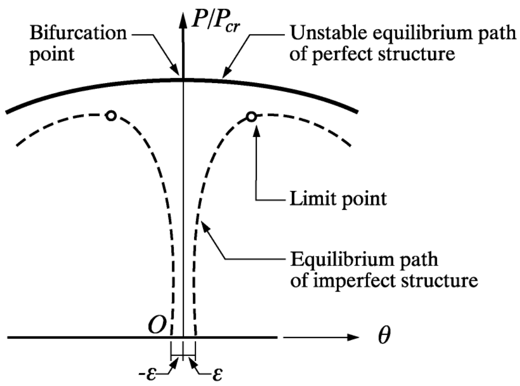


Figure g Load–deflection diagram for the structure of Fig. f.

For $k_e L^2 = 8k_r$ relations (c), (d) and (i) reduce to

$$P = \frac{2k_r}{L}\left(2\cos\theta + \frac{\theta}{\sin\theta}\right) \tag{z}$$

$$P_{cr} = \frac{6k_r}{L} \tag{za}$$

$$\Pi_s = 2k_r(2\sin^2\theta + \theta^2) - 2PL(1 - \cos\theta) \tag{zb}$$

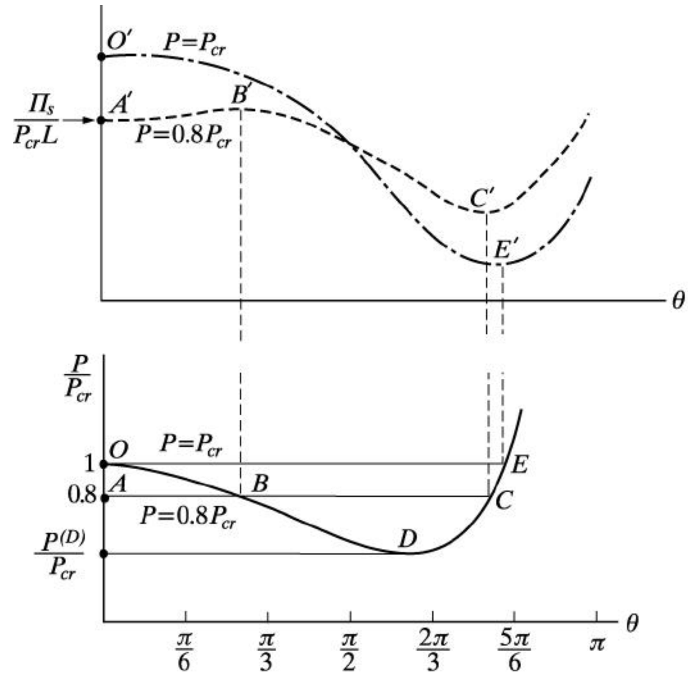


Figure h Load–deflection diagram for the structure of Fig. a with $K_e L^2 = 8\,K_r$.

In Fig. h we plot P/P_{cr} versus θ [relation (z)] and $\Pi_s/P_{cr}L$ versus θ [relation (zb)] for the structure under consideration; referring to this figure, we see that when the load reaches the value $P = 0.8P_{cr}$, the total potential energy of the structure is a relative minimum at its straight-line configuration specified by point A on the load–rotation diagram. Consequently, for this value of P the straight-line configuration is stable. However, if sufficient energy was applied to the structure by a transverse force or by a bending moment, the structure could snap through to the configuration specified by point C on the load–rotation diagram of Fig. h since this configuration is more stable than that at point A. Referring to Fig. h, we see that at $P = P_{cr}$ the total potential energy of the is in a state of unstable equilibrium in its straight-line configuration and it will snap through to the stable equilibrium configuration at point E. This example illustrates the importance of knowing the postbuckling path of a structure when designing it. Although the straight-line configurations of the structure are stable up to $P < P_{cr}$, for values of $P < P^{(D)}$, the structure may snap through to a more stable bent configuration at which it may not be able to perform the task for which it has been designed. Thus, the failure load for the structure under consideration is not P_{cr} but the much lower load $P^{(D)}$.

Computation of the critical load at buckling by investigating only the beginning of buckling

If we investigate only the beginning of buckling, the angle θ in Figs. b and c is very small and, consequently, $\sin\theta$ is approximately equal to the angle θ while $\cos\theta$ is approximately equal to unity. Taking this into account relation (a) reduces to

$$PL\theta - \frac{k_e L^2}{2}\theta - 2k_r\theta = 0 \tag{zc}$$

This equation has the following solution:

$$\theta = 0 \quad \text{for all choice of } k_e,\ k_r,\ P \text{ and } L$$

$$\theta \neq 0 \qquad P = \frac{k_e L}{2} + \frac{2k_r}{L} = P_{cr} \tag{zd}$$

It is clear that this approach gives only the value of the critical load at buckling.

Imperfect structure

Up to this point we have assumed that the structure under consideration is perfect. That is, it is perfectly straight in its undeformed state and the external compressive axial force is perfectly centroidal. This implies that the springs are unstretched and unstrained when the structure is in its straight line configuration. In what follows we assume that the structure, under consideration, is imperfect in the sense that the two springs are unstrained when the rigid bars AB and BC have a small rotation which in Fig. i we denote by ϵ. The postbuckling configuration of the structure is specified by the angle θ measured from its straight-line configuration (see Fig. i). The free-body diagram of bar AB in its deformed configuration is shown in Fig. j; referring to this figure, from the equilibrium of bar AB, we have

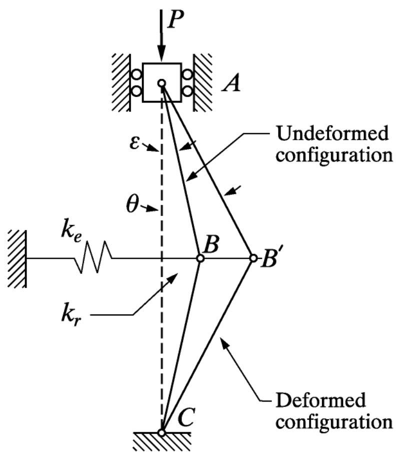


Figure i Imperfect structure.

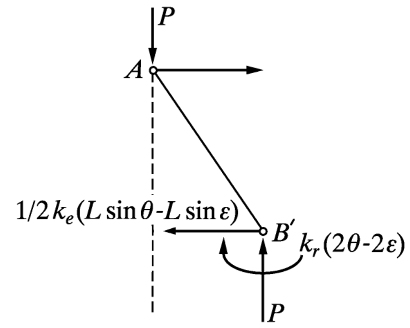


Figure j Free-body diagram of bar AB of the imperfect structure of Fig. i in a deformed configuration.

$$\sum M_2^{(A)} = \frac{k_e}{2}(L\sin\theta - L\sin\epsilon)L\cos\theta + k_r(2\theta - 2\epsilon) - PL\sin\theta = 0 \qquad \text{(ze)}$$

The solution of this equation is

$$P = \frac{k_e L\cos\theta}{2} - \frac{k_e L\sin\epsilon\cos\theta}{2\sin\theta} - \frac{k_r(2\theta - 2\epsilon)}{L\sin\theta} \qquad \text{(zf)}$$

In Fig. e we plot relation (zf) for $k_eL^2 = 0$ and a value of ϵ; referring to this figure, we see that the load–deflection diagram of the imperfect structure under consideration is constantly rising and stable ($\delta P\delta\theta > 0$). However, as the value of the force P acting on the imperfect structure approaches the critical value of the force, acting on the perfect structure ($\epsilon=0$) the rate of increase of the deflection of the imperfect structure becomes large. Moreover, the loading path for the imperfect structure of Fig. i approaches the postbuckling path for the perfect structure of Fig. d. Usually when the force acting on the imperfect structure reaches the value of P_{cr} for the perfect structure, the deflection of the imperfect structure is large and the structure cannot perform the task which it has been designed to perform. Thus, P_{cr} is an upper bound of the maximum load the structure can carry. From Fig. e we see that the effect of the initial imperfection on the ability of the structure of Fig. d to resist the applied force is small. That is, the critical load at buckling of the structure of Fig. d is *imperfection insensitive*.

In Fig. g we plot relation (zf) for $k_r = 0$ and a value of ϵ; referring to this figure, we see that the load–deflection diagram for the imperfect structure has a limit point. When the external force reaches the value corresponding to the limit point, the structure reaches a state of unstable equilibrium and collapses. The values of the external force corresponding to the limit point depends on the value of the initial imperfection ϵ and it is considerably less than P_{cr} of the perfect structure. That is, the critical load of the structure of Fig. f ($k_r = 0$) is *imperfection sensitive*.

Example 2 Consider the structure shown in Fig. a, consisting of two identical slender

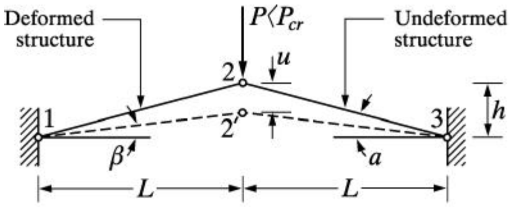


Figure a Geometry and loading of the structure.

members of constant cross section, pinned together at one end, while their other end is pinned to a rigid support. The members of the structure are made from an isotropic, linearly elastic material. That is, the relationship between the axial component of the Lagrangian stress tensor and the axial component of the Lagrangian strain tensor (2.13) of a particle of a structure is linear. Consequently, since the members of the structure are in a state of uniaxial stress ($\sigma_{11} \neq 0$, $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$), we have

$$\sigma_{11} = E\epsilon_{11} \tag{a}$$

where E is the modulus of elasticity of the material from which the members of the structure are made and ϵ_{11} is the Langrangian component of strain of a particle of a member of the structure in the direction of its axis. The structure is subjected to a concentrated transverse force at joint 2. Compute the displacement of joint 2 as a function of the force P and of the angle α. The latter specifies the geometry of the undeformed structure. Use the theory of moderate rotations to compute the displacement of joint 2 of the structure.

Solution The undeformed and deformed configurations of the structure are shown in Fig. a. Joint 2 is displaced to point 2′. The change of length per unit length of a particle of a member of the structure in the direction of its axis is equal to

$$E_{11} = \frac{L/\cos\beta - L/\cos\alpha}{L/\cos\alpha} = \frac{\cos\alpha}{\cos\beta} - 1 \tag{b}$$

Moreover, referring to Fig. a, from geometric considerations, we obtain

$$\begin{aligned} \cos\alpha &= \frac{L}{\sqrt{L^2 + h^2}} & \sin\alpha &= \frac{h}{\sqrt{L^2 + h^2}} \\ \sin\beta &= \frac{h - u}{\sqrt{L^2 + (h-u)^2}} & \cos\beta &= \frac{L}{\sqrt{L^2 + (h-u)^2}} \end{aligned} \tag{c}$$

Substituting the first and the fourth of the above relations into (b), we obtain the following non-linear relation between the unit elongation E_{11} and the displacement u:

$$E_{11} = \sqrt{\frac{L^2 + (h-u)^2}{L^2 + h^2}} - 1 = \sqrt{1 + \frac{u^2 - 2hu}{L^2 + h^2}} - 1 \tag{d}$$

We now proceed to analyze the structure using the theory of moderate rotations. In

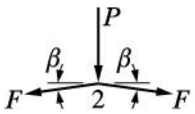


Figure b Free-body diagram of joint 2 of the structure of Fig. a in its deformed configuration.

this theory, the unit elongations or shrinkages and the unit shears are negligible compared to unity. Thus, the unit elongation or shrinkage E_{11} is approximately equal to the normal component ϵ_{11} of the Langrangian strain. That is,

$$\epsilon_{11} \approx E_{11} = \sqrt{\frac{L^2 + (h - u)^2}{L^2 + h^2}} - 1 = \sqrt{1 + \frac{u^2 - 2hu}{L^2 + h^2}} - 1 \tag{e}$$

Moreover, the change of the area of the cross sections of the members of the structure is disregarded, as compared to their original area A. Thus, denoting by F the axial centroidal force acting the cross sections of each member, we have

$$\sigma_{11} = \frac{F}{A} \tag{f}$$

However, in the theory of moderate rotations the rotations are assumed of higher order of magnitude than the unit elongations or shrinkages and the unit shears. Thus, in drawing the free-body diagrams of segments of the structure their rotation is taken into account. Consequently, referring to Fig. b from the equilibrium of joint 2, we have:

$$\sum \bar{F} = 0 = 2F\sin\beta + P \qquad \text{or} \qquad F = -\frac{P}{2\sin\beta} \tag{g}$$

Substituting relation (g) into (f) and the resulting relation into (a), we obtain

$$\epsilon_{11} = \frac{F}{EA} = -\frac{P}{2EA\sin\beta} \tag{h}$$

Substituting relations (h) into (e) and using the third of relations (c), we get

$$\begin{aligned}\frac{P}{EA} &= -\frac{2(h-u)}{\sqrt{L^2 + (h-u)^2}}\left[\frac{\sqrt{L^2 + (h-u)^2}}{L^2 - h^2} - 1\right] \\ &= -\frac{2(h-u)}{\sqrt{L^2 + (h-u)^2}}\left[\sqrt{1 + \frac{u^2}{L^2 + h^2} - \frac{2hu}{L^2 + h^2}} - 1\right]\end{aligned} \tag{i}$$

This is the nonlinear, force–displacement relation obtained on the basis of the theory of moderate rotations. It is plotted in Fig. c for $h/L = 0.01$, and $h/L = 0$; referring to this figure, it can be seen that for values of h/L different than zero, the displacement of point 2 of the structure increases monotonically as the force increases from zero. When the force reaches its critical value at the limit point $A(P_{cr}/EA \approx 3.85 \times 10^{-7})$, the structure reaches a state of unstable equilibrium and jumps (buckles) to its stable equilibrium configuration at point E (see Fig. c). For values above the critical value, the displacement of point 2 of the structure increases monotonically as the force increases. The structure does not reach another state of unstable equilibrium until its members yield. If the force decreases after reaching its value at point F, the displacement decreases monotonically

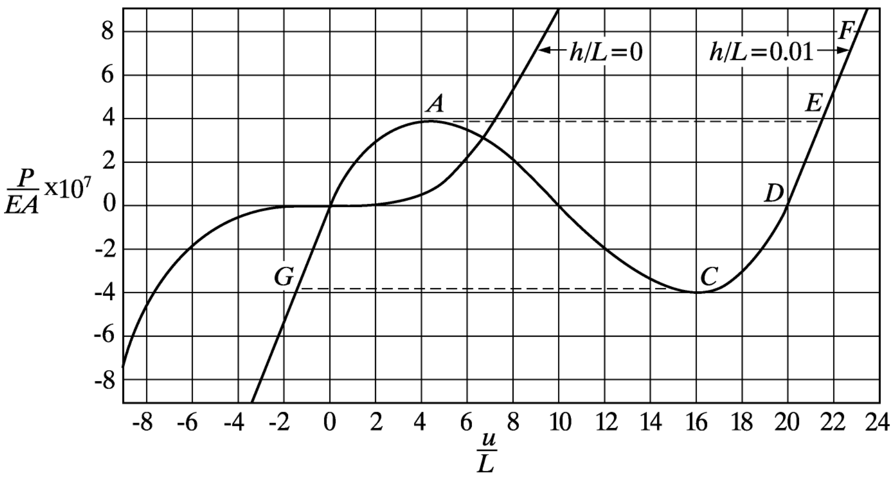


Figure c Force–displacement relations, on the basis of the theory of moderate rotations.

until the force vanishes at point *D*. If the structure is subsequently subjected to an upward force, the displacement at point 2 decreases monotonically until the force reaches its critical value at point *C* (see Fig. c) and the structure jumps (buckles) to its stable configuration at point *G*.

When h/L is equal to zero (see Fig. d), the structure is kinematically unstable. Its members cannot resist a transverse force before rotating. Thus, as soon as a transverse force is applied to this structure, its members rotate instantaneously without deforming. Moreover, as can be seen from Fig. c, for small values of the force, the rate of increase of the deformation is very large. However, the structure does not necessarily fail. In fact, as the force increases the rotation and the deformation of the members of the structure increases and, thus, their orientation is more favorable for resisting the transverse force. For instance, suppose that the structure of Fig. d is made from American steel A36 (yield stress σ_{11} = 248 MPa; modulus of elasticity E = 210 GPa). Thus, yielding will occur when the Lagrangian component of strain [$\epsilon_{11} = \sigma_{11}/E = 248/210(10^3)$] in the members of the structure is equal to 1.181×10^{-3}. The corresponding displacement of point 2 of the structure maybe obtained from relation (b) as

$$\left.\frac{u}{L}\right|_{\text{yielding}} = \sqrt{\epsilon_{11}^2 + 2\epsilon_{11}} = 4.8615 \times 10^{-2}$$

Substituting this result into relation (i), the force required to produce yielding of the members of the structure is equal to

$$\left.\frac{P}{AE}\right|_{\text{yielding}} = \frac{2u}{L}\left[1 - \frac{1}{\sqrt{1 + (u^2/L^2)}}\right] = 1.147 \times 10^{-4} > \frac{P_{cr}}{AE} = 3.85(10^{-7})$$

Thus, the phenomena illustrated in Fig. c occur for values of the force P/AE considerably below that which causes the members of the structure to yield.

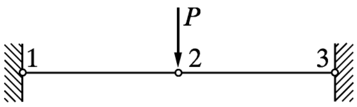


Figure d Kinematically unstable structure.

18.6 Buckling of Structures Having Infinite Degree of Freedom — The Direct Equilibrium Approach

When the theory of moderate rotations is applied to an infinite degree of freedom structure, the resulting differential equations are difficult to solve. For this reason, in this text we consider only prismatic line members (columns) subjected to conservative compressive axial component of tractions on their end surfaces.

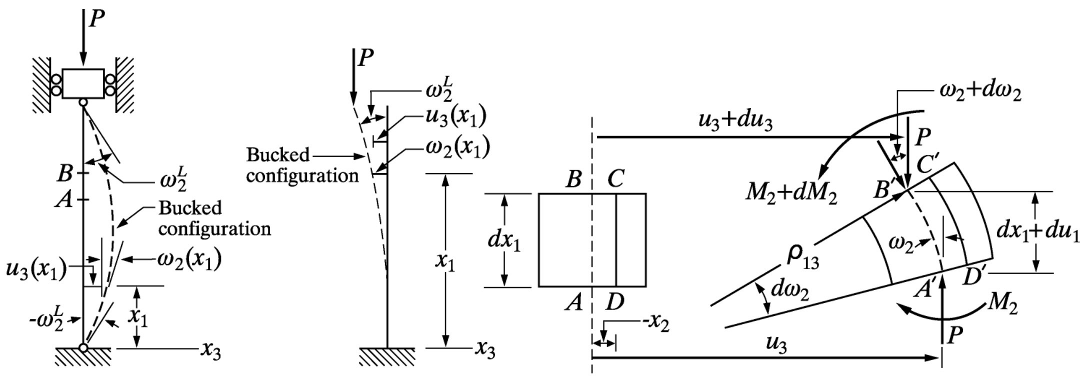
Consider a prismatic column in a stress-free, strain-free state of mechanical and thermal equilibrium at the uniform temperature T_o. The column reaches a second state of mechanical and thermal equilibrium at the uniform temperature T_o due to the application on each of its end surfaces $x_1 = 0$ nd $x_1 = L$ of compressive axial component of tractions whose resultant acts at the centroid of each of its end surfaces and we denote it by P. Referring to relation (18.4), we assume that

$$\sigma_{22} \approx P_{22} \approx 0 \qquad\qquad \sigma_{33} \approx P_{33} \approx 0 \tag{18.11}$$

Taking into account relations (18.11) and (18.5), the stress–strain relations of the form (3.48) reduce to

$$\sigma_{11} \approx E\epsilon_{11} \approx EE_{11} \tag{18.12}$$

In order to simplify our presentation we assume that the shear center of the cross sections of the column under consideration coincides with their centroid and that the x_2 and x_3 axes are principal centroidal such that $I_2 < I_3$. Thus, in its buckled configuration the column bents about its x_2 axis. Referring to Fig. 18.13c consider a fiber of the column of infinitesimal length dx_1 extending in its undeformed state from point D to point C. In the buckled state of the column the ends of this fiber move to points D' and C' (see Fig. 18.13d). From geometric considerations, denoting by dx_s and dx_s^o the deformed length of fibers CD and AB, respectively, we have



(a) Cantilever column (b) Simply supported column (c) Element of the underformed column (d) Element of the buckled column

Figure 18.13 Buckling of a column.

$$D'C' = dx_s = (\rho_{13} + x_3)d\omega_2 \tag{18.13}$$

$$A'B' = dx_s^o = \rho_{13}d\omega_2 \tag{18.14}$$

where $\rho_{13}(x_1)$is the radius of curvature of the elastic curve at point A and ω_2 is the angle the tangent to the elastic curve at point A' makes with the x_1 axis (see Fig. 18.13d). In obtaining the above relations we have explicitly assumed that plane sections normal to the axis of the column before deformation remain plane after deformation. Moreover, we have assumed that the change of the transverse component of displacement $\hat{u}_3(x_1, x_2, x_3)$ with x_2 and x_3 is negligible. That is, $\hat{u}_3(x_1, x_2, x_3) = u_3(x_1)$. The unit elongations E_{11} and E^o_{11} are
defined as

$$E_{11} = \frac{dx_s - dx_1}{dx_1} = (\rho_{13} + x_3)\frac{d\omega_2}{dx_1} - 1 \tag{18.15a}$$

$$E_{11}^o = \frac{dx_s^o - dx_1}{dx_1} = \rho_{13}\frac{d\omega_2}{dx_1} - 1 \tag{18.15b}$$

where E^o_{11} is the unit elongation of the particles of the axis of the column.

The moment of the distribution of traction acting on a cross section of the column which before deformation was normal to its axis, referring to relations (18.6) (18.12) and (18.15a), is equal to†

$$M_2 = \iint_A P_{11}x_3 dA = \iint_A EE_{11}x_3 dA = E\iint_A x_3(\rho_{13} + x_3)\frac{d\omega_2}{dx_1}dA = EI_2\frac{d\omega_2}{dx_1} \tag{18.17}$$

Referring to Fig. 18.13d, from the equilibrium of a deformed element of the column, we have

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† In the literature an attempt has been made to retain the effect of the axial unit shrinkage E_{11} of the column in a modified theory of moderate rotations. In this case, referring to relation (2.11a), we have

$$2\epsilon_{11} = (1 + E_{11})^2 - 1 = E_{11}(2 + E_{11})$$

Using the above and relations (18.4) and (18.12) and disregarding E^2_{11} as compared to unity, it seems that one must use the following expression for the component P_{11} of traction:

$$P_{11} = (1 + E_{11})\sigma_{11} = (1 + E_{11})E\epsilon_{11} = (1 + E_{11})E\frac{E_{11}}{2}(2 + E_{11}) = E(E_{11} + \frac{3}{2}E_{11}^2) \tag{18.16a}$$

Using the above relation and (18.15a), we get

$$\begin{aligned} M_2 &= \iint_A P_{11}x_3 dA = E\iint_A (E_{11} + \frac{3}{2}E_{11}^2)x_3 dA \\ &= EI_2\frac{d\omega_2}{dx_1} + \frac{3E}{2}\iint_A \left[[\rho_{13}^2 + 2x_3\rho_{13} + x_3^2]\left(\frac{d\omega_2}{dx_1}\right)^2 + 1 - 2(\rho_{13} + x_3)\frac{d\omega_2}{dx_1}\right]x_3 dA \\ &= EI_2\frac{d\omega_2}{dx_1} + \frac{3E}{2}\left[2\rho_{13}I_2\left(\frac{d\omega_2}{dx_1}\right)^2 - 2I_2\frac{d\omega_2}{dx_1}\right] = -2EI_2\frac{d\omega_2}{dx_1} + 3EI_2(E_{11}^o + 1)\frac{d\omega_2}{dx_1} = EI_2\frac{d\omega_2}{dx_1}(1 + 3E_{11}^o) \end{aligned} \tag{18.16b}$$

$$\sum M_2^{(B')} = dM_2 - Pdu_3 = 0 \qquad \text{or} \qquad \frac{dM_2}{dx_1} - \frac{Pdu_3}{dx_1} = 0 \tag{18.18}$$

Notice that relation (18.18) is also valid when the force has an eccentricity. That is, it is applied at $x_3 = -\epsilon$. Referring to Fig. 18.13a and b, from geometric considerations, we obtain

$$\sin\omega_2 = -\frac{du_3}{dx_s} = -\frac{du_3}{dx_1}\left(\frac{1}{1 + E_{11}}\right)$$

or

$$\frac{du_3}{dx_1} = -\sin\omega_2(1 + E_{11}) \approx -\sin\omega_2 \tag{18.19}$$

Substituting relations (18.17) and (18.19) into (18.18), we get

$$\frac{d}{dx_1}\left(EI_2\frac{d\omega_2}{dx_1}\right) + P\sin\omega_2 = 0 \tag{18.20}$$

When EI_2 is constant, relation (18.20) reduces to

$$EI_2\frac{d^2\omega_2}{dx_1^2} + P\sin\omega_2 = 0 \tag{18.21}$$

This is a non-linear equation whose solution, when subjected to the appropriate boundary conditions of a column gives the rotation $\omega_2(x_1)$ as a function of the axial compressive force P. By inspection we see that for initially straight columns with homogeneous boundary conditions, for any value of the force P, one solution of equation (18.21) is $\omega_2(x_1) = 0$. This solution represents the straight-line configuration of the column. However, this configuration of the column becomes unstable for values of the force P greater than certain critical value P_{cr} and a second stable configuration exists, which can be established by finding the second solution of equation (18.21). Therefore, the diagram of the force P versus the rotation ω_2^L of the end $x_1 = L$ of the column under consideration has a bifurcation point at $P = P_{cr}$.

Notice that equation (18.21) is also valid for initially straight columns made from an isotropic, linearly elastic material subjected to eccentric compressive axial forces at their ends. However, $\omega_2(x_1) = 0$ is not a solution of this boundary value problem because it does not satisfy its non-homogeneous boundary condition $M_2(L) = P\epsilon$, where ϵ is the eccentricity of the forces.

18.6.1 Solution of the Non-Linear Equation (18.21)

In this subsection we present a method for finding the second solution of equation (18.21) in the form of a transcendental relation between $\omega_2(x_1)$ and P. For this purpose notice that

$$\frac{d^2\omega_2}{dx_1^2} = \frac{d\omega_2}{dx_1}\frac{d\left(\dfrac{d\omega_2}{dx_1}\right)}{d\omega_2}$$

Substituting the above relation into (18.21), we get

$$\frac{d\omega_2}{dx_1} d\left(\frac{d\omega_2}{dx_1}\right) = -\frac{P}{EI_2} \sin \omega_2 d\omega_2$$

Integrating the above relation, we have

$$\frac{1}{2}\left(\frac{d\omega_2}{dx_1}\right)^2 = \frac{P}{EI_2} \cos \omega_2 + C_1 \tag{18.22}$$

We assume that the end $x_1 = L$ of the column is either unsupported or simply supported. On the basis of this assumption, referring to relation (18.17), the constant C_1 is evaluated from the following condition:

$$\text{at } x_1 = L \qquad \left.\frac{d\omega_2}{dx_1}\right|_{\omega_2 = \omega_2^L} = \frac{M_2}{EI_2} = 0 \tag{18.23}$$

From relations (18.22) and (18.23), we obtain

$$C_1 = -\frac{P}{EI_2} \cos \omega_2^L \tag{18.24}$$

Substituting relation (18.24) into (18.22), we get

$$\frac{d\omega_2}{dx_1} = \pm \sqrt{\frac{2P}{EI_2}\left(\cos \omega_2 - \cos \omega_2^L\right)} \tag{18.25}$$

Inasmuch as ω_2 is positive when it represents counterclockwise rotation, for the simply supported or the cantilever column under consideration, (see Fig. 18.13) $d\omega_2/dx_1$ is always positive. Consequently, the negative sign must be disregarded in relation (18.25).

Simply Supported Column

The rotation of a simply supported column at $x_1 = 0$ is equal to $-\omega_2^L$. Taking this into account, from relation (18.25), we have

$$x_1 = \int_{-\omega_2^L}^{\omega_2} \frac{d\omega_2}{\sqrt{\frac{2P}{EI_2}\left(\cos \omega_2 - \cos \omega_2^L\right)}} = \int_{-\omega_2^L}^{\omega_2} \frac{d\omega_2}{\sqrt{\frac{4P}{EI_2}\left[\sin^2\left(\frac{\omega_2^L}{2}\right) - \sin^2\left(\frac{\omega_2}{2}\right)\right]}} \tag{18.26}$$

$$0 \le x_1 \le \frac{L}{2}$$

In obtaining the above relation we took into account that $\cos \omega_2 = 1 - 2\sin(\omega_2/2)$.

In what follows we compute the deflection $-u_3(L/2)$ of the middle span of the column under consideration. For this purpose we introduce the following notation:

$$\sin\left(\frac{\omega_2^L}{2}\right) = k \qquad \sin\left(\frac{\omega_2}{2}\right) = -k \sin g = -\sin\left(\frac{\omega_2^L}{2}\right) \sin g \tag{18.27}$$

Thus,

$$\cos\left(\frac{\omega_2}{2}\right) = \sqrt{1 - \sin^2\left(\frac{\omega_2}{2}\right)} = \sqrt{1 - k^2\sin^2 g} \tag{18.28}$$

$$\sqrt{\sin^2\left(\frac{\omega_2^L}{2}\right) - \sin^2\left(\frac{\omega_2}{2}\right)} = \sqrt{k^2 - k^2\sin^2 g} = k\cos g \tag{18.29}$$

$$d\omega_2 = -\frac{2k\cos g\,dg}{\cos\left(\frac{\omega_2}{2}\right)} = -\frac{2k\cos g\,dg}{\sqrt{1 - k^2\sin^2 g}} \tag{18.30}$$

Noting that ω_2 varies from $-\omega_2^L$ to ω_2^L, from the second of relations (18.27) we see that the quantity sin*g* varies from 1 to -1 and thus, g varies from $\pi/2$ to $-\pi/2$. Substituting relations (18.29) and (18.21) into (18.26), we get

$$x_1 = -\sqrt{\frac{EI_2}{P}}\int_{g'=\pi/2}^{g'=g}\frac{dg'}{\sqrt{1 - k^2\sin^2 g'}} = \sqrt{\frac{EI_2}{P}}\int_{g'=g}^{g'=\pi/2}\frac{dg'}{\sqrt{1 - k^2\sin^2 g'}} \qquad 0 \le x_1 \le \frac{L}{2} \tag{18.31}$$

Using the condition that

$$\text{at } x_1 = \frac{L}{2} \qquad \omega_2 = 0 \qquad \text{and} \qquad g = 0 \tag{18.32}$$

from relation (18.31), we obtain

$$\frac{L}{2} = \sqrt{\frac{EI_2}{P}}\int_{g'=0}^{g'=\pi/2}\frac{dg'}{\sqrt{1 - k^2\sin^2 g'}} = \sqrt{\frac{EI_2}{P}}K(k) \tag{18.33}$$

The integral $K(k)$ is known as a complete elliptic integral of the first kind and its value depends only on the parameter $k = \sin(\omega_2^L/2)$. Numerical values of $K(k)$ for different values of k are given in some engineering handbooks†. Relation (18.33) is a transcendental relation between the force P and the angle of rotation ω_2^L of the end $x_1 = L$ of the column. For any chosen value of the angle ω_2^L the corresponding value of k can be computed from the first of relations (18.27) and used in the available tables to obtain the value of the integral $K(k)$ which when substituted into relation (18.33) gives the value of P required to produce the chosen value of ω_2^L.

At the beginning of buckling ($P = P_{cr}$) the angle ω_2^L is very small therefore k is also very small and the term $k^2\sin^2 g$ is negligible compared to unity. Thus, relation (18.33) reduces to

$$P_{cr} = \frac{\pi^2 EI_2}{L^2} \tag{18.34}$$

In what follows, we proceed to compute the deflection of the column at $x_1 = L/2$. For

† See for example, Byrd, P.F. and Friedman, M.D., *Handbook of Elliptic Integrals for Engineers and Scientists,* 2nd edition, Springer-Verlag, Berlin, 1971.

this purpose, referring to relation (18.19) and using relation (18.25), we have

$$du_3 = -\sin\omega_2 dx_1 = -\frac{\sin\omega_2 d\omega_2}{\dfrac{d\omega_2}{dx_1}} = -\frac{\sin\omega_2 d\omega_2}{\sqrt{\dfrac{2P}{EI_2}(\cos\omega_2 - \cos\omega_2^L)}}$$

$$= -\frac{\sin\omega_2 d\omega_2}{\sqrt{\dfrac{4P}{EI_2}\left[\sin^2\left(\dfrac{\omega_2^L}{2}\right) - \sin^2\left(\dfrac{\omega_2}{2}\right)\right]}} \tag{18.35}$$

Integrating relation (18.35) and taking into account that $\omega_2(L/2) = 0$, we obtain

$$u_3\left(\frac{L}{2}\right) = \int_0^{u_3(L/2)} du_3 = -\int_{-\omega_2^L}^{0} \frac{\sin\omega_2 d\omega_2}{\sqrt{\dfrac{4P}{EI_2}\left[\sin^2\left(\dfrac{\omega_2^L}{2}\right) - \sin^2\left(\dfrac{\omega_2}{2}\right)\right]}} \tag{18.36}$$

Using the first of relations (18.28) and (18.27), we get

$$\sin\omega_2 = 2\sin\left(\frac{\omega_2}{2}\right)\cos\left(\frac{\omega_2}{2}\right) = -2k\sin g\sqrt{1 - k^2\sin^2 g} \tag{18.37}$$

Substituting relation (18.37), (18.29) and (18.30) into (18.36) and taking into account that at $\omega_2 = \omega_2^L$, the parameter g is equal to $\pi/2$ while at $\omega_2 = 0$ the parameter g is equal to zero, we have

$$u_3\left(\frac{L}{2}\right) = -\sqrt{\frac{EI_2}{P}}\int_0^{\pi/2} 2k\sin g\, dg = 2k\sqrt{\frac{EI_2}{P}} \tag{18.38}$$

For any chosen value of ω_2^L, the parameter k can be computed from the first of relations (18.27) and substituted into relation (18.38) to compute the maximum deflection at $x_1 = L/2$ of the column under consideration as function of the force P.

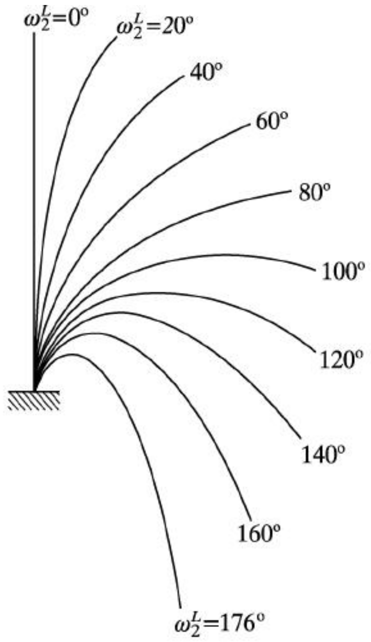
Cantilever Column

The rotation ω_2 of a cantilever column at $x_1 = 0$ is equal to zero. Taking this into account from relation (18.25), we get

$$x_1 = \int_0^{\omega_2} \frac{d\omega_2}{\sqrt{\dfrac{4P}{EI_2}\left[\sin^2\left(\dfrac{\omega_2^L}{2}\right) - \sin^2\left(\dfrac{\omega_2}{2}\right)\right]}} \qquad 0 \le x_1 \le L \tag{18.39}$$

We introduce the following notation

$$\sin\left(\frac{\omega_2^L}{2}\right) = k \qquad \sin\left(\frac{\omega_2}{2}\right) = k\sin\phi = \sin\left(\frac{\omega_2^L}{2}\right)k\sin\phi \tag{18.40}$$



Figure† **18.14** Shape of the elastic curve of a cantilever column for various values of ω_2^L.

Noting that ω_2 varies from 0 to ω_2^L, from the second of relations (18.40), we see that the quantity $\sin\phi$ varies from 0 to 1 and thus, ϕ varies from 0 to $\pi/2$. Using relations (18.40) and following a procedure similar to the one employed for the simply supported beam, from relation (18.39), we obtain

$$L = \sqrt{\frac{EI_2}{P}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \sqrt{\frac{EI_2}{P}}\ K(k) \tag{18.41}$$

Relation (18.41) is a transcendental relation between the force P and the angle of rotation ω_2^L of the unsupported end of the cantilever column. Values of P/P_{cr} obtained from relation (18.41) for various values of ω_2^L are tabulated in Table 18.1; referring to this table we see that ω_2^L increases as P increases. Consequently, the postbuckling configurations of the column under consideration are stable. The shapes of the elastic curve of a cantilever column are shown in Fig. 18.14 for various values of ω_2^L.

Table 18.1† Shape of the elastic curve of a cantilever column for various values of ω_2^L.

ω_2^L	0°	20°	40°	60°	80°	100°	120°	140°	160°	176°
P/P_{cr}	1	1.015	1.063	1.152	1.293	1.518	1.884	2.541	4.029	9.116
x_1^L / L	1	0.970	0.881	0.741	0.560	0.349	0.123	-0.107	0.340	-0.577
u_3^L / L	0	0.220	0.422	0.593	0.719	0.792	0.803	0.750	0.625	0.421

† Taken from Timoshenko, S.P. and Gere, J.M., *Theory of Elastic Stability,* McGraw-Hill, New York, 1961, p.79.

At the beginning of buckling ($P = P_{cr}$) the angle ω_2^L is very small; therefore, k is also very small and the term $k^2 \sin^2 \phi$ in relation (18.41) is negligible compared to unity. Thus, relation (18.41) reduces to

$$P_{cr} = \frac{\pi^2 EI_2}{4L^2} \tag{18.42}$$

Comments

Referring to relations (18.34) and (18.42) we see that the critical force at buckling of the columns of Fig. 18.13a and b, is proportional to the flexural rigidity EI_2 and inversely proportional to the square of the length of the column. It is not affected by the yield stress of the material from which the column is made.

18.7 Buckling of Structures Having Infinite Degree of Freedom — The Stationary Total Potential Energy Approach

The equation of equilibrium (18.20) can be established by applying the theorem of stationary total potential energy which is proved in Section 13.17 for the linear theory of elasticity and it is extended to the non-linear theory of elasticity and the theory of moderate rotations in Section 18.2. For the columns of Fig. 18.13a and b, referring to relation (18.9) and taking into account relations (18.5) and (18.12), we have

$$U_s = \frac{1}{2}\sigma_{11}\epsilon_{11} = \frac{1}{2}EE_{11}^2 \tag{18.43}$$

Substituting relation (18.43) into (18.7), we get

$$\Pi_s = \iiint_V \frac{1}{2}EE_{11}^2 dV + Pu_1^L = \int_o^L \left[\frac{E}{2}\iint_A E_{11}^2 dA + P\frac{du_1}{dx_1}\right]dx_1 \tag{18.44}$$

Referring to Fig. 18.13 from geometric consideration using relation (18.14), we obtain

$$du_1 = dx_s^o \cos\omega_2 - dx_1 = \left(\rho_{13}\frac{d\omega_2}{dx_1}\cos\omega_2 - 1\right)dx_1 \tag{18.45}$$

when referring to Fig. 18.13d, dx_s^o is the length of the curves A′B′. From relation (18.45), we obtain

$$\frac{du_1}{dx_1} = \rho_{13}\frac{d\omega_2}{dx_1}\cos\omega_2 - 1 \tag{18.46}$$

Substituting relations (18.15a) and (18.46) into (18.44), integrating over the area of the cross section of the column and using relation (18.15b), we get

$$\Pi_s = \int_o^L \left[\frac{E}{2}\iint_A \left[(\rho_{13} + x_3)\frac{d\omega_2}{dx_1} - 1\right]^2 dA + P\left[\rho_{13}\frac{d\omega_2}{dx_1}\cos\omega_2 - 1\right]\right]dx_1$$

$$= \int_o^L \left[\frac{E}{2}\iint_A \left[(\rho_{13}^2 + x_3^2 + 2\rho_{13}x_3)\left(\frac{d\omega_2}{dx_1}\right)^2 + 1 - 2(\rho_{13} + x_3)\frac{d\omega_2}{dx_1}\right]dA\right.$$

$$+ P[(1 + E_{11}^o)\cos\omega_2 - 1]\Big]dx_1$$

$$= \int_o^L \left[\frac{EA\rho_{13}^2}{2}\left(\frac{d\omega_2}{dx_1}\right)^2 + \frac{EI_2}{2}\left(\frac{d\omega_2}{dx_1}\right)^2 + \frac{EA}{2} - \rho_{13}A\left(\frac{d\omega_2}{dx_1}\right) \right.$$

$$\left. + P[(1 + E_{11}^o)\cos\omega_2 - 1]\right]dx_1$$

$$= \int_o^L F\,dx_1 \tag{18.47}$$

where

$$F = \frac{EA(1 + E_{11}^o)^2}{2} + \frac{EI_2}{2}\left(\frac{d\omega_2}{dx_1}\right)^2 + \frac{EA}{2} - A(1 + E_{11}^o) + P[(1 + E_{11}^o)\cos\omega_2 - 1] \tag{18.48}$$

The theorem of stationary total potential energy of the non-linear theory of elasticity states that a function $\omega_2(x_1, P)$ which renders the total potential energy of the column of Fig. 18.13a or b stationary specifies a configuration of equilibrium of that column. This implies that a function $\omega_2(x_1, P)$ which renders the first variation of the total potential energy of the column equal to zero, specifies a configuration of equilibrium of that column. Referring to relation (18.48), we get

$$\begin{aligned} \frac{\partial F}{\partial \omega_2} &= -P(1 + E_{11}^o)\sin\omega_2 \approx -P\sin\omega_2 \\ \frac{\partial F}{\partial \omega_2^{/}} &= EI_2\frac{d\omega_2}{dx_1} \end{aligned} \tag{18.49}$$

where $\omega_2^{/}$ stands for $d\omega_2/dx_1$. Referring to relation (E.20) of Appendix E and using relations (18.49), the Euler–Lagrange differential equation for the functional (18.47) is

$$\frac{\partial F}{\partial \omega_2} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial \omega_2^{/}}\right) = -P\sin\omega_2 - \frac{\partial}{\partial x_1}\left(EI_2\frac{d\omega_2}{dx_1}\right) = 0 \tag{18.50}$$

As expected relation (18.50) is identical to (18.20). It is the necessary relation that the function $\omega_2(x_1, P)$ must satisfy in order that the functional Π_s given by relation (18.47), assumes stationary values. Referring to relations (E.22) of Appendix E we see that in order that the functional Π_s given by relation (18.47), assumes stationary values the following boundary conditions must be satisfied at the ends $x_1 = 0$ and $x_1 = L$ of the column:

$$\delta\omega_2 = 0 \qquad \text{or} \qquad \frac{\partial F}{\partial \omega_2^{/}} = EI_2\frac{d\omega_2}{dx_1} = M_2 = 0 \tag{18.51}$$

These boundary conditions are satisfied by the columns of Fig. 18.13a and b.

In order to establish the postbuckling path is stable or unstable we compute the second variation of Π_s. For this purpose referring to relation (18.50) we find

$$\frac{\partial^2 F}{\partial^2 \omega_2} = -P[(1 + E_{11}^o)\cos\omega_2] \qquad \frac{\partial^2 F}{\partial\omega_2 \partial\omega_2^{\prime}} = 0 \qquad \frac{\partial^2 F}{\partial^2 \omega_2^{\prime}} = EI_2 \tag{18.52}$$

Referring to relation (E.16), using relation (18.52) integrating by parts and using relations (18.51) and (18.46), we get

$$\begin{aligned}\delta^{(2)}\Pi_s &= \int_o^L \left[-P\,|\,(1 + E_{11}^o)\cos\omega_2\,|\,(\omega_2)^2 + EI_2\left[\delta(\omega_2^{\prime})\right]^2\right]dx_1 \\ &= -\int_o^L \left[EI_2 \frac{\partial^2\delta\omega_2}{\partial x_1^2}\delta\omega_2 + P(1 + E_{11}^o)\cos\omega_2(\delta\omega_2)^2\right]dx_1 + EI_2 \frac{\partial^2\delta\omega_2}{\partial x_1^2}\Bigg]_o^L \\ &= -\int_o^L \delta\left[EI_2\frac{\partial^2\omega_2}{\partial x_1^2} + P(1 + E_{11})\sin\omega_2\right]\delta\omega_2\,dx_1 = 0\end{aligned} \tag{18.53}$$

Since $\delta^2\Pi_s$ vanishes, we must check the third variation of Π_s. For this purpose from relations (18.52) we find

$$\frac{\partial^3 F}{\partial\omega_2^3} = P(1 + E_{11}^o)\sin\omega_2 \qquad \frac{\partial^3 F}{\partial^3\omega_2^{\prime}} = 0 \tag{18.54}$$

Substituting relations (18.54) into (E.9) of Appendix E, we get

$$\delta^{(3)}\Pi_s = \int_o^L P(1 + E_{11}^o)\sin\omega_2(\delta\omega_2^3)dx_1 > 0 \tag{18.55}$$

Thus, the postbuckling path of the columns of Fig. 18.13a and b is a stable equilibrium path. The integrand of the integral in relation (18.55) is always positive for the columns under consideration, because if ω_2 is negative $\delta\omega_2$ must also be negative while if ω_2 is positive $\delta\omega_2$ must also be positive. That is, the absolute value of ω_2 can only increase. Relation (18.55) indicates that the structure in its postbuckling configurations, as expected is in a state of stable equilibrium.

18.8 Determination of the Critical Load at Buckling of Infinite Degree of Freedom Structures by Investigating the Beginning of Buckling

At the beginning of buckling the external force acting on a column is equal to its critical value. Moreover, not only the unit elongations or shrinkages and the unit shears are negligible compared to unity, but also the rotations are negligible compared to unity. Thus,

$$\sin\omega_2 \approx \omega_2 \tag{18.56}$$

Taking relation (18.56) into account, relations (18.19) and (18.21) reduce to

$$\omega_2 = -\frac{du_3}{dx_1} \tag{18.57}$$

$$EI_2 \frac{d^2\omega_2}{dx_1^2} + P_{cr}\omega_2 = 0 \tag{18.58}$$

The solution of equation (18.58) has the following form:

$$\omega_2(x_1) = A\cos\left(\sqrt{\frac{P_{cr}}{EI_2}}x_1\right) + B\sin\left(\sqrt{\frac{P_{cr}}{EI_2}}x_1\right) \tag{18.59}$$

The constants A and B are evaluated from the boundary conditions of the column.

Cantilever column

The boundary conditions of a perfectly straight cantilever column subjected to a perfectly centroidal compressive axial force at its unsupported end are

$$\text{at } x_1 = 0 \qquad \omega_2 = 0 \qquad \text{hence} \qquad A = 0 \tag{18.60}$$

$$\text{at } x_1 = L \qquad \frac{d\omega_2}{dx_1} = \frac{M_2}{EI_2} = 0 \qquad \text{hence} \qquad B\cos\left(L\sqrt{\frac{P_{cr}}{EI_2}}\right) = 0 \tag{18.61}$$

From relation (18.61), we get

$$P_{cr} = \frac{n^2\pi^2 EI_2}{4L^2} \qquad n = 1,3,5,\ldots \tag{18.62}$$

Taking into account relations (18.60) and (18.62) relation (18.59) reduces to

$$\omega_2(x_1) = B\sin\left(\frac{n\pi}{2L}x_1\right) \qquad n = 1,3,5,\ldots \tag{18.63}$$

For n = 1, 3, 5, ... we obtain an infinite number of critical loads and corresponding buckled shapes. However, the column buckles when the load reaches its lowest (n = 1) critical value and thus, only the critical load and buckled shape for n = 1 is of practical interest. The constant B in relation (18.63) represents the rotation ω_2^L at the unsupported end of the column which is unspecified.

Substituting relation (18.63) with n = 1 into (18.57), we get

$$\frac{du_3}{dx_1} = -B\sin\left(\frac{\pi x_1}{L}\right) \tag{18.64}$$

Integrating the above relation, we obtain

$$u_3(x_1) = -B\int_o^{x_1}\sin\left(\frac{\pi x_1}{L}\right)dx_1 = C\left[\cos\left(\frac{\pi x_1}{L}\right) - 1\right] \tag{18.65}$$

Simply supported column

The boundary conditions for a perfectly straight simply supported column subjected to perfectly centroidal axial compressive forces are

$$\text{at } x_1 = 0 \qquad \frac{d\omega_2}{dx_1} = \frac{M_2}{EI_2} = 0 \qquad \text{hence} \qquad B = 0 \tag{18.66}$$

$$\text{at } x_1 = \frac{L}{2} \qquad \omega_2\left(\frac{L}{2}\right) = 0 \qquad \text{hence} \qquad A\cos\left(\frac{L}{2}\sqrt{\frac{P_{cr}}{EI_2}}\right) = 0 \tag{18.67}$$

Consequently,

$$\cos\left(\frac{L}{2}\sqrt{\frac{P_{cr}}{EI_2}}\right) = 0 \tag{18.68}$$

or

$$P_{cr} = \frac{n^2\pi^2 EI_2}{L^2} \qquad n = 1, 3, 5, \ldots \tag{18.69}$$

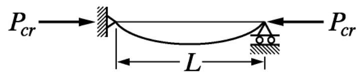
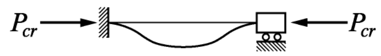
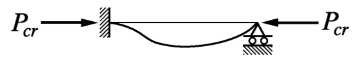
Taking into account relations (18.66) and (18.69) relation (18.59) reduces to

$$\omega_2(x_1) = A\cos\left(\frac{n\pi}{L}x_1\right) \qquad n = 1, 3, 5, \ldots \tag{18.70}$$

Substituting relation (18.70) with $n = 1$ into (18.57) integrating the resulting relation and taking into account that $u_3(0) = 0$, we obtain

$$u_3(x_1) = -C\sin\left(\frac{\pi x_1}{L}\right) \tag{18.71}$$

Table 18.2 Critical load at buckling and buckled shapes of columns with various boundary conditions.

Boundary Conditions	Critical Load	Buckled Shape
Fixed supported at both ends	$\dfrac{\pi^2 EI}{L^2}$	
Fixed at both ends	$\dfrac{4\pi^2 EI}{L^2}$	
Fixed at one end and simply supported at the other	$\dfrac{2.04\pi^2 EI}{L^2}$	
Fixed at one end and unsupported at the other	$\dfrac{\pi^2 EI}{4L^2}$	

Comments

From the presentation in this section it is apparent that the problem of studying the beginning of buckling of a perfectly prismatic, perfectly straight column subjected to perfectly centroidal compressive axial forces involves the solution of the differential equation (18.58) with appropriate boundary conditions. Solutions of this equation exist only for certain values of $P = P_{cr}$. These values of P are called *eigenvalues* or *characteristic values*. The corresponding functions $\omega_2(x_1)$ are called *eigenfunctions* or *normal buckling modes*. They specify only the shape of the elastic curve of the column not the magnitude of its deformation. Such problems are known as eigenvalue problems.

In Table 18.2 we give the critical load at buckling and the buckled shapes of columns with different boundary conditions.

18.9 Columns Subjected to Eccentric Axial Compressive Forces at Their Ends

Consider the simply supported column shown in Fig. 18.15 made from an isotropic, linearly elastic material subjected to compressive axial forces at its ends with an eccentricity $x_3 = \epsilon$ and $x_2 = 0$. The x_2 and x_3 axes are principal centroidal. For any value of the applied forces the column has an unique stable equilibrium configuration. Moreover, for small values of the applied forces the deformation of the imperfect column is within the range of validity of the theory of small deformations. Therefore, we can disregard the effect of the change of the geometry of the column due to its deformation on its internal actions. Thus, the moment at any cross section of the column can be approximated by

$$M_2 = P\epsilon \tag{18.72}$$

Furthermore, the moment acting on a cross section of the column and the component of translation u_3 of this cross section are related by relation (9.32a). That is,

$$M_2 = -EI_2 \frac{d^2 u_3}{dx_1^2} \tag{18.73}$$

I_2 is the moment of inertia of the cross section of the column with respect to its x_2 axis. Substituting relation (18.73) into (18.72), we get

$$EI_2 \frac{d^2 u_3}{dx_1^2} + P\epsilon = 0 \tag{18.74}$$

The solution of this equation is

$$u_3 = -\frac{P\epsilon x_1^2}{2EI_2} + Ax_1 + B \tag{18.75}$$

where the constants A and B are evaluated from the boundary conditions of the simply supported column. That is,

$$\begin{aligned} u_3(0) &= 0 \qquad \text{thus } B = 0 \\ u_3(L) &= 0 \qquad \text{thus } A = \frac{P\epsilon L}{2EI_2} \end{aligned} \tag{18.76}$$

Consequently, the solution of the differential equation (18.74) becomes

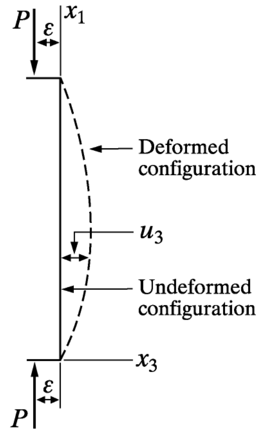


Figure 18.15 Simply supported column subjected to eccentric axial compressive forces at its ends.

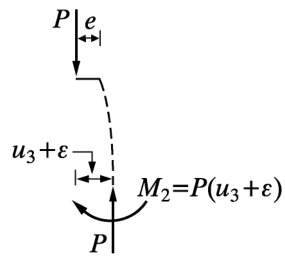


Figure 18.16 Free-body diagram of a portion of the column.

$$u_3 = \frac{P\epsilon x_1}{2EI_2}(L - x_1) \tag{18.77}$$

When the eccentricity ϵ is different than zero, the component of translation u_3 of the column increases as the force increases. However, for values of the external force which induce a component of translation u_3 at the middle point of the column whose magnitude approaches the value of ϵ, solution (18.77) does not represent a good approximation of u_3 because the deformation of the column is not any longer in the range of validity of the theory of small deformation. In this case a non-linear theory such as the theory of moderate rotations must be used in order to find the deformed configuration of the column. That is, the solution of the non-linear differential equation (18.21) must be established which satisfies the non-homogeneous boundary condition $M_2(L) = P\epsilon$. In the literature this problem is simplified by retaining only the effect of change of the geometry of the column, due to its deformation, on the magnitude of its internal moment. That is, referring to Fig. 18.16, the moment at any point of the column is taken equal to

$$M_2 = P(\epsilon + u_3) \tag{18.78}$$

Notice that in relation (18.78), the component of translation u_3 is a function of the force P; consequently, the moment is not a linear function of the force P. Substituting relation (18.73) into (18.78) and differentiating twice the resulting relation, we get the following linear differential equation:

$$\frac{d^4u_3}{dx_1^4} + \frac{P}{EI_2}\frac{d^2u_3}{dx_1^2} = 0 \tag{18.79}$$

Equation (18.79) can be obtained by substituting relation (18.57) into (18.58) and differentiating the resulting relation. This equation is valid only for values of P and ϵ which cause small values of the component of translation u_3, that is, for values of P somewhat smaller that the critical force at buckling of the perfect column. The solution of equation (18.79) is

$$u_3 = A \sin\left(\sqrt{\frac{P}{EI_2}}\, x_1\right) + B \cos\left(\sqrt{\frac{P}{EI_2}}\, x_1\right) + C x_1 + D \tag{18.80}$$

where A, B, C and D are constants of integration which are evaluated from the following conditions at the ends of the column:

$$\text{(a)}\quad u_3(0) = 0 \qquad\qquad \text{(b)}\quad u_3(L) = 0$$

$$\text{(c)}\quad \left.\frac{d^3u_3}{dx_1^2}\right|_{x_1=0} = -\frac{P\epsilon}{EI_2} \qquad\qquad \text{(d)}\quad \left.\frac{d^3u_3}{dx_1^2}\right|_{x_1=L} = -\frac{P\epsilon}{EI_2} \tag{18.81}$$

Substituting relation (18.80) into (18.81), we get

$$B = -D = \epsilon \qquad C = 0 \qquad A = \frac{\left(1 - \cos\left(L\sqrt{\dfrac{P}{EI_2}}\right)\right)\epsilon}{\sin\left(L\sqrt{\dfrac{P}{EI_2}}\right)} = \epsilon \tan\left(\frac{L}{2}\sqrt{\frac{P}{EI_2}}\right) \tag{18.82}$$

Substituting the value of the constants (18.82) into solution (18.80), we have

$$u_1(x_1) = \epsilon\left[\left(\tan\frac{L}{2}\sqrt{\frac{P}{EI_2}}\right)\sin\left(\sqrt{\frac{P}{EI_2}}\, x_1\right) + \cos\left(\sqrt{\frac{P}{EI_2}}\, x_1\right) - 1\right] \tag{18.83}$$

In the above result, the relation between the component of translation u_3 and the force P is not linear. Notice that when $\epsilon \neq 0$, the first term in the bracket of relation (18.83) becomes infinite for certain values of the axial force P satisfying the following relation:

$$\tan\left(\frac{L}{2}\sqrt{\frac{P}{EI_2}}\right) = \infty \tag{18.84}$$

That is, for the values of P satisfying relation (18.84) the transverse component of displacement $u_3(x_1)$ becomes infinite and thus, relation (18.83) is not valid. The values of P satisfying relation (18.84) are identical to the critical force at buckling of the perfect column. That is,

$$P_{cr} = \frac{n^2\pi^2 EI_2}{L^2} \qquad\qquad n = 1, 2, 3, \ldots \tag{18.85}$$

It should be emphasized that although relation (18.83) represents a better approximation of the component of translation $u_3(x_1)$ than relation (18.77), both relations do not represent a satisfactory approximation of $u_3(x_1)$ for values of P close to P_{cr} of the perfect column.

The maximum deflection occurs at the midpoint of the column, and referring to relation (18.83), it is equal to

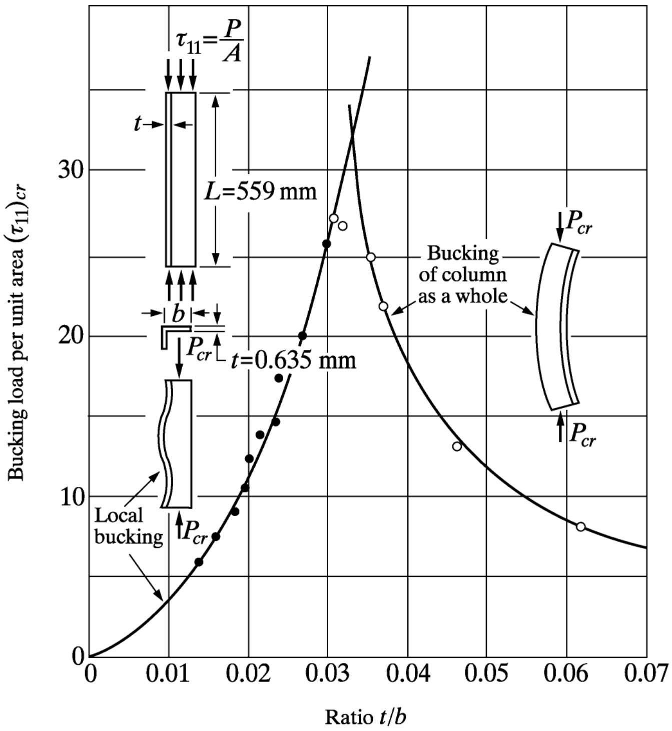


Figure 18.17 Load for local and total buckling of columns made from 245 TR aluminum (E = 74.5 Gpa)[†].

$$(u_3)_{\max} = u_3\left(\frac{L}{2}\right) = \epsilon\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI_2}}\right) - 1\right] \tag{18.86}$$

The maximum bending moment occurs at the midpoint of the column where its deflection is maximum and it is equal to

$$(M_2)_{\max} = P(\epsilon + (u_3)_{\max}) = P\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI_2}}\right) \tag{18.87}$$

The maximum compressive stress occurs on the concave side of the column at $x_1 = L/2$. Thus, using relation (18.87), we have

$$(\tau_{11})_{\max} = -\frac{P}{A} - \frac{(M_2)_{\max}(c_3)}{EI_2} = -\frac{P}{A}\left[1 + \frac{\epsilon c_3}{r_2^2}\sec\left(\frac{L}{2r_2}\sqrt{\frac{P}{EA}}\right)\right] \tag{18.88}$$

where c_3 is the distance from the x_2 axis to the extreme fiber on the concave side of the column, while $r_2 = \sqrt{I_2/A}$ is the radius of gyration of the cross sections of the column with respect to the x_2 axis. Equation (18.88) is known as the *secant formula* for an

† Taken from Bridget, F.J., Jerome C.C. and Vosseller, A.B., Some new experiments in buckling of thin-walled construction, *Transactions of the American Society of Mechanical Engineers*, No. 56, 1934, p. 569-578.

eccentrically loaded column. It can be used to obtain the maximum stress in simply supported columns subjected to eccentric compressive axial forces at their ends provided that the value of P is not very close to P_{cr} of the corresponding perfect column.

18.10 Local Buckling of Columns

Columns whose cross sections are formed by two or more thin-wall parts (i.e., wide-flanges, angles, I-beams, channels) may exhibit local buckling of their flange or of their web at values of the compressive forces below the critical value at buckling of the column as a whole (global buckling). In Fig. 18.17, we show the results of experiments performed on aluminum columns whose cross section is an angle with equal legs of length b and thickness t = 0.635 mm. For values of the ratio t/b greater than 0.033 the columns buckle as a whole while for small values of the ratio t/b the one leg of the columns buckles at values of the compressive forces below the critical value at buckling of the column as a whole.

Local buckling of a flange or of the web of a column may not cause the immediate collapse of the column. However, it will cause the column to buckle as a whole at values of the compressive forces less than the computed critical forces at buckling. In practice local buckling of columns is avoided by choosing columns whose cross sections are sufficiently thick.

18.11 Problems

1. to **2.** Establish the postbuckling behavior and the critical load at buckling of the perfectly straight rigid bar shown in Fig. 18P1 subjected to a perfectly centroidal axial force P. Establish the equation of equilibrium using both the direct equilibrium method and the method of stationary total potential energy. Repeat with the bar of Fig. 18P2.

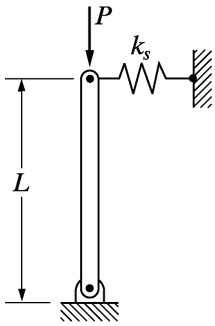


Figure 18P1

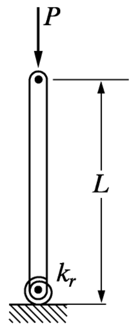


Figure 18P2

APPENDICES

Appendix A

Mechanical Properties of Materials

Material	Specific Weight (kN/m^3)	Modulus of† Elasticity E (GPa)	Poisson's Ratio ν (GPa)	Yield Stress† τ^Y (MPa)	Ultimate Stress† τ^u (MPa)	Coefficient of Thermal Expansion (10^{-6}/°C)
Aluminum (pure)	26.6	70	0.33	20	70	23
Aluminum alloys						
2014-T6	28	73	0.33	410	480	23
6061-T6	26	70	0.33	270	310	23
7075-T6	28	72	0.33	480	550	23
Brass	82–85	96–100	0.34	70–550	200–620	20.9
Brick (comp.)	17–22	10–24	—	550	7–70	5–7
Bronze	80–86	96–120	0.34	82–690	200–830	18–21
Cast iron (ten.)	68–72	83–170	0.2–0.3	120–290	69–480	9.9–12.0
Cast iron (comp.)	68–72	83–170	0.2–0.3	—	50–200	9.9–12.0
Concrete (comp.)						
Low strength	23	18	0.1–0.2	—	14	11
Medium strength	23	25	0.1–0.2	—	28	11
High strength	23	30	0.1–0.2	—	41	11
Copper (pure)	87	110–120	0.33–0.36	330	380	16.6–17.6
Glass	24–28	48–83	0.20–0.27	—	30–1000	5–11
Nickel	87	210	0.31	140–620	310–760	13
Rubber	9–13	0.0007–0.004	0.45–0.50	1–7	7–20	130–200

† The yield stress, the ultimate stress and the modulus of elasticity are for both compression and tension unless otherwise stated.

Material	Specific Weight (kN/m^3)	Modulus of† Elasticity E (GPa)	Poisson's ratio ν (GPa)	Yield Stress† τ^Y (MPa)	Ultimate Stress† τ^u (MPa)	Coefficient of Thermal Expansion (10^{-6}/°C)
Steel						
High strength	77	190–210	0.27–0.30	340–1000	550–800	14
Machine	77	190–210	0.27–0.30	340–700	700–1900	17
Spring	77	190–210	0.27–0.30	400–1600	400–1000	17
Stainless	77	190–210	0.27–0.30	240–700	900	17
Tool	77	190–210	0.27–0.30	520	340–830	12
Steel structural	77	190–210	0.27–0.30	200–700	400	12
ASTM A36	77	190–210	0.27–0.30	250	500	12
ASTM A572	77	190–210	0.27–0.30	340	830	12
ASTM A514	77	190–210	0.27–0.30	700	—	—
Titanium (pure)	44	110	0.33	400	500	8–10
Titanium (alloy)	44	110–120	0.33	760–900	900–970	8–10
Timber (air dry) (bending)						
Ash	5.3–6.3	10–11	—	40–70	50–100	—
Douglas fir	4.7–5.5	11–13	—	30–50	50–80	—
Oak	6.3–7.1	11–12	—	40–60	50–100	—
Southern pine	5.5–6.3	11–14	—	40–60	50–100	—
Wrought iron	72–77	190	0.3	210	340	12
Stone (comp.)						
Granite	26	40–70	0.2–0.3	—	70–280	5–9
Limestone	20–28	20–70	0.2–0.3	—	20–200	5–9
Marble	26–28	50–100	0.2–0.3	—	50–180	5–9

† The yield stress, the ultimate stress and the modulus of elasticity are for both compression and tension unless otherwise stated.

Appendix B

Stress–Strain Relations for Orthotropic and Isotropic Materials

B.1 Stress–Strain Relations for Orthotropic, Linearly Elastic Materials

Certain materials, such as fiber reinforced composites, have a microstructure which is symmetric with respect to three mutually perpendicular planes. Consequentlly, their elastic constants are also symmetric with respect to these planes. These materials are called *orthotropic*. In this Appendix we prove that the stress–strain relations for orthotropic, linearly elastic materials involve only nine constants when referred to axes which are perpendicular to the planes of symmetry of their microstructure. Moreover, in this Appendix we show that the stress–strain relations for isotropic, linearly elastic materials involve only two constants.

We consider a body made from an orthotropic, linearly elastic material and we choose the x_1, x_2, x_3 axes in a way that the x_1x_2, x_1x_3, x_2x_3 planes are parallel to the planes of symmetry of the microstructure of the material. Thus, the coefficients in the stress–strain relations (3.39) or (3.40) must be symmetric with respect to these planes. Symmetry of these coefficients with respect to the x_2x_3 plane implies that they remain the same when the system of axes to which the stress–strain relations (3.39) or (3.40) are referred,

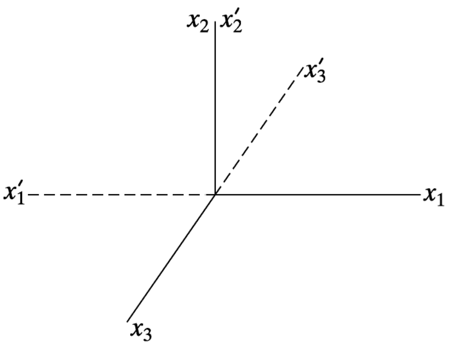


Figure B.1 System of axes x_1', x_2', x_3' obtained from the system of axes x_1, x_2, x_3 by rotating it by 180° about the x_2 axis.

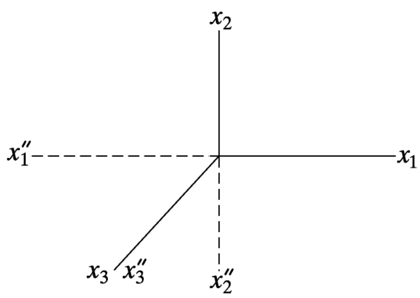


Figure B.2 System of axes x_1'', x_2'', x_3'' obtained from the system of axes x_1, x_2, x_3 by rotating it by 180° about the x_3 axis.

changes from x_1, x_2, x_3 to x_1', x_2, x_3 (see Fig. B.1). Moreover, symmetry of the coefficients of the stress–strain relations (3.39) or (3.40) with respect to the x_1x_2 plane implies that they remain the same when the system of axes to which these relations are referred changes from x_1, x_2, x_3 to x_1, x_2, x_3' (see Fig. B.1). Thus, if the coefficients of relations (3.39) and (3.40) are symmetric with respect to the x_1x_2 and x_2x_3 planes, they remain the same when the system of axes to which these relations are referred changes from x_1, x_2, x_3 to x_1', x_2', x_3'. The latter is obtained by rotating the system of axes x_1, x_2, x_3 by 180° about the x_2 axis, that is, referring to Fig. B.1, the stress–strain relations (3.39), can be written as follows with respect to the system of axes x_1', x_2', x_3':

$$\begin{aligned}
\tau'_{11} &= C_{1111}e'_{11} + C_{1122}e'_{22} + C_{1133}e'_{33} + C_{1112}e'_{12} + C_{1113}e'_{13} + C_{1123}e'_{23} \\
\tau'_{22} &= C_{2211}e'_{11} + C_{2222}e'_{22} + C_{2233}e'_{33} + C_{2212}e'_{12} + C_{2213}e'_{13} + C_{2223}e'_{23} \\
\tau'_{33} &= C_{3311}e'_{11} + C_{3322}e'_{22} + C_{3333}e'_{33} + C_{3312}e'_{12} + C_{3313}e'_{13} + C_{3323}e'_{23} \\
\tau'_{12} &= C_{1211}e'_{11} + C_{1222}e'_{22} + C_{1233}e'_{33} + C_{1212}e'_{12} + C_{1213}e'_{13} + C_{1223}e'_{23} \\
\tau'_{13} &= C_{1311}e'_{11} + C_{1322}e'_{22} + C_{1333}e'_{33} + C_{1312}e'_{12} + C_{1313}e'_{13} + C_{1323}e'_{23} \\
\tau'_{23} &= C_{2311}e'_{11} + C_{2322}e'_{22} + C_{2333}e'_{33} + C_{2312}e'_{12} + C_{2313}e'_{13} + C_{2323}e'_{23}
\end{aligned} \tag{B.1}$$

Referring to Fig. B.1 the transformation matrix of the system of axes x_1', x_2', x_3' with respect to the system of axes x_1, x_2, x_3 is

$$[\,\Lambda_s\,] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{B.2}$$

Substituting relation (B.2) into a transformation relations (2.46a), the components of stress referred to the system of axes x_1', x_2', x_3' are related to the components of stress referred to the system of axes x_1, x_2, x_3 by the following relation:

$$[\,\tau'\,] = \begin{bmatrix} \tau'_{11} & \tau'_{12} & \tau'_{13} \\ \tau'_{21} & \tau'_{22} & \tau'_{23} \\ \tau'_{31} & \tau'_{32} & \tau'_{33} \end{bmatrix} = [\,\Lambda_s\,]\,[\,\tau\,]\,[\,\Lambda_s\,]^T = \begin{bmatrix} \tau_{11} & -\tau_{12} & \tau_{13} \\ -\tau_{21} & \tau_{22} & -\tau_{23} \\ \tau_{31} & -\tau_{32} & \tau_{33} \end{bmatrix} \tag{B.3}$$

Moreover, substituting relation (B.2) into the transformation relation (2.46b) the components of strain referred to the system of axes x_1', x_2', x_3' are related to the components of strain referred to the system of axes x_1, x_2, x_3 by the following relation:

$$[\,e'\,] = \begin{bmatrix} e'_{11} & e'_{12} & e'_{13} \\ e'_{21} & e'_{22} & e'_{23} \\ e'_{31} & e'_{32} & e'_{33} \end{bmatrix} = [\,\Lambda_s\,]\,[\,e\,]\,[\,\Lambda_s\,]^T = \begin{bmatrix} e_{11} & -e_{12} & e_{13} \\ -e_{21} & e_{22} & -e_{23} \\ e_{31} & -e_{32} & e_{33} \end{bmatrix} \tag{B.4}$$

Substituting relation (B.3) and (B.4) into (B.1), we obtain

$$
\begin{aligned}
\tau_{11} &= C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33} - C_{1112}e_{12} + C_{1113}e_{13} - C_{1123}e_{23} \\
\tau_{22} &= C_{2211}e_{11} + C_{2222}e_{22} + C_{2233}e_{33} - C_{2212}e_{12} + C_{2213}e_{13} - C_{2223}e_{23} \\
\tau_{33} &= C_{3311}e_{11} + C_{3322}e_{22} + C_{3333}e_{33} - C_{3312}e_{12} + C_{3313}e_{13} - C_{3323}e_{23} \\
-\tau_{12} &= C_{1211}e_{11} + C_{1222}e_{22} + C_{1233}e_{33} - C_{1212}e_{12} + C_{1213}e_{13} - C_{1223}e_{23} \\
\tau_{13} &= C_{1311}e_{11} + C_{1322}e_{22} + C_{1333}e_{33} - C_{1312}e_{12} + C_{1313}e_{13} - C_{1323}e_{23} \\
-\tau_{23} &= C_{2311}e_{11} + C_{2322}e_{22} + C_{2333}e_{33} - C_{2312}e_{12} + C_{2313}e_{13} - C_{2323}e_{23}
\end{aligned}
\tag{B.5}
$$

Comparing relations (B.5) with (3.49), we see that

$$
\begin{aligned}
C_{1112} = C_{1123} = C_{2212} = C_{2223} = C_{3312} = C_{3323} = 0 \\
C_{1211} = C_{1222} = C_{1233} = C_{2311} = C_{2322} = C_{2333} = 0 \\
C_{1213} = C_{2313} = C_{1312} = C_{1323} = 0
\end{aligned}
\tag{B.6}
$$

Following a reasoning similar to the one presented above we may conclude that if the coefficients of the stress–strain in relations (3.39) or (3.40) are symmetric with respect to x_1x_3 and x_2x_3 planes, they remain the same when the system of axes to which these relations are referred changes from x_1, x_2, x_3 to x_1'', x_2'', x_3''. The latter is obtained by rotating the system of axes x_1, x_2, x_3 by 180° about the x_3 axis (see Fig. B.2). That is, referring to Fig. B.2, with respect to the system of axes x_1'', x_2'', x_3'' the stress–strain (3.39) can be written as

$$
\begin{aligned}
\tau''_{11} &= C_{1111}e''_{11} + C_{1122}e''_{22} + C_{1133}e''_{33} + C_{1112}e''_{12} + C_{1113}e''_{13} + C_{1123}e''_{23} \\
\tau''_{22} &= C_{2211}e''_{11} + C_{2222}e''_{22} + C_{2233}e''_{33} + C_{2212}e''_{12} + C_{2213}e''_{13} + C_{2223}e''_{23} \\
\tau''_{33} &= C_{3311}e''_{11} + C_{3322}e''_{22} + C_{3333}e''_{33} + C_{3312}e''_{12} + C_{3313}e''_{13} + C_{3323}e''_{23} \\
\tau''_{12} &= C_{1211}e''_{11} + C_{1222}e''_{22} + C_{1233}e''_{33} + C_{1212}e''_{12} + C_{1213}e''_{13} + C_{1223}e''_{23} \\
\tau''_{13} &= C_{1311}e''_{11} + C_{1322}e''_{22} + C_{1333}e''_{33} + C_{1312}e''_{12} + C_{1313}e''_{13} + C_{1323}e''_{23} \\
\tau''_{23} &= C_{2311}e''_{11} + C_{2322}e''_{22} + C_{2333}e''_{33} + C_{2312}e''_{12} + C_{2313}e''_{13} + C_{2323}e''_{23}
\end{aligned}
\tag{B.7}
$$

Referring to Fig. B.2 the transformation matrix of the system of axes x_1'', x_2'', x_3'' with respect to the system of axes x_1, x_2, x_3 is

$$
[\,\Lambda_s\,] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\tag{B.8}
$$

Substituting relation (B.8) into the transformation relations (2.46a) the components of stress referred to the x_1'', x_2'', x_3'' axes are related to the components of stress referred to the x_1, x_2, x_3 axes by the following relation:

$$
[\,\tau''\,] = \begin{bmatrix} \tau''_{11} & \tau''_{12} & \tau''_{13} \\ \tau''_{21} & \tau''_{22} & \tau''_{23} \\ \tau''_{31} & \tau''_{32} & \tau''_{33} \end{bmatrix} = [\,\Lambda_s\,]\,[\,\tau\,]\,[\,\Lambda_s\,]^T = \begin{bmatrix} \tau_{11} & \tau_{12} & -\tau_{13} \\ \tau_{21} & \tau_{22} & -\tau_{23} \\ -\tau_{31} & -\tau_{32} & \tau_{33} \end{bmatrix}
\tag{B.9}
$$

Moreover, substituting relation (B.8) into the transformation relations (2.46a) the components of strain referred to the x_1'', x_2'', x_3'' axes are related to the components of strain referred to the x_1, x_2, x_3 axes by the following relation:

$$[e''] = \begin{bmatrix} e''_{11} & e''_{12} & e''_{13} \\ e''_{21} & e''_{22} & e''_{23} \\ e''_{31} & e''_{32} & e''_{33} \end{bmatrix} = [\Lambda_s]\,[e]\,[\Lambda_s]^T = \begin{bmatrix} e_{11} & e_{12} & -e_{13} \\ e_{21} & e_{22} & -e_{23} \\ -e_{31} & -e_{32} & e_{33} \end{bmatrix} \tag{B.10}$$

Substituting relation (B.9) and (B.10) into (B.7) we obtain,

$$\begin{aligned}
\tau_{11} &= C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33} + C_{1112}e_{12} - C_{1113}e_{13} - C_{1123}e_{23} \\
\tau_{22} &= C_{2211}e_{11} + C_{2222}e_{22} + C_{2233}e_{33} + C_{2212}e_{12} - C_{2213}e_{13} - C_{2223}e_{23} \\
\tau_{33} &= C_{3311}e_{11} + C_{3322}e_{22} + C_{3333}e_{33} + C_{3312}e_{12} - C_{3313}e_{13} - C_{3323}e_{23} \\
\tau_{12} &= C_{1211}e_{11} + C_{1222}e_{22} + C_{1233}e_{33} + C_{1212}e_{12} - C_{1213}e_{13} - C_{1223}e_{23} \\
-\tau_{13} &= C_{1311}e_{11} + C_{1322}e_{22} + C_{1333}e_{33} + C_{1312}e_{12} - C_{1313}e_{13} - C_{1323}e_{23} \\
-\tau_{23} &= C_{2311}e_{11} + C_{2322}e_{22} + C_{2333}e_{33} + C_{2312}e_{12} - C_{2313}e_{13} - C_{2323}e_{23}
\end{aligned} \tag{B.11}$$

Comparing relations (B.11) with (3.39) we see that

$$C_{1113} = C_{2213} = C_{3313} = C_{1223} = C_{2312} = C_{1311} = C_{1322} = C_{1333} = 0 \tag{B.12}$$

Substituting relations (B.6) and (B.12) into (3.39), the stress–strain relations for an orthotropic, linearly elastic materials are

$$\begin{aligned}
\tau_{11} &= C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33} \\
\tau_{22} &= C_{2211}e_{11} + C_{2222}e_{22} + C_{2233}e_{33} \\
\tau_{33} &= C_{3311}e_{11} + C_{3322}e_{22} + C_{3333}e_{33} \\
\tau_{12} &= C_{1212}e_{12} \\
\tau_{13} &= C_{1313}e_{13} \\
\tau_{23} &= C_{2323}e_{23}
\end{aligned} \tag{B.13}$$

B.2 Stress–Strain Relations for Isotropic, Linearly Elastic Materials

The coefficients of the stress–strain relations for isotropic materials are independent of the system of axes to which the components of stress and strain are referred. In the previous section we consider a system of axes x_1', x_2', x_3' or x_1'', x_2'', x_3'' obtained by

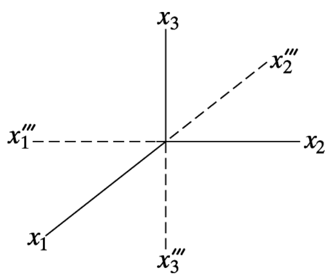


Figure B.3 System of axes x_1''', x_2''', x_3''' obtained from the system of axes x_1, x_2, x_3 by rotating it first about the x_1 axis by 180° and then about the x_3''' axis by 90°.

rotating the system of axes x_1, x_2, x_3 by 180° either about the x_2 or about the x_3 axis, respectively. We have shown that in order that each one of the coefficients of the stress-strain relations referred to the system of axis x_1', x_2', x_3' and x_1'', x_2'', x_3'' is equal to the corresponding coefficients of the stress-strain relations referred to the system of axes x_1, x_2, x_3, the stress–strain relations must have the form (B.13).

In this section we establish the additional relations which must exist between the coefficients of the stress–strain relations (B.13) in order to be independent of the system of axes to which the components of stress and strain are referred. In order to accomplish this, we consider the stress–strain relations referred to the system of axes x_1''', x_2''', x_3''' shown in Fig. B.3. Since the microstructure of an isotropic material is symmetric with respect to the planes normal to these axes we have

$$
\begin{aligned}
\tau'''_{11} &= C_{1111}e'''_{11} + C_{1122}e'''_{22} + C_{1133}e'''_{33} \\
\tau'''_{22} &= C_{2211}e'''_{11} + C_{2222}e'''_{22} + C_{2233}e'''_{33} \\
\tau'''_{33} &= C_{3311}e'''_{11} + C_{3322}e'''_{22} + C_{3333}e'''_{33} \\
\tau'''_{12} &= C_{1212}e'''_{12} \\
\tau'''_{13} &= C_{1313}e'''_{13} \\
\tau'''_{23} &= C_{2323}e'''_{23}
\end{aligned}
\tag{B.14}
$$

Referring to Fig. B.3, the transformation matrix of the system of axes x_1''', x_2''', x_3''' with respect to to the system of axes x_1, x_2, x_3 is

$$
[\, \Lambda_s \,] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{B.15}
$$

Substituting relation (B.15) into the transformation relations (2.46a), the components of stress referred to the system of axes x_1''', x_2''', x_3''' are related to the components of stress referred to the system of axes x_1, x_2, x_3 by the following relation:

$$
[\,\tau'''] = \begin{bmatrix} \tau'''_{11} & \tau'''_{12} & \tau'''_{13} \\ \tau'''_{21} & \tau'''_{22} & \tau'''_{23} \\ \tau'''_{31} & \tau'''_{32} & \tau'''_{33} \end{bmatrix} = [\Lambda_s]\,[\,\tau]\,[\Lambda_s]^T = \begin{bmatrix} \tau_{22} & \tau_{12} & \tau_{23} \\ \tau_{21} & \tau_{11} & \tau_{31} \\ \tau_{23} & \tau_{13} & \tau_{33} \end{bmatrix} \tag{B.16}
$$

Moreover, substituting relation (B.15) into the transformation relations (2.46a), the components of strain referred to the system of axes x_1''', x_2''', x_3''' are related to the components of strain referred to the system of axes x_1, x_2, x_3 by the following relation:

$$
[\,e'''] = \begin{bmatrix} e'''_{11} & e'''_{12} & e'''_{13} \\ e'''_{21} & e'''_{22} & e'''_{23} \\ e'''_{31} & e'''_{32} & e'''_{33} \end{bmatrix} = [\Lambda_s]\,[\,e]\,[\Lambda_s]^T = \begin{bmatrix} e_{22} & e_{12} & e_{23} \\ e_{21} & e_{11} & e_{31} \\ e_{23} & e_{13} & e_{33} \end{bmatrix} \tag{B.17}
$$

Substituting relation (B.16) and (B.17) into (B.14), we obtain

$$\begin{aligned}
\tau_{11} &= C_{2211}e_{22} + C_{2222}e_{11} + C_{2233}e_{33} \\
\tau_{22} &= C_{1111}e_{22} + C_{1122}e_{11} + C_{1133}e_{33} \\
\tau_{33} &= C_{3311}e_{22} + C_{3322}e_{11} + C_{3333}e_{33} \\
\tau_{12} &= C_{1212}e_{12} \\
\tau_{23} &= C_{1313}e_{23} \\
\tau_{13} &= C_{2323}e_{13}
\end{aligned} \tag{B.18}$$

Comparing relations (B.18) with (B.13), we see that

$$\begin{aligned}
C_{1111} &= C_{2222} & C_{3311} &= C_{3322} \\
C_{1122} &= C_{2211} & C_{2323} &= C_{1313} \\
C_{1133} &= C_{2233}
\end{aligned} \tag{B.19}$$

Referring to Fig. B.4, the transformation matrix of the system of the axes with respect to the system of axes x_1, x_2, x_3 is

$$[\,\Lambda_s\,] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{B.20}$$

Following a procedure analogous to the one employed above, we obtain

$$\begin{aligned}
&C_{1111} = C_{2222} = C_{3333} = A \\
&C_{1133} = C_{1122} = C_{2233} = C_{2211} = C_{3311} = C_{3322} = \lambda \\
&C_{2323} = C_{1313} = C_{1212} = 2G
\end{aligned} \tag{B.21}$$

Using relations (B.19) and (B.21), the stress–strain relations (B.13) reduce to

$$\begin{aligned}
\tau_{11} &= Ae_{11} + \lambda(e_{22} + e_{33}) \\
\tau_{22} &= Ae_{22} + \lambda(e_{11} + e_{33}) \\
\tau_{33} &= Ae_{33} + \lambda(e_{22} + e_{11}) \\
\tau_{12} &= 2Ge_{12} \\
\tau_{13} &= 2Ge_{13} \\
\tau_{23} &= 2Ge_{23}
\end{aligned} \tag{B.22}$$

These relations can be solved for e_{11}, e_{22}, e_{33} to give

$$\begin{aligned}
e_{11} &= \frac{1}{E}[\,\tau_{11} - \nu(\tau_{22} + \tau_{33})\,] \\
e_{22} &= \frac{1}{E}[\,\tau_{22} - \nu(\tau_{11} + \tau_{33})\,] \\
e_{33} &= \frac{1}{E}[\,\tau_{33} - \nu(\tau_{11} + \tau_{22})\,]
\end{aligned} \tag{B.23}$$

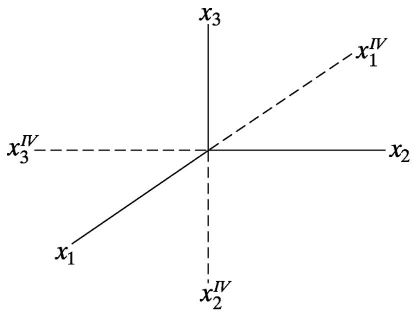


Figure B.4 System of axes x_1^{IV}, x_2^{IV}, x_3^{IV} obtained by rotating the system of axes $x1$, $x2$, $x3$ first about the axis x_2^{IV} by 180° and then about the axis x_1^{IV} by 90°.

$$e_{12} = \frac{\tau_{12}}{2G} \qquad e_{13} = \frac{\tau_{13}}{2G} \qquad e_{23} = \frac{\tau_{23}}{2G} \tag{B.23}$$

where

$$E = \frac{A(1+\nu)(1-2\nu)}{(1-\nu)} \qquad \nu = \frac{\lambda}{\lambda + A} \tag{B.24}$$

Solving relations (B.24) for A and λ, we get

$$A = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{B.25}$$

Thus, the stress-strain relations (B.22) can be written as

$$\begin{aligned}
\tau_{11} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{11} + \nu(e_{22} + e_{33})] \\
\tau_{22} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{22} + \nu(e_{11} + e_{33})] \\
\tau_{33} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)e_{33} + \nu(e_{22} + e_{11})] \\
\tau_{12} &= 2Ge_{12} \qquad \tau_{13} = 2Ge_{13} \qquad \tau_{23} = 2Ge_{23}
\end{aligned} \tag{B.26}$$

For a state of uniaxial stress ($\tau_{11} \neq 0$, $\tau_{22} = \tau_{33} = \tau_{12} = \tau_{13} = \tau_{23} = 0$), relations (B.23) reduce to

$$e_{11} = \frac{\tau_{11}}{E} \qquad e_{22} = -\frac{\nu\tau_{11}}{E} = -\nu e_{11} \qquad e_{33} = -\frac{\nu\tau_{11}}{E} = -\nu e_{11} \tag{B.27}$$

Comparing relations (3.3) and (3.5) with (B.27), we see that E is the modulus of elasticity and ν is Poisson ratio. Relations (B.23) and (B.26) are the stress–strain relations for an isotropic, linearly elastic material.

In what follows we show that the material constants G, E and ν are related. In order to establish the relation among the three constants, we consider a particle of a body made from an isotropic, linearly elastic material and we refer our discussion to the principal axes $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ of the components of stress acting on this particle. The shearing component of stress τ_{12}' acting on a plane specified by the unit vector $\mathbf{i}_1' = \lambda_{11}\mathbf{i}_1 + \lambda_{12}\mathbf{i}_2 + \lambda_{13}\mathbf{i}_3$ in the direction of the unit vector $\mathbf{i}_2' = \lambda_{21}\mathbf{i}_1 + \lambda_{22}\mathbf{i}_2 + \lambda_{23}\mathbf{i}_3$ may be expressed in terms of the principal components of stress τ_1, τ_2, τ_3, using the transformation of relations (2.49a). That is,

$$\tau'_{12} = \tau_1\lambda_{11}\lambda_{21} + \tau_2\lambda_{12}\lambda_{22} + \tau_3\lambda_{13}\lambda_{23} \tag{B.28}$$

Moreover, the shearing component of strain e_{12}' of the particle under consideration referred to the directions specified by the unit vectors $\mathbf{i}_1'$ and $\mathbf{i}_2'$ can be written as

$$e'_{12} = e_1\lambda_{11}\lambda_{21} + e_2\lambda_{12}\lambda_{22} + e_3\lambda_{13}\lambda_{23} \tag{B.29}$$

The fourth of the stress–strain relations (B.26) referred to the system of axes x_1', x_2', x_3' becomes

$$\tau'_{12} = 2Ge'_{12} \tag{B.30}$$

Substituting relations (B.28) and (B.29) into (B.30), we obtain

$$2G(e_1\lambda_{11}\lambda_{21} + e_2\lambda_{12}\lambda_{22} + e_3\lambda_{13}\lambda_{23}) = \tau_1\lambda_{11}\lambda_{21} + \tau_2\lambda_{12}\lambda_{22} + \tau_3\lambda_{13}\lambda_{23} \tag{B.31}$$

Substituting the stress-strain relations (B.23) in the above and simplifying, we obtain

$$\begin{aligned}
&\tau_1\left[\lambda_{11}\lambda_{21}\left(\frac{2G - E}{2G}\right) - \nu(\lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23})\right] \\
&+ \tau_2\left[\lambda_{12}\lambda_{22}\left(\frac{2G - E}{2G}\right) - \nu(\lambda_{11}\lambda_{21} + \lambda_{13}\lambda_{23})\right] \\
&+ \tau_3\left[\lambda_{13}\lambda_{23}\left(\frac{2G - E}{2G} + \nu\right) - \nu(\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22})\right] = 0
\end{aligned} \tag{B.32}$$

Using the first of relations (1.25b) the above relation reduces to

$$\left(\frac{2G - E}{2G} + \nu\right)(\tau_1\lambda_{11}\lambda_{21} + \tau_2\lambda_{12}\lambda_{22} + \tau_3\lambda_{13}\lambda_{23}) = 0 \tag{B.33}$$

Relation (B.33) is valid for any state of stress. Consequently, the first term of the product on the left side of the above relation must vanish. That is,

$$G = \frac{E}{2(1 + \nu)} \tag{B.34}$$

Thus, the stress–strain relations for an isotropic, linearly elastic material involve only two independent elastic constants.

Relations (B.23) may be obtained directly as follows:

Consider a body subjected to a loading inducing an uniaxial state of stress on its particles ($\tau_{11}^{(1)} \neq 0$, $\tau_{22}^{(1)} = \tau_{33}^{(1)} = \tau_{12}^{(1)} = \tau_{13}^{(1)} = \tau_{23}^{(1)} = 0$). Referring to relations (3.3) and (3.5) the corresponding components of strain are

$$e_{11}^{(1)} = \frac{\tau_{11}^{(1)}}{E} \qquad e_{22}^{(1)} = -\frac{\nu\tau_{11}^{(1)}}{E} \qquad e_{33}^{(1)} = -\frac{\nu\tau_{11}^{(1)}}{E} \qquad e_{12}^{(1)} = e_{13}^{(1)} = e_{23}^{(1)} = 0 \tag{B.35}$$

Consider the same body subjected to a second loading inducing a uniaxial state of stress ($\tau_{22}^{(2)} \neq 0,\ \tau_{11}^{(2)} = \tau_{33}^{(2)} = \tau_{12}^{(2)} = \tau_{13}^{(2)} = \tau_{23}^{(2)} = 0$). The corresponding components of strain are

$$e_{11}^{(2)} = -\frac{\nu\tau_{22}^{(2)}}{E} \qquad e_{22}^{(2)} = \frac{\tau_{22}^{(2)}}{E} \qquad e_{33}^{(2)} = -\frac{\nu\tau_{22}^{(2)}}{E} \qquad e_{12}^{(2)} = e_{13}^{(2)} = e_{23}^{(2)} = 0 \tag{B.36}$$

Finally, consider the same body subjected to a third loading inducing an uniaxial state of stress ($\tau_{33}^{(3)} \neq 0,\ \tau_{22}^{(3)} = \tau_{11}^{(3)} = \tau_{12}^{(3)} = \tau_{13}^{(3)} = \tau_{23}^{(3)} = 0$). The corresponding components of strain are

$$e_{11}^{(3)} = -\frac{\nu\tau_{33}^{(3)}}{E} \qquad e_{22}^{(3)} = -\frac{\nu\tau_{33}^{(3)}}{E} \qquad e_{33}^{(3)} = \frac{\tau_{33}^{(3)}}{E} \qquad e_{12}^{(3)} = e_{13}^{(3)} = e_{23}^{(3)} = 0 \tag{B.37}$$

When the three loadings are applied on the body under consideration simultaneously the resulting components of strain $e_{ij}(i, j = 1, 2, 3)$ are equal to the sum of the components of strain given by relations (B.35) to (B.37). That is, denoting by $\tau_{ij}(i, j = 1, 2, 3)$ the sum of the components of stress $\tau_{ij}^{(1)}$, $\tau_{ij}^{(2)}$ and $\tau_{ij}^{(3)}(i, j = 1, 2, 3)$ we have

$$\begin{aligned}
e_{11} &= e_{11}^{(1)} + e_{11}^{(2)} + e_{11}^{(3)} = \frac{1}{E}[\ \tau_{11} - \nu(\tau_{22} + \tau_{33})\ ] \\
e_{22} &= e_{11}^{(1)} + e_{11}^{(2)} + e_{11}^{(3)} = \frac{1}{E}[\ \tau_{22} - \nu(\tau_{11} + \tau_{33})\ ] \\
e_{33} &= e_{11}^{(1)} + e_{11}^{(2)} + e_{11}^{(3)} = \frac{1}{E}[\ \tau_{33} - \nu(\tau_{11} + \tau_{22})\ ]
\end{aligned} \tag{B.38}$$

Appendix C

Centroid, Moments and Products of Inertia of Plane Surfaces

C.1 The Centroid of a Plane Surface

Consider a plane surface of area A located in the x_1x_2 plane. The centroid of this surface is defined as the point whose coordinates $(\bar{x}_1, \bar{x}_2)$ with respect to two mutually perpendicular axes x_1 and x_2 are

$$\bar{x}_i = \frac{\iint_A x_i \, dA}{\iint_A dA} \qquad (i = 1, 2) \tag{C.1}$$

The integral $\iint_A x_i \, dA$ is known as the *first moment of the plane surface of area A about the x_j axis* ($i, j = 1, 2$; $i \neq j$). Any set of two mutually perpendicular axes located on a plane surface and having as origin its centroid is called *centroidal set of axes* for this surface. As can be seen from relation (C.1) *the first moment of a surface about one of its centroidal axis vanishes*. That is,

$$\iint_A \bar{x}_i \, dA = 0 \qquad (i = 1, 2) \tag{C.2}$$

If the boundary of a plane surface is an irregular curve, the integral in relation (C.1) may have to be evaluated numerically.

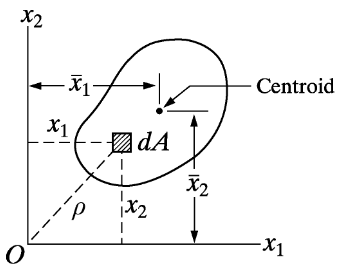
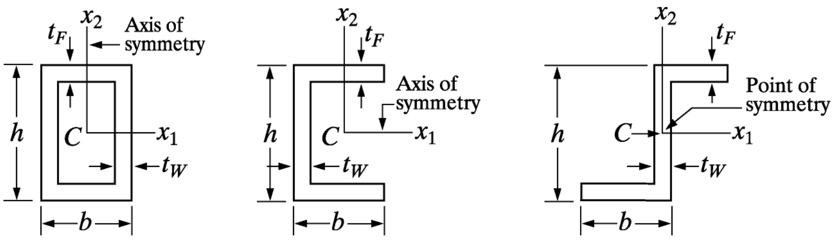


Figure C.1 Centroid of a plane surface.



(a) Plane surface with axis of symmetry (b) Plane surface with point of symmetry

Figure C.2 Plane surfaces with axis or point of symmetry.

If a plane surface has an axis of symmetry x_j for every infinitesimal area dA with a positive coordinate $x_i(i \neq j)$ measured from the axis of symmetry x_j, there is an area dA with a negative coordinate x_i and vice versa (see Fig. C.2a). Thus, the first moment of the surface about its axis of symmetry is zero. Moreover, if a plane surface has a point of symmetry for every infinitesimal area dA with a position vector **r** measured from the point of symmetry, there is an area dA with a position vector $-$**r** and vice versa (see Fig. C.2b). Hence, the first moment of a surface about any axis passing through its point of symmetry is zero. Consequently, *if a surface has an axis or a point of symmetry, its centroid lies on the axis of symmetry or is the point of symmetry*. Thus, if an area has two axes of symmetry, its centroid is the intersection of the two axes of symmetry. In the table of the inside of the back cover of the book we give the coordinates of the centroids of certain plane surfaces.

If a plane surface can be subdivided into parts whose centroids are known, then its centroid can be found by noting that the first moment of a plane surface about an axis is equal to the sum of the first moments of its parts about the same axis. Thus, if a plane surface is subdivided into n parts, we have

$$\bar{x}_i = \frac{\iint_{A_1} x_i \, dA + \iint_{A_2} x_i \, dA + \cdots + \iint_{A_n} x_i \, dA}{A_1 + A_2 + \cdots + A_n} \qquad (i = 1, 2) \tag{C.3}$$

where A_j (j = 1, 2..., n) is the area of the j^{th} part. Referring to relation (C.1), the coordinates of the centroid of the j^{th} part satisfy the following relation:

$$\iint_{A_j} x_i \, dA = A_j \bar{x}_i^{(j)} \tag{C.4}$$

Substituting relation (C.4) into (C.3), we obtain

$$\bar{x}_i = \frac{\sum_{j=1}^{n} \bar{x}_i^{(j)} A_j}{\sum_{j=1}^{n} A_j} \tag{C.5}$$

C.2 Moments and Products of Inertia of a Plane Surface

Referring to Fig. C.1, the moments of inertia I_{11} and I_{22} of a plane surface of area A about the x_1 and x_2 axes, respectively, are defined as

$$I_{11} = \iint_A x_2^2 \, dA \qquad I_{22} = \iint_A x_1^2 \, dA \tag{C.6}$$

Moreover, the products of inertia I_{12} and I_{21} of a plane surface of area A about the x_1, x_2 axes are defined as

$$I_{12} = I_{21} = \iint_A x_1 \, x_2 \, dA \tag{C.7}$$

Furthermore, referring to Fig. C.1, the polar moment of inertia I_p of a plane surface of area A about a point O is defined as

$$I_p = \iint_A \rho^2 \, dA \tag{C.8}$$

where ρ is the distance from point O to the element of area dA. If point O is the origin of the axis of reference we have

$$\rho^2 = x_1{}^2 + x_2{}^2 \tag{C.9}$$

Substituting relation (C.9) into (C.8) and using (C.6), we obtain

$$I_p = \iint_A (x_1^2 + x_2^2) \, dA = I_{11} + I_{22} \tag{C.10}$$

From their definition, it is apparent that the moment of inertia of a plane surface with respect to any axis in the plane of the surface is positive, while its product of inertia with respect to a set of two mutually perpendicular axes in the plane of the surface could be positive or negative. Moreover, the polar moment of inertia with respect to any point O in the plane of the surface is positive.

The moment of inertia of a composite surface with respect to an axis in its plane is the sum of the moments of inertia of its parts with respect to the same axis. Moreover, the product of inertia of a composite surface with respect to a set of two mutually perpendicular axes in its plane is the sum of the products of inertia of its parts with respect to the same set of axes. Furthermore, the polar moment of inertia of a composite surface with respect to a point on its plane is the sum of the polar moments of inertia of its parts with respect to the same point.

Consider a plane surface having an axis of symmetry. The product of inertia of this surface with respect to two mutually perpendicular axes, one of which is the axis of

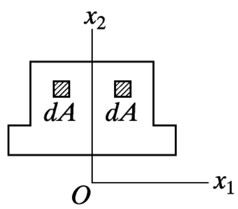


Figure C.3 Plane surface with an axis of symmetry.

symmetry of the surface, vanishes. This becomes apparent by referring to Fig. C.3 and noting that for every dA with positive x_2 there is a dA with the same x_1 and negative x_2.

In the inside of the back cover of the book we give the moments and product of inertia about a set of centroidal axes for certain plane surfaces of engineering interest.

The *radius of gyration* of a surface of area A with respect to the axis x_i (i =1, 2) is defined as

$$r_i = \sqrt{\frac{I_{ii}}{A}} \quad (i = 1, 2) \tag{C.11}$$

C.3 The Parallel Axes Theorems

In this section we derive the relation between the moment of inertia of a plane surface with respect to any axis in its plane and its moment of inertia with respect to a parallel to it centroidal axis. Moreover, we derive the relation between the product of inertia of a plane surface with respect to a system of two mutually perpendicular axes in the plane of the surface and its product of inertia with respect to a parallel to it system of two mutually perpendicular centroidal axes. Referring to Fig. C.4, we have

$$I_{11} = \iint_A x_2^2\, dA = \iint_A (\bar{x}_2 + a_2)^2\, dA = \iint_A \bar{x}_2^2\, dA + a_2^2 \iint_A dA + 2a_2 \iint_A \bar{x}_2\, dA \tag{C.12a}$$

$$I_{22} = \iint_A x_1^2\, dA = \iint_A (\bar{x}_1 + a_1)^2\, dA = \iint_A \bar{x}_1^2\, dA + a_1^2 \iint_A dA + 2a_1 \iint_A \bar{x}_1\, dA \tag{C.12b}$$

$$\begin{aligned} I_{12} &= \iint_A x_1 x_2\, dA = \iint_A (\bar{x}_1 + a_1)(\bar{x}_2 + a_2)\, dA \\ &= \iint_A \bar{x}_1 \bar{x}_2\, dA + a_1 a_2 \iint_A dA + a_1 \iint_A \bar{x}_2\, dA + a_2 \iint_A \bar{x}_1\, dA \end{aligned} \tag{C.13}$$

The first integral on the right side of relation (C.12a) is the moment of inertia $\bar{I}_{11}$ of the surface with respect to the centroidal axis $\bar{x}_1$. The second integral is equal to the area of the surface, while the third integral represents the first moment of the surface with respect to the centroidal axis $\bar{x}_1$ and consequently vanishes. Moreover, the first integral on the right side of relation (C.13) is the product of inertia $\bar{I}_{12}$ of the

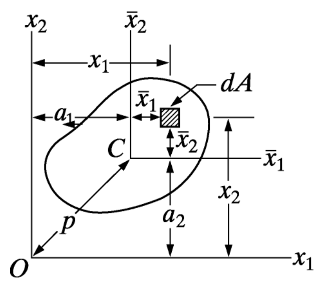


Figure C.4 Plane surface with two parallel systems of axes in its plane.

surface with respect to the centroidal axes $\bar{x}_1, \bar{x}_2$. The third and fourth integrals represent the first moments of the surface with respect to the centroidal axes $\bar{x}_1$ and $\bar{x}_2$, respectively, and consequently vanish. Therefore, relations (C.12) and (C.13) reduce to

$$I_{11} = \bar{I}_{11} + Aa_2^2 \tag{C.14}$$

$$I_{22} = \bar{I}_{22} + Aa_1^2 \tag{C.15}$$

$$I_{12} = \bar{I}_{12} + Aa_1a_2 \tag{C.16}$$

Relations (C.14) to (C.16) are called the *parallel axis theorems for the moments and product of inertia*. If it is required to find the moment of inertia I'_{11} of a plane surface with respect to a non-centroidal axis x_1' when its moment of inertia I_{11} is known with respect to another non-centroidal axis x_1 which is parallel to x_1', we must apply the parallel axis theorem (C.14) twice. First, we use the theorem to find the moment of inertia $\bar{I}_{11}$ of the surface about the centroidal axis $\bar{x}_1$ which is parallel to x_1'. Then we use the theorem a second time to find the moment of inertia I'_{11} from $\bar{I}_{11}$. We follow a similar procedure if it is required to find the product of inertia I'_{12} of a plane surface with respect to a system of non-centroidal mutually perpendicular axes x_1', x_2' when its product of inertia, I_{12}, is known with respect to another system of non-centroidal axes x_1, x_2 which is parallel to the system of axes x_1', x_2'. Referring to Fig. C.5, we have

$$\bar{I}_{11} = I_{11} - Aa_2^2 \qquad I'_{11} = \bar{I}_{11} + Ac_2^2 = I_{11} + A\,(c_2^2 - a_2^2) \tag{C.17}$$

and

$$\bar{I}_{12} = I_{12} - Aa_1a_2\; I'_{12} = \bar{I}_{12} + Ac_1\,c_2 = I_{12} + A\,(c_1c_2 - a_1a_2) \tag{C.18}$$

The parallel axes theorems are extremely useful for finding moments and products of inertia of composite sections. This is illustrated by the following example.

Example 1 Determine the moments and product of inertia with respect to the set of centroidal axes x_1x_2 of the plane surface shown in Fig. a.

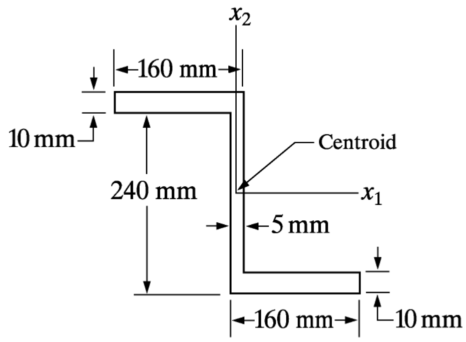


Figure a Geometry of the plane surface.

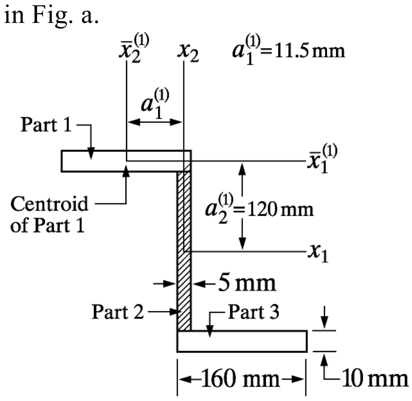


Figure b Plane surface subdivided into three rectangular parts.

Solution We subdivide the plane surface under consideration into three rectangular parts as shown in Fig. b. Each one of the moments and the product of inertia of the surface with respect to the system of axes x_1, x_2 are equal to the sum of the corresponding quantities of the three parts about the same system of axes. Noting that each one of the moments and the product of inertia of part 3 with respect to the x_1, x_2 axes are equal to the corresponding quantity of part 1 with respect to the same axes we have

$$
\begin{aligned}
I_{11} &= I_{11}^{(1)} + I_{11}^{(2)} + I_{11}^{(3)} = 2I_{11}^{(1)} + I_{11}^{(2)} \\
I_{22} &= I_{22}^{(1)} + I_{22}^{(2)} + I_{22}^{(3)} = 2I_{22}^{(1)} + I_{22}^{(2)} \\
I_{12} &= I_{12}^{(1)} + I_{12}^{(2)} + I_{12}^{(3)} = 2I_{12}^{(1)} + I_{12}^{(2)}
\end{aligned} \tag{a}
$$

where $I_{ij}^{(k)}$ $(i, j = 1, 2)$ $(k = 1, 2, 3)$ are the moments and the product of inertia of part k of the surface with respect to the system of axes x_1, x_2. Denoting by $I_{ij}^{(k)}$ a moment or the product of inertia of part k of the surface with respect to its own centroidal axis $\bar{x}_1^{(k)}$ and $\bar{x}_2^{(k)}$, we have

$$
\begin{aligned}
I_{11}^{(1)} &= \bar{I}_{11}^{(1)} + A_1[a_2^{(1)}]^2 = \frac{(160)(10)^3}{12} + (160)(10)(120)^2 = 230.53(10^5)\ \mathrm{mm}^4 \\
I_{22}^{(1)} &= \bar{I}_{22}^{(1)} + A_1[a_1^{(1)}]^2 = \frac{(160)^3(10)}{12} + (160)(10)(-77.5)^2 = 130.23(10^5)\ \mathrm{mm}^4 \\
I_{12}^{(1)} &= \bar{I}_{12}^{(1)} + A_1 a_1^{(1)} a_2^{(1)} = A_1 a_1^{(1)} a_2^{(1)} = (160)(10)(120)(-77.5) = -148.8(10^5)\ \mathrm{mm}^4 \\
I_{11}^{(2)} &= \bar{I}_{11}^{(2)} = \frac{(230)^3\ (5)}{12} = 50.70(10^5)\ \mathrm{mm}^4 \\
I_{22}^{(2)} &= \bar{I}_{22}^{(2)} = \frac{(230)(5)^3}{12} = 0.024(10^5)\ \mathrm{mm}^4 \\
I_{12}^{(2)} &= \bar{I}_{12}^{(2)} = 0
\end{aligned} \tag{b}
$$

Substituting relations (c) into (b), we obtain

$$
\begin{aligned}
I_{11} &= 2(230.53)(10^5) + 50.70(10^5) = 511.76(10^5)\ \mathrm{mm}^4 \\
I_{22} &= 2(130.23)(10^5) + 0.024(10^5) = 260.48(10^5)\ \mathrm{mm}^4 \\
I_{12} &= 2(-148.8)(10^5) = -297.6(10^5)\ \mathrm{mm}^4
\end{aligned} \tag{c}
$$

C.4 Transformation of the Moments and Product of Inertia of a Plane Surface upon Rotation of the Axes to Which They Are Referred

The values of the moments and product of inertia of a plane surface depend upon the position of the origin of the system of axes to which they are referred and upon the orientation of this system of axes. In the previous sections we have established how the values of the moments and product of inertia of a plane surface change as the system of axes to which they are referred is translated. In this section we establish how the values of the moments and product of inertia of a plane surface change as the system of two mutually perpendicular axes x_1, x_2 to which they are referred is rotated about an axis perpendicular to its plane at its origin.

Consider the following matrix of moments and products of inertia of a plane surface with respect to two mutually perpendicular axes x_1, x_2 located in its plane:

$$\begin{bmatrix} I_{11} & -I_{12} \\ -I_{21} & I_{22} \end{bmatrix} \tag{C.19}$$

Moreover, consider the following matrix of moments and products of inertia of the same plane surface, with respect to two mutually perpendicular axes x_1', x_2' located in its plane and having the same origin as the system of axes x_1, x_2:

$$\begin{bmatrix} I'_{11} & -I'_{12} \\ -I'_{21} & I'_{22} \end{bmatrix} \tag{C.20}$$

In what follows we show that the elements of these matrices are components of a plane symmetric tensor of the second rank called the *inertia tensor*.

The position of a point in space may be specified by a position vector **r** with respect to a fixed point O. The components of this vector with respect to a rectangular system of axes having as it origin point O are the coordinates of the point with respect to that system of axes. Thus, for a point on the x_1x_2 plane, we have

$$\mathbf{r} = \mathrm{x}_1\mathbf{i}_1 + \mathrm{x}_2\mathbf{i}_2 = \mathrm{x}'_1\mathbf{i}'_1 + \mathrm{x}'_2\mathbf{i}'_2$$

Consequently, referring to relations (1.37) and (1.36a), we have

$$\begin{aligned} x_1' &= (\cos\phi_{11})x_1 + (\sin\phi_{11})x_2 \\ x_2' &= (-\sin\phi_{11})x_1 + (\cos\phi_{11})x_2 \end{aligned} \tag{C.21}$$

Substituting relations (C.21) into the expressions [see relation (C.6) and (C.7)] for the elements of array (C.20), we obtain

$$\begin{aligned} I'_{11} &= \iint_A (x_2')^2\, dA = \iint_A [-(\sin\phi_{11})x_1 + (\cos\phi_{11})x_2]^2\, dA \\ &= \sin^2\phi_{11}\iint_A x_1^2\, dA + \cos^2\phi_{11}\iint_A x_2^2\, dA - 2\sin\phi_{11}\cos\phi_{11}\iint_A x_1 x_2\, dA \\ &= \sin^2\phi_{11}\, I_{22} + \cos^2\phi_{11}\, I_{11} - \sin 2\phi_{11}\, I_{12} \\ &= \frac{1}{2}(I_{11} + I_{22}) + \frac{1}{2}(I_{11} - I_{22})\cos 2\phi_{11} - I_{12}\sin 2\phi_{11} \end{aligned}$$

$$\begin{aligned} I'_{22} &= \iint_A (x_1')^2\, dA = \iint_A [(\cos\phi_{11})x_1 + (\sin\phi_{11})x_2]^2\, dA \\ &= \cos^2\phi_{11}\iint_A x_1^2\, dA + \sin^2\phi_{11}\iint_A x_2^2\, dA + 2\sin\phi_{11}\cos\phi_{11}\iint_A x_1 x_2\, dA \\ &= \frac{1}{2}(I_{11} + I_{22}) - \frac{1}{2}(I_{11} - I_{22})\cos 2\phi_{11} + I_{12}\sin 2\phi_{11} \end{aligned}$$

$$I'_{12} = \iint_A x'_1 x'_2 \, dA = \iint_A [(\cos\phi_{11})x_1 + (\sin\phi_{11})x_2][-(\sin\phi_{11})x_1 + (\cos\phi_{11})x_2] \, dA$$

$$= -\cos\phi_{11}\sin\phi_{11}\iint_A x_1^2 \, dA + \cos\phi_{11}\sin\phi_{11}\iint_A x_2^2 \, dA + (-\sin^2\phi_{11} + \cos^2\phi_{11})\iint_A x_1 x_2 \, dA$$

$$= \frac{1}{2}(I_{11} - I_{22})\sin 2\phi_{11} + \cos 2\phi_{11} \, I_{12} \tag{C.22}$$

Comparing relations (C.22) with (1.116), we see that the elements of the arrays (C.19) and (C.20) transform upon rotation of the system of axes to which they are referred in accordance to relations (1.116). Consequently, they are components of a plane symmetric tensor of the second rank known as the *inertia tensor of the planar surface* under consideration. Therefore, they have the properties of the components of such tensors established in Sections 1.3 to 1.12. From those properties the one that has the greatest practical application is that there exists at least one set of two mutually perpendicular axes, called *principal*, with respect to which the tensor assumes its diagonal form; that is, the product of inertia vanishes. It is often convenient to refer our calculations to principal axes in order to avoid the presence of the product of inertia in the results and, thus, it is important to know how to establish these axes.

In what follows we illustrate how to establish the direction of the principal axes and the values of the principal moments of inertia of a plane surface.

Example 2 Using Mohr's circle establish the principal axes and the principal moments of inertia at the centroid of the plane surface shown in Fig. a.

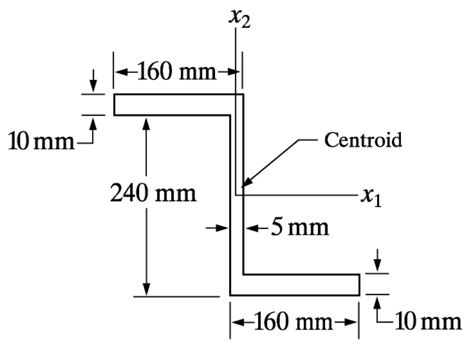


Figure a Geometry of the plane surface.

Solution

The moments and product of inertia of the plane surface of Fig. a with respect to the set of centroidal axes x_1, x_2 have been computed in the example of the previous section. Referring to this example we have

$$I_{11} = 511.76(10^5) \text{ mm}^4$$

$$I_{22} = 260.48(10^5) \text{ mm}^4$$

$$I_{12} = -297.60(10^5) \text{ mm}^4$$

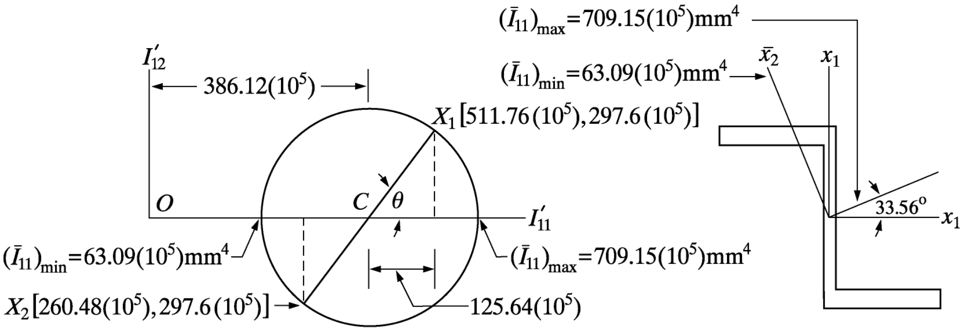


Figure b Mohr's circle. **Figure c** Results.

Thus, the tensor of the moments and product of inertia is

$$[\,I\,] = 10^5 \begin{bmatrix} 511.76 & 297.60 \\ 297.60 & 260.48 \end{bmatrix} \tag{a}$$

In Fig. b, we draw the two axes of reference — the I'_{11} axis and the I'_{12} axis. Then we plot point X_1 [511.76(10^5), 297.60(10^5)] whose coordinates are the components of the tensor I_{11} and $-I_{12}$. Moreover, we plot point X_2 [260.48(10^5) - 297.60(10^5)] whose coordinates are I_{22} and I_{12}. Points X_1 and X_2 lie on the ends of a diameter of Mohr's circle. The center of this circle lies on the I'_{11} axis at distance ½ [$(I_{11} + I_{22})$] = 386.12(10^5) mm^4 from the origin. Moreover, from geometric considerations, the radius of the Mohr's circle is

$$R = 10^5 \sqrt{\left(\frac{511.76 - 260.48}{2}\right)^2 + (297.6)^2} = 323.03(10^5) \tag{b}$$

Referring to Fig. b, we have

$$\tan\theta = \frac{297.6}{125.64} = 2.36867$$

Therefore,

$$\theta = 67.11° \qquad \phi_{11} = 33.56° \tag{c}$$

The direction of the principal axes is shown in Fig. c. The maximum and minimum value of the moments of inertia are computed on Fig. b. They are also given in Fig. c.

C.5 Problems

1. Using the definition of the centroid (C.1), determine the coordinates $\bar{x}_1$ and $\bar{x}_2$ to the centroid of the trapezoid of Fig. CP1.

Ans. $\bar{x}_1 = \dfrac{a^2+b^2+ab+2ac+bc}{3(a+b)} \qquad \bar{x}_2 = \dfrac{h}{3}\left(\dfrac{2a+b}{a+b}\right)$

2. Calculate the coordinate $\bar{x}_2$ of the centroid of the channel section shown in Fig. CP2, if a = 240 mm, b = 20 mm and c = 80 mm. Ans. $\bar{x}_2$ = 30 mm

3. Determine the coordinates $\bar{x}_1$ and $\bar{x}_2$ of the centroid of the surface shown in Fig. CP3.

Ans. $\bar{x}_1 = 5a/12$, $\bar{x}_2 = 5a/12$

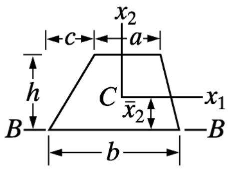


Figure CP1

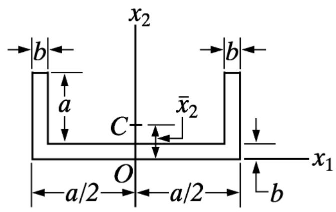


Figure CP2

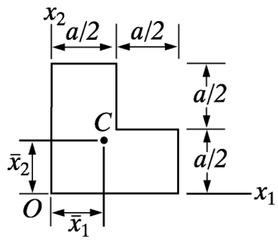


Figure CP3

4. Determine the coordinate $\bar{x}_2$ of the centroid of the surface shown in the Fig. CP4.

Ans. $\bar{x}_2 = 41.25$ mm

5. Determine the coordinates $\bar{x}_1$ and $\bar{x}_2$ of the centroid of the area shown in the Fig. CP5.

Ans. $\bar{x}_1 = 55.61$ mm, $\bar{x}_2 = 47.89$ mm

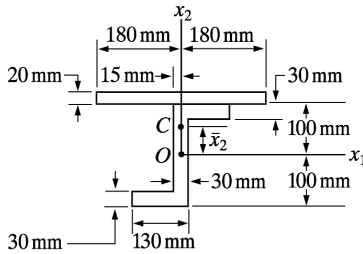


Figure CP4

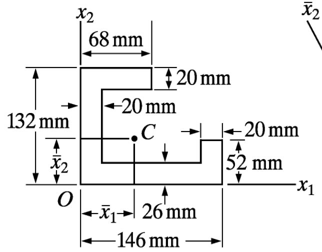


Figure CP5

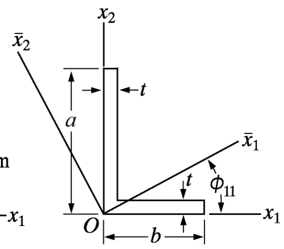


Figure CP6

6. Calculate the moments of inertia I'_{11} and the product of inertia I'_{12} for the angle section shown in Fig. CP6. (Assume a = 160 mm, b = 120 mm, t = 10 mm and ϕ_{11}= 30°).

Ans. $I'_{11} = 1085.61(10^4)\ \text{mm}^4$ $I'_{22} = 864.37(10^4)\ \text{mm}^4$ $I'_{12} = 391.1(10^4)\ \text{mm}^4$

7. Consider the angle section shown in Fig. CP6 with a = 240 mm, b = 160 mm and t = 40 mm. Using Mohr's circle, determine the orientation of the principal axes through the origin O and the magnitudes I_1 and I_2 of the principal moments of inertia with respect to axes with origin point O.

Ans. $I_1 = 151.36(10^6)\ \text{mm}^4$ $I_2 = 47.36(10^6)\ \text{mm}^4$ $\tilde{\phi}_{11}$ = clockwise

8. to 10. Using Mohr's circle determine the orientation of the principal centroidal axes and the principal moments of inertia of the surface shown in Fig. CP3. Repeat with the surfaces of Figs. CP4 and CP5.

Ans. 8 $I_1 = 5a^4/64$, $I_2 = 7a^4/192$, $\tilde{\phi}_{11}$ = 45° counterclockwise

Ans. 9 $I_1 = 147.05(10^6)\text{mm}^4$, $I_2 = 80.01(10^6)\text{mm}^4$, $\tilde{\phi}_{11}$ = 40.05°clockwise

Ans. 10 $I_1 = 217.22(10^5)\text{mm}^4$, $I_2 = 86.46(10^5)\text{mm}^4$, $\tilde{\phi}_{12}$ = 45.07°clockwise

Appendix D

Method of Finite Differences†

D.1 Introduction

Problems involving the determination of a function of one or more variables satisfying a differential equation which cannot be integrated in closed form are solved using an approximate method. The most commonly used approximate methods are numerical as, for example, the finite difference and the finite element methods. These methods lead to numerical values of the unknown function of the problem at certain pivotal points of its domain.

In the method of finite differences the derivatives in the governing differential equation and in the boundary conditions of a problem are replaced by approximate expressions involving the values of the unknown function at certain pivotal points. In this Appendix we establish such expressions for a function $f(x)$ and $f(x_1, x_2)$ and we investigate the order of their error.

D.2 Approximations of the Derivatives of a Function $f(x)$ Using Central Differences

Consider a smooth function $f(x)$ and denote by $f_k (k = 1, 2, 3, \ldots, K)$ its values at the equally spaced pivotal points $x^k (k = 0, 1, 2, \ldots, K)$ and by h the distance between two adjacent pivotal points. Moreover, denote by $f_{k+1/2}$ and $f_{k-1/2}$ the values of $f(x)$ at the middle points of the intervals from x^k to x^{k+1} and from x^{k-1} to x^k, respectively. The first central difference of $f(x)$ at point x^k is denoted by δf_k and is defined as

$$\delta f_k = f(x^k + \frac{h}{2}) - f(x^k - \frac{h}{2}) = f(x^{k+\frac{1}{2}}) - f(x^{k-\frac{1}{2}}) = f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}} \tag{D.1a}$$

The second central difference at $x = x^k$ is the central difference of the first central difference. That is

$$\delta^2 f_k = \delta(\delta f_k) = \delta f_{k+\frac{1}{2}} - \delta f_{k-\frac{1}{2}} = (f_{k+1} - f_k) - (f_k - f_{k-1}) = f_{k+1} - 2f_k + f_{k-1} \tag{D.1b}$$

Similarly,

---

† For a more detailed presentation see Salvadore, M. and Baron, M., *Numerical Methods in Engineering,* Prentice-Hall, New York, 1952.

$$\delta^3 f_k = f_{k+\frac{3}{2}} - 3f_{k+\frac{1}{2}} + 3f_{k-\frac{1}{2}} - f_{k-\frac{3}{2}} \tag{D.1c}$$
$$\delta^4 f_k = f_{k+2} - 4f_{k+1} + 6f_k - 4f_{k-1} + f_{k-2}$$

From relations (D.1) we may deduce that the coefficients of the pivotal values of a function in the expression for its n^{th} central difference are the coefficients of the binomial expansion of $(a - b)^n$. Moreover, from relations (D.1) we see that the expressions for the odd central differences of a function $f(x)$ at point x^k involve the values of this function at the intermediate points $x^k + h/2$, $x^k - h/2$, $x^k + 3h/2$, $x^k - 3h/2$, etc. which are not known. In order to obtain for an odd central difference of a function $f(x)$ at point x^k, an expression which involves only values of this function at pivotal points, this central difference is approximated by the average of the central differences of the function at points $x^k + h/2$ and $x^k - h/2$. For example, the average first central difference of a function $f(x)$ at point x^k is denoted by $\mu\delta f_k$ and referring to relation (D.1a), it is equal to

$$\mu\delta f_k = \frac{1}{2}(\delta f_{k+\frac{1}{2}} + \delta f_{k-\frac{1}{2}}) \approx \frac{1}{2}(f_{k+1} - f_{k-1}) \tag{D.2}$$

In what follows we denote the n^{th} derivative of a function $f(x)$ by the n^{th} power of the symbol D. That is,

$$\left.\frac{d^n f}{dx^n}\right|_{x=x^k} = D^n f(x^k) = D^n f_k \tag{D.3}$$

Expanding the function $f(x + h)$, in Taylor series about point x^k using relation (D.3) and the series expansion for e^{hD}, we get

$$f_{k+1} = f(x^k + h) = f(x^k) + \frac{h}{1!}\left.\frac{df}{dx}\right|_{x=x^k} + \frac{h^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=x^k} + \frac{h^3}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=x^k} + \frac{h^4}{4!}\left.\frac{d^4 f}{dx^4}\right|_{x=x^k} + \cdots$$
$$= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \frac{h^4D^4}{4!} + \cdots\right) f_k = e^{hD} f(x^k) = e^{hD} f_k \tag{D.4}$$

Similarly, we can show that

$$f_{k-1} = f(x^k - h) = e^{-hD} f(x^k) = e^{-hD} f_k \tag{D.5}$$

Using relations (D.4) and (D.5), relation (D.2) can be rewritten as

$$\mu\delta f_k \approx \frac{1}{2}\left(e^{hD} f_k - e^{-hD} f_k\right) = \sin h(hD) f_k \tag{D.6}$$

or using the Taylor series expansion of $\sin h^{-1}(\mu\delta)$, we have

$$hDf_k = \left[\sin h^{-1}(\mu\delta)\right] f_k = \left[\mu\delta - \frac{(\mu\delta)^3}{6} + \frac{3(\mu\delta)^5}{40} + \cdots\right] f_k \tag{D.7}$$

The operators δ and μ are related. Their relation may be established by computing the average difference of μf_k. That is, referring to relations (D.2) and (D.1b), we have

$$\mu^2 f_k = \frac{1}{2}(\mu f_{k+\frac{1}{2}} + \mu f_{k-\frac{1}{2}}) = \frac{1}{2}\left[\frac{1}{2}(f_{k+1} + f_k) + \frac{1}{2}(f_k + f_{k-1})\right] = \frac{1}{4}(f_{k+1} + 2f_k + f_{k-1}) = \left(\frac{\delta^2}{4} + 1\right) f_k \tag{D.8}$$

or

$$\mu^2 = \frac{\delta^2}{4} + 1 \tag{D.9}$$

It can be shown that the central difference operators δ, $\mu\delta$, δ^2, $\mu\delta^3$, δ^4, etc. and the differential operator D satisfy the laws of algebra and consequently they can be treated as if they were numbers. Thus, using relation (D.9) to eliminate even powers of μ, relation (D.7) may be rewritten as

$$hDf_k = \mu\left(\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} - \cdots\right) f_k \tag{D.10a}$$

Taking powers of hD and eliminating even powers of μ using relation (D.9), we obtain

$$h^2D^2 f_k = (hD)(hD)f_k = \left(\delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} + \cdots\right) f_k \tag{D.10b}$$

$$h^3D^3 f_k = (hD)(h^2D^2)f_k = \mu\left(\delta^3 - \frac{\delta^5}{4} + \frac{7\delta^7}{120} - \cdots\right) f_k \tag{D.10c}$$

$$h^4D^4 f_k = (hD)^2(h^2D^2)f_k = \left(\delta^4 - \frac{\delta^6}{6} + \frac{7\delta^8}{240} - \cdots\right) f_k \tag{D.10d}$$

	k-3	k-2	k-1	k	k+1	k+2	k+3
$2hD$ =			-1	0	1		
h^2D^2 =			-1	-2	1		
$2h^3D^3$ =		-1	2	0	-2	1	
h^4D^4 =		1	-4	6	-4	1	

$[e=O(h^2)]$

(a)

	k-3	k-2	k-1	k	k+1	k+2	k+3
$12hD$ =		1	-8	0	8	-1	
$12h^2D^2$=		-1	16	-30	16	-1	
$8h^3D^3$ =	1	-8	13	0	-13	8	-1
$6h^4D^4$ =	-1	12	-39	56	-39	12	-1

$[e=O(h^4)]$

(b)

Figure D.1 Approximations of the most commonly used derivatives of a function $f(x)$ using central differences.

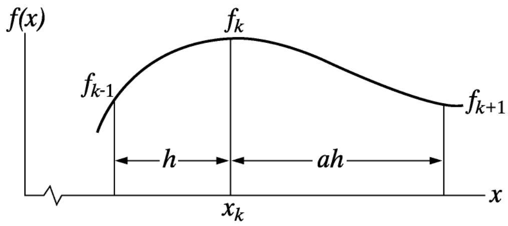


Figure D.2 Unequal spacing of pivotal points.

Approximations for the most commonly used ordinary derivatives using central differences are given in Fig. D.1. Those obtained by retaining only the first term in relations (D.10) are given in Fig. D.1a while those obtained by retaining the first two terms in relations (D.10) are given in Fig. D.1b.

The order of the error in the central difference approximations for the derivatives of a function of one variable can be established by rederiving these approximations using Taylor series. The Taylor series of a function $f(x + h)$ about point x^k is given by

$$f(x^k + h) = f(x^k) + h\left.\frac{df}{dx}\right|_{x=x^k} + \frac{h^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=x^k} + \frac{h^3}{3!}\left.\frac{d^3f}{dx^3}\right|_{x=x^k} + \cdots \tag{D.11}$$

Referring to Fig. D.2 and applying relation (D.11), we obtain

$$f_{k+1} = f(x^k + \alpha h) = f_k + \alpha h\left.\frac{df}{dx}\right|_{x=x^k} + \frac{\alpha^2 h^2}{2}\left.\frac{d^2f}{dx^2}\right|_{x=x^k} + \frac{\alpha^3 h^3}{6}\left.\frac{d^3f}{dx^3}\right|_{x=x^k} + \frac{\alpha^4 h^4}{24}\left.\frac{d^4f}{dx^4}\right|_{x=x^k} + \cdots \tag{D.12a}$$

$$f_{k-1} = f(x^k - h) = f_k - h\left.\frac{df}{dx}\right|_{x=x^k} + \frac{h^2}{2}\left.\frac{d^2f}{dx^2}\right|_{x=x^k} + \frac{h^3}{6}\left.\frac{d^3f}{dx^3}\right|_{x=x^k} + \frac{h^4}{24}\left.\frac{d^4f}{dx^4}\right|_{x=x^k} + \cdots \tag{D.12b}$$

Subtracting relation (D.12b) from (D.12a), we get

$$\left.\frac{df}{dx}\right|_{x=x^k} = \frac{f_{k+1} - f_{k-1}}{h(1 + \alpha)} + (\text{error})_{1c} \tag{D.13}$$

where

$$(\text{error})_{1c} = \frac{h(1-\alpha)}{2}\left.\frac{d^2f}{dx^2}\right|_{x=x^k} - \frac{h^2}{6}\left(\frac{1+\alpha^3}{1+\alpha}\right)\left.\frac{d^3f}{dx^3}\right|_{x=x^k} + \frac{h^3(1-\alpha)(1+\alpha^2)}{24}\left.\frac{d^4f}{dx^4}\right|_{x=x^k} + \cdots \tag{D.14}$$

From relations (D.13) and (D.14) we see that the error in the approximation for the first derivative of a function using central differences approaches zero as fast as h if $\alpha \neq 1$ and as fast as h^2 if $\alpha = 1$; that is, if the pivotal points are evenly spaced.

Eliminating df/dx between relations (D.12), we obtain

$$\left.\frac{d^2f}{dx^2}\right|_x = \frac{2}{\alpha h^2(1 + \alpha)}[\alpha f_{k+1} - (1 + \alpha)f_k + f_{k-1}] + (\text{error})_{2c} \tag{D.15}$$

where

$$(\text{error})_{2c} = \frac{(1-\alpha)h}{3}\frac{d^3f}{dx^3}\bigg|_{x=x^k} - \frac{(1+\alpha^3)h^2}{12(1+\alpha)}\frac{d^4f}{dx^4}\bigg|_{x=x^k} + \cdots \tag{D.16}$$

From relations (D.15) and (D.16), we see that the error in the approximation for the second derivative of a function $f(x)$ using central differences approaches zero as fast as h if $\alpha \neq 1$ and as fast as h^2 if $\alpha = 1$, that is, if the pivotal points are evenly spaced.

D.3 Approximations of the Derivatives of a Function $f(x)$ Using Forward and Backward Differences

Consider a smooth function $f(x)$ and denote by $f_k(k = 0, 1, 2, 3, \ldots, K)$ its values at the equally spaced $K+1$ pivotal points $x_k(k = 0, 1, 2, 3, \ldots, K)$ and by h the distance between two adjacent pivotal points. The first forward difference of $f(x)$ at the pivotal point x^k is denoted by Δf_k and is defined as

$$\Delta f_k = f(x^k + h) - f(x^k) = f(x^{k+1}) - f(x^k) = f_{k+1} - f_k \tag{D.17}$$

while the first backward difference of $f(x)$ is denoted by ∇f_k and it is defined as

$$\nabla f_k = f(x^k) - f(x^k - h) = f(x^k) - f(x^{k-1}) = f_k - f_{k-1} \tag{D.18}$$

Following a procedure similar to that employed in the previous section, we obtain the formulas shown in Figs. D.3 and D.4, respectively, for the approximations of the derivatives of a function $f(x)$ using forward and backward differences.

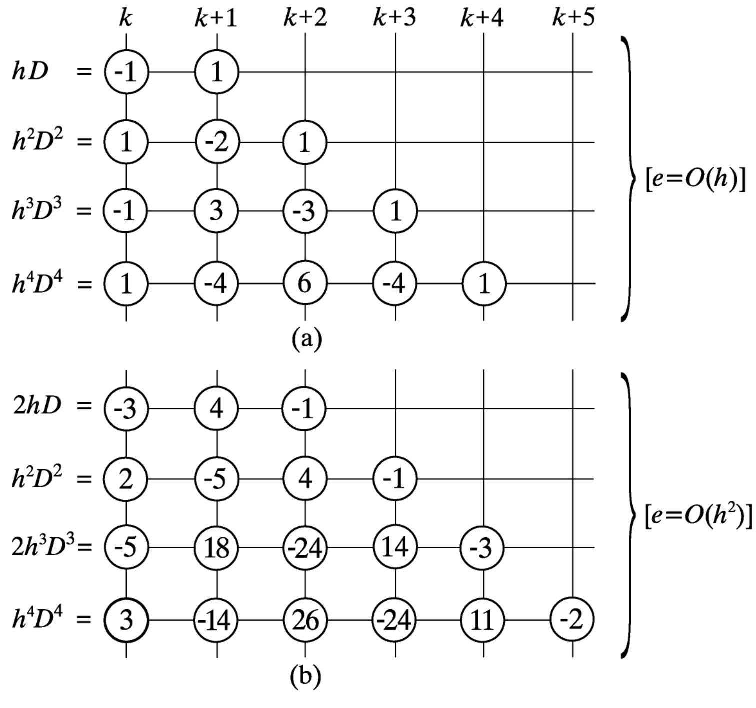


Figure D.3 Approximations of the derivatives of a function $f(x)$ using forward differences.

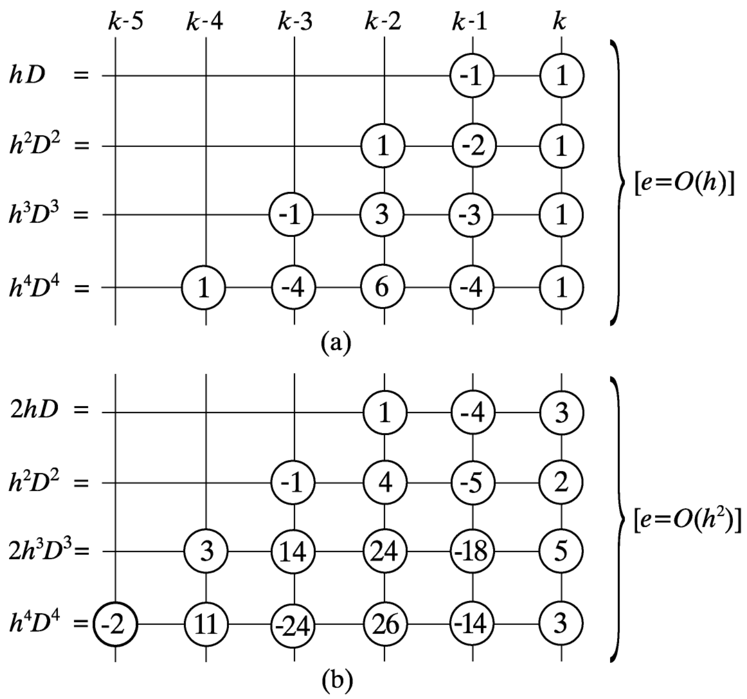


Figure D.4 Approximations of the derivatives of a function $f(x)$ using backward differences.

D.4 Solution of Boundary Value Problems Involving the Determination of a Function of One Variable, Using Finite Differences

In this section we consider boundary value problems which involve the determination of a function of one variable $f(x)$ that satisfies an ordinary differential equation at every point of a line domain and appropriate boundary conditions at its two end points. In order to obtain numerical solutions of such problems, using finite differences to approximate the derivatives involved in their differential equation and their boundary conditions, we adhere to the following steps:

STEP 1 We subdivide the domain of the problem into K intervals (usually equal) by $K + 1$ pivotal points $k = 0, 1, 2, \ldots, K$.

STEP 2 We replace the derivatives appearing in the differential equation whenever possible by their central difference approximations since their accuracy is greater than that of the forward or backward difference approximations. We apply the resulting algebraic equation to each pivotal point of the domain of the problem and we obtain a set of linear algebraic equations involving the values of the function $f(x)$ at the pivotal points of the domain of the problem as well as certain pivotal points outside the domain.

STEP 3 We replace the derivatives appearing in the boundary conditions of the problem by their central, forward or backward difference approximations. When central differences with error approaching zero as fast as h^2 are used to approximate the derivatives in the differential equation and either forward or backward differences are used to approximate the derivatives in the boundary conditions, the error of the latter should also approach zero as fast as h^2. The boundary conditions are used to define the function $f(x)$ at points located outside the domain of the problem, that is, to express values of the function $f(x)$ at points located outside the domain of the problem in terms of its values at points located inside the domain of the problem.

STEP 4 We use the relations obtained in step 3 to eliminate from the algebraic equations obtained in step 2 the values of the function $f(x)$ at points outside the domain of the problem. Moreover, we solve the resulting algebraic equations to obtain the values of the function at the pivotal points of the domain of the problem.

D.5 Approximation of the Partial Derivatives Using Central Differences

The approximations of the partial derivatives of a function of two or more variables by central differences may be obtained directly from Fig. D.1. Consider a function of two variables $f(x_1, x_2)$. We subdivide the domain of this function into a network of identical rectangular subdomains of dimensions h_1 and h_2 called the finite difference mesh (see Fig. D.5). We denote by $f_{m,n}$ the values of the function $f(x_1, x_2)$ at the mesh point whose coordinates are $x_1 = x_1^m$ and $x_2 = x_2^n$. Moreover, we denote by $D_1^k f_{m,n}$ and by $D_2^k f_{m,n}$ the k^{th} partial derivatives of a function $f(x_1, x_2)$ with respect to x_1 and x_2, respectively, at the mesh point whose coordinates are $x_1 = x_1^m$ and $x_2 = x_2^n$. Thus, referring to Fig. D.1 the central differential approximations of the partial derivatives of $f(x_1, x_2)$ are

$$
\begin{aligned}
2h_1 D_1 f_{m,n} &\approx f_{m+1,n} - f_{m-1,n} \\
h_1^2 D_1^2 f_{m,n} &\approx f_{m+1,n} - 2f_{m,n} + f_{m-1,n} \\
2h_1^3 D_1^3 f_{m,n} &\approx f_{m+2,n} - 2f_{m+1,n} + 2f_{m-1,n} - f_{m-2,n} \\
h_1^4 D_1^4 f_{m,n} &\approx f_{m+2,n} - 4f_{m+1,n} + 6f_{m,n} - 4f_{m-1,n} + f_{m-2,n}
\end{aligned}
\tag{D.19a}
$$

$$
\begin{aligned}
2h_2 D_2 f_{m,n} &\approx f_{m,n+1} - f_{m,n-1} \\
h_2^2 D_2^2 f_{m,n} &\approx f_{m,n+1} - 2f_{m,n} + f_{m,n-1} \\
2h_2^3 D_2^3 f_{m,n} &\approx f_{m,n+2} - 2f_{m,n+1} + 2f_{m,n-1} - f_{m,n-2} \\
h_2^4 D_2^4 f_{m,n} &\approx f_{m,n+2} - 4f_{m,n+1} + 6f_{m,n} - 4f_{m,n-1} + f_{m,n+2}
\end{aligned}
\tag{D.19b}
$$

The error in the above approximations approaches zero as fast as h^2.

Referring to relations (D.19) the Laplacian $\nabla^2 f$ of the function $f(x_1, x_2)$ may be approximated as

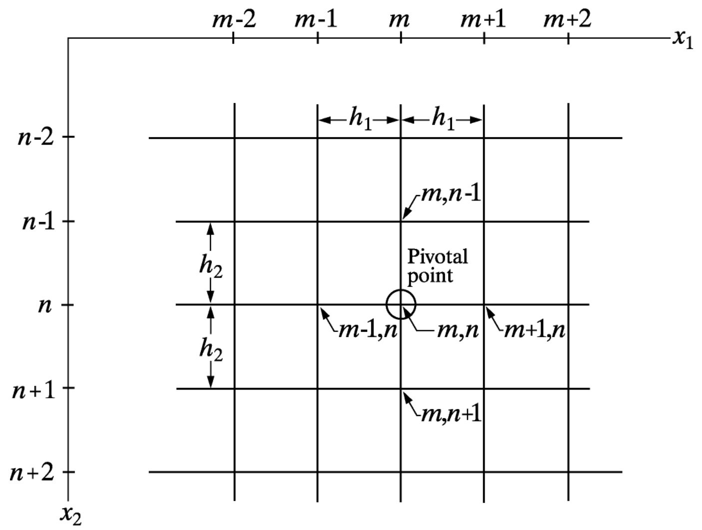


Figure D.5 Constant rectangular finite difference mesh.

$$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial f}{\partial x_2^2} = \frac{1}{h_1^2}(f_{m+1,n} - 2f_{m,n} + f_{m-1,n}) + \frac{1}{h_2^2}(f_{m,n+1} - 2f_{m,n} + f_{m,n-1}) \tag{D.20}$$

For a mesh of equal spacing $h_1 = h_2 = h$ relation (D.20) reduces to

$$\nabla^2 f = \frac{f_{m+1,n} + f_{m-1,n} + f_{m,n-1} + f_{m,n-1} - 4f_{m,n}}{h^2} \tag{D.21}$$

The central difference approximation for the second mixed derivative at point $x_1 = x_1^m$, $x_2 = x_2^n$ of the function $f(x_1, x_2)$ with respect to x_1 and x_2 is equal to the central difference approximation of the first derivative with respect to x_2 of the central difference approximation of the first derivative of $f(x_1, x_2)$ with respect to x_1. That is,

$$\begin{aligned} \frac{\partial^2 f}{\partial x_1 \partial x_2} &= D_{12} f_{m,n} = \frac{1}{2h_2}\left(D_1 f_{m,n+1} - D_1 f_{m,n-1}\right) \\ &= \frac{1}{4h_2 h_1}\left[(f_{m+1,n+1} - f_{m-1,n+1}) - (f_{m+1,n-1} - f_{m-1,n-1})\right] \end{aligned} \tag{D.22}$$

Similarly, the central difference approximation of the fourth mixed derivative of $f(x_1, x_2)$ is equal to the central difference approximation of the second derivative with respect to x_2 of the central difference approximation of the second derivative of $f(x_1, x_2)$ with respect to x_1. That is,

$$\begin{aligned} \frac{\partial^4 f}{\partial x_1^2 \partial x_2^2} &\approx \frac{1}{h_2^2}\left(D_1^2 f_{m,n+1} - 2D_1^2 f_{m,n} + D_1^2 f_{m,n-1}\right) \\ &\approx \frac{1}{h_2^2 h_1^2}[(f_{m+1,n+1} - 2f_{m,n+1} + f_{m-1,n+1}) - 2(f_{m+1,n} - 2f_{m,n} + f_{m-1,n}) \end{aligned}$$

$$
\begin{aligned}
&+ (f_{m+1,n-1} - 2f_{m,n-1} + f_{m-1,n-1}] \\
&= \frac{1}{h_2^2 h_1^2}(f_{m+1,n+1} - 2f_{m,n+1} + f_{m-1,n+1} - 2f_{m+1,n} + 4f_{m,n} - 2f_{m-1,n} \\
&+ f_{m+1,n-1} - 2f_{m,n-1} + f_{m-1,n-1})
\end{aligned}
\tag{D.23}
$$

Finally, for a mesh of equal spacing $h_1 = h_2 = h$ referring to relation (D.20), the function $\nabla^4 f$ may be approximated as

$$
\begin{aligned}
\nabla^4 f &= \frac{\partial^4 f}{\partial x_1^4} + 2\frac{\partial^4 f}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 f}{\partial x_2^4} = \nabla^2(\nabla^2 f) \approx \frac{1}{h^2}(\nabla^2 f_{m,n+1} + \nabla^2 f_{m,n-1} - 4\nabla^2 f_{m,n} \\
&\qquad + \nabla^2 f_{m+1,n} + \nabla^2 f_{m-1,n} \\
&= \frac{1}{h^4}(f_{m+1,n+1} + f_{m-1,n+1} - 4f_{m,n+1} + f_{m,n+2} + f_{m,n}) \\
&+ \frac{1}{h^4}(f_{m+1,n-1} + f_{m-1,n-1} - 4f_{m,n-1} + f_{m,n} + f_{m,n-2}) \\
&- \frac{4}{h^4}(f_{m+1,n} + f_{m-1,n} - 4f_{m,n} + f_{m,n+1} + f_{m,n-1}) \\
&+ \frac{1}{h^4}(f_{m+2,n} + f_{m,n} - 4f_{m+1,n} + f_{m+1,n+1} + f_{m+1,n-1}) \\
&+ \frac{1}{h^4}(f_{m,n} + f_{m-2,n} - 4f_{m-1,n} + f_{m-1,n+1} + f_{m-1,n-1}) \\
&= \frac{1}{h^4}(f_{m,n+2} + f_{m,n-2} + f_{m+2,n} + f_{m-2,n}) \\
&+ \frac{2}{h^4}(f_{m+1,n+1} + f_{m-1,n+1} + f_{m+1,n-1} + f_{m-1,n-1}) \\
&- \frac{8}{h^4}(f_{m,n+1} + f_{m,n-1} + f_{m+1,n} + f_{m-1,n}) + 20f_{m,n}
\end{aligned}
\tag{D.24}
$$

The approximations of the operators ∇^2 and ∇^4, for a square mesh, are shown in Fig. D.6a and D.6b.

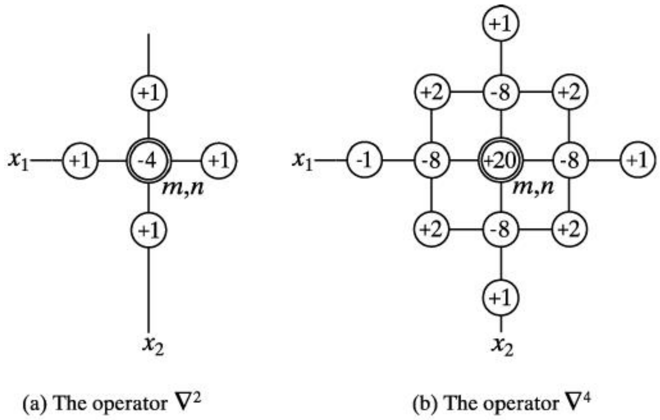


Figure D.6 Approximations to the operators ∇^2 and ∇^4 for a square mesh.

D.6 Numerical Integration

In this section we derive approximate formulas for single and double numerical integration. The formulas for single integration are based on the interpretation of the definite integral $\int_a^b f(x)dx_1$ as the area under the curve $x_2 = f(x_1)$ between the coordinates $x_1 = a$ and $x_1 = b$. The formulas for double integration are based on the interpretation of the definite integral $\int_c^d \left[\int_a^b f(x_1, x_2)dx_1\right]dx_2$ as the volume under the surface $x_3 = f(x_1, x_2)$ between the lines $x_1 = a$, $x_1 = b$, $x_2 = c$, and $x_2 = d$.

D.6.1 Numerical Integration of Single Integrals

Consider the definite integral

$$I = \int_a^b f(x_1)dx_1 \tag{D.25}$$

extended over the straight line from $x_1 = a$ to $x_1 = b$. In order to evaluate this integral numerically referring to Fig D.7 we subdivide the interval from point a to b into m segments of equal length h by $m + 1$ points $x_1^{(i)}(i = 0, 1, \ldots, m)$, where $x_1^{(0)} = a$ and $x_1^{(m)} = b$. We assume that at these points, the values f_o, $f_1, \ldots, f_m$ of the function $f(x_1)$ can be established. The best known formula for numerical integration of the integral $\int_a^b f(x_1)\,dx_1$ is Simpson's rule and is derived in what follows.

The function $f(x_1)$ between any three points $x_1^{(2i)}$, $x_1^{(2i+1)}$ and $x_1^{(2i+2)}[i = 0, 1, 2, \ldots,$

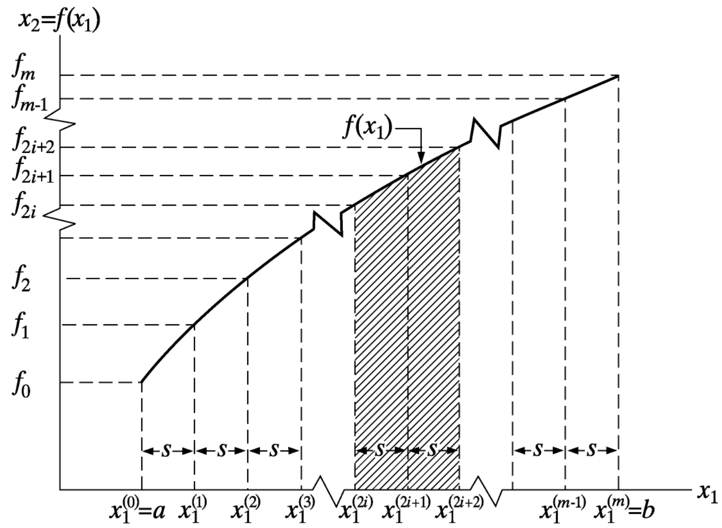


Figure D.7 Numerical integration of a function.

$(m - 2)/2]$ where m is an even number, is approximated by a second degree parabola passing through these points. That is, in the interval $x_1^{(2i)} \le x_1 \le x_1^{(2i+2)}$, we have

$$f(x_1) = a_i x_1^2 + b_i x_1 + c_i \tag{D.26}$$

where the coefficients a_i, b_i and c_i are obtained by requiring that $f(x_1)$ assumes the values of f_{2i}, f_{2i+1} and f_{2i+2} at points $x_1^{(2i)}$, $x_1^{(2i+1)}$ and $x_1^{(2i+2)}$, respectively. That is, referring to Fig. D.7, we have

$$\begin{aligned}
f_{2i} &= a_i (x_1^{(2i)})^2 + b_i x_1^{(2i)} + c_i \\
f_{2i+1} &= a_i (x_1^{(2i+1)})^2 + b_i x_1^{(2i+1)} + c_i \\
f_{2i+2} &= a_i (x_1^{(2i+2)})^2 + b_i x_1^{(2i+2)} + c_i
\end{aligned} \tag{D.27}$$

Solving relations (D.27) simultaneously and noting that $h = x_1^{(2i+1)} - x_1^{(2i)} = x_1^{(2i+2)} - x_1^{(2i+1)}$, we get

$$\begin{Bmatrix} a_i \\ b_i \\ c_i \end{Bmatrix} = \frac{1}{2h^2} \begin{bmatrix} 1 & -2 & 1 \\ -(2x_1^{(2i)} + h) & 4(x_1^{(2i)} + h) & -(2x_1^{(2i)} + 3h) \\ x_1^{(2i)}(x_1^{(2i)} + h) & -2x_1^{(2i)}(x_1^{(2i)} + 2h) & (x_1^{(2i)} + 2)(x_1^{(2i)} + h) \end{bmatrix} \begin{Bmatrix} f_{2i+2} \\ f_{2i+1} \\ f_{2i} \end{Bmatrix} \tag{D.28}$$

The area A_{2i} under the curve $x_2 = f(x_1)$ from line $x_1 = x_1^{(2i)}$ to line $x_1 = x_1^{(2i+2)}$ is equal to

$$\begin{aligned}
A_{2i} &= \int_{x_1^{(2i)}}^{x_1^{(2i+2)}} f(x_1)\,dx_1 = \int_{x_1^{(2i)}}^{x_1^{(2i+2)}} (a_i x_1^2 + b_i x_i + c_i)\,dx_1 \\
&= \frac{1}{3} a_i \left[(x_1^{(2i+2)})^3 - (x_1^{(2i)})^3 \right] + \frac{1}{2} b_i \left[(x_1^{(2i+2)})^2 - (x_1^{(2i)})^2 \right] \\
&\quad + c_i (x_1^{(2i+2)} - x_1^{(2i)})
\end{aligned} \tag{D.29}$$

Substituting relations (D.28) into (D.29) and simplifying, we obtain

$$A_{2i} = \int_{x_1^{(2i)}}^{x_1^{(2i+2)}} f(x_1)\,dx_1 = \frac{h}{3}(f_{2i} + 4f_{2i+1} + f_{2i+2}) \tag{D.30}$$

Applying the above formula for $i = 0, 1, 2, \ldots, (m - 2)/2$, we obtain

$$\begin{aligned}
\int_a^b f(x_1)\,dx_1 = \sum_{i=1}^{(m-2)/2} A_{2i} &= \frac{h}{3}(f_o + 4f_1 + 2f_2 + 4f_3 + 2f_4 \\
&\quad + \cdots + 2f_{m-2} + 4f_{m-1} + f_m)
\end{aligned} \tag{D.31}$$

D.6.2 Numerical Integration of Double Integrals

Consider the double integral

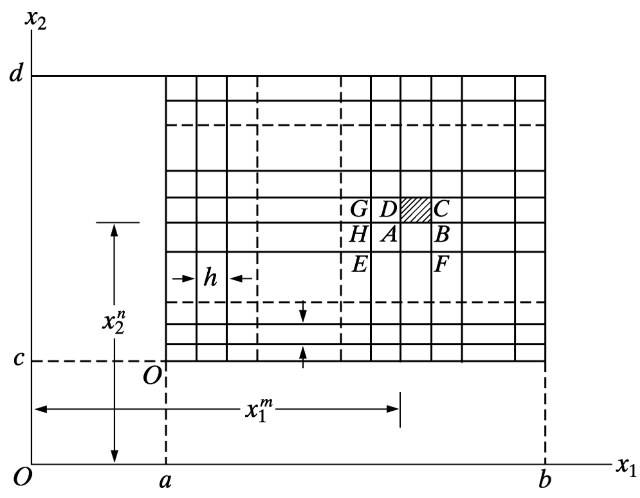


Figure D.8 Domain of the function $f(x_1, x_2)$ subdivided into equal rectangles.

$$I = \int_c^d \left[\int_a^b f(x_1, x_2)\, dx_1 \right] dx_2 \tag{D.32}$$

extended over a rectangle of edges $x_1 = a$, $x_1 = b$, $x_2 = c$, and $x_2 = d$. In order to evaluate this integral numerically, referring to Fig. D.8 we subdivide the domain of the function into a number (S_1 x S_2) of equal rectangles of sides $h_1 = (b - a)/S_1$ and $h_2 = (d - c)/S_2$. We denote by $f_{m,n}$ the value of the function $f(x_1, x_2)$ at the mesh point $x_1 = x_1^m$ and $x_2 = x_2^n$.

D.6.3 Simpson's Rule

The value $\Pi_{m,n}$ of the integral (D.32) extended over four rectangles (see rectangles *EFBCDGHE* in Fig. D8) meeting at point x_1^m, x_2^n is obtained by means of two successive applications of Simpson's rule in the x_1 and x_2 directions. Thus, referring to relation (D.31) and Fig. D.8 we have

$$\begin{aligned}
\Pi_{m,n} &= \int_{x_2^{n-1}}^{x_2^{n+1}} \left[\int_{x_1^{m-1}}^{x_1^{m+1}} f(x_1, x_2)\, dx_1 \right] dx_2 \approx \int_{x_2^{n-1}}^{x_2^{n+1}} \frac{h_1}{3} [f_{m-1}(x_2) + 4f_m(x_2) + f_{m+1}(x_2)] dx_2 \\
&\approx \frac{h_1 h_2}{9} [f_{m-1,n-1} + f_{m+1,n-1} + f_{m-1,n+1} + f_{m+1,n+1} \\
&\quad + 4(f_{m,n+1} + f_{m,n-1} + f_{m-1,n} + f_{m+1,n}) + 16 f_{m,n}]
\end{aligned} \tag{D.33}$$

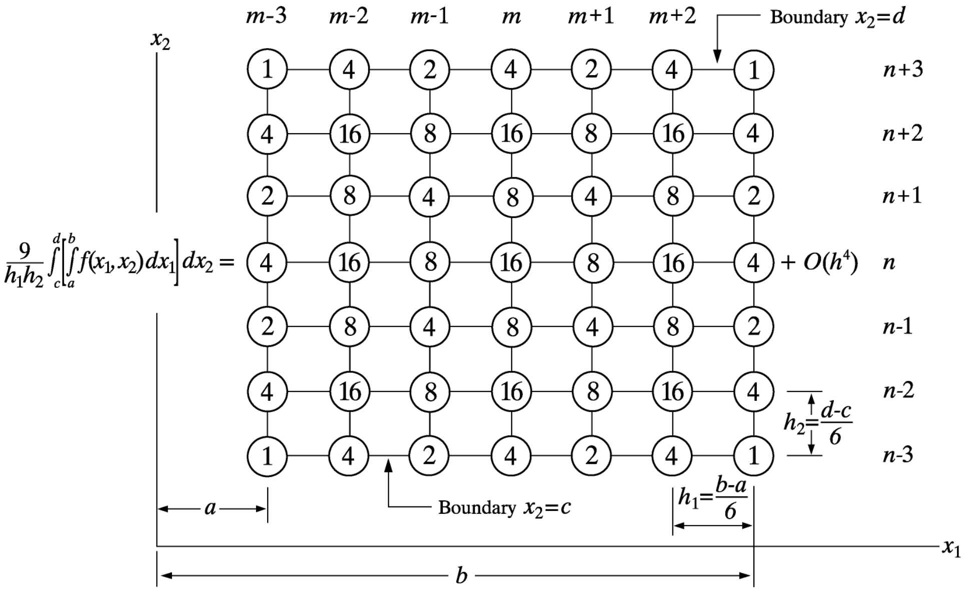


Figure D.9 Schematic representation of the approximation to a surface integral using Simpson's rule.

Adding the value of $\Pi_{m,n}$ over the domain of the function, we obtain the approximate value of the integral I in terms of the values of the function $f(x_1, x_2)$ at the mesh points. It can be shown that the error in this approximation approaches zero as fast as h^4. The value of the integral I is represented schematically in Fig. D.9.

Appendix E

Elements of Calculus of Variations†

E.1 Introduction

In this Appendix we present a brief review of the elements of calculus of variations which are required for the variational formulation of boundary value problems.

The calculus of variations is a mathematical discipline which deals with problems that require the determination of one or more functions of one or more variables which make a given definite integral of a functional of these functions assume stationary values. As an example suppose that we have to design a frictionless chute between points A and B located in a vertical plane $(x_1\ x_2)$ such that a body sliding under the action of its own weight goes from point A to point B in the shortest interval of time. x_1 is the horizontal coordinate. We denote by I^* the time required for the body to slide from point A to point B on a chute whose geometry is specified by the curve $x_2 = x_2(x_1)$. That is,

$$I^* = \int_A^B \frac{ds}{v} = \int_A^B \frac{\sqrt{dx_1^2 + dx_2^2}}{v} = \int_{x_1 = x_1^{(A)}}^{x_1 = x_1^{(B)}} \frac{\sqrt{1 + (x_2')^2}}{v} dx_1$$

where v is the speed of the body, s is the distance along the chute and $x_2' = dx_2/dx_1$. Employing the principle of conservation of energy, denoting by v_o the initial velocity of the body at point $A(x_1^{(A)}, x_2^{(A)})$, and by v the velocity of the body at point (x_2, x_1), we have Therefore,

$$\frac{mv_o^2}{2} - mgx_2^{(A)} = \frac{mv^2}{2} - mgx_2$$

$$v = \sqrt{v_o^2 - 2g(x_2^{(A)} - x_2)}$$

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† For a more detailed and rigorous approach see

1. Dym, C.L. and Shames, I.H., *Solid Mechanics a Variational Approach,* McGraw-Hill, New York, 1973, Chapter 2.
2. Courant, R., *Differential and Integral Calculus,* Vol. II, Blackie and Son, London, 1936.
3. Gelfand, I.M. and Fomin S.V., *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, NJ, 1963.
4. Fox, C., *An Introduction to the Calculus of Variations*, Oxford University Press, London, 1954.

Substituting the above relation into the expression for I^*, we get

$$I^* = \int_{x_1 = x_1^{(A)}}^{x_1 = x_1^{(B)}} \frac{\sqrt{1 + (x_2')^2}}{\sqrt{v_o^2 - 2g(x_2^{(A)} - x_2)}} dx_1$$

Thus, we must find which one from all the curves $x_2 = x_2(x_1)$ between points A and B renders the above integral a minimum. It can be shown that the chute should take the shape of a cycloid. This problem was posed by Johann Bernoulli in 1696 and was solved by Jahann and Jacob Bernoulli, by Sir Isaac Newton and by the French mathematician L'Hospital.

E.2 Variation of a Function

Consider the following single parameter family of slightly differing functions $u^*(x_1)$ which have continuous second derivatives in the internal $a \le x_1 \le b$ and are defined as

$$u^*(x_1) = u(x_1) + \epsilon\eta(x_1) \tag{E.1}$$

where $\eta(x_1)$ is a given function having continuous second order derivatives in the interval $a \le x_1 \le b$ and ε is a small parameter. The functions $u^*(x_1)$ may or may not take identical prescribed values at points $x_1 = a$ and $x_1 = b$ If the functions $u^*(x_1)$ take identical values at these points, the function $\eta(x_1)$ vanishes at points $x_1 = a$ and $x_1 = b$. We call the functions $u^*(x_1)$ the varied functions. We define the *variations of the functions* $u(x_1)$ as

$$\delta u = u^*(x_1) - u(x_1) \equiv \varepsilon\eta(x_1) \tag{E.2}$$

where δ-delta is called *the variational operator*. The variation δu of a function $u(x_1)$ represents an arbitrary infinitesimal change of the function. A family of functions $u^*(x_1)$ is plotted in Fig. E.1; referring to this figure, it is apparent that the variational operator indicates an arbitrary change in the value of $u(x_1)$ at a point. Moreover, the variation δu of a function is different than its differential du. Both represent change of the function $u(x_1)$. However, du is the infinitesimal change of the value of the function $u(x_1)$ caused by the infinitesimal change dx_1 of the independent variable, while δu is an infinitesimal change of $u(x_1)$ which produces a new function $u(x_1) + \delta u$. The independent variable x_1 does not change in the variation process.

On the basis of relation (E.2) and agreeing to use as varied functions the derivatives du^*/dx_1, the variation of the derivative of a function $u(x_1)$ at a point is given as

$$\delta\left(\frac{du}{dx_1}\right) = \frac{du^*}{dx_1} - \frac{du}{dx_1} = \frac{d(\epsilon\eta)}{dx_1} = \frac{d(\delta u)}{dx_1} \tag{E.3}$$

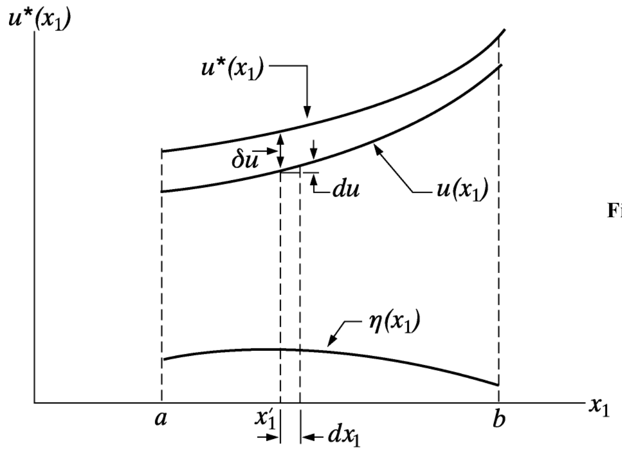

Figure E.1 Variation of a function.

Moreover, if we agree to use as the varied functions the integrals $\int_a^b u*(x_1)\,dx_1$, the variation of the integral of a function $u(x_1)$ is given as

$$\delta \int_a^b u\,dx_1 = \int_a^b u^*\,dx_1 - \int_a^b u\,dx_1 = \int_a^b \epsilon\eta\,dx_1 = \int_a^b \delta u\,dx_1 \tag{E.4}$$

Thus, it is apparent that the variational operator is commutative with the differential and integral operators.

Consider a functional† $F(x_1, u, u', u'')$ which is a continuously differentiable function of the variable x_1, the dependable variable $u(x_1)$ and its derivatives u' and u''. The variation of F is defined as

$$\delta F = F^*(x_1,\ u + \delta u,\ u' + \delta u',\ u'' + \delta u'') - F(x_1,\ u,\ u',\ u'') \tag{E.5}$$

Expanding $F^*(x_1,\ u + \delta u,\ u' + \delta u',\ u'' + \delta u'')$ in a Taylor series†† about u, u', and u'' we obtain

† A functional is a function of a function whose values are dependent on the function used in the functional.

†† Consider the function $F(x,y,z)$ pocessing partial derivatives of third order at a point (x_o,y_o,z_o) with respect to all combinations of the variables x, y and z. Then this function can be approximated in the neighborhood of (x_o,y_o,z_o) by a Taylor series, as

$$\begin{aligned}
F(x,y,z) = {} & F(x_0,y_0,z_0) + (x-x_0)\frac{\partial F}{\partial x}\bigg|_0 + (y-y_0)\frac{\partial F}{\partial y}\bigg|_0 + (z-z_0)\frac{\partial F}{\partial z}\bigg|_0 + \frac{1}{2!}\bigg[(x-x_0)^2\frac{\partial^2 F}{\partial x^2}\bigg|_0 \\
& + (y-y_0)^2\frac{\partial^2 F}{\partial y^2}\bigg|_0 + (z-z_0)^2\frac{\partial^2 F}{\partial z^2}\bigg|_0 + (x-x_0)(y-y_0)\frac{\partial^2 F}{\partial x\partial y}\bigg|_0 + (x-x_0)(z-z_0)\frac{\partial^2 F}{\partial x\partial z}\bigg|_0 + (y-y_0)(z-z_0)\frac{\partial^2 F}{\partial y\partial z}\bigg|_0\bigg] \\
& + \frac{1}{3!}\bigg[(x-x_0)^3\frac{\partial^3 F}{\partial x^3}\bigg|_0 + (y-y_0)^3\frac{\partial^3 F}{\partial y^3}\bigg|_0 + (z-z_0)^3\frac{\partial^3 F}{\partial z^3}\bigg|_0 + (x-x_0)(y-y_0)^2\frac{\partial^3 F}{\partial x\partial y^2}\bigg|_0 + (x-x_0)(z-z_0)^2\frac{\partial^3 F}{\partial x\partial z^2}\bigg|_0 \\
& + (y-y_0)(z-z_0)^2\frac{\partial^3 F}{\partial y\partial z^2}\bigg|_0 + (y-y_0)(x-x_0)^2\frac{\partial^2 F}{\partial x^2\partial y}\bigg|_0 + (z-z_0)(x-x_0)^2\frac{\partial^3 F}{\partial x^2\partial z}\bigg|_0 + (z-z_0)(y-y_0)^2\frac{\partial^3 F}{\partial y^2\partial z}\bigg|_0\bigg]
\end{aligned} \tag{E.6}$$

$$\delta F = F^*(x_1, u+\delta u, u'+\delta u', u''+\delta u'') - F(x_1, u, u', u'') = \delta^{(1)}F + \delta^{(2)}F + \delta^{(3)}F + 0(\epsilon^4)$$

where $0(\epsilon^4)$ represents the terms containing products of four or more variations of u, u' and u''. The quantities $\delta^{(1)}F$, $\delta^{(2)}F$ and $\delta^{(3)}F$ are called the *first, second and third variations of function* $F(x_1, u, u', u'')$, respectively, and are defined as

$$\delta^{(1)}F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u'' \tag{E.7}$$

$$\begin{aligned}\delta^{(2)}F = {} & \left(\frac{\partial^2 F}{\partial^2 u}\right)(\delta u)^2 + \left(\frac{\partial^2 F}{\partial u \partial u'}\right)\delta u \delta u' + \left(\frac{\partial^2 F}{\partial u \partial u''}\right)\delta u \delta u'' + \left(\frac{\partial^2 F}{\partial^2 u'}\right)(\delta u')^2 \\ & + \left(\frac{\partial^2 F}{\partial u' \partial u''}\right)\delta u' \delta u'' + \left(\frac{\partial^2 F}{\partial^2 u''}\right)(\delta u'')^2\end{aligned} \tag{E.8}$$

$$\begin{aligned}\delta^{(3)}F = {} & \left(\frac{\partial^3 F}{\partial^3 u}\right)(\delta u)^3 + \left(\frac{\partial^3 F}{\partial^2 u \partial u'}\right)(\delta u)^2\delta u' + \left(\frac{\partial^3 F}{\partial u \partial^2 u'}\right)\delta u(\delta u')^2 \\ & + \left(\frac{\partial^3 F}{\partial^2 u \partial u''}\right)(\delta u)^2\delta u'' + \left(\frac{\partial^3 F}{\partial u \partial^2 u''}\right)\delta u(\delta u'')^2 + \left(\frac{\partial^3 F}{\partial^3 u'}\right)(\delta u')^3 \\ & + \left(\frac{\partial^3 F}{\partial^2 u' \partial u''}\right)(\delta u')^2\delta u'' + \left(\frac{\partial^3 F}{\partial u' \partial^2 u''}\right)\delta u'(\delta u'')^2 + \left(\frac{\partial^3 F}{\partial^3 u''}\right)(\delta u'')^3\end{aligned} \tag{E.9}$$

If we consider small variations of u, u' and u'' the terms $\delta^{(2)}F$, $\delta^{(3)}F$ and $0(\varepsilon^4)$ in relation (E.6) are small compared to $\delta^{(1)}F$ and thus in this case this relation reduces to

$$\delta F \approx \delta^{(1)}F \tag{E.10}$$

Notice that at a fixed value of $x_1(dx_1 = 0)$, the first variation of the function $F(x_1, u, u', u'')$ has the same form as its total differential dF. That is,

$$dF = \frac{\partial F}{\partial u}du + \frac{\partial F}{\partial u'}du' + \frac{\partial F}{\partial u''}du'' \tag{E.11}$$

This suggests that the variational operator δ acts like a differential operator with respect to the dependent variables. Indeed the rules of variation are analogous to the corresponding rules of differentiation. That is, the following relations are valid:

$$\delta(F^{(1)} \pm F^{(2)}) = \delta F^{(1)} \pm \delta F^{(2)} \tag{E.12a}$$

$$\delta(F^{(1)}F^{(2)}) = \delta F^{(1)}F^{(2)} + F^{(1)}\delta F^{(2)} \tag{E.12b}$$

$$\delta\left(\frac{F^{(1)}}{F^{(2)}}\right) = \frac{\delta F^{(1)}F^{(2)} - F^{(1)}\delta F^{(2)}}{(F^{(2)})^2} \tag{E.12c}$$

$$\delta(F^n) = nF^{n-1}\delta F \tag{E.12d}$$

E.3 The First, Second and Third Variations of a Functional

Consider the following integral of a known functional $F(x_1, u_1, u_1', u_1'')$ of the function $u(x_1)$:

$$I = \int_a^b F(x_1, u, u', u'')dx_1 + (K_1 u)_{x_1=a}^{x_1=b} + (K_2 u'')_{x_1=a}^{x_1=b} \tag{E.13}$$

where K_1 and K_2 are constants. It is apparent that the value of the functional I at any point x_1 depends on the function $u(x_1)$. Using relations (E.2) and (E.6) the variation of I is given as

$$\begin{aligned}\delta I = I^*(u) - I(u) &= \int_a^b F^*(x_1, u+\delta u, u'+\delta u', u''+\delta u'')dx_1 - \int_a^b F(x_1, u, u', u'')dx_1 \\ &+ (K_1\delta u)_{x_1=a}^{x_1=b} + (K_2\delta u')_{x_1=a}^{x_1=b} = \delta^{(1)}I + \delta^{(2)}I + \delta^{(3)}I + 0(\delta^3)\end{aligned} \tag{E.14}$$

where referring to relations (E.6) to (E.9) we have

$$\delta^{(1)}I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial u''}\delta u''\right)dx_1 + (K_1\delta u)_{x_1=a}^{x_1=b} + (K_2\delta u')_{x_1=a}^{x_1=b} \tag{E.15}$$

$$\begin{aligned}\delta^{(2)}I = \int_a^b &\left[\left(\frac{\partial^2 F}{\partial u^2}\right)(\delta u)^2 + \left(\frac{\partial^2 F}{\partial u\partial u'}\right)\delta u\delta u' + \left(\frac{\partial^2 F}{\partial u\partial u''}\right)\delta u\delta u'' + \left(\frac{\partial^2 F}{\partial^2 u'}\right)(\delta u')^2\right. \\ &\left. + \left(\frac{\partial^2 F}{\partial u'\partial u''}\right)\delta u'\delta u'' + \left(\frac{\partial^2 F}{\partial^2 u''}\right)(\delta u'')^2\right]dx_1\end{aligned} \tag{E.16}$$

$$\begin{aligned}\delta^{(3)}I = \int_a^b &\left(\frac{\partial^3 F}{\partial^3 u}\right)(\delta u)^3 + \left(\frac{\partial^3 F}{\partial^2 u\partial u'}\right)(\delta u)^2\delta u' + \left(\frac{\partial^3 F}{\partial u\partial^2 u'}\right)\delta u(\delta u')^2 \\ &+ \left(\frac{\partial^3 F}{\partial^2 u\partial u''}\right)(\delta u)^2\delta u'' + \left(\frac{\partial^3 F}{\partial u\partial^2 u''}\right)\delta u(\delta u'')^2 + \left(\frac{\partial^3 F}{\partial^3 u'}\right)(\delta u')^3 \\ &+ \left(\frac{\partial^3 F}{\partial^2 u'\partial u''}\right)(\delta u')^2\delta u'' + \left(\frac{\partial^3 F}{\partial u'\partial^2 u''}\right)\delta u'(\delta u'')^2 + \left(\frac{\partial^3 F}{\partial^3 u''}\right)(\delta u'')^3\end{aligned} \tag{E.17}$$

$\delta^{(1)}I$, $\delta^{(2)}I$ and $\delta^{(3)}I$ are known as the first, the second and the third variations, respectively, of the functional defined by relation (E.13).

E.4 Stationary Values† of a Functional

It can be shown†† that a function $u(x_1)$ which makes the functional I stationary (maximum, minimum or inflection point) must render its first variation equal to zero. That is,

$$\delta^{(1)}I = 0 \tag{E.18}$$

Moreover, it can be shown that if the function $u(x_1)$ which makes the functional $I(u)$ stationary renders its second variation positive (negative), then this function renders the functional I a local minimum (maximum). Integrating by parts the integral in relation (E.15), we get

$$\begin{aligned}\delta^{(1)}I = 0 = &\int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx_1^2}\left(\frac{\partial F}{\partial u''}\right)\right]\delta u\, dx_1 + \left[\frac{\partial F}{\partial u'} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u''}\right)\right]\delta u\Bigg|_{x_1=a}^{x_1=b} \\ &+ \frac{\partial F}{\partial u''}\delta u''\Bigg|_{x_1=a}^{x_1=b} + (K_1 \delta u)_{x_1=a}^{x_1=b} + (K_2 \delta u')_{x_1=a}^{x_1=b} = 0\end{aligned} \tag{E.19}$$

Since δu is arbitrary, we may choose it so that δu and $\delta u''$ vanish at $x_1 = a$ and $x_1 = b$ while inside the region $a < x_1 < b$ could assume any value. For this choice relation (E.19) reduces to

$$0 = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx_1^2}\left(\frac{\partial F}{\partial u''}\right)\right]\delta u\, dx_1$$

According to the fundamental lemma of the calculus of variations†††, the term in brackets must vanish at any point in the interval $a < x_1 < b$. Consequently,

$$\frac{\partial F}{\partial u} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx_1^2}\left(\frac{\partial F}{\partial u''}\right) = 0 \tag{E.20}$$

This relation is referred to as the Euler or the Euler–Lagrange differential equation for the functional I. It is the necessary condition which $u(x_1)$ must satisfy in order to make the functional I stationary. However, referring to relation (E.19) we see that

† We say that a function has a stationary value at a certain point if the rate of change of the function in every possible direction from this point vanishes.

†† See:

1. Courant, R., *Differential and Integral Calculus,* Vol. II, Blackie and Son, London, 1936.
2. Dym, C.L. and Shames, I.H., *Solid Mechanics a Variational Approach* McGraw-Hill, New York, 1973, Chapter 2.

††† The fundamental lemma of the calculus of variations states: A continuous function $\phi(x_1)$ vanishes identically in the internal $a \le x \le b$ if the integral $\int_a^b \phi(x_1)\,\eta(x_1)\,dx_1$ vanishes for all functions $\eta(x_1)$ which

have continuous second derivatives and vanish at $x_1=a$ and $x_1=b$.
there are additional necessary conditions for making I stationary. That is, a function $u(x_1)$ must be such that at points $x_1 = a$ and $x_1 = b$ satisfy the following condition:

$$\left[\frac{\partial F}{\partial u'} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u''}\right)\right]\delta u\Bigg|_{x_1=a}^{x_1=b} + \frac{\partial F}{\partial u''}\delta u'\Bigg|_{x_1=a}^{x_1=b} + (K_1\delta u)_{x_1=a}^{x_1=b} + (K_2\delta u')_{x_1=a}^{x_1=b} = 0 \tag{E.21}$$

Relation (E.21) is satisfied if one condition from each of the following pairs is satisfied at each end point $x_1 = a$ and $x_1 = b$ of the interval $a < x_1 < b$.

$$\begin{aligned} \delta u = 0 \quad &\text{or} \quad \frac{\partial F}{\partial u'} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u''}\right) = -K_1 \\ \delta u' = 0 \quad &\text{or} \quad \frac{\partial F}{\partial u''} = -K_2 \end{aligned} \tag{E.22}$$

The boundary conditions $\delta u = 0$ and $\delta u' = 0$ are called *essential*, while the remaining two boundary conditions (E.22) are called *natural*.

In case K_2 is equal to zero and the function F does not involve second order derivatives of $u(x_1)$, the partial derivative of the function F with respect to u'' is equal to zero. Thus, in this case the Euler–Lagrange equation (E.20) for the functional I reduces to

$$\frac{\partial F}{\partial u} - \frac{d}{dx_1}\left(\frac{\partial F}{\partial u'}\right) = 0 \tag{E.23}$$

Moreover, relation (E.21) becomes

$$\frac{\partial F}{\partial u'}\delta u\Bigg|_{x_1=a}^{x_1=b} = -(K_1\delta u)_{x_1=a}^{x_1=b} \tag{E.24}$$

Relation (E.24) is satisfied if

$$\delta u = 0 \quad \text{or} \quad \frac{\partial F}{\partial u'}\Bigg|_{x_1=a}^{x_1=b} = -K_1 \tag{E.25}$$

The boundary condition $\delta u = 0$ is called essential while the boundary condition $\partial F/\partial u' = -K_1$ is called natural.

We have started with the functional I and we have established the differential equation (E.20) and the boundary conditions (E.22) that must be satisfied by the function $u(x_1)$ which renders this functional stationary. It can be shown that the function $u(x_1)$ which satisfies the Euler–Lagrange differential equation (E.20) and the boundary conditions (E.22) renders the functional stationary. We can begin with the strong form of a boundary value problem and we can establish the functional which has the differential equation of this boundary value problem as its Euler–Lagrange equation. However, it is not always possible to find a functional whose Euler–Lagrange equation is the differential equation of a given boundary value problem. Only when the differential equation of a boundary value problem satisfies certain conditions, such as a functional exists[†]. Nevertheless, if for a given boundary value problem such a functional exists, we can have an alternative equivalent formulation of this boundary value problem. That is, its solution could be sought either as the function which satisfies the differential equation and the given essential and natural boundary conditions or as the function which satisfies the given essential boundary conditions and renders the functional described previously stationary. The latter formulation of a boundary value problem is called *variational* and it is suited for establishing approximate solutions of such a boundary value problem (see Section 13.17.2).

† Shame, I.H. and Dym, C.L., *Energy and Finite Element Methods in Structural Mechanics*, McGraw-Hill, New York, 1985, Section 3.11, p. 151.

Appendix F

Derivation of the Expression for the Plane Stress Function $X(x_1, x_2, x_3)$

Consider a prismatic body in a state of plane stress. That is, in general the components of stress acting on its particles have the form indicated by relations (7.39). The body is subjected to body forces which are obtained from a potential function $V(x_2, x_3)$ on the basis of the following relations:

$$B_2(x_2, x_3) = -\frac{\partial V}{\partial x_2} \qquad B_3(x_2, x_3) = -\frac{\partial V}{\partial x_3}$$

Substituting these relations into the equations of equilibrium (2.69) and taking into account relations (7.39) we obtain

$$\frac{\partial V}{\partial x_1} = 0$$

$$\frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} - \frac{\partial V}{\partial x_2} = 0 \tag{F.1}$$

$$\frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} - \frac{\partial V}{\partial x_3} = 0$$

From the first of the above relations we see that in order to maintain a state of plane stress in a prismatic body the potential V and, consequently, the components of specific body force must be a function only of x_2 and x_3. From the second of relations (F.1) we can conclude that there exists a function $X_1(x_1, x_2, x_3)$ such that

$$\tau_{22} = \frac{\partial X_1(x_1, x_2, x_3)}{\partial x_3} + V \qquad \tau_{23} = -\frac{\partial X_1(x_1, x_2, x_3)}{\partial x_2} \tag{F.2}$$

From the third of relations (F.1) we can conclude that there exists a function $X_2(x_1, x_2, x_3)$ so that

$$\tau_{33} = \frac{\partial X_2}{\partial x_2} + V \qquad \tau_{23} = -\frac{\partial X_2}{\partial x_3} \tag{F.3}$$

From the second of relations (F.2) and (F.3) we obtain

$$\frac{\partial X_1}{\partial x_2} = \frac{\partial X_2}{\partial x_3} \tag{F.4}$$

Thus, there exists a function $X(x_1, x_2, x_3)$ such that

$$X_1 = \frac{\partial X}{\partial x_3} \qquad X_2 = \frac{\partial X}{\partial x_2} \tag{F.5}$$

Substituting the above relations into (F.2) and (F.3), we have

$$\begin{aligned} \tau_{22}(x_1, x_2, x_3) &= \frac{\partial^2 X(x_1, x_2, x_3)}{\partial x_3^2} + V \\ \tau_{33}(x_1, x_2, x_3) &= \frac{\partial^2 X(x_1, x_2, x_3)}{\partial x_2^2} + V \\ \tau_{23}(x_1, x_2, x_3) &= -\frac{\partial^2 X(x_1, x_2, x_3)}{\partial x_2\, \partial x_3} \end{aligned} \tag{F.6}$$

In what follows we establish the restrictions which must be imposed on the plane stress function $X(x_1, x_2, x_3)$ in order to ensure that when the components of stress are substituted into the stress–strain relations for plain stress (3.52) give components of strain which when substituted into the strain–displacement relations (2.16) the resulting relations can be integrated to give a set of components of displacement. In order to accomplish this it is necessary and sufficient that the components of strain satisfy the equations of compatibility (2.65) at every point of the body. For simply connected bodies the satisfaction of the equations of compatibility by the components of strain is necessary and sufficient to ensure that the components of displacement are single-valued continuous functions of the space coordinates. For multiply connected bodies the satisfaction of the equations of compatibility by the components of strain is necessary and sufficient to ensure the integrability of the strain–displacement relations (2.16). However, the resulting components of displacement may or may not be singled-valued continuous functions of the space coordinates. In order to ensure that the components of displacement are single-valued continuous functions of the space coordinates the components of strain must satisfy in addition to the equations of compatibility certain other relations. Substituting relations (F.6) into the stress–strain relations for plane stress (3.52) we get

$$\begin{aligned} e_{11} &= -\frac{\nu I_1}{E} \qquad e_{22} = \frac{1+\nu}{E}\left(\frac{\partial^2 X}{\partial x_3^2} - \frac{\nu}{1+\nu} I_1 + V\right) \\ e_{33} &= \frac{1+\nu}{E}\left(\frac{\partial^2 X}{\partial x_2^2} - \frac{\nu}{1+\nu} I_1 + V\right) \qquad e_{23} = -\frac{(1+\nu)}{E}\frac{\partial^2 X}{\partial x_2\, \partial x_3} \qquad e_{21} = e_{31} = 0 \end{aligned} \tag{F.7}$$

Where I_1 is the first invariant of stress given by

$$I_1 = \tau_{22} + \tau_{33} = \frac{\partial^2 X}{\partial x_3^2} + \frac{\partial^2 X}{\partial x_2^2} + 2V = \nabla_1^2 X + 2V \tag{F.8}$$

From the above relation, we obtain

$$\nabla_1^2 I_1 = \nabla_1^4 X + 2\nabla_1^2 V \tag{F.9}$$

where $\nabla_1{}^2$ and $\nabla_1{}^4$ are defined by relations (7.16) and (7.18), respectively.

Substituting relations (F.7) into the third of the compatibility equations (2.63) and referring to relation (7.18), we get

$$\nabla_1^4 X - \frac{\nu}{1+\nu}\nabla_1^2 I_1 + \nabla_1^2 V = 0 \tag{F.10}$$

Using relation (F.9) to eliminate I_1 from the above and using the above to eliminate X from relation (F.9), we have

$$\nabla_1^4 X = -(1-\nu)\nabla_1^2 V \tag{F.11a}$$

$$\nabla_1^2 I_1 = \frac{\partial^2 I_1}{\partial x_2^2} + \frac{\partial^2 I_1}{\partial x_3^2} = (1+\nu)\nabla_1^2 V \tag{F.11b}$$

Notice, that X is a function of x_1, x_2 and x_3. Consequently, if ∇^2V is equal to zero, X is not a biharmonic function.

Substituting relations (F.7) into the fifth and sixth of the compatibility relations (2.63) and taking into account that V is not a function of x_1, we obtain

$$\frac{\partial^4 X}{\partial x_3^3 \partial x_1} - \left(\frac{\nu}{1+\nu}\right)\frac{\partial^2 I_1}{\partial x_3 \partial x_1} = -\frac{\partial^4 X}{\partial x_2^2 \partial x_3 \partial x_1} \tag{F.12a}$$

$$\frac{\partial^4 X}{\partial x_2^3 \partial x_1} - \left(\frac{\nu}{1+\nu}\right)\frac{\partial^2 I_1}{\partial x_2 \partial x_1} = -\frac{\partial^4 X}{\partial x_1 \partial x_3^2 \partial x_2} \tag{F.12b}$$

We find $(\partial^2 X/\partial^2 x_3 + \partial^2 X/\partial^2 x_2)$ from relation (F.8) and we substitute it into the above relations. Taking into account that V is not a function of x_1, we have†

$$\frac{\partial^2 I_1}{\partial x_3 \partial x_1} = 0 \qquad \frac{\partial^2 I_1}{\partial x_2 \partial x_1} = 0 \tag{F.13}$$

From the above relations we see that $\partial I_1/\partial x_1$ must be only a function of x_1. Therefore, the function $I(x_1, x_2, x_3)$ must have the following form:

$$I_1 = I_0(x_2, x_3) + F(x_1) \tag{F.14}$$

Substituting relation (F.14) into (F.11b), we get

$$\nabla_1^2 I_0 = (1+\nu)\nabla_1^2 V \tag{F.15}$$

Substituting relations (F.7) into the first and second of the compatibility equations (2.63),

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† If τ_{22}, τ_{33}, τ_{23} were assumed not to be functions of x_1, X and I would be functions only of x_2 and x_3 and, consequently, the second and third compatibility equations would have yielded

$$\frac{\partial^2 I_1}{\partial x_2^2} = \frac{\partial^2 I_1}{\partial x_3^2} = 0$$

This indicates that $I_1 = Ax_2 + Bx_3 + C$. Thus, referring to the first of relations (F.7) it is apparent that the component of strain e_{11} would have been a linear function of x_2 and x_3. There are not, however, many practical problems having solutions which satisfy this requirement.

we get

$$(1+\nu)\frac{\partial^4 X}{\partial x_3^2 \partial x_1^2} - \nu\left(\frac{\partial^2 I_1}{\partial x_2^2} + \frac{\partial^2 I_1}{\partial x_1^2}\right) = 0$$

$$(1+\nu)\frac{\partial^4 X}{\partial x_2^2 \partial x_1^2} - \nu\left(\frac{\partial^2 I_1}{\partial x_3^2} + \frac{\partial^2 I_1}{\partial x_1^2}\right) = 0 \qquad \text{(F.16)}$$

Using relations (F.11b), relation (F.16) can be written as

$$\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial^2 X}{\partial x_1^2} + \frac{\nu}{1+\nu} I_1\right) = \frac{\nu}{1+\nu}\frac{\partial^2 I_1}{\partial x_1^2} + \nu \nabla_1^2 V$$

$$\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 X}{\partial x_1^2} + \frac{\nu}{1+\nu} I_1\right) = \frac{\nu}{1+\nu}\frac{\partial^2 I_1}{\partial x_1^2} + \nu \nabla_1^2 V \qquad \text{(F.17)}$$

Adding the above relations, we have

$$\nabla_1^2\left(\frac{\partial^2 X}{\partial x_1^2}\right) + \frac{\nu}{1+\nu}\nabla_1^2 I_1 = \left(\frac{2\nu}{1+\nu}\right)\frac{\partial^2 I_1}{\partial x_1^2} + 2\nu \nabla_1^2 V \qquad \text{(F.18)}$$

Differentiating equation (F.8), we obtain

$$\frac{\partial^2}{\partial x_1^2}\left(\nabla_1^2 X\right) = \frac{\partial^2 I_1}{\partial x_1^2} \qquad \text{(F.19)}$$

Substituting relations (F.19) into (F.18), we get

$$\frac{\partial^2 I_1}{\partial x_1^2} + \frac{\nu}{1-\nu}\nabla_1^2 I_1 = \frac{2\nu(1+\nu)\nabla_1^2 V}{1-\nu} \qquad \text{(F.20)}$$

Substituting relation (F.11b) into (F.20), we have

$$\frac{\partial^2 I_1}{\partial x_1^2} = \frac{\nu(1+\nu)\nabla_1^2 V}{1-\nu} \qquad \text{(F.21)}$$

Substituting relation (F.14) into the above relation, we obtain

$$\frac{\partial^2 F(x_1)}{\partial x_1^2} = \frac{\nu(1+\nu)}{1-\nu}\nabla_1^2 V = C \qquad \text{(F.22)}$$

The left side of the above relation is a function of x_1 only while the right side is a function of x_2 and x_3 only. Consequently, both sides must be equal to a constant C. Thus for a state of plane stress the body force potential $V(x_2, x_3)$ must satisfy the following relation:

$$\nabla_1{}^2 V = \text{constant} \qquad \text{(F.23)}$$

Substituting equation (F.21) into (F.17), we get

$$\frac{\partial^2}{\partial x_3^2}\left(\frac{\partial^2 X}{\partial x_1^2} + \frac{\nu}{1+\nu}I_1\right) = \frac{\nu\nabla_1^2 V}{1-\nu} = \text{const}$$
$$\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 X}{\partial x_1^2} + \frac{\nu}{1+\nu}I_1\right) = \frac{\nu\nabla_1^2 V}{1-\nu} = \text{const} \tag{F.24}$$

Substituting relations (F.7) into the fourth of the compatibility relations (2.65) we obtain

$$\frac{\partial^2}{\partial x_2\,\partial x_3}\left[\frac{\partial^2 X}{\partial x_1^2} + \frac{\nu}{1+\nu}I_1\right] = 0 \tag{F.25}$$

Thus, from relations (F.24) and (F.25) we may conclude that

$$\frac{\partial^2 X}{\partial x_1^2} + \frac{\nu}{1+\nu}I_1 = \frac{\nu\,\nabla_1^2 V(x_2^2 + x_3^2)}{2(1-\nu)} + g(x_1, x_2, x_3) \tag{F.26}$$

where the functions $g(x_1, x_2, x_3)$ is an arbitrary function of x_1 and a linear function of x_2 and x_3. Integrating relation (F.26) with respect to x_1 and using (F.14), we obtain

$$X = \int\left(\int g\,dx_1\right)dx_1 + \frac{\nu}{4(1-\nu)}\nabla_1^2 V(x_2^2 + x_3^2)x_1^2 - \frac{\nu}{1+\nu}\left[\frac{I_1 x_1^2}{2} + \int\left[\int F(x_1)\,dx_1\right]dx_1\right]$$
$$+ x_1 Y_1(x_2, x_3) + Y_2(x_2, x_3) \tag{F.27}$$

As it was indicated previously, $g(x_1, x_2, x_3)$ is a linear function of x_2 and x_3, whereas, the components of stress are second derivatives of X, with respect to x_2 or x_3. Thus the first term on the right-hand side of the above equation may be disregarded inasmuch as it does not affect the components of stress (see relation F.6). Moreover, the term $-\frac{\nu}{(1-\nu)}\int\left[\int F(x_1)dx_1\right]dx_1$ does not involve x_2 and x_3 and, consequently, does not affect the components of stress and may be disregarded. Therefore,

$$X = Y_2\,(x_2, x_3) + x_1 Y_1\,(x_2, x_3) - \frac{\nu x_1^2 I_o(x_2, x_3)}{2(1+\nu)} + \frac{\nu\nabla_1^2 V(x_2^2 + x_3^2)x_1^2}{4(1-\nu)} \tag{F.28}$$

Operating both sides of relation (F.28) by ∇_1^2, taking into account that $\nabla_1^2 V$ is constant and using relation (F.15) we get

$$\nabla_1^2 X = \nabla_1^2 Y_2 + \frac{\nu(1 + \nu)x_1^2\nabla_1^2 V}{2(1-\nu)} + x_1\nabla_1^2 Y_1 \tag{F.29}$$

From relations (F.8) and (F.14) we have

$$\nabla_1^2 X = I_1(x_1, x_2, x_3) - 2V = I_o(x_2, x_3) + F(x_1) - 2V \tag{F.30}$$

Equating the right side of relations (F.29) and (F.30) we obtain

$$\nabla_1^2 Y_2(x_2, x_3) = I_o(x_2, x_3) + F(x_1) - 2V - \frac{\nu(1 + \nu)\nabla_1^2 V}{2(1 - \nu)} - x_1 \nabla_1^2 Y_1(x_2, x_3) \tag{F.31}$$

Consequently,

$$I_o(x_2, x_3) = \nabla_1^2 Y_2(x_2, x_3) + 2V \tag{F.32}$$

and

$$F(x_1) = \frac{\nu(1 + \nu)x_1^2 \nabla_1^2 V}{2(1 - \nu)} + x_1 \nabla_1^2 Y_1(x_2, x_3) \tag{F.33}$$

$$\nabla_1^2 Y_1 = \text{constant} \tag{F.34}$$

Taking the Laplacian of both sides of relation (F.32) and (F.34), we get

$$\nabla_1^2 I_o = \nabla_1^4 Y_2 + 2\nabla_1^2 V \tag{F.35}$$

Substituting relation (F.15) into (F.35), we have

$$\nabla_1^4 Y_2 = -(1-\nu)\nabla_1^2 V - x_1 \nabla_1^4 Y_1 \tag{F.36}$$

Substituting relation (F.32) into (F.28) and taking into account relation (F.36), we get

$$X(x_1, x_2, x_3) = Y_2(x_2, x_3) + x_1 Y_1(x_2, x_3) - \frac{\nu x_1^2 \nabla_1^2 Y_2}{2(1+\nu)} - \frac{\nu x_1^2 V}{(1+\nu)} + \frac{\nu \nabla_1^2 V}{4(1-\nu)}(x_2^2 + x_3^2)x_1^2 \tag{F.37}$$

Substituting relation (F.37) into (F.6) we see that the components of stress have the following form:

$$\begin{aligned}
\tau_{22} &= g_1(x_2, x_3) + x_1 f_1(x_2, x_3) + x_1^2 h_1(x_2, x_3) \\
\tau_{33} &= g_2(x_2, x_3) + x_1 f_2(x_2, x_3) + x_1^2 h_2(x_2, x_3) \\
\tau_{23} &= g_3(x_2, x_3) + x_1 f_3(x_2, x_3) + x_1^2 h_3(x_2, x_3)
\end{aligned} \tag{F.38}$$

From to relations (F.38) we may conclude that *in order to maintain a state of plane stress in a simply connected prismatic body the components of stress and consequently the components of traction* $\overset{n}{T}_2(x_1, x_2, x_3)$ *and* $\overset{n}{T}_3(x_1, x_2, x_3)$ *acting on its lateral surfaces must be the sum of two parts, one that varies linearly with* x_1 *and another that has an* x_1^2 *variation.* Thus, if a simply connected prismatic body has traction-free end surfaces and is subjected to traction $\overset{n}{T}_2(x_1, x_2, x_3)$ and $\overset{n}{T}_3(x_1, x_2, x_3)$ which have the form indicated by relations (F.38) and to components of specific body forces $B_2(x_2, x_3)$ and $B_3(x_2, x_3)$ ($B_1 = 0$) derivable from a potential $V(x_2, x_3)$ satisfying relation (F.23), then this body is in a state of plane stress. Its stress distribution can be established by finding expressions for the function $Y_2(x_1, x_2)$ satisfying relation (F.36) and for the function $Y_1(x_2, x_3)$ satisfying relation (F.34). These expressions involve two constants each, which are evaluated by requiring that when the functions $Y_2(x_2, x_3)$ and $Y_1(x_2, x_3)$ are substituted into relation (F.37) give a plane stress function $X(x_1, x_2, x_3)$ which when substituted in relations (F.6) gives components of stress, which when substituted into the traction–stress relations (2.75) give the specified components of traction on the lateral surface of the body.

Appendix G

Functions of Discontinuity

G.1 Definition of the Unit Step Function, the Dirac δ-Function, and the Doublet Function

Referring to Fig. G.1, the unit step function $\Delta(x - a)$ is defined as

$$\Delta(x - a) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases} \tag{G.1}$$

The unit step function is not defined at $x = a$. In case $a = 0$, the unit step function is denoted as $\Delta(x)$.

In general, for $b < a$ we have

$$\int_b^x f(x)\Delta(x - a)\,dx = \int_b^a f(x)\Delta(x - a)\,dx + \int_a^x f(x)\Delta(x - a)\,dx \tag{G.2}$$

Referring to the definition of the unit step function (G.1) it can be seen that the first integral on the right-hand side of relation (G.2) vanishes. Thus relation (G.2) reduces to

$$\int_b^x f(x)\Delta(x - a)\,dx = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(x)\,dx & \text{if } x > a \end{cases} = \Delta(x - a)\int_a^x f(x)\,dx \tag{G.3}$$

From relation (G.3) we obtain

$$\int_b^x \Delta(x - a)\,dx = (x - a)\Delta(x - a) \qquad b < a \tag{G.4}$$

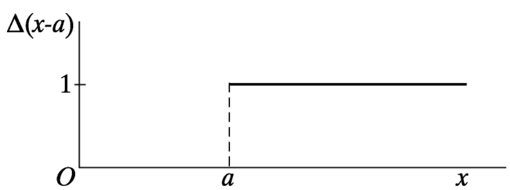


Figure G.1 Unit step function.

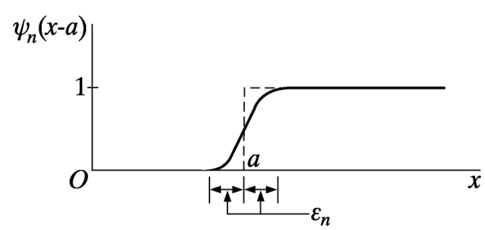


Figure G.2 Approximation of the unit step function by the continuous function $\psi_n(x - a)$.

$$\int_b^x (x-a)^k \Delta(x-a)\,dx = \frac{(x-a)^{k+1}}{k+1}\Delta(x_1-a) \qquad b<a \tag{G.5}$$

Referring to Fig. G.2, imagine that there exists a sequence of continuous functions $\psi_n(x-a)$ ($n = 1, 2, 3, \ldots$) which vanish for $x < a - \epsilon_n$ and are equal to unity for $x > a + \epsilon_n$ where ϵ_n is a small positive number. Moreover, assume that ϵ_n approaches zero as n approaches infinity. Then, as n increases and ϵ_n decreases, $\psi_n(x-a)$ approaches $\Delta(x-a)$. Consequently, $\Delta(x-a)$ may be considered as the limit of the sequence of continuous functions $\psi_n(x-a)$ as n approaches infinity. Thus,

$$\Delta(x-a) = \lim_{n\to\infty} \psi_n(x-a) \tag{G.6}$$

Consider the following sequence of functions, one of which is plotted in Fig. G.3

$$\phi_n(x-a) = \frac{d\psi_n(x-a)}{dx} \tag{G.7}$$

Notice that the cross-hatched area in Fig. G.3 is equal to unity for any value of ϵ_n. This may be shown as follows

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_n(x-a)\,dx &= \int_{x=a-\epsilon_n}^{x=a+\epsilon_n} \phi_n(x-a)\,dx = \int_{x=a-\epsilon_n}^{x=a+\epsilon_n} \frac{d\psi_n(x-a)}{dx}dx \\ &= \psi_n(x-a)\Big|_{x=a+\epsilon_n} - \psi_n(x-a)\Big|_{x=a-\epsilon_n} = 1\end{aligned} \tag{G.8}$$

For very small values of ϵ_n, the cross-hatched area in Fig. G.3 may be approximated by $2\epsilon_n \phi(x-a)$. Thus,

$$\int_{-\infty}^{\infty} \phi_n(x-a)\,dx = 2\epsilon_n \phi_n(x-a) = 1$$

Consequently, for large values of n (small values of ϵ_n), we have

$$\phi_n(x-a) = \frac{d\psi_n(x-a)}{dx} \approx \frac{1}{2\epsilon_n} \qquad a-\epsilon_n < x < a+\epsilon_n \tag{G.9}$$

It is apparent that in the limit as $n \to \infty$ the value of $\phi_n(x-a)$ in the region $a - \epsilon_n < x < a + \epsilon_n$ increases to infinity (see Fig. G.3). That is, in the limit as $n \to \infty$ the function $\phi_n(x-a)$ does not exist as an ordinary function. We call that limit the Dirac δ-function, that is,

$$\delta(x-a) = \lim_{n\to\infty} \phi_n(x-a) = \lim_{n\to\infty} \frac{d\psi_n(x-a)}{dx} = \lim_{\epsilon_n\to 0} \begin{cases} 0 & \text{if } a+\epsilon_n \le x \le a-\epsilon_n \\ \dfrac{1}{2\epsilon_n} & \text{if } a-\epsilon_n < x < a+\epsilon_n \end{cases} \tag{G.10}$$

If $a = 0$, the Dirac δ-function is denoted as $\delta(x)$.

We assume that $\delta(x - a)$ has the same properties as ordinary functions when used as an integrand. Using relations (G.10) and (G.8) we obtain

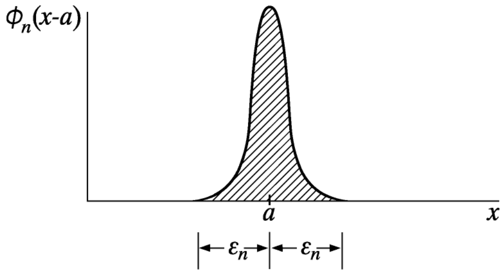


Figure G.3 Derivative of the function $\psi_n(x - a)$.

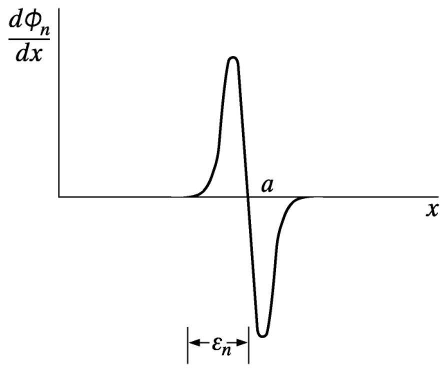


Figure G.4 Derivative of the function $\phi_n(x - a)$.

$$\int_{-\infty}^{\infty} \delta(x - a)\,dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} \phi_n(x - a)\,dx = 1 \tag{G.11}$$

Noting that $\delta(x - a)$ vanishes at every point except $x = a$. Taking into account relation (G.11) for any function $f(x)$ which is continuous at $x = a$ the following relation is valid

$$\int_{-\infty}^{\infty} \delta(x - a) f(x)\,dx = f(a) \tag{G.12}$$

Thus, the δ-function cannot be assigned any value at $x = a$ and vanishes at all other points. However, the integral $\int_{-\infty}^{\infty} \delta(x - a)\,dx$ is equal to unity. Hence, the δ-function is not an ordinary function, having definite values for every value of x, but rather, an entity possessing certain properties as, for instance, those given by relation (G.12)†. This relation indicates that $\delta(x - a)$ acts as a sieve selecting from all possible values of $f(x)$ its value at the point $x = a$.

Using relations (G.6), (G.7) and (G.10) we obtain

$$\begin{aligned}\Delta(x - a) &= \lim_{n\to\infty} \psi_n(x - a) = \lim_{n\to\infty} \int_{-\infty}^{x} d\psi_n(x - a) = \lim_{n\to\infty} \int_{-\infty}^{x} \phi_n(x - a)\,dx \\ &= \int_{-\infty}^{x} \delta(x - a)\,dx = \int_{b}^{x} \delta(x - a)\,dx\end{aligned} \tag{G.13}$$

where $b < a$. Moreover using relations (G.6), (G.7) and (G.10) we have

$$\frac{d\Delta(x - a)}{dx} = \lim_{n\to\infty} \frac{d\psi_n(x - a)}{dx} = \lim_{n\to\infty} \phi_n(x - a) = \delta(x - a) \tag{G.14}$$

Consider the sequence of functions $d\phi_n(x - a)/dx$, one of which is shown in Fig. G.4. We define the *doublet function* as

† For a more detailed discussion see Butkov, E., *Mathematical Physics*, Addison-Wesley, Reading, MA, 1973, p. 221; and Papoulis, A., *The Fourier Integral and Its Applications*, McGraw-Hill, New York, 1962, p. 269.

$$\delta^1(x - a) = \frac{d\delta(x - a)}{dx} = \lim_{n\to\infty} \frac{d\phi_n(x - a)}{dx} \tag{G.15}$$

On the basis of this definition we have

$$\int_{-\infty}^{x} \delta^1(x - a)\,dx = \int_{-\infty}^{x} d\delta(x - a) = \delta(x - a) \tag{G.16}$$

Moreover, for any function $f(x)$ which is continuous at $x = a$, using relation (G.15) and integrating by parts we have

$$\int_{-\infty}^{\infty} \delta^1(x - a) f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,d\delta(x - a) = f(x)\delta(x - a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x - a)\frac{df}{dx}dx = -\frac{df}{dx}\Big|_{x=a} \tag{G.17}$$

Following a procedure analogous to that employed in proving relation (G.17) it can be shown that

$$\int_{-\infty}^{\infty} \delta^k(x - a) f(x)\,dx = (-1)^k \frac{d^k f(x)}{dx^k}\Big|_{x=a} \tag{G.18}$$

G.2 Additional Properties of the Dirac δ-Function and of the Doublet Function

1. For any function $f(x)$ which is continuous at $x = 0$, the following relation is valid:

$$f(x + c)\delta(x) = f(c)\delta(x) \tag{G.19}$$

2. For any constant $c \neq 0$, the following relation is valid:

$$\delta(cx) = |c|^{-1}\delta(x) \tag{G.20}$$

3. As a consequence of property (G.20), it is apparent that $\delta(x)$ is an even function, that is

$$\delta(x) = \delta(-x) \tag{G.21}$$

4. The following relation is valid:

$$(x - a)\delta(x - a) = 0 \tag{G.22}$$

5. The following relation may be proven by differentiating relation (G.22) with respect to x:

$$\delta^1(x - a) = -\frac{\delta(x - a)}{(x - a)} \tag{G.23}$$

Similarly,

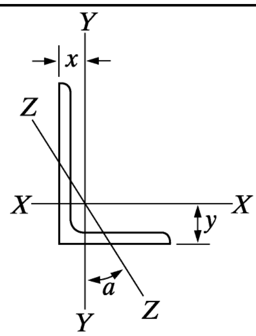
$$\delta^k(x - a) = (-1)^k k! \frac{\delta(x - a)}{(x - a)^k} \tag{G.24}$$

Appendix H

Properties of Rolled Shapes †

In this Appendix we present in S.I. units, the properties of some rolled shapes manufactured in the U.S.A.

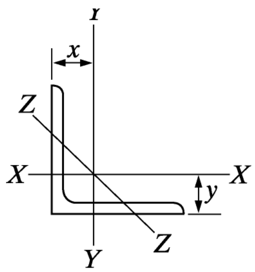
ANGLES
Unequal legs



Size and Thickness mm	Mass per Meter kg/m	Area mm²	Axis X-X I_x 10^6 mm⁴	S_x 10^3 mm³	r_x mm	y mm	Axis Y-Y I_y 10^6 mm⁴	S_y 10^3 mm³	r_y mm	x mm	Axis Z-Z r_z mm	tan a
L152 × 102 × 19	35.0	4470	10.1	102	47.5	52.5	3.65	49.0	28.6	27.5	21.9	0.435
× 12.7	24.0	3060	7.20	70.8	48.5	50.3	2.64	34.4	29.4	25.3	22.2	0.446
× 9.5	18.2	2320	5.56	54.0	49.0	49.1	2.06	26.4	29.8	24.1	22.4	0.452
L127 × 76 × 12.7	19.0	2420	3.93	47.6	40.3	44.4	1.06	18.6	20.9	19.0	16.3	0.355
× 9.5	14.5	1840	3.06	36.6	40.8	43.3	0.841	14.5	21.4	17.8	16.6	0.362
× 6.4	9.8	1260	2.14	25.2	41.2	42.1	0.598	10.1	21.8	16.6	16.8	0.369
L102 × 76 × 12.7	16.4	2100	2.12	31.1	31.8	33.9	1.00	18.1	21.8	20.9	16.2	0.536
× 9.5	12.6	1600	1.66	24.0	32.2	32.8	0.792	14.1	22.2	19.8	16.3	0.545
× 6.4	8.6	1100	1.17	16.6	32.6	31.6	0.564	9.83	22.6	18.6	16.5	0.552
L89 × 64 × 12.7	13.9	1780	1.36	23.3	27.6	30.6	0.581	12.7	18.1	18.1	13.7	0.491
× 9.5	10.7	1360	1.07	18.0	28.0	29.5					13.8	0.503
× 6.4	7.3	938	0.759	12.5	28.4	28.3	0.463	9.83	18.5	16.9	13.9	0.512
							0.333	6.91	18.8	15.8		
L76 × 51 × 12.7	11.5	1450	0.795	16.4	23.4	27.4					10.9	0.420
× 9.5	8.8	1120	0.632	12.7	23.8	26.2					10.9	0.434
× 6.4	6.1	772	0.453	8.90	24.2	25.1	0.283	7.84	14.0	14.9	11.1	0.446
							0.228	6.11	14.3	13.7		
L64 × 51 × 9.5	7.9	1000	0.388	9.10	19.7	21.3	0.166	4.32	14.7	12.6	10.8	0.610
× 6.4	5.4	695	0.280	6.39	20.1	20.2					10.8	0.621
							0.217	5.99	14.7	14.8		
							0.158	4.24	15.1	13.7		

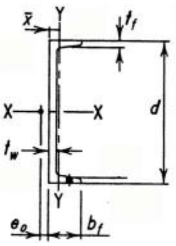
† Taken from the Metric Version of the second edition of LRFD Manual of Steel Construction of the American Institute of Steel Construction, Chicago, 1999.

ANGLES
Equal legs



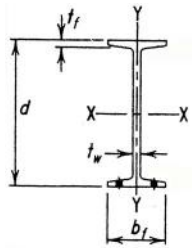
Size and Thickness mm	Mass per Meter kg/m	Area mm^2	Axis *X-X* and *Y-Y*				Axis *Z-Z*		r
			I $10^6\ mm^4$	*S* $10^3\ mm^3$	*r* mm	*x* or *y* mm	*x* mm	tan *a*	
L203 × 203 × 25.4	75.9	9670	36.9	258	61.8	60.0	39.7	1.000	
× 19	57.9	7350					40.0	1.000	
× 12.7	39.3	4990	28.9	199	62.7	57.8	40.4	1.000	
			20.2	137	63.6	55.5			
L152 × 152 × 25.4	55.7	7080					29.5	1.000	
× 19	42.7	5420	14.6	139	45.4	47.2	29.7	1.000	
× 15.9	36.0	4580	11.6	108	46.3	44.9	29.9	1.000	
× 11.1	25.6	3250	10.0	92.5	46.7	43.9	30.1	1.000	
× 9.5	22.2	2800	7.29	66.3	47.4	42.1	30.2	1.000	
			6.34	57.4	47.6	41.5			
L127 × 127 × 19	35.1	4470					24.7	1.000	
× 15.9	29.8	3790	6.54	74.0	38.3	38.6	24.8	1.000	
× 12.7	24.1	3060	5.66	63.2	38.6	37.5	25.0	1.000	
× 9.5	18.3	2320	4.68	51.7	39.1	36.5	25.1	1.000	
			3.63	39.6	39.6	35.3			
L102 × 102 × 19	27.5	3520					19.9	1.000	
× 15.9	23.4	2990	3.23	46.3	30.3	32.3	19.9	1.000	
× 12.7	19.0	2430	2.81	39.7	30.7	31.3	19.9	1.000	
× 9.5	14.6	1850	2.34	32.6	31.0	30.2	20.0	1.000	
× 6.4	9.8	1260	1.83	25.1	31.5	29.0	20.3	1.000	
			1.29	17.4	32.0	28.0			
L89 × 89 × 12.7	16.5	2100					17.4	1.000	
× 9.5	12.6	1600	1.52	24.5	26.9	26.9	17.4	1.000	
× 6.4	8.6	1100	1.19	18.8	27.3	25.8	17.6	1.000	
			0.845	13.1	27.7	24.6			
L76 × 76 × 12.7	14	1770					14.8	1.000	
× 9.5	10.7	1350	0.915	17.5	22.7	23.6	14.9	1.000	
× 6.4	7.3	932	0.725	13.6	23.2	22.5	15.0	1.000	
			0.517	9.50	23.6	21.4			
L64 × 64 × 12.7	11.4	1460					12.5	1.000	
× 9.5	8.7	1130	0.524	12.1	18.9	20.6	12.5	1.000	
× 6.4	6.1	778	0.419	9.40	19.3	19.4	12.6	1.000	
× 4.8	4.6	591	0.302	6.62	19.7	18.4	12.7	1.000	
			0.235	5.09	19.9	17.8			
L51 × 51 × 9.5	7.0	879					9.95	1.000	
× 6.4	4.7	612	0.202	5.80	15.2	16.2	9.94	1.000	
× 3.2	2.4	316	0.147	4.09	15.5	15.1	10.1	1.000	
			0.0806	2.17	16.0	13.9			

CHANNELS
AMERICAN STANDARD



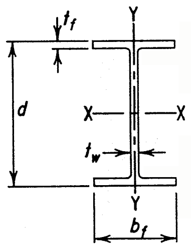
			Web	Flange				Axis X-X			Axis Y-Y		
Desig-nation	Area A	Depth d	Thickness t_w	Width b_t	Thickness t_t	$\bar{x}$	Shear Center Location e_0	$I/10^6$	$S/10^3$	r	$I/10^6$	$S/10^3$	r
mm x kg/m	mm^2	mm	mm	mm	mm	mm	mm	mm^4	mm^3	mm	mm^4	mm^3	mm
C380 × 74	9480	381	18.2	94	16.5	20.2	14.6	167	877	133	4.54	61.5	21.9
× 60	7570	381	13.2	89	16.5	19.7	19.3	144	756	138	3.79	54.7	22.4
×50.4	6430	381	10.2	86	16.5	19.9	22.5	131	688	143	3.34	50.5	22.8
C310 × 45	5690	305	13.0	80	12.7	17.0	15.4	67.2	441	109	2.09	33.3	19.2
× 37	4720	305	9.8	77	12.7	17.0	18.8	59.7	391	112	1.83	30.5	19.7
× 30.8	3920	305	7.2	74	12.7	17.4	21.7	53.4	350	117	1.57	27.7	20
C250 × 45	5670	254	17.1	76	11.1	16.3	8.97	42.7	336	86.8	1.58	26.5	16.7
× 37	4750	254	13.4	73	11.1	15.6	12.4	37.9	298	89.3	1.38	24	17
× 30	3780	254	9.6	69	11.1	15.3	16.0	32.6	257	92.9	1.14	21.2	17.4
× 22.8	2880	254	6.1	65	11.1	15.8	19.7	27.7	218	98.1	0.912	18.5	17.8
C230 × 30	3800	229	11.4	67	10.5	14.7	12.9	25.4	222	81.8	0.997	19.1	16.2
× 22	2840	229	7.2	63	10.5	14.9	17.3	21.2	185	86.4	0.796	16.5	16.7
× 19.9	2530	229	5.9	61	10.5	15.0	18.5	19.8	173	88.5	0.708	15.4	16.7
C200 × 27.9	3560	203	12.4	64	9.9	14.3	10.8	18.2	179	71.5	0.817	16.4	15.1
× 20.5	2600	203	7.7	59	9.9	13.9	15.1	14.9	147	75.7	0.620	13.7	15.4
× 17.1	2170	203	5.6	57	9.9	14.4	17.5	13.4	132	78.6	0.538	12.6	15.7
C180 ×22	2790	178	10.6	58	9.3	13.4	11.1	11.3	127	63.6	0.561	12.6	14.2
× 18.2	2310	178	8.0	55	9.3	13.1	13.3	10.0	112	65.8	0.470	11.2	14.3
× 14.6	1850	178	5.3	53	9.3	13.7	16.4	8.83	99.2	69.1	0.400	10.2	14.7
C150 × 19.3	2450	152	11.1	54	8.7	12.9	9.32	7.11	93.6	53.9	0.420	10.2	13.1
× 15.6	1980	152	8.0	51	8.7	12.5	12.0	6.21	81.7	56.0	0.347	9.01	13.2
× 12.3	1540	152	5.1	48	8.7	12.7	14.8	5.35	70.4	58.9	0.276	7.82	13.4
C130 × 12	1710	127	8.3	48	8.1	12.2	10.8	3.70	58.3	46.5	0.264	7.37	12.4
× 10.4	1310	127	4.8	47	8.1	13.0	15.3	3.25	51.2	49.8	0.229	6.74	13.2
C100 × 10.8	1370	102	8.2	43	7.5	11.4	9.38	1.90	37.3	37.2	0.172	5.44	11.2
× 8	1020	102	4.7	40	7.5	11.5	12.6	1.61	31.6	39.7	0.130	4.56	11.3
C75 × 8.9	1130	76.2	9.0	40	6.9	11.3	7.96	0.850	22.3	27.4	0.122	4.25	10.4
× 74	936	76.2	6.6	37	6.9	10.8	9.39	0.751	19.7	28.3	0.0948	3.62	10.1
× 6.1	765	76.2	4.3	35	6.9	10.8	11.3	0.671	17.6	29.6	0.0765	3.16	10.0

S SHAPES



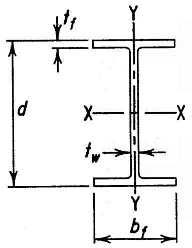
				Web	Flange		Elastic Properties					
							Axis X-X			Axis Y-Y		
Designation	Weight	Area A	Depth d	Thickness t_w	Width b_f	Thickness t_f	$I/10^6$	$S/10^3$	r	$I/10^6$	$S/10^3$	r
mm × kg/m	kN/m	mm^2	mm	mm	mm	mm	mm^4	mm^3	mm	mm^4	mm^3	mm
S610 × 149	1.46	19000	610	18.9	184	22.1	995	3260	229	20.2	215	32.3
× 134	1.31	17100	610	15.9	181	22.1	938	3080	234	19.0	206	33.0
× 119	1.17	15200	610	12.7	178	22.1	878	2880	240	17.9	198	34.0
S510 × 143	1.40	18200	516	20.3	183	23.4	700	2710	196	21.3	228	33.9
× 128	1.26	16400	516	16.8	179	23.4	658	2550	200	19.7	216	34.4
S510 × 112	1.10	14200	508	16.1	162	20.2	530	2090	193	12.6	152	29.5
× 98.2	0.963	12500	508	12.8	159	20.2	495	1950	199	11.8	145	30.4
S460 × 104	1.02	13300	457	18.1	159	17.6	385	1680	170	10.4	127	27.5
× 81.4	0.799	10400	457	11.7	152	17.6	333	1460	179	8.83	113	28.8
S380 × 74	0.726	9500	381	14	143	15.8	201	1060	145	6.65	90.8	26.1
× 64	0.628	8150	381	10.4	140	15.8	185	971	151	6.15	85.7	27.1
S310 × 74	0.726	9480	305	17.4	139	16.7	126	826	115	6.69	93.2	26.1
× 60.7	0.595	7730	305	11.7	133	16.7	113	741	121	5.73	83.6	26.8
S310 × 52	0.510	6650	305	10.9	129	13.8	95.3	625	120	4.19	63.6	24.8
× 47.3	0.464	6040	305	8.9	127	13.8	90.5	593	122	3.97	61.1	25.3
S250 × 52	0.510	6670	254	15.1	126	12.5	61.2	482	95.8	3.59	55.7	22.9
× 37.8	0.371	4820	254	7.9	118	12.5	51.1	402	103	2.86	47.5	24.1
S200 × 34	0.334	4370	203	11.2	106	10.8	26.8	264	78.3	1.83	33.8	20.2
× 27.4	0.269	3500	203	6.9	102	10.8	23.9	235	82.6	1.60	30.6	21.1
S150 × 25.7	0.252	3270	152	11.8	91	9.1	10.8	142	57.5	1.00	21.3	17.2
× 18.6	0.182	2370	152	5.9	85	9.1	9.11	120	62.0	0.782	18.0	18.0
S130 × 15	0.147	1890	127	5.4	76	8.3	5.07	79.8	51.8	0.513	13.2	16.3
S100 × 14.1	0.138	1800	102	8.3	71	7.4	2.82	55.3	39.6	0.383	10.5	14.4
× 11.5	0.113	1460	102	4.9	68	7.4	2.53	49.6	41.6	0.328	9.41	14.8
S75 × 11.2	0.110	1430	76	8.9	64	6.6	1.20	31.6	29.0	0.254	7.72	13.1
× 8.5	0.0834	1070	76	4.3	59	6.6	1.03	27.1	31.0	0.190	6.44	13.3

W SHAPES



Designation	Area A	Depth d	Web Thickness t_w	Flange Width b_f	Flange Thickness t_f	Elastic Properties Axis X-X $I/10^6$	Axis X-X $S/10^3$	Axis X-X r	Axis Y-Y $I/10^6$	Axis Y-Y $S/10^3$	Axis Y-Y r
mm × kg/m	mm^2	mm	mm	mm	mm	mm^4	mm^3	mm	mm^4	mm^3	mm
W920 × 446	57000	933	24.0	423	42.7	8470	18200	385	540	2550	97.3
× 201	25600	903	15.2	304	20.1	3250	7200	356	94.4	621	60.7
W840 × 299	38100	855	18.2	400	29.2	4790	11200	355	312	1560	90.5
× 176	22400	835	14.0	292	18.8	2460	5890	331	78.2	536	59.1
W760 × 257	32600	773	16.6	381	27.1	3420	8850	324	250	1310	87.6
× 147	18700	753	13.2	265	17.0	1660	4410	298	52.9	399	53.2
W690 × 217	27700	695	15.4	355	24.8	2340	6730	291	185	1040	81.7
× 125	16000	678	11.7	253	16.3	1190	3510	273	44.1	349	52.5
W530 × 150	19200	543	12.7	312	20.3	1010	3720	229	103	660	73.2
× 92	11800	533	10.2	209	15.6	552	2070	216	23.8	228	44.9
× 66	8370	525	8.9	165	11.4	351	1340	205	8.57	104	32.0
W460 × 158	20100	476	15.0	284	23.9	796	3340	199	91.4	644	67.4
× 113	14400	463	10.8	280	17.3	556	2400	196	63.3	452	66.3
× 74	9450	457	9.0	190	14.5	333	1460	188	16.6	175	41.9
× 52		450	7.6	152	10.8	212	942	179	6.34	83.4	30.9
	6630										
W360 × 551	70100	455	42.0	418	67.6	2260	9930	180	825	3950	108
× 216	27600	375	17.3	394	27.7	712	3800	161	283	1440	101
× 122	15500	363	13.0	257	21.7	365	2010	153	61.5	479	63.0
× 101	12900	357	10.5	255	18.3	302	1690	153	50.6	397	62.6
× 79	10100	354	9.4	205	16.8	227	1280	150	24.2	236	48.9
× 64	8140	347	7.7	203	13.5	178	1030	148	18.9	186	48.2
× 57.8	7220	358	7.9	172	13.1	161	899	149	11.1	129	39.2
× 44	5730	352	6.9	171	9.8	122	693	146	8.18	95.7	37.8
× 39	4980	353	6.5	128	10.7	102	578	143	3.75	58.6	27.4
× 32.9	4170	349	5.8	127	8.5	82.7	474	141	2.91	45.8	26.4

W SHAPES



Designation	Area A	Depth d	Web	Flange		Elastic Properties					
						Axis X-X			Axis Y-Y		
			Thickness t_w	Width b_f	Thickness t_f	$I/10^6$	$S/10^3$	r	$I/10^6$	$S/10^3$	r
mm × kg/m	mm^2	mm	mm	mm	mm	mm^4	mm^3	mm	mm^4	mm^3	mm
W310 × 143	18200	323	14.0	309	22.9	348	2150	138	113	731	78.8
× 107	13600	311	10.9	306	17.0	248	1590	135	81.2	531	77.3
× 74	9490	310	9.4	205	16.3	165	1060	132	23.4	228	49.7
× 60	7590	303	7.5	203	13.1	129	851	130	18.3	180	49.1
× 52	6670	317	7.6	167	13.2	118	744	133	10.3	123	39.3
× 44.5	5690	313	6.6	166	11.2	99.2	634	132	8.55	103	38.8
× 38.7	4940	310	5.8	165	9.7	85.1	549	131	7.27	88.1	38.4
× 32.7	4180	313	6.6	102	10.8	65.0	415	125	1.92	37.6	21.4
× 23.8	3040	305	5.6	101	6.7	42.7	280	119	1.16	23.0	19.5
W250 × 167	21300	289	19.2	265	31.8	300	2080	119	98.8	746	68.1
× 101	12900	264	11.9	257	19.6	164	1240	113	55.5	432	65.6
× 80	10200	256	9.4	255	15.6	126	984	111	43.1	338	65.0
× 67	8550	257	8.9	204	15.7	104	809	110	22.2	218	51.0
× 58	7420	252	8.0	203	13.5	87.3	693	108	18.8	185	50.3
× 44.8	5720	266	7.6	148	13.0	71.1	535	111	7.03	95.0	35.1
× 22.3	2850	254	5.8	102	6.9	28.9	228	101	1.23	24.1	20.8
W200 × 86	11100	222	13.0	209	20.6	94.7	853	92.4	31.4	300	53.2
× 71	9110	216	10.2	206	17.4	76.6	709	91.7	25.4	247	52.8
× 59	7560	210	9.1	205	14.2	61.1	582	89.9	20.4	199	51.9
× 41.7	5310	205	7.2	166	11.8	40.9	399	87.8	9.01	109	41.2
× 35.9	4580	201	6.2	165	10.2	34.4	342	86.7	7.64	92.6	40.8
× 26.6	3390	207	5.8	133	8.4	25.8	249	87.2	3.3	49.6	31.2
× 22.5	2860	206	6.2	102	8.0	20.0	194	83.6	1.42	27.8	22.3
× 19.3	2480	203	5.8	102	6.5	16.6	164	81.8	1.15	22.5	21.5
W150 × 37.1	4730	162	8.1	154	11.6	22.2	274	68.5	7.07	91.8	38.7
× 29.8	3790	157	6.6	153	9.3	17.2	219	67.4	5.56	72.7	38.3
× 24	3060	160	6.6	102	10.3	13.4	168	66.2	1.83	35.9	24.5
× 18	2290	153	5.8	102	7.1	9.17	120	63.3	1.26	24.7	23.5
× 13.5	1730	150	4.3	100	5.5	6.87	91.6	63.0		18.4	23.0
									0.918		
W130 × 28.1	3580	131	6.9	128	10.9	10.9	166	55.2		59.5	32.6
× 23.8	3010	127	6.1	127	9.1	8.80	139	54.1	3.81	49.0	32.1
									3.11		
W100 × 19.3	2480	106	7.1	103	8.8	4.77	90.0	43.9		31.3	25.5
									1.61		

Conversion of SI Units to USCS Units

Quantity	SI Unit	Conversion factor	USCS unit
Area	square meter, m^2 square millimeter, mm^2	10.76391 0.001550	square foot, ft^2 square inch, in^2
Energy	joule, J megajoule, MJ joule, J	0.737561 0.277778 0.0009478	foot-pound, ft·lb kilowatthour, kWh British thermal unit, Btu
Force	newton†, N kilonewton, kN	0.22481 0.22481	pound, lb kip (1000 pounds)
Length	meter, m millimeter, mm kilometer, km	3.28084 0.03937 0.6213722	foot, ft inch, in mile, mi
Mass	kilogram, kg	0.68522	slug, $lb{\cdot}s^2/ft$
Moment	newton-meter, N·m newton-meter, N·m kilonewton-meter, kN·m	0.73756 8.85073 0.73756	foot-pound, ft·lb inch-pound, in·lb foot-kip, ft·kip
Power	watt, W watt, W	0.737561 0.001341	foot-pound per second, ft·lb/s horsepower, hp
Stress (pressure)	pascal, Pa megapascal, MPa	0.0208854 145.04	pounds per square foot, lb/ft^2 pounds per square inch lb/in^2
Temperature	degrees Celsious, ^oC	$1.8^oC + 32$	degrees Farenheit, ^oF
Volume	cubic meter, m^3 cubic millimeter, mm^3	35.3147 61.0236×10^{-6}	cubic foot, ft^3 cubic inch, in^3

† A newton is the force required to accelerate a 1-kg mass by a constant acceleration of 1 m/s^2. A pascal is equal to 1 N/m^2 .

NOTE: To convert USCS units to SI units, divide by the conversion factor. For temperature, $^oC = (5/9)(^oF - 32)$.

SI Prefixes

Prefix	Symbol	Multiplication factor	Prefix	Symbol	Multiplication factor
tera	T	10^{12}	milli	m	10^{-3}
giga	G	10^{9}	micro	μ	10^{-6}
mega	M	10^{6}	nano	n	10^{-9}
kilo	k	10^{3}	pico	p	10^{-12}

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ISBN 0-8493-9899-1
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