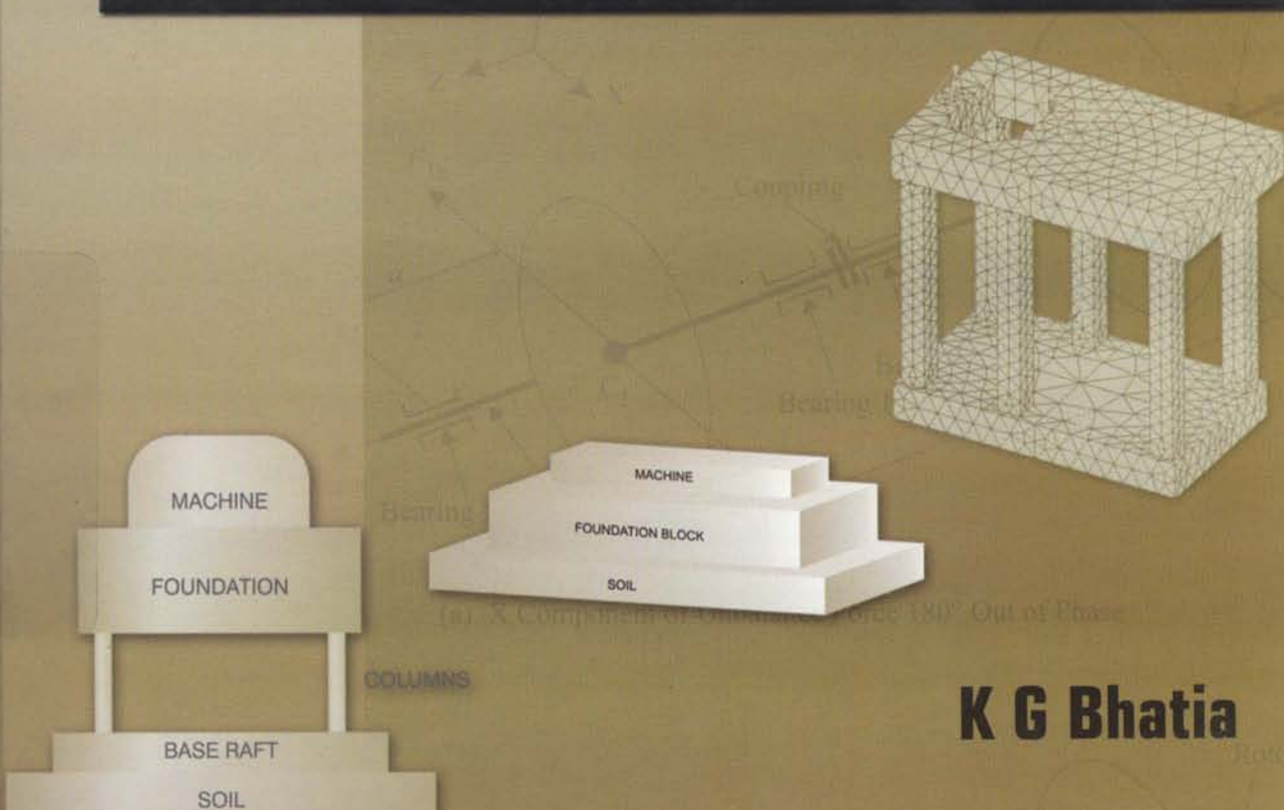
FOUNDATIONS FOR INDUSTRIAL MACHINES

Handbook for Practising Engineers



K G Bhatia

FOUNDATIONS FOR INDUSTRIAL MACHINES

Handbook for Practising Engineers

FOUNDATIONS FOR INDUSTRIAL MACHINES

- Rotary Machines
- Reciprocating Machines
- Impact Machines
- Vibration Isolation System

Handbook for Practising Engineers

K.G. BHATIA

Dr K G Bhatia

Foundations for Industrial Machines

First Edition 2008

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In the loving memory of

My parents

*Late **Shri Kartar Chand Bhatia** & Late **Smt. Leelawanti Bhatia***

who, in spite of sufferings on account of partition, struggled hard

and devoted their active life in upbringing me and my siblings

In expression of my gratitude to

My brother

*Shri **MG Bhatia**, who sacrificed his career*

in support of my studies

Dedicated to

My wife

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About the Handbook

The author has been engaged in designing, testing and review of machine foundations for various industrial projects viz. Petrochemicals, Refineries, Power plants etc. for the last about three decades.

The handbook is written primarily for practising engineers as well as for students at Post Graduate level. Handbook shares author's long experience on the subject and focuses on the improvements needed in the design process with the sole objective of making practising engineers physically understand and feel the dynamics of machine foundation system.

The handbook covers basic fundamentals necessary for understanding and evaluating dynamic response of machine foundation system. The author has also conducted extensive tests on ***machine foundation models*** *as well as on* ***prototypes****. For over two decades, the author has been associated with* ***Failure Analysis Studies*** *on various types of machines.*

Observations from all the above studies suggest ***need for improvement*** *in the* ***design of foundations*** *for better performance of machines. These include:*

- *a) More comprehensive evaluation of* ***Site Soil Data***
- *b) Better understanding of* ***Machine Data*** *and its use in foundation design*
- *c) Improvement in the* ***Design Philosophy*** *that suggests*
    - *i) Improvement in the* ***Modeling Technique***
    - *ii) Improvement in* ***Analysis Technique***

iii) *Improvement in* ***Structural Design*** *process, and*
iv) *Improvement in* ***Construction Technology***

It is the author's concerned observation that in most of the cases, due recognition to Machine, Foundation and Soil data is lacking. More often than not, ***machine data*** *as well as* ***soil data*** *is treated as black box and used in the design without its proper understanding. A better interaction between foundation designer and machine manufacturer would definitely improve the foundation performance and thereby machine performance. Over the years, author has observed that such interactions are lacking. It is the author's concerned opinion that such an interaction is not only desirable but essential too.*

It is anticipated that this handbook shall serve as a Reference Book. The author is confident that it shall bridge the knowledge gap and shall be beneficial to the practising engineers, students, academicians/researchers as well to the industry.

The text is divided in to 5 Parts.

Part I takes care of Theoretical Aspects

- *An overview providing basic familiarization with the subject is covered in* ***Chapter 1****.*
- *Necessary understanding of Theory of Vibration with specific application to machine foundation design is included in* ***Chapter 2*** *and* ***Chapter 3****.* ***Chapter 4 caters to*** *Basic Theory of Vibration Isolation.*

Part II caters to Design Parameters

- ***Chapter 5*** *provides reasonable coverage to* ***Soil Dynamics*** *and evaluation of Design Soil Parameters as applied to Machine Foundation Design.*
- *Desired emphasis has been given to Design Machine Parameters. Translation of* ***Machine Data*** *to* ***Design Data*** *is given in* ***Chapter 6****.*
- ***Chapter 7*** *is attributed to* ***Design Foundation Parameters.*** *It covers all those aspects related to foundation that play vital role in computing Dynamic Response.*

Part III deals with design of Foundations for Real Life Machines.

- ***Chapter 8*** *is devoted to Modeling and analysis including Finite Element Analysis. All possible aspects of modeling related to design of foundation have adequately been covered.*
- ***Chapter 9, 10 & 11*** *cover* ***Design*** *of* ***Foundation for real life Rotary Machines, Reciprocating Machines and Impact Type Machines respectively****.*

Part IV caters to Design of Foundations with Vibration Isolation System

- *A good amount of emphasis is given to Vibration Isolation of the Foundations. Design of* **Foundations with Isolation Devices** *is covered in* ***Chapter 12****.*

Part V caters to Construction Aspects and Case Studies *related to machine foundation*

- ***Construction Aspects*** *are covered in* ***Chapter 13****.*
- ***Case studies and observations*** *are given in* ***Chapter 14***

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K. G. Bhatia

FOREWORD

Improvement in manufacturing technology has provided machines of higher ratings with better tolerances and controlled behaviour. The increased dependence of society provides no room for failure and demands equipment and systems with higher performance reliability.

Each and every machine does require detailed vibration analysis providing insight in to the dynamic behaviour of machine, foundation and their components. Complete knowledge of machine excitation forces and associated frequencies, knowledge of load transfer mechanism from the machine to the foundation and foundation to soil is a must for correct evaluation of machine performance. Thus for a technically correct and economical solution, it calls for that the designer must have a fairly good knowledge of Dynamic Soil Parameters, Dynamic Foundation Parameters as well as Dynamic Machine Parameters. A close co-operation between manufacturer and the foundation designer is therefore a must. Development of analytical procedures backed by field monitoring, for evaluating dynamic response, is the need of the day.

It is fortunate that Dr K G Bhatia, a person of eminence, who is not only a well known research scientist in the field of Structural Dynamics but also an expert in the profession of Machine Foundation Design, Seismic Qualification of Machinery, Failure Analysis, Weight Optimization etc, has undertaken the challenging task to bring out this Handbook on Foundation for Industrial Machines. With his initial experience with M/s Engineers India Limited for about 4 years and his association with M/s Bharat Heavy Electrical Limited for about 24 years, Dr Bhatia has carried out machine foundation design for many projects and has also conducted field studies on many machines and their foundations.

It is this rich experience which Dr Bhatia has compiled and brought out in form of a Hand Book. He has touched all related aspects required for machine foundation design including Vibration Isolation System. Starting with basic fundamentals of vibration analysis, he has given due coverage to analytical aspects, modeling aspects, design aspects and also included foundation design for real life machines backed up by field measurements based on his own experience and study.

On behalf of the engineering community and on my own behalf, I wish to extend my hearty congratulations to Dr. Bhatia for having brought out this excellent Handbook. It is earnestly hoped that the book will be found useful by not only practising engineers but also by students, researchers and academicians.

K P Mathur

(Formerly) Executive Director
Project Engineering Management
Bharat Heavy Electricals Ltd,
New Delhi, India

PREFACE

This handbook reflects collection of author's works in analysis, design and field investigations during last about 30 years. The book, designed primarily for the practising engineers engaged in design of machine foundation, also provides a platform to students at Post Graduate level for developing professional skill in attaining desired proficiency in designing Foundations for Industrial Machines. Reference to problems, made at various stages, is for the real field problems. Emphasis, throughout, has been laid on applied analysis and design so as to provide deeper understanding to the reader about the physical understanding of the Dynamic Behaviour of Machine Foundation system. The text has been so arranged so as to provide an insight to the reader regarding the need for various design stages to complete the task.

The performance, safety and stability of machines depend largely on their design, manufacturing and interaction with environment. In principle, machine foundations should be designed such that the dynamic forces of machines are transmitted to the soil through the foundation in such a way that all kinds of harmful effects are eliminated. Many scientists have significantly contributed to the field of machine foundation laying greater emphasis on vibration response of both machine and foundation. The contributions to the practical and theoretical development of the subject, especially from authors like Geiger, Rauch, Barkan and Alexander Major, are noteworthy.

The design aids /methodologies provide insight to the dynamic behaviour of foundation and its elements for satisfactory performance of the machine thus suggesting the need for the complete knowledge of load transfer mechanism from the machine to the foundation and the knowledge of excitation forces and associated frequencies for correct evaluation of machine performance.

This book covers basic fundamentals necessary for understanding and evaluating dynamic response of machine foundation system. Stress is laid on detailed dynamic analysis for evaluating the response. Use of commercially available Finite Element packages, for analysis and design of the foundation, is recommended. The author has carried out extensive field investigations on many foundations and some of the findings are presented for comparison with analytical results.

This handbook is written with the sole objective of making the practising engineers physically understand and feel the dynamics of machine foundation system. Any suggestion from the readers that leads to improvement of the contents, style, etc of the handbook is welcome.

K. G. Bhatia

SYMBOLS AND NOTATIONS

α	*Amplitude Ratio*
α_{eff}	*Pile Influence Coefficient*
β	*Frequency Ratio of one frequency to other frequency*
β_x	*Frequency ratio* ω/p_x
β_y	*Frequency ratio* ω/p_y
β_z	*Frequency ratio* ω/p_z
β_θ	*Frequency ratio* ω/p_θ
β_ψ	*Frequency ratio* ω/p_ψ
β_ϕ	*Frequency ratio* ω/p_ϕ
δ_{st}	*Static Deflection*
δ_x	*Static deflection along X*
δ_y	*Static deflection along Y*
δ_z	*Static deflection along Z*

δ_θ	*Static deflection along* θ
δ_ψ	*Static deflection along* ψ
δ_ϕ	*Static deflection along* ϕ
ε	*Linear strain*
ϕ	*Rotation about Z-axis and also Phase angle*
γ	*Shear Strain, also Mass moment of Inertia ratio*
γ_{01}	*Site Shear Strain Value*
γ_{02}	*Design Shear Strain Value*
η	*Isolation Efficiency, also Efficiency of drop of hammer*
λ	*Mass ratio*
μ	*Magnification Factor*
μ_1, μ_2	*Magnification factors*
μ_x	*Magnification in X-direction*
μ_y	*Magnification in Y-direction*
μ_z	*Magnification in Z-direction*
μ_θ	*Magnification in* θ *-direction*
μ_ψ	*Magnification in* ψ *-direction*
μ_ϕ	*Magnification in* ϕ *-direction*
ν	*Poisson's ratio*
ν_s	*Poisson's Ratio for Soil*
ν_c	*Poisson's Ratio for Concrete*
θ	*Rotation about X-axis*
θ_0, ψ_0 & ϕ_0	*Rotations about X, Y & Z axes at DOF location 'O'*
ρ	*Mass density & also Total Amplitude*
ρ_s	*Mass Density of Soil*
ρ_c	*Mass density of Concrete*
σ	*Direct Stress*
σ_1, σ_2	*Static stress*
$\bar{\sigma}_{01}$	*Site Static Stress (Overburden Pressure)*
$\bar{\sigma}_{02}$	*Design Static Stress (Overburden Pressure)*
τ	*Pulse Duration*
ω	*Excitation Frequency*
$\omega_1\ \omega_2\ \omega_3$	*Excitation frequencies of machine 1, 2 and 3*
ψ	*Rotation about Y-axis*

ζ	*Damping Constant*
ζ_x	*Damping in X-direction*
ζ_y	*Damping in Y-direction*
ζ_z	*Damping in Z-direction*
ζ_θ	*Damping in θ -direction*
ζ_ψ	*Damping in ψ -direction*
ζ_ϕ	*Damping in ϕ -direction*
A	*Amplitude, also* *Area*
A_{01}	*Base Contact Area corresponding to site test method*
A_{02}	*Design Base Contact Area for the foundation*
A_b	*Base Contact Area with the Soil*
A_1, A_2, A_3	*Amplitudes of masses 1,2,3 etc.*
A_x	*Amplitude in X-direction*
A_y	*Amplitude in Y-direction*
A_z	*Amplitude in Z-direction*
A_θ	*Amplitude in θ -direction*
A_ψ	*Amplitude in ψ -direction*
A_ϕ	*Amplitude in ϕ -direction*
$A' \& A''$	*Amplitudes corresponding to 1st and 2nd mode resp.*
A_p	*Area of Piston, also* *Area of Plate*
B	*Width of Foundation*
b_x	*Mass Ratio in X-direction*
b_y	*Mass Ratio in Y-direction*
b_z	*Mass Ratio in Z-direction*
b_θ	*Mass Ratio in θ -direction*
b_ψ	*Mass Ratio in ψ -direction*
b_ϕ	*Mass Ratio in ϕ -direction*
c	*Viscous Damping*
$[c]$	*Damping Matrix*
C_u	*Coefficient of Uniform Compression of Soil*
C_{u01}	*Site Evaluated Coefficient of Uniform Compression of the soil*
C_{u02}	*Design Coefficient of Uniform Compression of the soil*

C_τ	*Coefficient of Uniform Displacement of Soil*
C_θ & C_ϕ	*Coefficient of Non-Uniform Compression of Soil*
C_ψ	*Coefficient of Non-Uniform Displacement of Soil*
C_r	*CG of Crank rod*
D	*Depth of pit*
d_{01}	*Effective depth (considered for site test)*
d_{02}	*Effective depth (considered for foundation design)*
d	*Pile Diameter*
E	*Elastic Modulus*
E_y	*Elastic Modulus in Y*
E_{01}	*Site Evaluated Dynamic Elastic Modulus*
E_{02}	*Design Dynamic Elastic Modulus*
E_c	*Elastic Modulus of concrete*
E_s	*Elastic Modulus of soil and also* *Elastic Modulus of Sheet Isolators*
E_p	*Elastic Modulus of pile*
E_{static}	*Static Elastic Modulus*
$E_{dynamic}$	*Dynamic Elastic Modulus*
e	*Rotor Eccentricity and also* *Coefficient of Restitution*
e_x & e_z	*Eccentricity in X & Z direction respectively.*
$F(t)$	*Dynamic Force*
F_x	*Force in X-direction*
F_y	*Force in Y-direction*
F_z	*Force in Z-direction*
F_{ox}	*Force at point 'O' in X-direction*
F_{oy}	*Force at point 'O' in Y-direction*
F_{oz}	*Force at point 'O' in Z-direction*
F_E	*Applied Excitation Force*
F_T	*Transmitted Force*
F_1, F_2	*Unbalance Force*
F_A	*Force at point A*
F_B	*Force at point B*
F_{1x}, F_{2x}	*X-Component of force* F_1 & F_2 *respectively*
F_{1y}, F_{2y}	*Y-Component of force* F_1 & F_2 *respectively*

f	*Frequency in Cycle per sec (Hz)*
f_n	*Natural Frequency in Cycle per sec (Hz)*
f_b	*Observed Frequency in Cycle per sec (Hz)*
f_{ck}	*Characteristic Compressive Strength of Concrete*
F_{cy}, F_{cz}	*Force at point C in Y & Z direction respectively*
g	*Acceleration due to gravity*
G	*Shear Modulus*
G_s	*Shear Modulus of Soil*
G_x	*Shear Modulus in X*
G_z	*Shear Modulus in Z*
G_{01}	*Site Evaluated Dynamic Shear Modulus*
G_{02}	*Design Dynamic Shear Modulus*
G_r	*Rotor Balance Grade*
H	*Depth of Foundation, Height of Portal Frame*
h	*Height of Centroid from Base, also* *Height of fall of hammer*
h_a , h_b & h_c	*Height of Centroid for Drive Machine, Driven Machine and Coupling from CG of Base RE point 'O'*
I_{xx}	*Moment of Inertia of an Area about X- axes*
I_{yy}	*Moment of Inertia of an Area about Y- axes*
I_{zz}	*Moment of Inertia of an Area about Z- axes*
I_b	*Moment of Inertia of Beam*
I_c	*Moment of inertia of Column*
J	*Polar Moment of Inertia*
k	*Stiffness*
k_s	*Stiffness of Soil*
$k, k_1, k_2 .etc$	*Stiffness*
$[k]$	*Stiffness Matrix*
k_{pv}	*Vertical Pile Stiffness*
k_{ph}	*Lateral Pile Stiffness*
k_v	*Vertical Stiffness*
k_h	*Horizontal Stiffness*
k_x	*Translational Stiffness along X-direction*
k_y	*Translational Stiffness along Y-direction*
k_z, k_h	*Translational Stiffness along Z-direction*

k_θ	Rotational Stiffness about X-direction
k_ψ	Rotational Stiffness about Y-direction
k_ϕ	Rotational Stiffness about Z-direction
k_b	Beam Stiffness Factor (I_b / L) Beam MI/Beam Span
k_c	Column Stiffness Factor (I_c / H) Col MI/Col Height
k_r	Stiffness ratio (k_b / k_c)
L	Length of Foundation, Span of Beam
l	Length of Crank rod
l_1	Distance of CG of connecting rod from its end 1 i.e. point A
l_2	Distance of CG of connecting rod from its end 2 i.e. point B
$M(t)$	Dynamic Moment
M_θ	Moment about X-axis
M_ψ	Moment about Y-axis
M_ϕ	Moment about Z-axis
$M_{o\theta}$	Moment at 'O' about X-axis
$M_{o\psi}$	Moment at 'O' about Y-axis
$M_{o\phi}$	Moment at 'O' about Z-axis
m	Mass
m_b	Mass of Block
m_m	Mass of Machine
m_o	Mass of Oscillator, also Mass of Tup (falling mass)
$[m]$	Mass Matrix
M_m	Mass Moment of Inertia
M_{mx}	Mass Moment of Inertia at Centroid about X-axis
M_{my}	Mass Moment of Inertia at Centroid about Y-axis
M_{mz}	Mass Moment of Inertia at Centroid about Z-axis
M_{mox}	Mass Moment of Inertia at 'O' about X-axis
M_{moy}	Mass Moment of Inertia at 'O' about Y-axis
M_{moz}	Mass Moment of Inertia at 'O' about Z-axis
$M20, M25, M30$	Grade of Concrete
m_r	Mass of Rotor, also Mass of Crank rod
m_{r1}, m_{r2}	Mass of rotor 1 & rotor 2 respectively

m_p	*Mass of Piston*
m_A	*Mass at point A*
m_B	*Mass at point B*
m_c	*Mass of connecting rod*
M_{cx}	*Moment at point C about X-axis.*
N	*Rotor speed (rpm)*
n	*Number of Piles, also* *Number of Cylinders (in Reciprocating Engine)*
p	*Natural Frequency in rad/sec, also* *Number of Springs, bearing pressure*
p_x	*Translational Natural Frequency in X-direction, also* *Uniform Shear Stress*
p_y	*Translational Natural Frequency in Y-direction, also* *Pressure in Vertical Y-direction*
p_z	*Translational Natural Frequency in Z-direction, also* *Uniform Shear Stress*
p_θ	*Rotational Natural Frequency about X-axis*
p_ψ	*Rotational Natural Frequency about Y-axis*
$p_{\psi r}$	*Torsional Shear Stress at radius r*
p_ϕ	*Rotational Natural Frequency about Z-axis*
p_{L1}, p_{L2}	*1st and 2nd Limiting Frequencies*
p_r	*Pressure/Load Intensity*
p_1, p_2, p_3 p_4, p_5, p_6	*First six natural frequency in ascending order*
p_s	*Pressure acting on the piston*
q	*Bearing capacity, also Number of Springs*
R	*Radius Vector*
r_0	*Equivalent Radius*
S_t	*Total settlement*
S_e	*Elastic settlement*
S_p	*Settlement after removal of load from plate*
s	*Pile Spacing*
TR	*Transmissibility Ratio*
t	*Time*
T	*Time Period*
V_s	*Shear Wave Velocity*
V_R	*Rayleigh Wave velocity*

V_c	*Compression Wave Velocity*
v	*Velocity*
v'_o	*Velocity of falling mass*
v_1	*Initial velocity imparted to the foundation after impact*
v_0, v_1	*Velocity after Impact*
v'_0, v'_1	*Velocity before Impact*
W_m	*Total Machine load*
W_{mi}	*Machine load at point 'i'*
$W_{f1}, W_{f2} \& W_{f3}$	*Weight of Foundation Block part 1, 2 &3 resp.*
W_{fi}	*Weight of Foundation Block 'i'*
$x, y \& z$	*Displacement in X, Y & Z-directions respectively*
$\bar{x}, \bar{y} \& \bar{z}$	*Overall centroid coordinates*
$x_0, y_0 \& z_0$	*Amplitudes along X, Y & Z direction at DOF location 'O'*
$\bar{x}_f, \bar{y}_f \& \bar{z}_f$	*Amplitudes along X, Y & Z direction at Center of Foundation Top*
$x_{fc}, y_{fc} \& z_{fc}$	*Amplitudes along X, Y & Z direction at Corners of Foundation Top*
$x_f, y_f \& z_f$	*Maximum Amplitudes in X, Y & Z direction at Foundation Top*
$x_{mi}, y_{mi} \& z_{mi}$	*X, Y and Z Coordinate of load point* W_{mi}
$\bar{x}_m \& \bar{z}_m$	*X & Z Coordinates of machine centroid*
x_1, x_2, x_3	*Displacements of masses* m_1, m_2, m_3 *in X-direction*
y_1, y_2, y_3	*Displacements of masses* m_1, m_2, m_3 *in Y-direction*
z_1, z_2, z_3	*Displacements of masses* m_1, m_2, m_3 *in Z-direction*
$\dot{x}, \dot{y} \& \dot{z}$	*1st Differential of* $x, y \& z$ *with respect to time*
$\ddot{x}, \ddot{y} \& \ddot{z}$	*2nd Differential of* $x, y \& z$ *with respect to time*
$\ddot{y}_g$	*Ground Acceleration in Y-Direction*
X, Y, Z	*X, Y & Z Coordinate axes*
$\bar{y}_f$	*Height of Foundation Centroid*
$\bar{y}_m$	*Height of Machine Centroid*
z_p	*Displacement of Piston from its extreme position*
$\ddot{z}_p$	*Acceleration of piston*

1

MACHINES AND FOUNDATIONS

- Overview
- Design Philosophy
- Foundation Types
- Tuning of Foundation
- Foundation Material
- Soil
- Vibration isolation
- Field Performance & Feedback

MACHINES AND FOUNDATIONS

The performance, safety and stability of machines depend largely on their design, manufacturing and interaction with environment. In principle machine foundations should be designed such that the dynamic forces of machines are transmitted to the soil through the foundation in such a way that all kinds of harmful effects are eliminated.

In the past, simple methods of calculation were used most often involving the multiplication of static loads by an estimated Dynamic Factor, the result being treated as an increased static load without any knowledge of the actual safety factor. Because of this uncertainty, the value of the adopted dynamic factor was usually too high, although practice showed that during operation harmful deformations did result in spite of using such excessive factors. **This necessitated a deeper scientific investigation of dynamic loading.** A more detailed study became urgent because of development of machines of higher capacities.

Machines of higher ratings gave rise to considerably higher stresses thereby posing problems with respect to performance and safety. This called for development partly in the field of vibration technique and partly in that of soil mechanics. Hence new theoretical procedures were developed for calculating the dynamic response of foundations.

It is well established that the cost of foundation is but a small fraction of that of the machine and inadequately constructed foundations may result in failures and shutdowns exceeding many times the cost of the capital investment required for properly designed and built foundations.

1.1 AN OVERVIEW

A brief review indicates that over the years, many scientists have contributed to the field of machine foundation design. Gieger in 1922 carried out investigations to determine the natural frequencies of foundations. Rauch in 1924 dealt with the machine and turbine foundation and contributed greatly to the practical and theoretical development of the science. A great emphasis was thus laid on to vibration problems in machine foundations. Timoshenko (1928) & Den Hartog (1934) dealt with many vibration problems in engineering practices. Later Wilson (1942), Arya

(1958), Norris (1959), Harris and Crede (1961) contributed a lot in the field of vibration. D.D Barkan (1938) published his findings on dynamic effects on machine foundation. His basic work on the results of theoretical and experimental investigation in the field of machine foundations affected by dynamic action was published in 1948 and translated into English in 1962. Alexander Major has also made a significant contribution in the field of machine foundation. His book on "vibration analysis and design of foundations for machines and turbines" published in 1962 (translated from Hungarian) had been a very useful tool to deal with machine foundation problems.

Based on the scientific investigations carried out in the last few decades it has been established **that it is not enough to base the design only on vertical loads multiplied by a dynamic factor**, even if this factor introduces a dynamic load many times greater than original one. It should be remembered that operation of the machines generated not only vertical forces, but also forces acting perpendicular to the axis; it is thus not enough to take into account the vertical load and to multiply it by a selected dynamic factor. It has also been found that the suitability of machine foundations depends not only on the forces to which they will be subjected to, but also on their behaviour when exposed to dynamic loads which depends on the speed on the machine and natural frequency of the foundation. Thus a vibration analysis became necessary. In other words, it can be said that each and every machine foundation does require detailed vibration analysis providing insight in to the dynamic behaviour of foundation and its components for satisfactory performance of the machine. The complete knowledge of load transfer mechanism from the machine to the foundation and also the complete knowledge of excitation forces and associated frequencies are a must for correct evaluation of machine performance.

The performance, safety and stability of machines depend largely on their design, manufacturing and interaction with environment. In principle machine foundations should be designed such that the dynamic forces of machines are transmitted to the soil through the foundation in such a way that all kinds of harmful effects are eliminated. Hence, all machine foundations, irrespective of size and type of machine, should be regarded as engineering problem and their design should be based on sound engineering practices. The dynamic loads from the machines causing vibrations must duly be accounted for to provide a solution, which is technically sound and economical. For a technically correct and economical solution, a close co-operation between manufacturer and the foundation designer is a must.

Vibration problems have been drawing attention of scientists and engineers, since decades, world over to find ways and means to have desired satisfactory performance of machines and to minimize failures. In the past, due importance was not given to the machine foundation design. Simple methods of calculation were used for strength design of the foundation by multiplying static loads with an estimated **Dynamic Factor**. This resulted in consideration of increased static loads without any knowledge of actual safety factor. Even with these so-called excessive loads, harmful effects were observed during operation. Based on the scientific investigations carried out in the last few decades it has been established that it is not enough to base the design on vertical loads only, multiplied by an arbitrary Dynamic Factor.

Improvement in manufacturing technology has provided machines of higher ratings with better tolerances and controlled behaviour. The increased dependence of society on machines provides no room for failure and demands equipment and systems with higher performance reliability. All problems could not be solved theoretically because a good amount of assumptions had to be made for the analysis and these assumptions needed validation through experiments. Laboratory and field measurements were thus introduced to determine carefully the effects of various parameters on the dynamic response of machine foundation. Thus a detailed vibration analysis became necessary. It was also realised that a careful dynamic investigation of soil properties is essential as the elastic properties of the soil exercise considerable influence on the design of the foundation.

It is obvious that the cost of machine foundation is a small fraction of that of the equipment and inadequately constructed foundations may result in failures and shutdowns whose cost itself may exceed many times the cost of the properly designed and built foundations. Though, advanced computational tools are available for precise evaluation of dynamic characteristics of machine–foundation system, their use in design office, which was limited in the past, has now been found to be quite common.

The machine foundation system can be modeled either as a two-dimensional structure or three-dimensional structure. For mathematical modeling and analysis, valid assumptions are made keeping in view the following:

- The mathematical model should be compatible to the Prototype structure within a reasonable degree of accuracy
- The mathematical model has got be such that it can be analysed with the available mathematical tools
- The influence of each assumption should be quantitatively known with regard to the response of the foundation

Vibration isolation techniques have also been used to reduce vibrations in the machines. Isolation leads to reduction in the transmissibility of the exciting forces from the machine to the foundation and vice-versa. Uses of vibration isolation devices is one of the methods by which one can achieve satisfactory performance which in turn can result in minimising failures and reduce downtime on account of high vibrations. However, for equipment on elevated foundations, it is desirable to have support structure stiffness sufficiently higher than overall stiffness of isolation system in order to get the desired isolation efficiency.

The support structure, a 3-D elevated structural system, possesses many natural frequencies. The vibration isolation system, comprising of machine, inertia block and the isolation devices, also has six modes of vibration having specific stiffness values corresponding to each mode of vibration. Hence the comparison between stiffness of structure and isolator becomes complex task. It is of interest to note that lateral stiffness of elevated structures is very much lower than its vertical stiffness. If this lower (lateral) stiffness is comparable to the stiffness of isolators, it certainly affects the overall stiffness and thereby the response of the machine foundation system. Hence, lateral stiffness of support structure must also be computed and considered while selecting the isolators. Finally it may be desirable to carry out detailed dynamic analysis of the complete system including substructure.

1.2 DESIGN PHILOSOPHY

Machine foundation system, **in broader sense**, comprises of machine, supported by foundation resting over soil subjected to dynamic loads i) generated by machine itself; ii) applied externally, or iii) caused by external sources and transmitted through soil. A typical system is as shown in Figure 1.2-1.

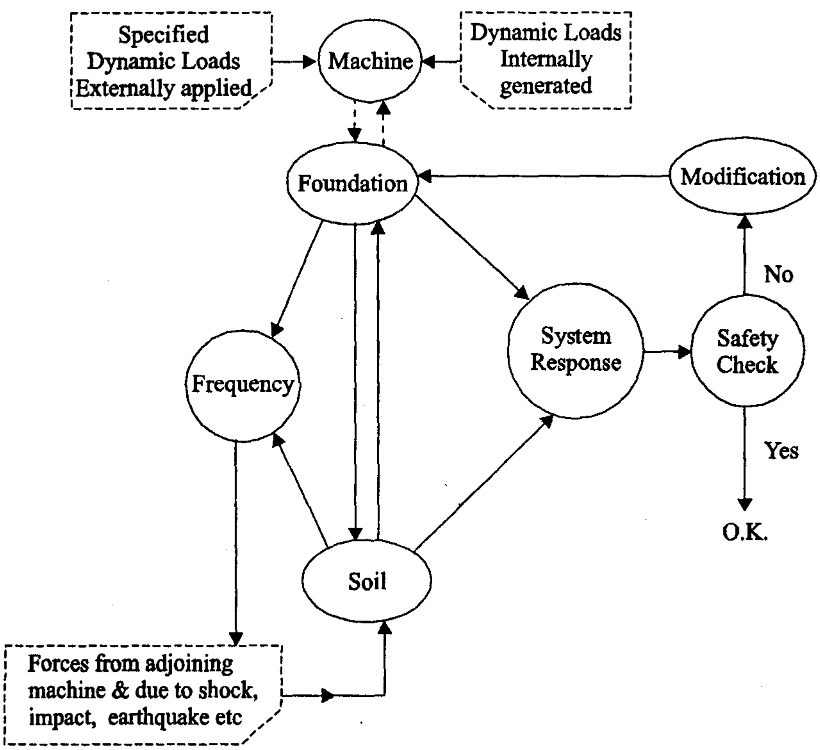


Figure 1.2-1 Machine Foundation System Qualification Subjected to Dynamic Loads

Irrespective of the source of dynamic load, the basic philosophy underlying design of machine foundation is that:

- The dynamic forces of machines are transmitted to the soil through the foundation in such a way that all kinds of harmful effects are eliminated and the amplitudes of vibration of the machine as well as that of the foundation are well within the specified limits.

- Foundation is structurally safe to withstand all static and dynamic forces generated by the machine. To accomplish these objectives, every foundation needs to be analysed for Dynamic Response, and thereafter for Strength Design

1.3 MACHINE FOUNDATION SYSTEM

In any machine foundation system, the **equipment (the machine)** is considered supported by a **foundation** and the foundation in turn rests on the **soil**. A typical machine foundation system is as shown in Figure 1.3-1.

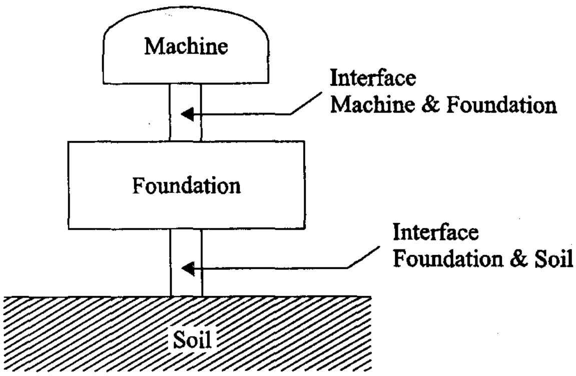


Figure 1.3-1 A Typical Machine Foundation System

At this stage it is necessary to address as to how the equipment, foundation and soil are interconnected.

- Machine could either be connected to the foundation directly through the foundation bolts, or it could be connected through isolation devices.
- Foundation could either be a solid block resting directly on the soil or it may be resting on the piles.
- The foundation could also be a frame structure (Frame Foundation) resting directly on the soil or it could be resting on the group of piles.

These interfaces, therefore, are essential to be appropriately addressed, for evaluating the dynamic response of the machine correctly. Thus, the three main constituents of machine foundation system that play significant role in overall controlling machine performance are, **machine, foundation and soil** and these need to be adequately addressed. **Modeling and Analysis** are adequately covered in **Chapter 8.**

1.4 MACHINES

Based on type of motion, the machines are broadly classified as:

a) Rotary Machines
b) Reciprocating Machines
c) Impact Type Machines

Based on the speed of operation, the machines are grouped as:

a) Very low speed machines (up to 100 rpm)

b) Low speed machines (100 to1500rpm)

c) Medium speed machines (1500 to 3000rpm)

d) High speed machines (3000 rpm and above)

For foundation design, broadly, the following information is needed:

- Geometric configuration of the machine
- Loads from machine: Mass of the stationary as well as rotating parts of the machine and load-transfer mechanism from the machine to the foundation
- Critical machine performance parameters: Critical speeds of rotors, balance grade and acceptable levels of amplitudes of vibration
- Dynamic forces generated by the machine: Forces generated under various operating conditions and their transfer mechanism to the foundation for dynamic response analysis
- Additional Forces: Forces generated under emergency or faulted conditions, Test condition, Erection condition & Maintenance condition of the machine, Forces due to bearing failure (if applicable) for strength analysis of the foundation

These parameters are covered in detail in **Chapter 6**.

1.5 FOUNDATION

Machine type and its characteristics do play a significant role while selecting the type of foundation. Most commonly used foundations in the industry are Block foundations and Frame foundations that are covered in this handbook.

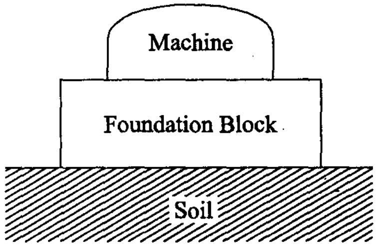


Figure 1.5-1 A Typical Block Foundation

1.5.1 Block Foundation

In this case, machine is mounted over a solid block, generally made of concrete. This block in turn rests directly on the soil. In this case both machine and foundation block are considered as non-elastic inertia bodies and the soil is treated as mass less elastic media i.e. having only stiffness and no inertia. Schematic view of a typical block foundation is shown in Figure 1.5-1.

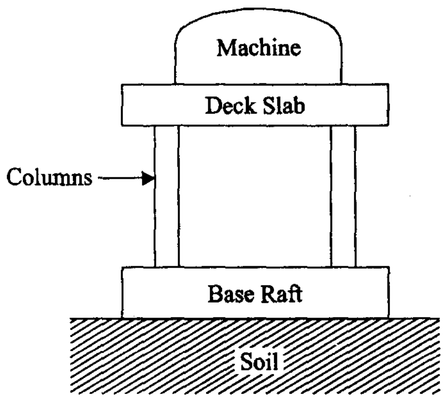


Figure 1.5-2 A Typical Frame Foundation

1.5.2 Frame Foundation

In this type of foundation, machine is supported on the deck slab. This deck slab in turn is supported on base raft through columns and base raft rests directly over soil or on group of piles. Size of deck slab, number of columns, height of columns above base raft etc. are primarily dependent on machine layout. In this case machine is treated as non-elastic inertia body whereas deck slab, and columns are considered as elastic inertia bodies and soil is considered as elastic media. In certain specific cases, base raft is also considered as elastic inertia body. Schematic view of a typical frame foundation is shown in Figure 1.5-2.

1.5.3 Tuning of the Foundation

Foundation, for which its vertical natural frequency is above the operating speed of the machine, is termed as **over-tuned foundation** or high-tuned foundation and the foundation, for which its vertical natural frequency is below the operating speed of the machine, is termed as **under-tuned foundation** or low-tuned foundation.

1.5.4 Foundation Material

Plain Concrete, Brick, Reinforced Cement Concrete, Pre-Stressed Concrete and Steel are the material employed for machine foundation construction. Foundations using steel structures have also been used for frame foundations. The sizes of structural members in steel foundations are less than those for RCC foundations and accordingly their space requirement is much less. As regards vibration, steel structures undoubtedly involve higher risk. Natural frequencies are low and the foundation is deeply under-tuned. The resistance to fire of a steel structure is lower than that of reinforced concrete one. Most high tuned foundations are built of reinforced concrete. Vibration amplitudes are reduced due to relatively higher damping present in the concrete.

1.5.5 Foundation Analysis and Design

Every foundation is analysed for its dynamic response and checked for strength and stability. Using the machine, soil and foundation parameters, amplitudes of vibration are computed at machine as well as foundation level. In addition foundation is designed for its strength and stability to withstand applicable static and dynamic forces. For this the dynamic forces of the machine are translated into equivalent static forces on the foundation. Strength check of the foundation is also done for forces due to environmental effects like wind & earthquake etc.

Should the strength analysis indicate need for change in the foundation size, a recheck on the dynamic analysis with the revised foundation size is a must. Typical foundation parameters needed for design of machine foundation system are:

- Foundation geometry
- Material properties i.e. mass density, dynamic modulus of elasticity, Poisson's ratio, coefficient of thermal expansion, etc.
- Strength parameters i.e. Yield stress, UTS, Allowable stress in compression, tension, bending and shear, etc.

These parameters in detail are covered in **chapter 7**. Construction aspects of these foundations are covered in **Chapter 13**.

1.6 SOIL

It is an established fact that the soil properties significantly influence the dynamic response of machine foundation system. Identical machines with identical foundations have been reported to behave differently in different soil conditions. For block foundation, the soil influence is predominant. The dynamic response largely depends upon mass of the machine, mass of the block, the geometry of the block and soil dynamic properties. However for frame foundations, it is generally reported that consideration of soil structure interaction i) induces additional modes pertaining to soil deformation with relatively low frequencies and ii) has a tendency to marginally enhance structural frequencies.

Soil system is a complex entity in itself and there are many uncertainties associated with its modeling. Correct evaluation of dynamic soil properties, however, is the most difficult task. These properties may vary from site to site, from location to location and from machine to machine as well as with variation of depth of foundation. Under the influence of dynamic forces, the foundation interacts with the soil activating dynamic soil structure interaction, which significantly influences the dynamic response of machine foundation system.

Depending upon type of analysis, soil is represented as an elastic half space with the help of equivalent soil springs represented by elastic sub-grade reaction coefficients. Typical soil parameters and dynamic properties of soil used in machine foundation design are:

E Young's Modulus of Elasticity

G Shear modulus

ν Poisson's ratio

ρ Mass density

ζ Soil damping

C_u Coefficients of uniform compression of the soil

C_ϕ Coefficients of non-uniform compression of the soil

C_τ Coefficients of uniform shear of the soil

C_ψ - Coefficients of non-uniform shear of the soil

The significant aspects of soil properties, which influence soil structure interaction, are: Energy Transfer Mechanism, Soil Mass Participation in Vibration of Foundations, Effect of Embedment of Foundation, Applicability of Hook's Law to Soil, Reduction in Permissible Soil Stress and Dynamic Soil Parameters.

These influences have suitably been addressed in **Chapter 5**.

1.7 VIBRATION ISOLATION

Isolation means reduction in the transmissibility of the exciting forces from the machine to the foundation and vice-versa. Vibration isolation devices have been used to achieve satisfactory performance. Isolation in broader sense includes the following:

- Control of transmission of dynamic forces from machine to the foundation and thereby to the adjoining structures and equipment
- Isolation of equipment from the vibration effects of the adjoining system
- Isolation from external forces like Earthquake Shock, Blast etc.

For cases, where a bunch of vibratory machines are to be mounted on a common elevated platform, vibration isolation may turn out to be a better proposition. Vibration Isolation Design for machine foundation systems includes, isolation requirement, isolation design, selection of isolation devices, influence of sub-structure (wherever applicable) on the response, etc.

Basic theory of Vibration Isolation is dealt in **Chapter 4** and the design of foundations with Vibration Isolation System is covered in **Chapter 12**.

1.8 FIELD PERFORMANCE AND FEED BACK

It goes without saying that proof of the pudding is in eating only. A feed back from the site for the machine's performance therefore is essential. The data needs to be recorded at frequent intervals at site, compiled over a period of time and feedback provided to design office for drawing necessary inferences from the same and use these for design updates.

It is the general practice in the industry to pay more attention only to those machines that do not perform well. More often than not, for every malfunction one keeps on trying modifications in the machine like better balancing, replacing bearings etc till satisfactory results are achieved. It is worth noting that every time the malfunction occurs the cause may not be machine alone but it could be foundation too. In certain cases, desired results could be achieved by correcting the source, which may be other than the machine.

In the opinion of the author, the data for healthy machines also need to be studied at regular interval and feedback given to designers. This will certainly help in improving design methodologies.

Some case studies are covered in **Chapter 14.**

PART - I

THEORY OF VIBRATION

Basic Understanding with Specific Application To Machine Foundation Design

2. Single Degree of Freedom System
3. Multi Degree of Freedom System
4. Vibration Isolation

2

SINGLE DEGREE OF FREEDOM SYSTEM

- Free and Forced Vibration
- Undamped System
- Damped System
- Equivalent SDOF System - Columns and Beams
- Dynamic Load Externally Applied
- Dynamic Load Internally Generated
- Impact Loads
- Impulsive Loads

- **Example Problems**

SINGLE DEGREE OF FREEDOM SYSTEM

In order to understand the dynamic behaviour of machine foundation system, the knowledge of theory of vibration is essential. Simplest system for basic understanding of vibration is a **spring mass system.** For understanding basic vibration, let us consider some of the aspects associated with vibration.

- Degree of **Freedom System:** Number of coordinates required to locate displaced position of the mass is called its **Degree Of Freedom** (DOF)
- **Single Degree of Freedom System:** A system is said to be a Single Degree of Freedom System (SDOF) when the displaced position of the mass is expressed by a single coordinate. For example, a one spring mass system as shown in Figure 2-1, is a Single Degree of Freedom (SDOF) system as the deformation of the spring takes place only in one direction and the displaced position of the mass m is defined by a single coordinate. For the system as shown in Figure 2-1, the degree of freedom is y coordinate of center of mass m
- **Free vibration:** A structural system, when disturbed from its position of equilibrium and released, oscillates about its mean position of equilibrium. This state of vibration of the structure without any external excitation force is termed as **free vibration**
- **Forced vibration:** If a system vibrates under the influence of an applied dynamic (time dependent) force, it is termed as **forced vibration** of the system
- **Damping:** Any engineering system, when disturbed from its position of rest, will show vibration, which will die out eventually with time. The process by which vibration steadily diminishes in amplitude is called damping. In other words it can be said that every physical system has inherent **damping** associated with it. If we ignore damping, the system is called **undamped system** and if damping is considered, it is called **damped system**

For better understanding of the dynamic behaviour of the SDOF system, **the spring properties are considered linear** and the presentation is developed in stages. Each stage contains its **Mathematical Treatment and Example Problems.** The stages considered are:

1) Free vibration
   - ❑ Undamped System
   - ❑ Damped System
2) Forced vibration

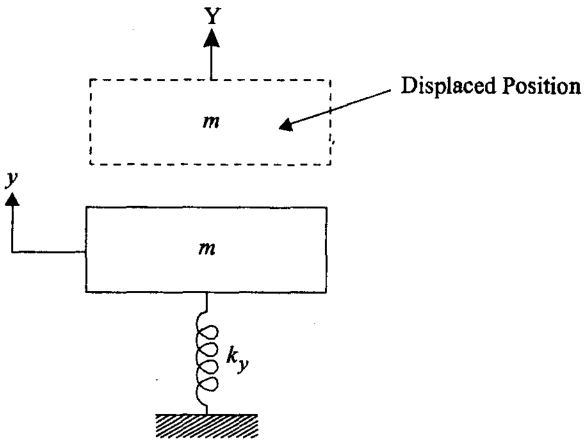


Figure 2-1 Single Degree of Freedom (SDOF) System

2.1 FREE VIBRATION

2.1.1 Undamped System - SDOF Spring Mass System

Let us consider two types of SDOF spring mass system:

i) System with Translational DOF and
ii) System with Rotational DOF

2.1.1.1 SDOF System - Spring Having Translational Stiffness

Unidirectional Translational Stiffness along Y-direction

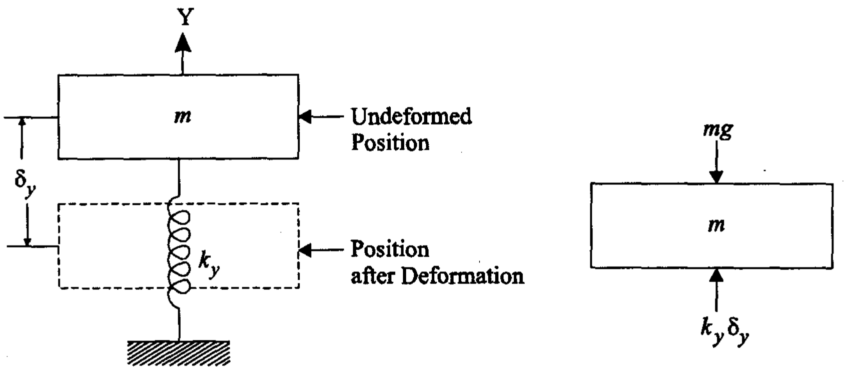
Consider one spring mass (SDOF) system without damping as shown in Figure 2.1.1-1. The system has mass m and unidirectional spring in y direction having stiffness k_y. Before we consider vibration of this SDOF system, let us consider the system under static equilibrium i.e. mass at rest position.

System at Rest i.e. Static Equilibrium Position: The gravity force acting on the mass is mg. Here g is acceleration due to gravity acting downward in (–) Y direction. Under this force, the spring deflects by an amount δ_y in (-) Y direction. This deformed position of the mass is termed as position of static equilibrium (also termed as mean position) and is shown in Figure 2.1.1-1 (a).

Let us now consider equilibrium of forces at DOF location i.e. center of mass location for this system at rest. The free body diagram showing forces acting on the mass is shown in Figure 2.1.1-1 (b). Considering equilibrium, we get

$$k_y \delta_y = mg \tag{2.1.1-1}$$

Solving, we get $$\delta_y = \frac{mg}{k_y} \tag{2.1.1-1a}$$



(a) System at Rest (b) Free Body Diagram

Figure 2.1.1-1 Undamped SDOF System-Static Equilibrium

System under Motion: Let us now impart a motion to the system at rest position. Let us disturb the mass by pulling/pushing it slightly (by an amount y) along Y and release it. The mass starts **oscillating about the mean position** i.e. position at rest as shown in Figure 2.1.1-2.

The displaced position of the mass at any instant of time t is shown in Figure 2.1.1-3. Consider that at any instant of time t, the position of the mass is at a distance y upward from the mean position as shown in Figure 2.1.1-3. Let us consider the equilibrium at the DOF location i.e. center of mass at mean position. Forces acting on the mass, as shown in Figure 2.1.1-3 (b) are:

a) Inertia force $m\ddot{y}$

b) Elastic resisting force (spring force) $k_y y$

It is important to note that all the internal forces i.e. inertia force and elastic resisting force oppose the motion. Accordingly their direction of application is in the direction opposite to direction of motion of the mass. For the displaced position of the mass in (+) Y direction, as shown in Figure, these forces act opposite to direction of motion i.e. (–) Y direction.

> ***It is to be noted that the motion of the mass is about position of static equilibrium i.e. the position where the gravity force and corresponding spring reaction force are in equilibrium. Hence these forces do not appear in the set of forces considered for dynamic equilibrium condition. In other words we can also say that the gravity force and the corresponding spring reaction force do not contribute to vibration of the spring mass system.***

Thus the net forces acting on the mass are only inertia force and elastic resisting force (spring reaction force). Free body diagram for the mass m under set of these forces is shown in Figure 2.1.1-.3 (b).

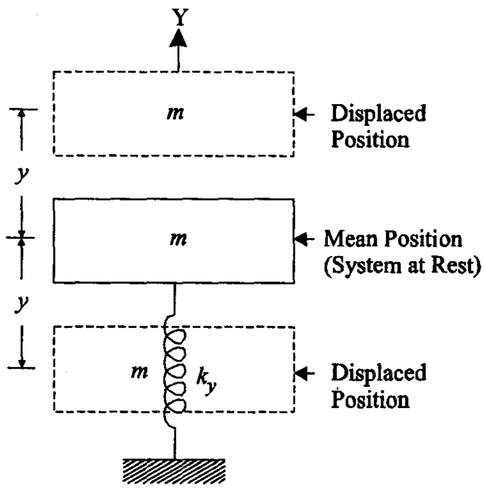


Figure 2.1.1-2 Undamped SDOF System-System under Motion

Considering the equilibrium of all forces acting on the mass at any instant t, the equation of motion is written as:

$$m\ddot{y} + k_y y = 0 \tag{2.1.1-2}$$

This equation is called **equation of motion** of free vibration of an undamped SDOF System.

The solution to equation of motion **(See SOLUTION 2.1.1-2)** gives **natural frequency** p_y and **amplitude of free vibration** ρ_y of the mass.

Natural frequency $$p_y = \sqrt{k_y / m} \text{ rad/s} \qquad (2.1.1\text{-}3)$$

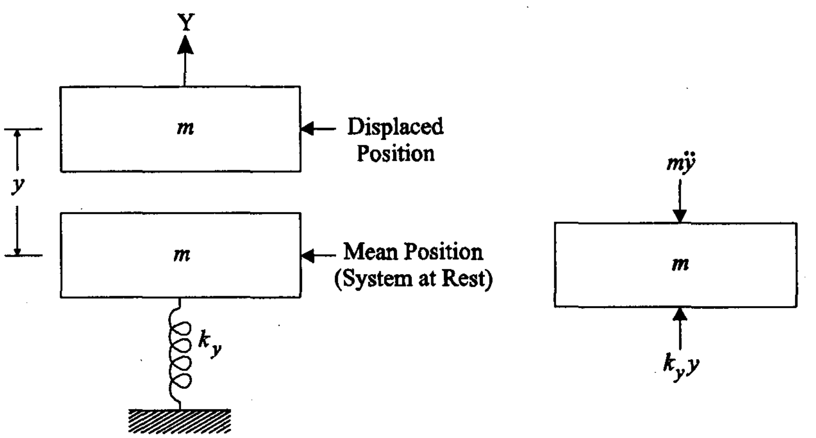
We can also express this in terms of static deflection δ_y and acceleration due to gravity g. Substituting equation (2.1.1-1) in (2.1.1-3) it gives

$$p_y = \sqrt{g / \delta_y} \text{ rad/s} \qquad (2.1.1\text{-}4)$$

Since system is undamped, it will continue vibrating indefinitely. The vibration motion of the mass depends upon initial conditions. For initial conditions $y(t) = y(0)$ & $\dot{y}(t) = \dot{y}(0)$ at time $t = 0$, the motion of the mass and maximum amplitude of free vibration are given as

$$y = y(0)\cos p_y t + \frac{\dot{y}(0)}{p_y}\sin p_y t \qquad (2.1.1\text{-}5)$$

Maximum amplitude as $$\rho_y = \sqrt{\left(y(0)^2 + \left(\frac{\dot{y}(0)}{p_y} \right)^2 \right)} \qquad (2.1.1\text{-}6)$$



(a) System Under Motion (b) Free Body Diagram

Figure 2.1.1-3 Undamped SDOF System-Free Vibration

Since system is undamped, it will continue vibrating indefinitely. The plot of the equation (2.1.1-5), showing motion of the mass for initial conditions $y(t) = y(0)$ & $\dot{y}(t) = \dot{y}(0)$ at time $t = 0$, is shown in Figure 2.1.1-4.

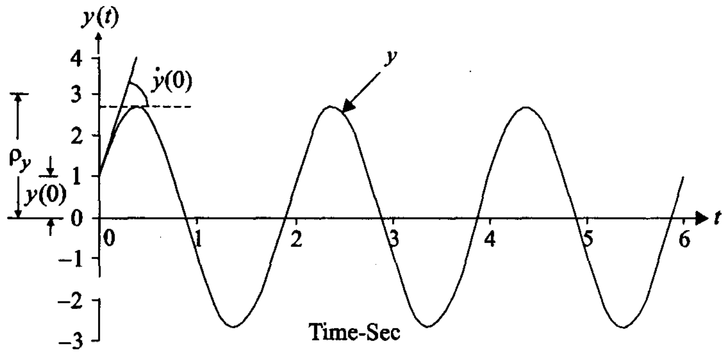


Figure 2.1.1-4 Free Vibration Response of SDOF System

SOLUTION 2.1.1-2

Rewriting equation (2.1.1-2) $$m\ddot{y} + k_y y = 0 \qquad (a)$$

For solution of the equation, let y *be represented as* $y = e^{st}$

Thus $$y = e^{st}; \qquad \ddot{y} = s^2 e^{st} \qquad (b)$$

Substituting equation (b) in equation (a), equation becomes $(ms^2 + k_y)e^{st} = 0$

Since exponent e^{st} *is not zero, therefore for solution to exist,* $(ms^2 + k_y) = 0$

This gives two values of s $$s = \pm i\sqrt{\frac{k_y}{m}}$$

Denoting $p_y = \sqrt{\frac{k_y}{m}}$ *, the solution takes the form* $$y = e^{\pm i p_y t} \qquad (c)$$

Here p_y *represents the frequency of free vibration or natural frequency in rad/s*

$$p_y = \sqrt{\frac{k_y}{m}} \text{ rad/s}; \qquad or \qquad f_y = \frac{1}{2\pi}\sqrt{\frac{k_y}{m}} \quad Hz \qquad (d)$$

Where f_y *is the cyclic frequency in cycles/sec (Hz)*

The solution (c) is thus written as $$y = A_1 e^{ip_y t} + A_2 e^{-ip_y t}$$

Using De Moivre's theorem, the equation is rewritten as

$$y = A\cos p_y t + B\sin p_y t \qquad (e)$$

The values of A & B are obtained using initial conditions.

Considering initial displacement $y(t) = y(0)$ *and velocity* $\dot{y}(t) = \dot{y}(0)$ *at time t=0 and substituting in (e), we get* $A = y(0)$ & $B = \dfrac{\dot{y}(0)}{p_y}$

Solution becomes

$$y = y(0)\cos p_y t + \frac{\dot{y}(0)}{p_y}\sin p_y t \qquad (f)$$

Maximum amplitude

$$\rho_y = \sqrt{\left(A^2 + B^2\right)} = \sqrt{\left(y(0)^2 + \left(\frac{\dot{y}(0)}{p_y}\right)^2\right)} \qquad (g)$$

For specific initial condition of $y(t) = y(0)$ & $\dot{y}(t) = 0$;

Equation (f) becomes

$$y = y(0)\cos p_y t \qquad (h)$$

It is to be noted that equation (2.1.1-2) $m\ddot{y} + k_y y = 0$ ***is the equation of motion for translational DOF*** y. ***For system with translational DOF*** x ***or*** z, ***this equation & its solution need to be modified by replacing*** y ***with*** x ***or*** z ***respectively.***

2.1.1.2 SDOF System - Spring Having Rotational Stiffness (Rocking Stiffness) Connected At CG of Base Area of the Block

Consider the block having mass m and Mass Moment of Inertia M_{mz} about centroid Z-axis passing through centroid C. A rotational spring having rotational stiffness k_ϕ is attached to the support point (center of the base point) O of the block as shown in Figure 2.1.1-5. The block is constrained such that it cannot move either in X or Y direction but it can **only rotate** in XY plane about Z-axis (perpendicular to plane of paper) passing through O. The centroid C is at a distance h from the center of the base of the block O. The DOF of the system is rotation ϕ at point O.

Static Equilibrium: Let us first consider the position of the mass at rest i.e. mean position of the mass. The gravity force mg is taken care of by reaction R at support point O.

System under motion: Let us disturb the block so as to cause it to rotate slightly about point O and then release the block. The block shall start oscillating about the mean position (i.e. position at rest) at point O in X-Y plane. Consider that at any instant of time t, the block position is rotated anti-clockwise by an angle ϕ as shown in Figure 2.1.1-6. Due to rotation, centroid point C moves to new location point C'. Rotation ϕ at O induces rotation ϕ and translation $h\phi$ at centroid C'

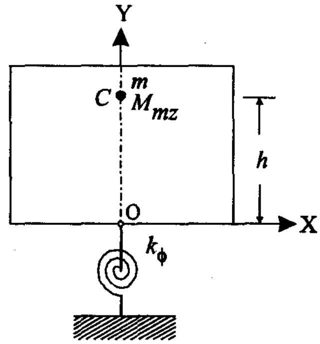


Figure 2.1.1-5 Undamped SDOF System with Rotational Spring attached at Center of Base of Block

Forces acting on the system are as shown in Figure 2.1.1-6. These are:

i) Rotational Inertia = $M_{mz}\ddot{\phi}$ acting clockwise (opposite to direction of motion) at centroid C'.

ii) Translational Inertia force (along direction perpendicular to line OC') = $mh\ddot{\phi}$ (opposite to direction of motion)

iii) Resisting moment offered by spring (clock wise) at O = $k_\phi\,\phi$

iv) Moment due to self-weight (anticlock wise) at O = $mg \times h\sin\phi = mgh\phi$

(For ϕ to be small $\sin\phi = \phi$)

Considering equilibrium of forces at DOF location i.e. at point O, we get

$$\Sigma Y = 0 \qquad mg - R = 0 \quad or \quad mg = R \tag{2.1.1-7}$$

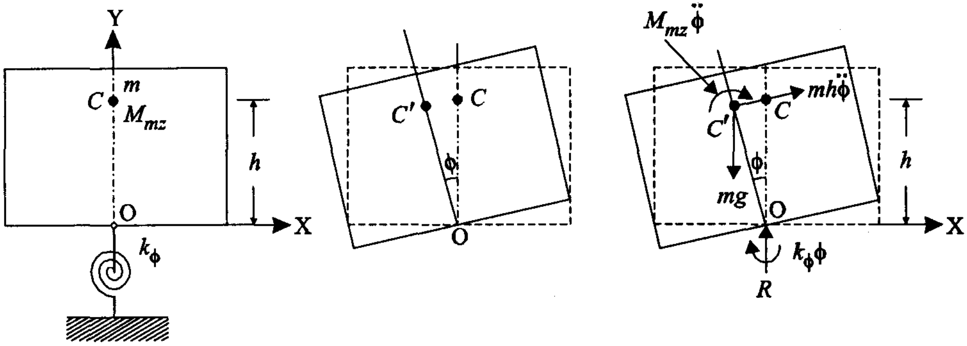
$$\Sigma M_z = 0 \qquad (M_{mz}\ddot{\phi} + mh\ddot{\phi} \times h) + k_\phi\phi - mgh\phi = 0 \tag{2.1.1-8}$$

Rearranging terms, we get

$$(M_{mz} + mh^2)\ddot{\phi} + (k_\phi - mgh)\phi = 0$$

$$\text{Or} \qquad M_{moz}\,\ddot{\phi} + (k_\phi - mgh)\,\phi = 0 \tag{2.1.1-9}$$

Here $M_{moz} = M_{mz} + mh^2$ is the mass moment of inertia of the block about Z-axis passing through support point O. This is the equation of motion of an undamped SDOF system with rotational DOF at point O.



(a) System at Rest (b) Rotation about O (c) Free body Diagram

Figure 2.1.1-6 Undamped SDOF System with Rotational Spring attached at center of Base of Block - Free Vibration

Solution to this equation **(see SOLUTION 2.1.1-9)** gives:

Motion of the block $$\phi = A\cos p_\phi t + B\sin p_\phi t \tag{2.1.1-10}$$

$$p_\phi = \sqrt{\frac{(k_\phi - mgh)}{M_{moz}}} \text{ rad/s} \tag{2.1.1-10a}$$

Here p_ϕ represents rotational natural frequency of the system and constants A & B are evaluated using initial conditions.

For all practical real life cases, it is seen that term mgh is negligible compared to k_ϕ and can be conveniently be ignored without any loss of accuracy in the frequency value. Hence the natural frequency becomes:

$$p_\phi = \sqrt{\frac{k_\phi}{M_{moz}}} \text{ rad/s} \tag{2.1.1-10b}$$

SOLUTION 2.1.1-9

Rewriting equation (2.1.1-9)

$$M_{moz}\ddot{\phi} + (k_\phi - mgh)\phi = 0 \tag{a}$$

For solution of the equation let ϕ *be represented as*

$$\phi = e^{st} \quad Thus \quad \ddot{\phi} = s^2 e^{st} \tag{b}$$

Substituting equation (b) in equation (a), equation becomes

$$(M_{moz}\, s^2 + (k_\phi - mgh))e^{st} = 0 \qquad (c)$$

Since exponent e^{st} is not zero, therefore for solution to exist,

$$(M_{moz}\, s^2 + (k_\phi - mgh)) = 0 \qquad (d)$$

This gives two values of s $\quad s = \pm i\sqrt{\dfrac{(k_\phi - mgh)}{M_{moz}}} \qquad (e)$

Denoting $\quad p_\phi = \sqrt{\dfrac{(k_\phi - mgh)}{M_{moz}}}$ *, the solution (b) takes the form*

$$\phi = e^{\pm i\, p_\phi\, t} \qquad (f)$$

This can be rewritten as $\quad \phi = A_1 e^{i\, p_\phi\, t} + A_2 e^{-i\, p_\phi\, t} \qquad (g)$

Using De Moivre's theorem, the equation is rewritten as

$$\phi = A\cos p_\phi\, t + B\sin p_\phi\, t \qquad (h)$$

The values of constants A & B are obtained using initial conditions.

It is to be noted that equation (2.1.1-9) $M_{moz}\, \ddot{\phi} + (k_\phi - mgh)\, \phi = 0$ is the equation of motion for Rocking about Z-axis i.e. DOF ϕ. For system with Rocking DOF θ, this equation & its solution need to be modified by replacing M_{mz} with M_{mx}, M_{moz} with M_{mox} and k_ϕ with k_θ i.e. appropriately replacing all ϕ parameters with θ parameters.

2.1.1.3 SDOF System - Spring Having Rotational Stiffness (Torsional Stiffness) Connected At CG of Base Area of the Block

Consider a block having mass m and Mass Moment of Inertia M_{my} about Y-axis passing through CG of the Base Area. A Torsional spring having rotational stiffness k_ψ is attached to the block at the CG of the Base area point O as shown in Figure 2.1.1-7. The block is constrained such that it can neither move in X nor in Z direction but it can **only rotate** in XZ plane about Y-axis passing through O.

Proceeding on the similar lines, we get equation of motion as

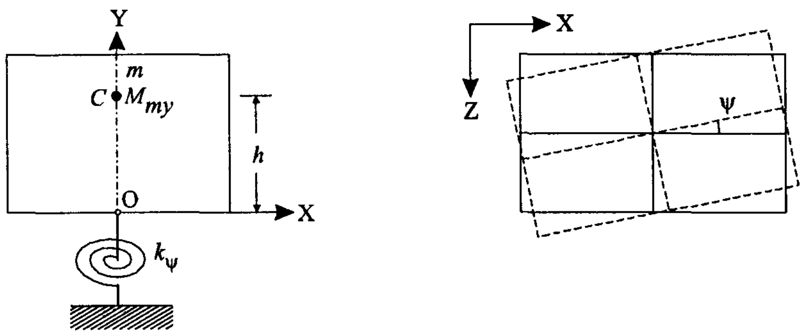
$$M_{moy}\, \ddot{\psi} + k_\psi\, \psi = 0 \qquad (2.1.1\text{-}11)$$

Here $M_{moy} = M_{my}$ is the mass moment of inertia of the block about Y-axis passing through support point O. This is the equation of motion of an undamped SDOF system with Torsional DOF at point O.

Solution to this equation gives:

$$\text{Motion of the block} \qquad \psi = A\cos p_\psi t + B\sin p_\psi t \qquad (2.1.1\text{-}12)$$

Here $p_\psi = \sqrt{\dfrac{k_\psi}{M_{moy}}}$ rad/s represents Torsional natural frequency of the system. Constants A & B are evaluated using initial conditions.



(a) System at Rest (b) Rotation about Y-axis

Figure 2.1.1-7 Undamped SDOF System with Torsional Spring attached at center of Base of Block- Free Vibration

2.1.1.4 Equivalent SDOF Systems

Any physical system, for the purpose of analysis, needs to be modeled mathematically and the model must represent the prototype nearly truly i.e. the mathematical model and prototype must be equivalent.

For mathematical modeling, the machine is generally considered as rigid body consisting of only mass whereas the foundation is considered a) as a rigid body having only mass as in the case of block foundation and b) as an elastic body having both mass and stiffness for the case where machine is supported on structural system comprising of beams and columns.

Block Foundation: In case of block foundation resting directly over soil the foundation rigidity is much higher compared to that of the soil. Thus the foundation is considered as a rigid body. Hence only mass of the block needs to be accounted for.

Machine supported on Structural System: For cases, where machine is supported over a column, a beam, a portal frame or their combination, contribution of both mass and stiffness of the support system become significant. For such systems, equivalent structural mass should be added to machine mass for computation of frequency and response. Frequency computation based on massless springs would turn out to be erroneous. It is therefore desirable to develop equivalent systems, which include influence of mass content of support system.

Following basic systems are considered and equivalent SDOF system developed for each of these:

i. A column supporting the mass
ii. A cantilever beam supporting the mass
iii. A simply supported beam supporting the mass
iv. A fixed beam supporting the mass
v. A Portal Frame supporting the mass
   a) Vertical Motion
   b) Transverse Motion
vi. A Rigid Beam Supported by number of columns – Lateral Motion

2.1.1.4.1 A column supporting the mass – axial motion

Consider a mass m supported by a column having cross-section area A, Elastic modulus E height h and mass m_c as shown in Figure 2.1.1-8 (a). Consider that the system is constrained to move only in vertical Y direction.

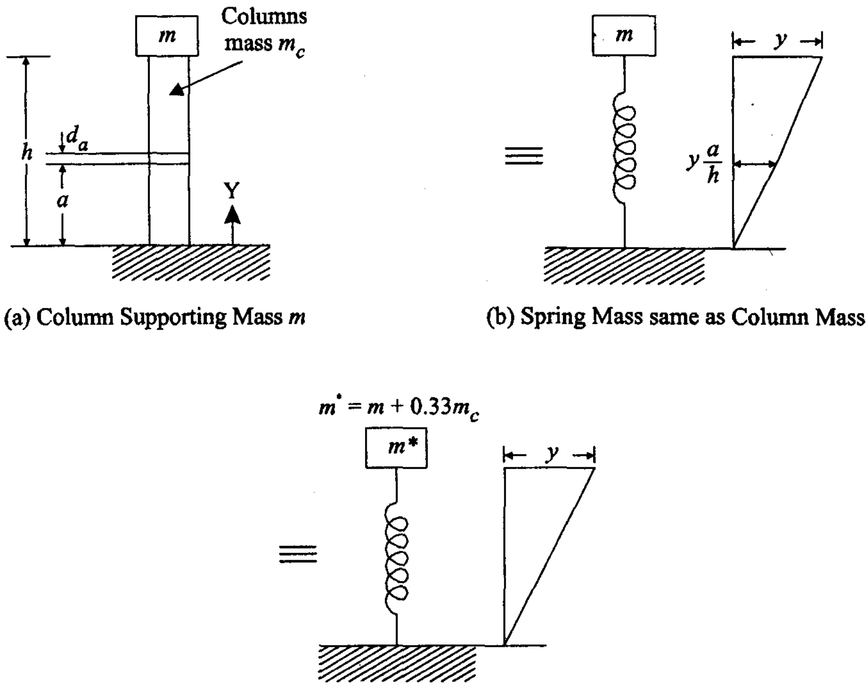
Column stiffness in Y direction = axial stiffness of column = $k_y = \dfrac{E \times A}{h}$ N/m

Representing column by an equivalent spring, the system represents a SDOF spring mass system as shown in Figure 2.1.1-8 (b), where spring is not a mass-less spring but has mass same as that of column i.e. mass m_c. This system is identical to the system shown in Figure 2.1.1-1 except that in this case the spring (i.e. column) has a mass m_c.

If we neglect the column mass (i.e. neglect the spring mass), we get

Natural frequency is $p = \sqrt{\dfrac{k_y}{m}}$ rad/s (2.1.1-13)

This is same as given by equation (2.1.1-3).



(a) Column Supporting Mass m

(b) Spring Mass same as Column Mass

(c) Column represented as massless Spring

Figure 2.1.1-8 Column Supporting Mass-Equivalent Mass

Now **let us include the column mass i.e. consider the spring having mass** m_c. Consider that the mass is displaced from its position of equilibrium and released. The spring mass system exhibits vibration. Let the displacement of the mass at any instant of time t be y. Consider that the variation of the displacement at spring top y to the displacement at spring bottom zero is linear. This represents column deformation y at the top to zero deformation at the column base. The displacement is shown in Figure 2.1.1-8 (b).

Consider a small element of the column at a distance a from fixed end.

Vertical displacement at this point of the column $= y\dfrac{a}{h}$

Mass of this element of column $= \dfrac{m_c}{h}da$

Kinetic energy of the spring mass system $E = \frac{1}{2}m(\dot{y})^2 + \int_0^h \frac{1}{2}\left(\frac{m_c}{h}da\right)\left(\dot{y}\frac{a}{h}\right)^2$

Here $\dot{y}$ represents the velocity

Simplifying this equation becomes

$$E = \frac{1}{2}m(\dot{y})^2 + \int_0^h \frac{1}{2}\left(\frac{m_c}{h}da\right)\left(\dot{y}\frac{a}{h}\right)^2 = \frac{1}{2}m(\dot{y})^2 + \frac{1}{2}\left(\frac{m_c}{h}\frac{1}{h^2}\right)(\dot{y})^2\int_0^h a^2 da$$

$$= \frac{1}{2}m(\dot{y})^2 + \frac{1}{2}\frac{m_c}{h}\frac{1}{h^2}(\dot{y})^2\frac{h^3}{3} = \frac{1}{2}\left\{m + \frac{m_c}{3}\right\}(\dot{y})^2 = \frac{1}{2}\left\{m^*\right\}(\dot{y})^2 \quad (2.1.1\text{-}14)$$

$$m^* = m + m_c^* = m + \frac{m_c}{3} \quad (2.1.1\text{-}15)$$

Here m_c^* is termed as **generalised mass** of the column, &

m^* is termed as **equivalent mass or generalised mass** of the total system.

This indicates that if one third of the column mass is added to the mass m then the spring could be considered as massless. The model thus becomes as shown in Figure 2.1.1-8(c).

The natural frequency of the system thus becomes

$$p = \sqrt{\frac{k}{m^*}} = \sqrt{\frac{k}{\left(m + \frac{m_c}{3}\right)}} \text{ rad/s} \quad (2.1.1\text{-}16)$$

2.1.1.4.2 A Column supporting the mass – lateral motion or a Cantilever Beam supporting the mass – flexural deformation

Lateral motion of a column supporting the mass or flexural deformation of a cantilever beam supporting the mass is the same. Let us therefore analyse the case as a cantilever beam.

Consider a mass m supported by a cantilever beam having cross-section area A, Elastic Modulus E, length L and mass m_b as shown in Figure 2.1.1-9. Consider that the beam deforms only in X-Y plane thus constraining the system such that mass m moves only in vertical Y direction.

Now consider that the beam mass system is displaced from its position of equilibrium and released, the system shall exhibit vibration.

Beam stiffness in bending: Representing beam by an equivalent spring, the system represents a SDOF spring mass system. This system is identical to the system shown in Figure 2.1.1-1 except that in this case the spring (i.e. beam) has a mass m_b. From the principles of statics, one can write equation of deflection curve of the cantilever beam.

Consider a small element of the beam d_a at a distance a from fixed end.

Mass of this element of beam $dm = \frac{m_b}{L} da$

The deflection of the beam δ_a at this location

$$\delta_a = \frac{(mg)}{6EI}\left(3a^2 L - a^3\right) \tag{2.1.1-17}$$

For a = L, we get deflection at the free end of the beam =

$$\delta_L = \frac{(mg)L^3}{3EI} \tag{2.1.1-18}$$

Beam stiffness (in bending) $k_y = \frac{mg}{\delta} = \frac{3EI}{L^3}$ N/m (2.1.1-19)

Neglecting beam mass, the natural frequency is $p = \sqrt{\frac{k_y}{m}}$ rad/s that is same as given by equation (2.1.1-3).

If the mass of the beam m_b is small (but not negligible) in comparison to the applied mass m it can be assumed with sufficient accuracy & with good level of confidence that the deflection curve of the beam during vibration has the same shape as the static deflection curve under influence of mass *m*.

Denoting the deflection of the beam at the free end as y, we get from equation (2.1.1-18)

$$y = \delta_L = \frac{(mg)L^3}{3EI}$$

Substituting in equation (2.1.1-17) we get δ_a in terms of y as $\delta_a = \frac{y}{2L^3}\left(3a^2 L - a^3\right)$

KE of the system

$$E=\frac{1}{2}m(\dot{y})^2+\int_0^L\frac{1}{2}\left(\frac{m_b\,da}{L}\right)\left\{\frac{\dot{y}}{2L^3}\left(3a^2L-a^3\right)\right\}^2$$

$$=\frac{1}{2}m(\dot{y})^2+\frac{1}{2}(\dot{y})^2\frac{m_b}{L}\left(\frac{1}{2L^3}\right)^2\int_0^L\left\{\left(3a^2L-a^3\right)\right\}^2da$$

Simplifying further, it gives:

$$E=\frac{1}{2}m(\dot{y})^2+\frac{1}{2}(\dot{y})^2\frac{m_b}{L}\left(\frac{1}{2L^3}\right)^2\frac{33}{35}L^7$$

$$=\frac{1}{2}m(\dot{y})^2+\frac{1}{2}(\dot{y})^2\frac{33}{140}m_b\quad=\frac{1}{2}\left(m+\frac{33}{140}m_b\right)(\dot{y})^2 \tag{2.1.1-20}$$

$$=\frac{1}{2}\left(m^*\right)(\dot{y})^2$$

Here $m^*=m+m_b^*=m+0.23\,m_b$ & $\dot{y}$ represents the velocity (2.1.1-21)

Where m_b^* is termed as **generalised mass** of the beam, &

m^* is termed as **equivalent mass or generalised mass** of the total system.

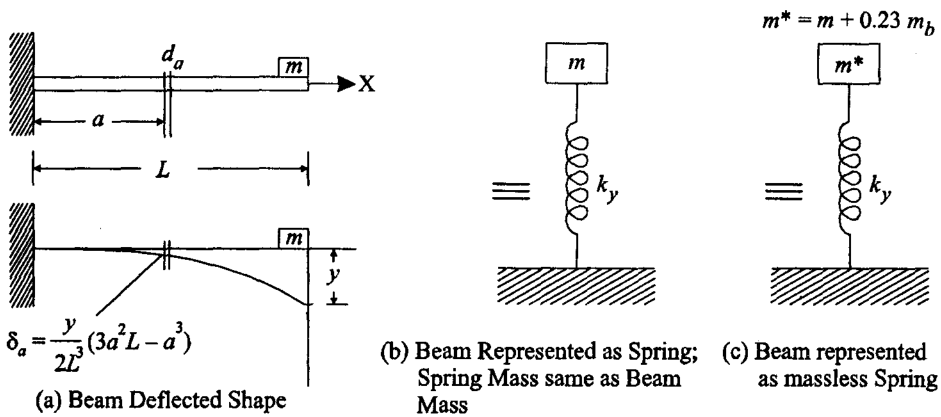


Figure 2.1.1-9 Cantilever Beam Supporting Mass *m* - Equivalent Mass

This indicates that if $(33/140\approx0.23)$ of the beam mass m_b is added to the applied mass *m* then the spring could be considered as massless. The model thus becomes as shown in Figure 2.1.1-9(c).

The natural frequency of the system thus becomes

$$p = \sqrt{\frac{k}{m^*}} = \sqrt{\frac{k}{\left(m + \frac{33}{140} m_b\right)}} \quad \text{rad/s} \qquad (2.1.1\text{-}22)$$

2.1.1.4.3 A simply supported beam supporting the mass

a) Mass at beam center location

Consider a mass m supported by a simply supported beam at center. The beam has cross-section area A, Elastic modulus E, length L and mass m_b as shown in Figure 2.1.1-10(a). Consider that the beam deforms only in X-Y plane thus constraining the system such that mass m moves only in vertical Y direction. Now consider that the beam mass system is displaced from its position of equilibrium and released, the system shall exhibit vibration.

Beam stiffness in bending: Representing beam by an equivalent spring, as shown in Figure 2.1.1-10(b) the system represents a SDOF spring mass system. This system is identical to the system shown in Figure 2.1.1-1 except that in this case the spring (i.e. beam) has a mass m_b.

From the principles of statics, one can write equation of deflection curve of the simply supported beam. Consider a small element of the beam da at a distance a from fixed end as shown.

Mass of this element of beam $\qquad dm = \frac{m_b}{L} da$

The deflection of the beam δ_a at this location

$$\delta_a = \frac{mg}{48EI}\left(3aL^2 - 4a^3\right) \qquad (2.1.1\text{-}23)$$

For $a = L/2$, we get deflection at the beam center

$$\delta_{L/2} = \frac{(mg)L^3}{48EI} \qquad (2.1.1\text{-}24)$$

Beam stiffness at beam center (in bending)

$$k_y = \frac{mg}{\delta_{L/2}} = \frac{48EI}{L^3} \quad \text{N/m} \qquad (2.1.1\text{-}25)$$

Neglecting beam mass, the natural frequency is $p = \sqrt{\frac{k_y}{m}}$ rad/s that is same as given by equation (2.1.1-3).

If the mass of the beam m_b is small (but not negligible) in comparison to the applied mass m it can be assumed with sufficient accuracy & with good level of confidence that the deflection curve of the beam during vibration has the same shape as the static deflection curve under influence of mass m.

Denoting deflection at the center of the beam as y, we get from equation (2.1.1-24)

$$y = \frac{(mg)L^3}{48EI}$$

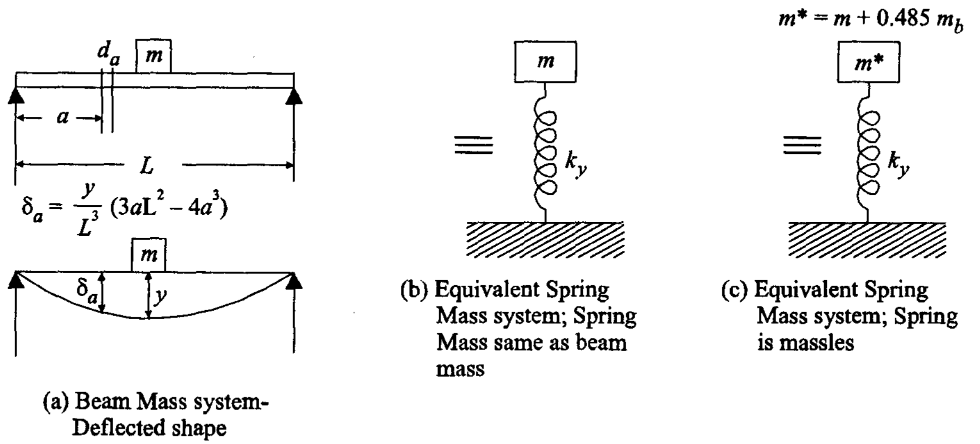


Figure 2.1.1-10 Simply Supported Beam - Mass at Center - Equivalent Mass

Substituting in equation (2.1.1-23) we get δ_a in terms of y as $\delta_a = \frac{y}{L^3}\left(3aL^2 - 4a^3\right)$

KE of the system

$$E = \frac{1}{2}m(\dot{y})^2 + 2\int_0^{L/2}\frac{1}{2}\left(\frac{m_b\,da}{L}\right)\left\{\frac{\dot{y}}{L^3}\left(3aL^2 - 4a^3\right)\right\}^2$$

Solving we get

$$= \frac{1}{2}\left(m + \frac{17}{35}m_b\right)(\dot{y})^2 \quad = \frac{1}{2}\left(m + 0.485m_b\right)(\dot{y})^2 = \frac{1}{2}\left(m^*\right)(\dot{y})^2 \qquad (2.1.1\text{-}26)$$

Where $m^* = m + m_b^* = m + 0.485m_b$

Here $\dot{y}$ represents the velocity of the mass

Here $m_b^* \,\&\, m^*$ are termed as **generalised mass** of the beam and **generalised mass** of the total system respectively.

This indicates that if $(17/35 \approx 0.485)$ of the beam mass m_b is added to the applied mass m then the spring could be considered as massless. The model thus becomes as shown in Figure 2.1.1-10(c).

The natural frequency of the system thus becomes

$$p = \sqrt{\frac{k}{m^*}} = \sqrt{\frac{k}{\left(m + \frac{17}{35} m_b\right)}} \text{ rad/s} \qquad (2.1.1\text{-}27)$$

b) Mass m at Beam Center and another Mass m_1 at off-center– Beam is mass less

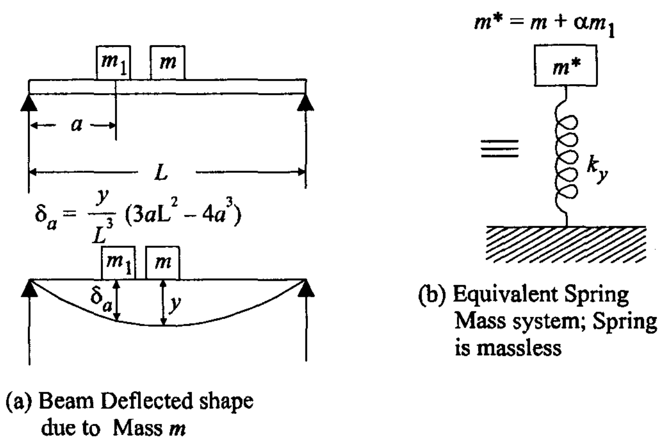


Figure 2.1.1-11 Simply Supported Beam - One Mass at Center and Other mass at Off-center -Beam Massless- Equivalent Mass of System

Consider mass m supported at beam center and another mass m_1 supported at a distance a from support as shown in Figure 2.1.1-11. In this case mass participation of the beam will depend upon **ratio a/L.**

Following the procedure as in case (a) above, we get Kinetic energy equation of the system as:

$$E=\frac{1}{2}m(\dot{y})^2+\frac{1}{2}(\dot{y})^2 m_1\left\{\frac{1}{L^3}\left(3aL^2-4a^3\right)\right\}^2=\frac{1}{2}(\dot{y})^2\left\{m+m_1\left\{\frac{3a}{L}-4\left(\frac{a}{L}\right)^3\right\}^2\right\}$$

$$E=\frac{1}{2}m^*(\dot{y})^2 \qquad (2.1.1\text{-}28)$$

$$\textit{Where} \quad m^*=m+\alpha\, m_1; \quad \alpha=\left\{\frac{3a}{L}-4\left(\frac{a}{L}\right)^3\right\}^2$$

Plot of α vs. a/L is shown in Figure 2.1.1-12

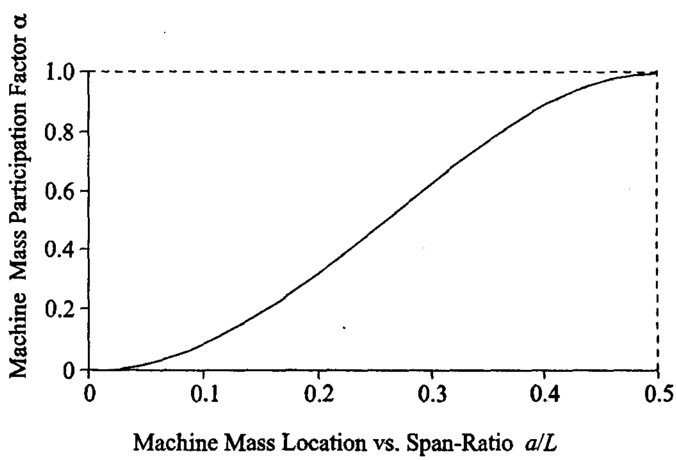


Figure 2.1.1-12 Simply Supported Beam-Mass at Off-center Location-Machine Mass Participation Factor-Beam Massless

2.1.1.4.4 A fixed beam supporting the mass - Consider Mass at beam center location

Consider a mass m supported by a fixed beam at center. The beam has cross-section area A, Elastic modulus E, length L and mass m_b as shown in Figure 2.1.1-15(a). Consider that the beam deforms only in X-Y plane thus constraining the system such that mass m moves only in vertical Y direction.

Now consider that the beam mass system is displaced from its position of equilibrium and released, the system shall exhibit vibration.

Beam stiffness in bending: Representing beam by an equivalent spring, as shown in Figure 2.1.1-15(b) the system represents a SDOF spring mass system. This system is identical to the system shown in Figure 2.1.1-1 except that in this case the spring (i.e. beam) has a mass m_b.

From the principles of statics, one can write equation of deflection curve of the fixed beam. Consider a small element of the beam 'da' at a distance a from fixed end as shown.

Mass of this element of beam $dm = \frac{m_b}{L} da$

The deflection of the beam at this location

$$\delta_a = \frac{mga^2}{48EI}(3L-4a) = \frac{4ya^2}{L^3}(3L-4a) \quad (2.1.1\text{-}29)$$

For $a = L/2$, we get deflection at the beam center =

$$y = \delta_{L/2} = \frac{(mg)L^3}{192EI} \quad (2.1.1\text{-}30)$$

Beam stiffness at beam center (in bending)

$$k_y = \frac{mg}{\delta_{L/2}} = \frac{192EI}{L^3} \text{ N/m} \quad (2.1.1\text{-}31)$$

Neglecting beam mass, the natural frequency is $p = \sqrt{\frac{k_y}{m}}$ rad/s that is same as given by equation (2.1.1-3).

Substituting equation (2.1.1-30) in equation (2.1.1-29) we get

$$\delta_a = \frac{4ya^2}{L^3}(3L-4a) \quad (2.1.1\text{-}32)$$

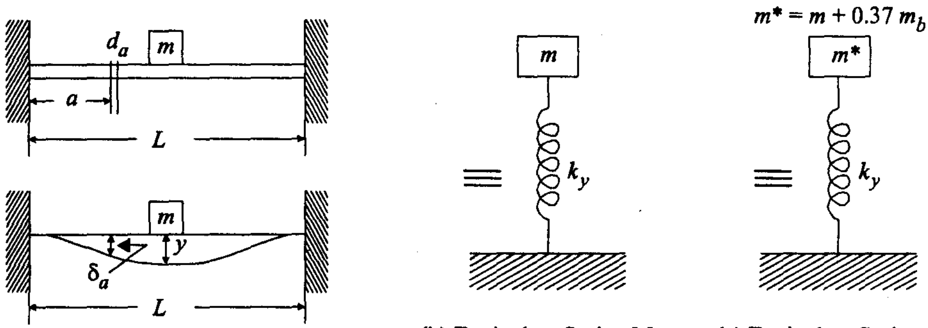
Consider that the deflection curve of the beam during vibration has the same shape as the static deflection curve under influence of mass m.

KE of the system $$E = \frac{1}{2}m(\dot{y})^2 + 2\int_0^{L/2} \frac{1}{2}\left(\frac{m_b\, da}{L}\right)\left\{\frac{4\dot{y}a^2}{L^3}(3L-4a)\right\}^2$$

Solving the equation gives us $E = \frac{1}{2}\left(m + \frac{13}{35}m_b\right)(\dot{y})^2 = \frac{1}{2}\left(m^*\right)(\dot{y})^2$ (2.1.1-33)

Where $m^* = m + m_b^* = m + 0.37m_b$ *&* $\dot{y}$ *represents velocity of the system*

Here $m_b^* \& m^*$ are termed as **generalised mass** of the beam and **generalised mass** of the total system respectively.



(a) Beam Mass System Deflected Shape

(b) Equivalent Spring Mass System; Spring Mass Same as Beam Mass

(c) Equivalent Spring Mass System; Spring is Massles

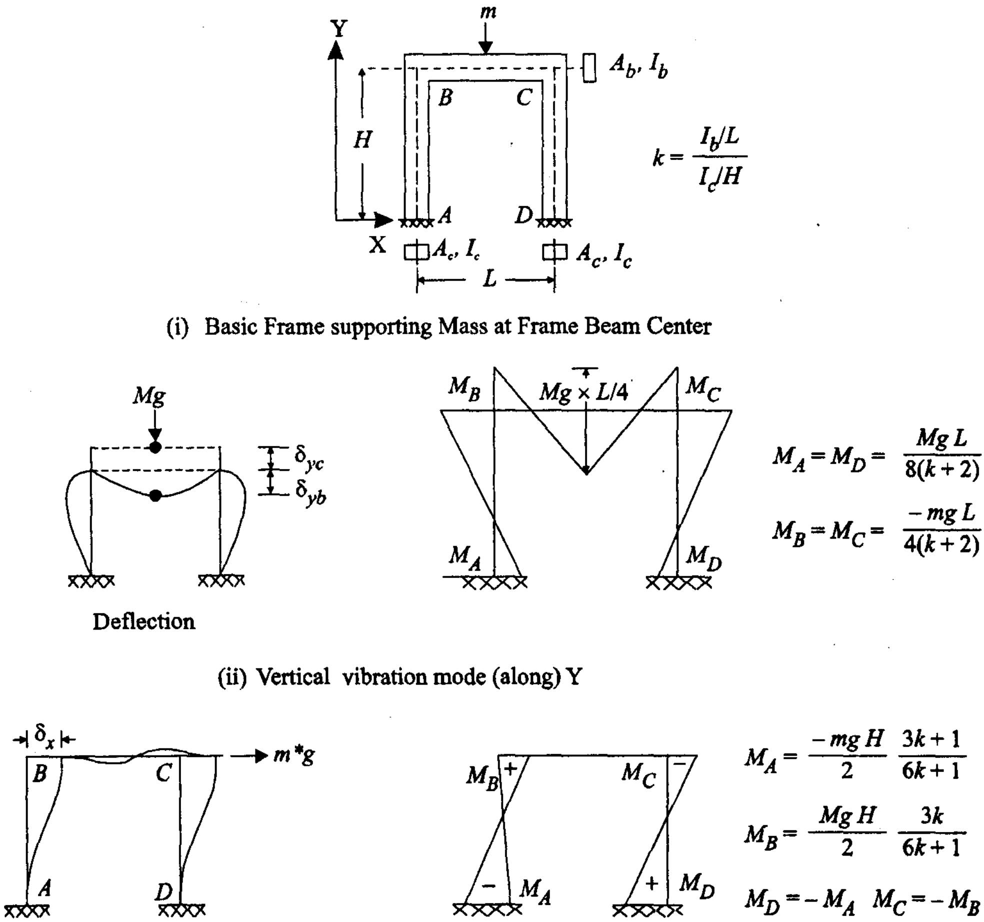
Figure 2.1.1-13 Fixed Beam - Mass at Center - Equivalent Mass

2.1.1.4.5 A Portal Frame supporting the mass

Consider a portal frame supporting mass m at beam center as shown in Figure 2.1.1-14. Material and section properties are as under:

Elastic Modulus of Material (Both column & Beam)	E
Mass density of the material	ρ
Span of Beam is	L
Height of Frame	H
Area of Beam Crossection	A_b
Area of Column Crossection	A_c
Moment of Inertia Beam Crossection	I_b
Moment of Inertia Column Crossection	I_c

Consider that portal frame is constrained to move only in X-Y plane. Possible motion directions are a) motion along Y and b) motion along X.



(i) Basic Frame supporting Mass at Frame Beam Center

(ii) Vertical vibration mode (along) Y

(iii) Transverse vibration mode (along) X

Figure 2.1.1-14 Portal Frame with Machine Mass *m* at Beam Center - Deflection and Bending Moments - Vibration in Vertical & Translational Mode

To represent the motion as SDOF system, the frame shall have one DOF for each of the motion along X and Y. For motion along Y the DOF is δ_y and for motion along X the DOF is δ_x. Let us consider these cases one by one.

Beam Stiffness factor $k_b = (I_b / L)$

Column Stiffness factor $k_c = (I_c / H)$

Beam to Column Stiffness ratio $k = \dfrac{k_b}{k_c} = \dfrac{I_b/L}{I_c/H}$

a) Motion along Y (Vertical Motion)

Degree of Freeedom δ_y

This has two components, one pertaining to beam deformation δ_{yb} and other pertaining to column deformation δ_{yc}.

i) Beam Deformation δ_{yb} :

Machine Mass on the frame Beam m

Mass of Frame Beam $m_b = \rho \times A_b \times L$

Generalised mass of Frame Beam (for vertical motion) $m_b^* = 0.45 \times m_b$

Note: For simply supported beam the factor for equivalent mass is 0.485 (see equation 2.1.1-26) and for fixed fixed beam this is close to 0.37 (see equation 2.1.1-33). For a frame this value is taken as 0.45 (close to average).

Total Effective mass on Frame Beam for deflection δ_{yb} $m^* = m + 0.45\, m_b$

Considering bending moment diagram of beam alone (as shown in the Figure), and using basics of theory of structures, we get:

a) Deflection due to span moment $\dfrac{m^* g\, L}{4}$ $\delta_{yb1} = \dfrac{m^* g\, L^3}{48\, E\, I_b}$

b) Due to support moments $\dfrac{m^* g\, L}{4(k+2)}$ $\delta_{yb2} = \dfrac{m^* g\, L}{4(k+2)} \times \dfrac{L^2}{8\, E\, I_b}$

Net beam deflection at center $\delta_{yb} = \delta_{yb1} - \delta_{yb2} = \dfrac{m^* g\, L^3}{48\, E\, I_b} - \dfrac{m^* g\, L}{4(k+2)} \times \dfrac{L^2}{8\, E\, I_b}$

$$\delta_{yb} = \frac{m^* g\, L^3}{96\, E\, I_b} \times \frac{2k+1}{k+2} \quad \text{Here } m^* = m + 0.45\, m_b \qquad (2.1.1\text{-}34)$$

ii) Column Deformation δ_{yc} :

Mass of each column $m_c = \rho \times A_c \times H$

Generalised Mass of each column (equation 2.1.1-15) $m_c^* = 0.33\, m_c$

Effective Mass on frame column top causing deflection δ_{yc} of the columns:

$$m^* = (m + m_b) + 2 \times 0.33 \times m_c$$

Vertical deflection of columns

$$\delta_{yc} = \frac{m^* \times g}{2 \times (EA_c / H)}; \quad \text{Here } m^* = (m + m_b) + 2 \times 0.33 \times m_c \qquad (2.1.1\text{-}35)$$

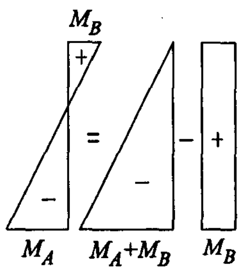
Overall vertical deflection at mass location $\delta_y = \delta_{yb} + \delta_{yc}$

Natural Frequency $p_y = \sqrt{\frac{g}{\delta_y}}$ rad/s (2.1.1-36)

b) Motion along X (Transverse Motion)

For transverse motion along X, consider toatal mass acting on the frame acting along X. For this motion only columns undergo flexural deformation and beam moves like a rigid body.

Generalised mass of each column top (equation 2.1.1-21) $m_c^* = 0.23\, m_c$



$$M_A = \frac{-mgH}{2} \frac{3k+1}{6k+1}$$

$$M_B = \frac{mgH}{2} \frac{3k}{6k+1}$$

$$M_A + M_B = \frac{-mgH}{2}$$

Deflection $\delta_x = \delta_x(M_A + M_B) - \delta_x(M_B)$

Total Effective mass on Frame column top (both columns) causing transverse deflection

$$m^* = \{(m + m_b) + 0.23 \times 2 \times m_c\} \qquad (2.1.1\text{-}37a)$$

Lateral deflection at column top this load applied along X:

Considering bending moment diagram of column (as shown in sketch above), we get:

$$\delta_x = \delta_{x(M_A+M_B)} - \delta_{x(M_B)}$$

$$\delta_x = \left\{\frac{m^* \times g \times H}{2} \times \frac{H^2}{3EI_c}\right\} - \left\{\frac{m^* \times g \times H}{2} \times \frac{3k}{6k+1} \times \frac{H^2}{2EI_c}\right\}$$

$$\delta_x = \frac{m^* \times g \times H^3}{12EI_c} \frac{2+3k}{1+6k} \qquad (2.1.1\text{-}37b)$$

Natural Frequency $\quad p_x = \sqrt{\frac{g}{\delta_x}}$ rad/s (2.1.1-38)

Representing in terms of mass and stiffness, we get:

Lateral Stiffness $\quad k_x = \frac{m^* \times g}{\delta_x} = \frac{12EI_c}{H^3} \frac{1+6k}{2+3k}$ (2.1.1-39)

Mass $\quad m^* = \{(m + m_b) + 0.23 \times 2 \times m_c\}$ (2.1.1-40)

Natural Frequency

$$p_x = \sqrt{\frac{k_x}{m^*}} = \sqrt{\frac{g}{\delta_x}} \text{ rad/s}$$ (Same as equation 2.1.1-38) (2.1.1-41)

2.1.2 Damped System

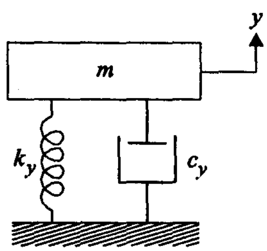


Figure 2.1.2-1 Damped SDOF System

It is well known that most engineering systems, during their vibratory motion, encounter resistance in the form of damping. Any engineering system, when disturbed from its position of rest, will show vibration, which will die out eventually with time. The process by which vibration steadily diminishes in amplitude is called damping. There are various forms of damping viz. air damping, coulomb damping, viscous damping, internal damping etc. and for detailed mathematical treatment to damping; readers are advised to refer to any standard text/reference book on structural dynamics/ vibration.

For the application specific to machine foundation, let us consider damping as viscous damping where, the resisting force is proportional to velocity. Damping, denoted by c_y, is represented by a dashpot. Damped SDOF System is shown in Figure 2.1.2-1.

The system under motion is shown in Figure 2.1.2-2 (a). Internal forces acting on the mass at any instant of time t are:

- Inertia force proportional to acceleration
- Damping force proportional to velocity
- Spring force proportional to displacement

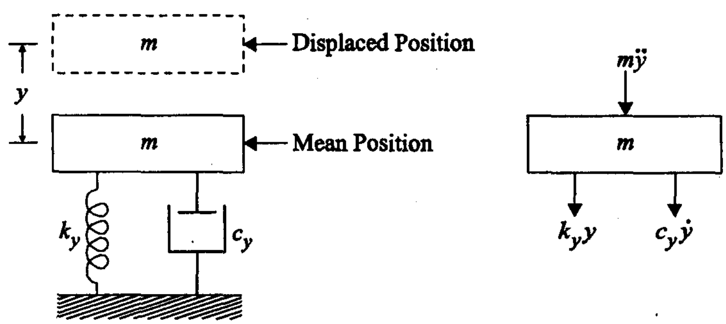
These forces are shown in the free body diagram as shown in Fig. 2.1.2-2b. It is to be noted that all the internal forces oppose the motion i.e. if mass is moving towards positive y direction, the forces are directed towards negative y direction.

Considering the equilibrium of the forces acting on the mass, equation of motion is written as:

$$m\ddot{y} + c_y\dot{y} + k_y y = 0 \qquad (2.1.2\text{-}1)$$

Solution to this equation suggests three types of systems

i) Critically Damped System
ii) Over-damped System
iii) Under-damped System



(a) System Under Motion (b) Free Body Diagram

Figure 2.1.2-2 Damped SDOF System under Motion

Consider the solution to be of the form

$$y = e^{st} \tag{2.1.2-1a}$$

$$\dot{y} = se^{st}\,; \quad \ddot{y} = s^2 e^{st} \tag{2.1.2-1b}$$

Substituting equation (2.1.2-1a & b) in equation (2.1.2-1), the equation becomes

$$(ms^2 + c_y s + k_y)e^{st} = 0 \tag{2.1.2-1c}$$

Since e^{st} is non-zero, for solution to exist

$$(ms^2 + c_y s + k_y) = 0 \tag{2.1.2-1d}$$

This gives two roots of s $\quad s_{1,2} = -\frac{c_y}{2m} \pm \sqrt{\left\{\frac{c_y}{2m}\right\}^2 - \frac{k_y}{m}}$

Substituting $\quad p_y = \sqrt{\frac{k_y}{m}}$, it gives

$$s_{1,2} = -\frac{c_y}{2m} \pm \sqrt{\left\{\frac{c_y}{2m}\right\}^2 - p_y^2} \tag{2.1.2-1e}$$

The solution to the equation of motion becomes

$$y = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{2.1.2-1f}$$

This expression represents three types of motion depending upon whether the radical in equation (2.1.2-1e) is zero, positive or negative. Thus:

i) If radical is zero, this represents critically damped system
ii) If radical is positive, it represents over damped system, and
iii) If radical is negative, it represents under damped system

Let us examine these cases one by one.

2.1.2.1 Critically Damped System

Rewriting equation (2.1.2-1e) $\quad s_{1,2} = -\frac{c_y}{2m} \pm \sqrt{\left\{\frac{c_y}{2m}\right\}^2 - p_y^2}$

For critically damped system, the radical in equation (2.1.2-1e) is zero, then

$$\sqrt{\left\{\frac{c_y}{2m}\right\}^2 - p_y{}^2} = 0 \ ; \qquad \text{This gives} \qquad c_y = 2mp_y$$

This damping is termed as **Critical Damping.**

Denoting critical damping as c_{ycr} , $c_{ycr} = 2mp_y$; $\frac{c_{ycr}}{2m} = p_y$ (2.1.2-2)

$$s_{1,2} = -\frac{c_y}{2m} = -\frac{c_{ycr}}{2m} = p_y \qquad (2.1.2\text{-}3)$$

Since there is only one value of s , the solution therefore, becomes:

$$y = (A_1 + A_2\, t)\, e^{-\frac{c_y}{2m}t}$$

or $$y = (A_1 + A_2\, t)\, e^{-p_y t} \qquad (2.1.2\text{-}4)$$

Constants A_1 & A_2 are evaluated using initial conditions.

Equation (2.1.2-4) indicates decaying amplitude y with time. It can be seen that the value of y reaches zero quickly and the mass comes to rest. In other words the mass shall not have any oscillation.

The damping value for which the radical becomes zero is termed as **Critical Damping** of the system.

$$\dot{y} = e^{-p_y t}\left\{A_1(-p_y) + A_2\left(1 - p_y t\right)\right\} \qquad (2.1.2\text{-}5)$$

Constants A_1 and A_2 are evaluated using initial conditions.

Considering initial displacement $y(t) = y(0)$ and velocity $\dot{y}(t) = \dot{y}(0)$ at time t=0 and substituting in to equation (2.1.2-4)& (2.1.2-5), it gives

$$A_1 = y(0) \ ; \quad A_2 = \dot{y}(0) + p_y y(0)$$

Equation (2.1.2-4) thus becomes

$$y = \left\{y(0) + \left(\dot{y}(0) + p_y y(0)\right)t\right\}e^{-p_y t} \qquad (2.1.2\text{-}6)$$

Equation (2.1.2-6) thus represents solution for critically damped system.

For a typical case of initial zero velocity i.e. $\dot{y}(0) = 0$

Equation (2.1.2-6) becomes

$$y = \{y(0)(1 + p_y t)\} e^{-p_y t}$$

$$\text{or} \quad \frac{y}{y(0)} = (1 + p_y t) e^{-p_y t} \qquad (2.1.2\text{-}7)$$

Defining damping coefficient as ratio of damping c_y to critical damping c_{ycr} and representing it by ζ_y

$$\zeta_y = \frac{c_y}{c_{ycr}} \text{ ; Since } c_{ycr} = 2mp_y \qquad c_y = 2mp_y\zeta_y \quad \frac{c_y}{2m} = p_y\zeta_y \qquad (2.1.2\text{-}8)$$

Plot of equation (2.1.2-7) is shown in Figure 2.1.2-3

It is seen from this figure that the free vibration response of a critically damped system does not show any oscillation about the zero deflection position, instead the displacement quickly returns to zero (depending upon its exponential decay term).

In other words the critically damped system has the smallest amount of damping for which no oscillation takes place.

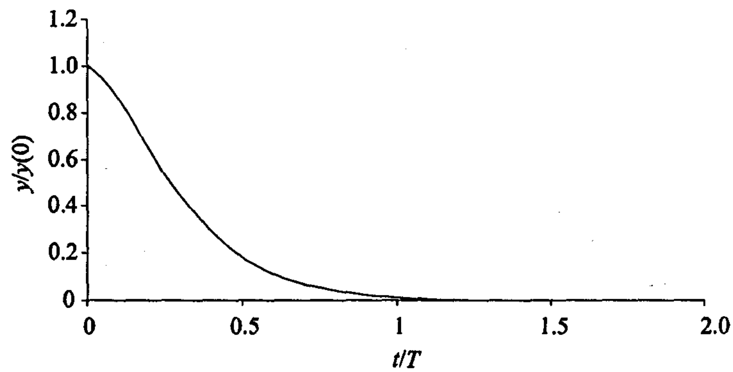


Figure 2.1.2-3 Response of Critically Damped System - Initial velocity $\dot{y}(0) = 0$

2.1.2.2 Over Damped System

Rewriting equation (2.1.2-1e) $\qquad s_{1,2} = -\frac{c_y}{2m} \pm \sqrt{\left\{\frac{c_y}{2m}\right\}^2 - p_y^2}$

For over damped system, the radical in this equation is positive

Substituting $c_y = 2mp_y\zeta_y$ from equation (2.1.2-8), equation becomes

$$s_{1,2} = -p_y\zeta_y \pm p_y\sqrt{\zeta_y^{\,2} - 1} \qquad (2.1.2\text{-}9)$$

For radical to be positive, $\zeta_y > 1$

In other words, the system is said to be over damped when ζ_y is greater than unity i.e. damping of the system is more than its critical damping.

Substituting values of s_1 & s_2 from equation (2.1.2-9), solution for over damped system, equation (2.1.2-1*f*) becomes

$$y = A_1 e^{\left(-p_y\zeta_y - p_y\sqrt{\zeta_y^{\,2}-1}\right)t} + A_2 e^{\left(-p_y\zeta_y + p_y\sqrt{\zeta_y^{\,2}-1}\right)t} \qquad (2.1.2\text{-}10)$$

The constants A_1 & A_2 are determined based on initial conditions.

Equation (2.1.2-10) indicates that the system does not vibrate and returns to equilibrium position at a relatively slower rate compared to critically damped system.

Differentiating equation (2.1.2-10), it gives

$$\dot{y}(t) = \left\{-p_y\zeta_y - p_y\sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}A_1 e^{\left\{-p_y\zeta_y - p_y\sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}t}$$
$$+\left\{-p_y\zeta_y + p_y\sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}A_2 e^{\left\{-p_y\zeta_y + p_y\sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}t} \qquad (2.1.2\text{-}11)$$

For a typical case of initial displacement of $y = y(0)$ and initial velocity of $\dot{y}(0) = 0$ and substituting in equation (2.1.2-10) & (2.1.2-11), it gives

$$A_1 = y(0)\frac{\left\{-\zeta_y + \sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}}{2\sqrt{\left(\zeta_y^{\,2}-1\right)}} \quad ; \quad A_2 = y(0)\frac{\left\{\zeta_y + \sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}}{2\sqrt{\left(\zeta_y^{\,2}-1\right)}} \qquad (2.1.2\text{-}12)$$

Substituting equation (2.1.2-12) in equation (2.1.2-10), the equation becomes

$$\frac{y(t)}{y(0)} = \frac{\left\{-\zeta_y + \sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}}{2\sqrt{\left(\zeta_y^{\,2}-1\right)}} e^{\left\{-\zeta_y - \sqrt{\left(\zeta_y^{\,2}-1\right)}\right\}p_y t}$$

$$+\frac{\left\{\zeta_y+\sqrt{\left(\zeta_y{}^2-1\right)}\right\}}{2\sqrt{\left(\zeta_y{}^2-1\right)}}e^{\left\{-\zeta_y+\sqrt{\left(\zeta_y{}^2-1\right)}\right\}p_y t} \qquad (2.1.2\text{-}13)$$

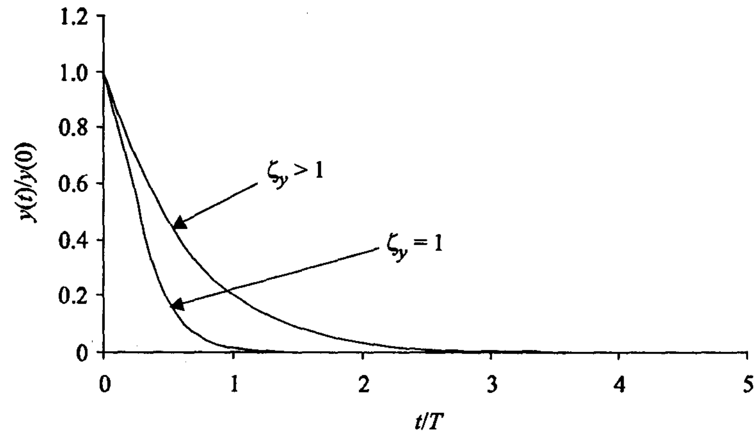


Fig 2.1.2-4 Response of Critically Damped System ($\zeta_y = 1$) & Over Damped System ($\zeta_y > 1$)

Plot of equation (2.1.2-13) is shown in Figure 2.1.2-4. Figure also shows plot of critically damped system for comparison.

Since structural systems having damping greater than critical damping are normally not encountered in practice, and neither there is any application to the machine foundation design, the details are not discussed further.

2.1.2.3 Under-Damped System

Rewriting equation (2.1.2-1e) $$s_{1,2}=-\frac{c_y}{2m}\pm\sqrt{\left\{\frac{c_y}{2m}\right\}^2-p_y{}^2} \qquad (2.1.2\text{-}14)$$

For **under damped system**, the radical in the equation is negative

Substituting $c_y = 2mp_y\zeta_y$ from equation (2.1.2-8), equation becomes

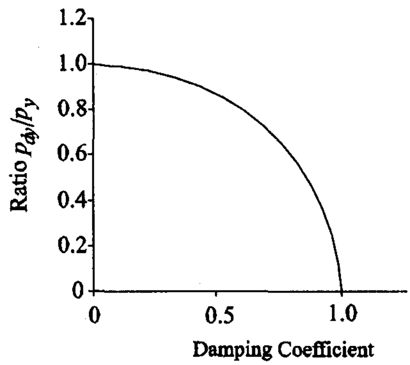


Figure 2.1.2-5 Damped Frequency vs. Damping Coefficient

$$s_{1,2} = -p_y\zeta_y \pm p_y\sqrt{\zeta_y^2 - 1} \qquad (2.1.2\text{-}15)$$

For radical to be negative, $\zeta_y < 1$ i.e. the system is said to be under damped when ζ_y is less than unity i.e. damping of the system is less than its critical damping.

Denoting $p_{dy} = p_y\sqrt{\left(1-\zeta_y^2\right)}$ (2.1.2-16)

Equation (2.1.2-15) becomes $s_{1,2} = -p_y\zeta_y \pm ip_{dy}$ (2.1.2-17)

Here p_{dy} represents damped natural frequency of the system.

Plot of equation (2.1.2-16) is shown in Figure 2.1.2-5. It is seen that for damping values up to 20%, there is hardly any appreciable change in damped frequency.

Substituting values of s_1 & s_2 from equation (2.1.2-17), solution for under damped system, equation (2.1.2-1*f*) becomes

$$y = e^{-p_y\zeta_y t}\left(A_1 e^{-p_{dy}t} + A_2 e^{+p_{dy}t}\right) \qquad (2.1.2\text{-}18)$$

This equation can also be expressed in trigonometric function as .

$$y = e^{-p_y\zeta_y t}\left(A\cos p_{dy}t + B\sin p_{dy}t\right) \qquad (2.1.2\text{-}19)$$

Differentiating we get:

$$\dot{y} = -p_y\zeta_y\, e^{-p_y\zeta_y t}\left(A\cos p_{dy}t + B\sin p_{dy}t\right) + e^{-p_y\zeta_y t}\left(-p_{dy}A\sin p_{dy}t + p_{dy}B\cos p_{dy}t\right) \tag{2.1.2-20}$$

The constants A & B are determined based on initial conditions.

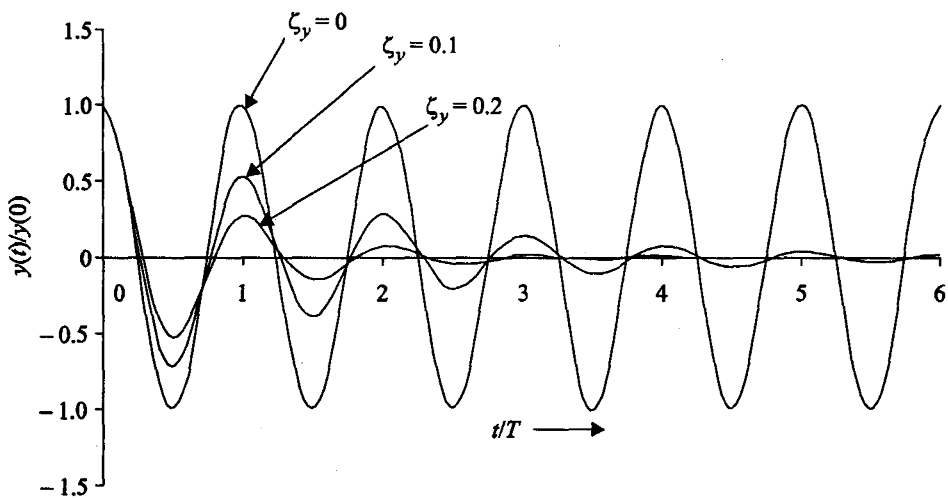


Figure 2.1.2-6 Variation of Amplitude with Damping

Considering initial displacement $y(t) = y(0)$ and velocity $\dot{y}(t) = \dot{y}(0)$ at time t=0 and substituting in equations (2.1.2-19) & (2.1.2-20), we get

$$A = y(0) \quad ; \quad B = \frac{1}{p_{dy}}\left(\dot{y}(0) + p_y\zeta_y\, y(0)\right)$$

Substituting for A & B in equation (2.1.2-19), the solution becomes

$$y(t) = e^{-p_y\zeta_y t}\left(y(0)\cos p_{dy}t + \frac{(\dot{y}(0) + p_y\zeta_y\, y(0))}{p_{dy}}\sin p_{dy}t\right) \tag{2.1.2-21}$$

For a particular case of initial velocity $\dot{y}(0) = 0$, the solution becomes

$$\frac{y(t)}{y(0)} = e^{-p_y \zeta_y t}\left\{(\cos p_{dy} t + \frac{p_y \zeta_y}{p_{dy}} \sin p_{dy} t)\right\} \qquad (2.1.2\text{-}22)$$

In this equation $e^{-p_y \zeta_y t}$ indicates exponential decay of the motion and rate of decay depends upon value of damping. The other part within parenthesis indicates harmonic motion with damped frequency p_{dy}.

Plot of equation (2.1.2-22) for various values of damping coefficient is shown in Figure 2.1.2-6. It is seen that with 10% damping, motion practically diminishes after about 4 cycles. For comparison sake, free vibration motion for over damped, critically damped, under damped & undamped system is shown in Figure 2.1.2-7

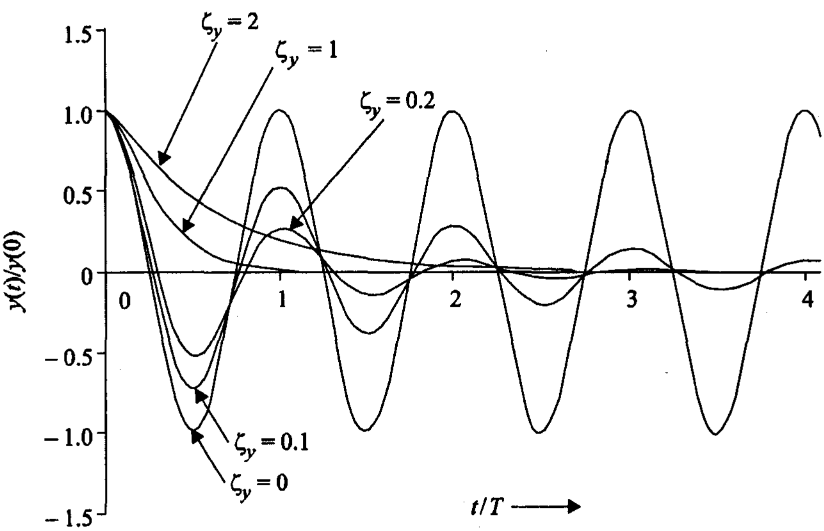


Figure 2.1.2-7 Free Vibration Response for Undamped System $\zeta_y = 0$, Under-Damped $\zeta_y < 1$, Critically Damped $\zeta_y = 1$ & Over-Damped $\zeta_y > 1$

2.2 FORCED VIBRATION

A structural system, when subjected to time dependent excitation force, is set to motion. This state of vibration of the structure is termed as **forced vibration.** For machine foundation application, more often than not, the machine internally generates the excitation force. However in some cases, this force could also come from external sources. It is of interest to note that whether the excitation force is applied externally or generated internally, the structure always vibrates with the frequency of the excitation force.

2.2.1 Undamped System - Dynamic Force Externally Applied

2.2.1.1 System having Translational Stiffness & Dynamic Force Externally Applied

A SDOF system subjected to dynamic excitation (whether internally generated or externally applied) is a simplest form of representation of a machine foundation system. The mass represents mass of machine and foundation whereas elasticity of the bedding (soil or support system) is represented by spring stiffness k_y and the system experiences a dynamic excitation force $F(t)$.

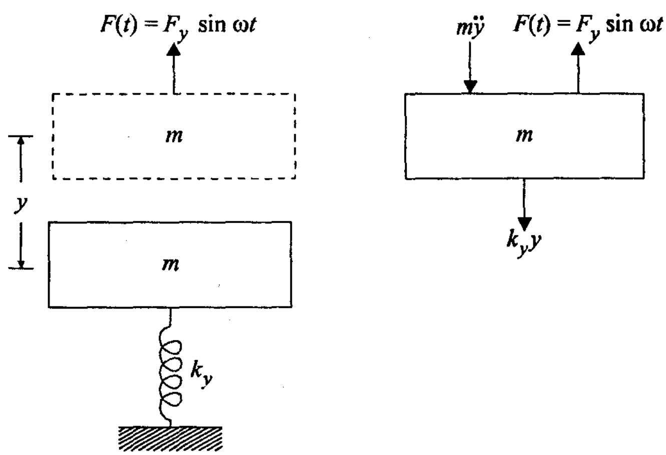
As a machine foundation designer, the basic interests are:

- To compute natural frequency
- To compute vibration of the mass i.e. response under dynamic force $F(t)$
- To compute force transmitted to the ground/ fixed support base

Consider an undamped SDOF system subjected to externally applied dynamic excitation force $F(t)$ as shown in Figure 2.2.1-1. Consider the applied excitation force to be harmonic with excitation frequency ω. Let this excitation force be $F(t) = F_y \sin \omega t$ and let it be applied to the mass 'm' as shown. Considering equilibrium of forces, equation of motion is written as

$$m\ddot{y} + k_y y = F_y \sin \omega t \qquad (2.2.1\text{-}1)$$

This is the **equation of motion** for forced vibration of Undamped SDOF System.



(a) System under Motion (b) Free Body Diagram

Figure 2.2.1-1 Forced Vibration - Undamped SDOF System

Solution to this equation **(See SOLUTION (2.2.1-1))** gives total response as:

$$y(t) = \underbrace{A\cos p_y t + B\sin p_y t}_{\text{Complimentary solution}} + \underbrace{\frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\sin\omega\ t}_{\text{Particular Solution}} \tag{2.2.1-2}$$

Complimentary solution represents transient vibration whereas particular solution represents steady-state vibration. Constants A & B are evaluated based on given initial conditions. It is seen that system vibrates with natural frequency $p_y = \sqrt{k_y/m}$ during transient phase (see equation 2.1.1-3) and with natural frequency ω during steady-state phase.

For initial conditions $y = y(0)$ & $\dot{y} = \dot{y}(0)$, solution becomes

$$y(t) = \underbrace{y(0)\cos p_y t + \left[\frac{\dot{y}(0)}{p_y} - \delta_y \frac{\beta_y}{\left(1-\beta_y^2\right)}\right]\sin p_y t}_{\substack{\text{complementary solution} \\ \text{(Transient Vibration)}}} + \underbrace{\delta_y \frac{1}{\left(1-\beta_y^2\right)}\sin\omega\ t}_{\substack{\text{particular solution} \\ \text{(Steady State Vibration)}}} \tag{2.2.1-3}$$

Here $\delta_y = F_y/k_y$ represents the static deflection due to applied force F_y

For specific initial condition of $y(0) = 0$ & $\dot{y}(0) = 0$, we get

Complementary solution

$$y_c(t) = -\frac{F_y}{k_y}\frac{\beta_y}{\left(1-\beta_y^2\right)}\sin p_y t \tag{2.2.1-4a}$$

Or $$\frac{y_c(t)}{(F_y/k_y)} = -\frac{\beta_y}{\left(1-\beta_y^2\right)}\sin p_y t \tag{2.2.1-4}$$

Plot of equation (2.2.1-4) is shown in Figure 2.2.1-2

Particular solution (Steady State Response)

$$y_p(t) = \frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\sin\omega\ t \tag{2.2.1-5a}$$

Or $$\frac{y_p(t)}{(F_y/k_y)} = \frac{y_p(t)}{\delta_y} = \frac{1}{\left(1-\beta_y^2\right)}\sin\omega\ t;\quad \delta_y = \frac{F_y}{k_y} \tag{2.2.1-5b}$$

Total Response becomes

$$y(t) = y_c(t) + y_p(t) = \frac{F_y}{k_y} \frac{1}{\left(1-\beta_y^2\right)} \left(\sin \omega t - \beta_y \sin p_y t\right)$$
$$= \delta_y \frac{1}{\left(1-\beta_y^2\right)} \left(\sin \omega t - \beta_y \sin p_y t\right) \tag{2.2.1-6a}$$

Total Response Amplification is given by:

$$\frac{y(t)}{(F_y/k_y)} = \frac{y_c(t)}{(F_y/k_y)} + \frac{y_p(t)}{(F_y/k_y)} \tag{2.2.1-6}$$

Plot of these three equations (2.2.1-4, 5 & 6) is as shown in Figure 2.2.1-3.

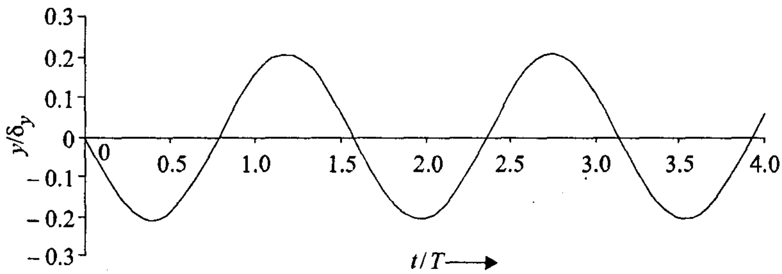


Figure 2.2.1-2 Undamped Transient Response - Forced Vibration

Force transmitted to support

The force $F_{Ty}(t)$ transmitted to support in Y-direction is only the spring reaction force in Y-direction, i.e. $F_{Ty}(t) = k_y\, y(t)$. Considering that every physical system possesses inherent damping, the transient response dies out with time (as we shall see later in this chapter) and the transmitted force to the support is only on account of steady-state response i.e. $F_{Ty}(t) = k_y\, y_p(t)$, where y_p is the steady-state response.

Substituting for y_p from equation (2.2.1-5), we get:

$$F_{Ty}(t) = F_y \frac{1}{\left(1-\beta_y^2\right)} \sin \omega t$$

Maximum value of transmitted force $$F_{Ty} = F_y \frac{1}{\left(1-\beta_y^2\right)} \tag{2.2.1-7}$$

From the above, we can summarize as:

Natural Frequency $p_y = \sqrt{k_y/m}$ rad/s

Steady state response $y_p(t) = \delta_y \frac{1}{\left(1-\beta_y^2\right)} \sin \omega t$

Max. Force transmitted to support $F_{Ty} = F_y \frac{1}{\left(1-\beta_y^2\right)}$

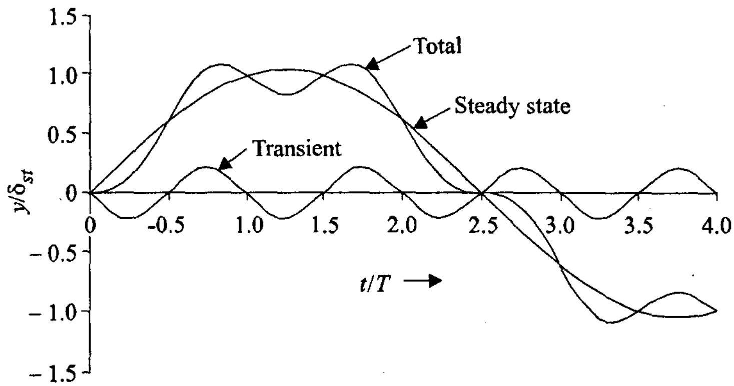


Figure 2.2.1-3 Forced Vibration Response - Undamped SDOF System

SOLUTION (2.2.1-1)

Rewriting equation (2.2.1-1)

$$m\ddot{y} + k_y y = F_y \sin \omega t \qquad (a)$$

Solution to this equation has two parts

i) Complimentary solution represents Transient response of the system

ii) Particular solution represents Steady-state response of the system

Complimentary solution:

It's the free vibration response of the system.

Setting RHS of equation (2.2.1-1) equal to zero, equation becomes

$$m\ddot{y}_c + k_y y_c = 0 \qquad (b)$$

Here subscript 'c' in y_c refers to complimentary solution

System vibrates with its natural frequency p_y *and the response is given as*

$$y_c(t) = A\cos p_y t + B\sin p_y t \tag{c}$$

For solution of this equation - see solution 2.1.1-2 equation (f)

Particular solution:

Particular solution is the forced response of the system. The system vibrates with the frequency of the excitation force i.e. frequency ' ω *'.*

Equation of motion, i.e. equation (a) is rewritten as

$$m\ddot{y}_p + k_y y_p = F_y \sin\omega t \tag{d}$$

Here subscript p *in* y_p *refers to particular solution*

Thus for particular solution of equation (d), let solution be of the form

$$y_p(t) = C\sin\omega t \tag{e}$$

Differentiating twice we get $\ddot{y}_p = -\omega^2 C\sin\omega t$

Substituting in (d), we get

$$-m\omega^2 C\sin\omega t + k_y C\sin\omega t = F_y \sin\omega t$$

Or $$(-\omega^2 + p_y^2)C\sin\omega t = \frac{F_y}{m}\sin\omega t$$

Here $p_y = \sqrt{\frac{k_y}{m}}$ *denotes natural frequency.*

Denoting ratio of excitation frequency to natural frequency as $\beta_y = \frac{\omega}{p_y}$ *and substituting, we get*

$$C = \frac{F_y}{m}\frac{1}{(-\omega^2 + p_y^2)} = \frac{F_y}{m\,p_y^2}\frac{1}{(-\beta_y^2 + 1)} = \frac{F_y}{k_y}\frac{1}{(1-\beta_y^2)}$$

Substituting in equation (e), solution becomes

$$y_p(t) = \frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\sin\omega t \tag{f}$$

The complete solution i.e. total response therefore becomes $y(t) = y_c(t) + y_p(t)$

$$y(t) = \underbrace{A\cos p_y t + B\sin p_y t}_{\text{Complimentary solution}} + \underbrace{\frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\sin\omega t}_{\text{Particular Solution}} \qquad (g)$$

Let us evaluate constants A & B for initial conditions $y = y(0)$ & $\dot{y} = \dot{y}(0)$ *and examine the response.*

Differentiating equation (g), we get

$$\dot{y}(t) = -p_y A\sin p_y t + p_y B\cos p_y t + \omega\frac{F_y}{k_y}\frac{1}{(1-\beta_y^2)}\cos\omega t \qquad (h)$$

Substituting initial conditions in equations (g) & (h), we get

$$A = y(0) \qquad ; \qquad B = \frac{\dot{y}(0)}{p_y} - \beta_y\frac{F_y}{k_y}\frac{1}{(1-\beta_y^2)}$$

Substituting for A & B in equation (g), solution becomes

$$y(t) = \underbrace{y(0)\cos p_y t + \left[\frac{\dot{y}(0)}{p_y} - \frac{F_y}{k_y}\frac{\beta_y}{\left(1-\beta_y^2\right)}\right]\sin p_y t}_{\substack{\text{complementary solution}\\ \text{(Transient Vibration)}}} + \underbrace{\frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\sin\omega t}_{\substack{\text{particular solution}\\ \text{(Steady State Vibration)}}}$$

$$y(t) = \underbrace{y(0)\cos p_y t + \left[\frac{\dot{y}(0)}{p_y} - \delta_y\frac{\beta_y}{\left(1-\beta_y^2\right)}\right]\sin p_y t}_{\substack{\text{complementary solution}\\ \text{(Transient Vibration)}}} + \underbrace{\delta_y\frac{1}{\left(1-\beta_y^2\right)}\sin\omega t}_{\substack{\text{particular solution}\\ \text{(Steady State Vibration)}}} \qquad (i)$$

Here $\delta_y = F_y/k_y$ *represents the static deflection of the mass due to force* F_y

For specific initial condition of $y(0) = 0$ & $\dot{y}(0) = 0$, *we get*

Complementary solution $$y_c(t) = -\frac{F_y}{k_y}\frac{\beta_y}{\left(1-\beta_y^2\right)}\sin p_y t \qquad (j)$$

Particular solution $$y_p(t) = \frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\sin\omega t \qquad (k)$$

Since $\delta_y = F_y/k_y$ *represents static deflection of the mass, term* $y(t)/(F_y/k_y)$ *thus represents response amplification.*

2.2.1.2 System having Rotational Stiffness & Dynamic Moment Externally Applied

Rocking about Z-axis

Consider the undamped SDOF system having rotational stiffness (about Z - axis) in X-Y plane as shown in Figure 2.2.1-4. System mass is m and M_{mz} is the mass moment of inertia about Z-axis passing through centroid C located at a height h above point O. Consider that a dynamic moment $M(t) = M_\phi \sin \omega t$ is applied externally at point O.

Proceeding on the similar lines, we get equation of motion as:

$$M_{moz}\, \ddot{\phi} + (k_\phi - mgh)\, \phi = M_\phi \sin \omega t \qquad (2.2.1\text{-}8)$$

Here $M_{moz} = M_{mz} + mh^2$ is the mass moment of inertia of the system about Z-axis passing through CG of base area point O.

We can write the solution to the equation as:

$$\phi(t) = \underbrace{A\cos p_\phi t + B\sin p_\phi t}_{\text{Complimentary solution}} + \underbrace{\frac{M_\phi}{(k_\phi - mgh)}\frac{1}{(1-\beta_\phi^2)}\sin \omega t}_{\text{Particular Solution}}$$

Here $p_\phi = \sqrt{\dfrac{k_\phi - mgh}{M_{moz}}}$ is the rocking natural frequency and $\beta_\phi = \omega/p_\phi$ represents frequency ratio of operating frequency to natural frequency.

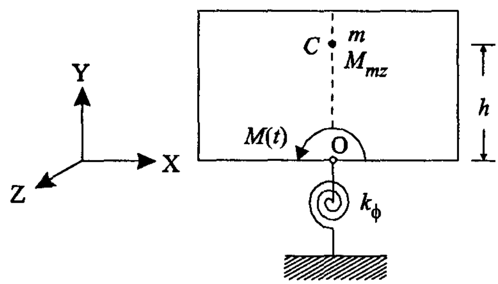


Figure 2.2.1-4 Undamped SDOF System - Rotational Spring Attached at Center of Base of Block - Dynamic Moment $M(t) = M_\phi \sin \omega t$ applied at point 'O'

As mentioned in § **2.2.1.2**, the term mgh is negligible compared to k_ϕ and can be conveniently be ignored for all practical real life cases, without any loss of accuracy in the response. Hence the response becomes:

$$\phi(t) = \underbrace{A\cos p_\phi t + B\sin p_\phi t}_{\text{Complimentary solution}} + \underbrace{\frac{M_\phi}{k_\phi}\frac{1}{\left(1-\beta_\phi^2\right)}\sin\omega t}_{\text{Particular Solution}} \qquad (2.2.1\text{-}9)$$

The steady State response (Particular solution) is given as:

$$\phi_p(t) = \frac{M_\phi}{k_\phi}\frac{1}{\left(1-\beta_\phi^2\right)}\sin\omega t \qquad (2.2.1\text{-}10)$$

Rocking about X-axis

Similarly for system having rotational stiffness k_θ (about X-axis) in Y-Z plane and a dynamic moment $M(t) = M_\theta \sin\omega t$ about X-axis applied at CG of Base area point O, we get

$$M_{mox}\ \ddot{\theta} + (k_\theta - mgh)\ \theta = M_x \sin\omega t \qquad (2.2.1\text{-}11)$$

We can write the solution to the equation as:

$$\theta(t) = \underbrace{A\cos p_\theta t + B\sin p_\theta t}_{\text{Complimentary solution}} + \underbrace{\frac{M_\theta}{k_\theta}\frac{1}{\left(1-\beta_\theta^2\right)}\sin\omega t}_{\text{Particular Solution}} \qquad (2.2.1\text{-}12)$$

Here $p_\theta = \sqrt{\dfrac{k_\theta}{M_{mox}}}$ is the rocking natural frequency (**see § 2.1.1.2**) and $\beta_\theta = \omega/p_\theta$ represents frequency ratio of operating frequency to natural frequency.

The steady State response (Particular solution) $\theta_p(t)$ is given as:

$$\theta_p(t) = \frac{M_\theta}{k_\theta}\frac{1}{\left(1-\beta_\theta^2\right)}\sin\omega t \qquad (2.2.1\text{-}13)$$

Torsional Motion about Y-axis

Consider a system having Torsional stiffness k_ψ (about Y-axis) in X-Z plane and a dynamic moment $M(t) = M_\psi \sin\omega t$ is applied at CG of Base area point O about Y-axis.

Proceeding on the similar lines, we get

$$M_{moy}\,\ddot{\psi}+k_{\psi}\,\psi=M_{\psi}\sin\omega t \tag{2.2.1-14}$$

We can write the solution to the equation as:

$$\psi(t)=\underbrace{A\cos p_{\psi}\,t+B\sin p_{\psi}\,t}_{\text{Complimentary solution}}+\underbrace{\frac{M_{\psi}}{k_{\psi}}\frac{1}{\left(1-\beta_{\psi}^{2}\right)}\sin\omega\,t}_{\text{Particular Solution}}$$

Steady state response is given as

$$\psi(t)=\delta_{\psi}\frac{1}{\left(1-\beta_{\psi}^{2}\right)}\sin\omega t\quad;\quad\delta_{\psi}=\frac{M_{\psi}}{k_{\psi}} \tag{2.2.1-15}$$

Here $p_{\psi}=\sqrt{\dfrac{k_{\psi}}{M_{moy}}}$ is the rocking natural frequency (see **§ 2.1.1.3**) and $\beta_{\psi}=\omega/k_{\psi}$ represents frequency ratio of operating frequency to natural frequency.

The steady State response (Particular solution) is given as:

$$\psi_{p}(t)=\frac{M_{\psi}}{k_{\psi}}\frac{1}{\left(1-\beta_{\psi}^{2}\right)}\sin\omega\,t \tag{2.2.1-16}$$

2.2.2 Damped System - Dynamic Force Externally Applied

Consider damped SDOF System with excitation force $F_y\sin\omega t$ as shown in Figure. 2.2.2-1. Considering equilibrium of forces (see free body diagram)

Equation of motion is written as

$$m\,\ddot{y}+c_y\,\dot{y}+k_y\,y=F_y\,\sin\omega t \tag{2.2.2-1}$$

Denoting $\left({}^{c_y}\!/_{m}\right)=2p_y\zeta_y$; $\left({}^{k_y}\!/_{m}\right)=p_y^2$; $\dfrac{\omega}{p_y}=\beta_y$ *and.* $\delta_{st}=\left({}^{F_y}\!/_{k_y}\right)$

The equation becomes

$$\ddot{y}+2p_y\,\zeta_y\,\dot{y}+p_y^2y=\frac{F_y}{m}\sin\omega\,t=\frac{F_y}{k}m\,p_y^2\sin\omega\,t=\delta_y p_y^2\sin\omega\,t \tag{2.2.2-2}$$

Solution to this equation **(See equation (*k*) - Solution 2.2.2-2)** gives total response as:

$$y(t) = \underbrace{e^{-p_y \zeta_y t}(A\cos p_{dy}t + B\sin p_{dy}t)}_{\substack{\text{Complimentary Solution} \\ \text{Transient Response}}} + \underbrace{\delta_y \frac{1}{\sqrt{(1-\beta_y^2)^2 + (2\beta_y \zeta_y)^2}} \sin(\omega t - \phi)}_{\substack{\text{Particular Solution} \\ \text{Steady-state response}}} \qquad (2.2.2\text{-}3)$$

Here $\delta_y = \left(F_y / k_y \right)$ is the static deflection of the spring mass system.

Constants A, & B, are evaluated based on initial conditions.

The first term on RHS (equation 2.2.2-3) represents Transient Response and the second term represents Steady-State Response of the system.

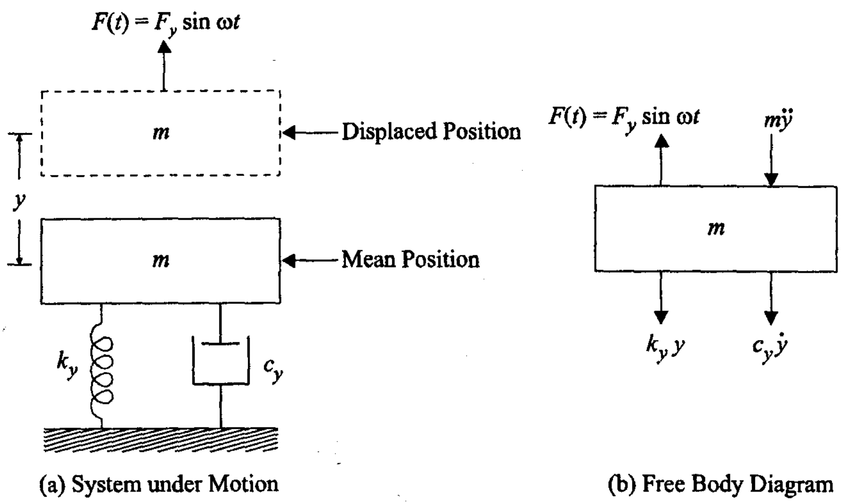


Figure 2.2.2-1 Forced Vibration-Damped SDOF System Translational Stiffness in Y-direction - Dynamic force $F(t) = F_y \sin \omega t$ applied along Y

For a specific case of initial conditions $y(0)=0$ **and** $\dot{y}(0)=0$ **at** $t=0$**, we get:**

Transient Response as **(see equation (q) -Solution 2.2.2-2)**

$$y_c(t)=\delta_y\, e^{-p_y\zeta_y t}\left\{\begin{array}{l}\dfrac{(2\zeta_y\beta_y)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\cos p_{dy}t \\ +\dfrac{1}{\sqrt{(1-\zeta_y^2)}}\dfrac{\zeta_y(2\zeta_y\beta_y)-\beta_y(1-\beta_y^2)}{(1-\beta_y{}^2)^2+(2\zeta_y\beta_y)^2}\sin p_{dy}t\end{array}\right\} \tag{2.2.2-4}$$

Steady-state response as **(see equation (*s*) - Solution 2.2.2-2)**

$$y_p(t)=\delta_y\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}}\sin(\omega t-\phi) \tag{2.2.2-5}$$

Here $\delta_y=\dfrac{F_y}{k_y}$ Represents static deflection under the force F_y and

$\phi=\tan^{-1}\left(\dfrac{2\beta_y\zeta_y}{(1-\beta_y^2)}\right)$ Represents the phase angle **(see equation (t) - Solution 2.2.2-2)**, which gives the time by which the steady state response lags behind the excitation force.

Total Response thus becomes $\quad y(t)=y_c(t)+y_p(t) \quad$ (2.2.2-6)

Transient response plot (equation 2.2.2-4) for $\zeta_y=0.2$ is shown in Figure 2.2.2-2.

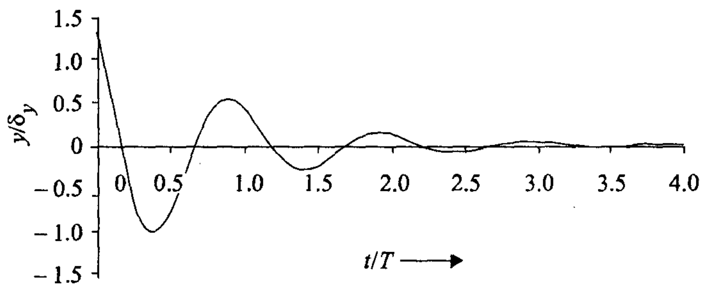


Figure 2.2.2-2 Damped SDOF System $\zeta_y=0.2$ - Forced Vibration - Transient Response

It is seen that transient response dies out with time in a few cycle. Thus it is the steady state response that is really important.

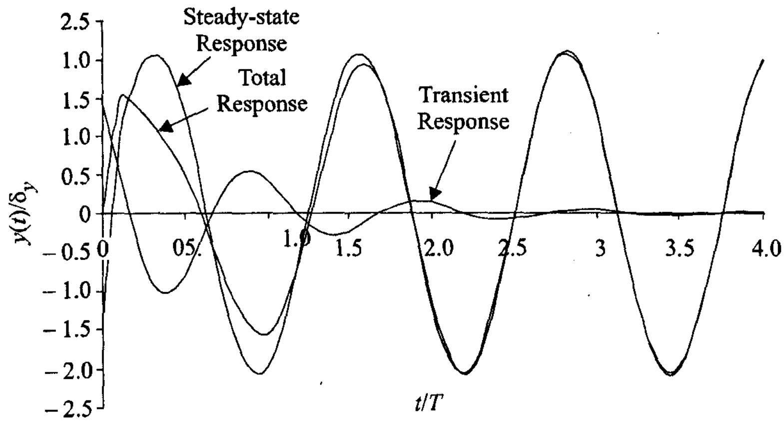


Figure 2.2.2-3 Forced Vibration Response - Damped SDOF System

Plot of equation (2.2.2-5) giving Steady-state Response and equation (2.2.2-6) giving Total Response is shown in Figure 2.2.2-3. Transient response is shown for comparison only.

Rewriting equation (2.2.2-5) in terms of response amplification, we get

$$\frac{y_p(t)}{\delta_y} = \mu_y \sin(\omega t - \phi) \tag{2.2.2-7}$$

Where

$$\mu_y = \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \tag{2.2.2-8}$$

μ_y Represents **Dynamic Magnification Factor along Y**

Plot of **Dynamic Magnification Factor** (equation 2.2.2-8) is shown in Figure 2.2.2-4

Force transmitted to support:

Since the system is damped system, the force $F_T(t)$ transmitted to support is force due to spring reaction + force due to damping. Thus we get

$$F_T(t) = k_y y_p + c_y \dot{y}_p \tag{2.2.2-9}$$

From equation (2.2.2-5), integrating $y_p(t)$, we get

$$\dot{y}_p(t) = \delta_y \omega \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} \cos(\omega t - \phi) \tag{2.2.2-10}$$

Substituting (2.2.2-5) and (2.2.2-10) in equation (2.2.2-9), we get

$$F_T(t) = k_y\,\delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}}\sin(\omega t-\phi)+c_y\,\delta_y\,\omega\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}}\cos(\omega t-\phi)$$

Using relationships $k_y\,\delta_y = F_y$; $c_y\omega\,\delta_y = 2mp_y\zeta_y\,\omega\frac{F_y}{k_y} = 2\,\beta_y\,\zeta_y$ and simplifying, we get

$$F_T(t) = F_y\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}}\left\{\sin(\omega t-\phi)+2\,\beta_y\,\zeta_y\cos(\omega t-\phi)\right\}$$

Maximum value of transmitted force thus becomes:

$$F_T = F_y\frac{\sqrt{1+(2\,\beta_y\,\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}} = F_y\;\mu_y\sqrt{1+(2\,\beta_y\,\zeta_y)^2} \qquad (2.2.2\text{-}11)$$

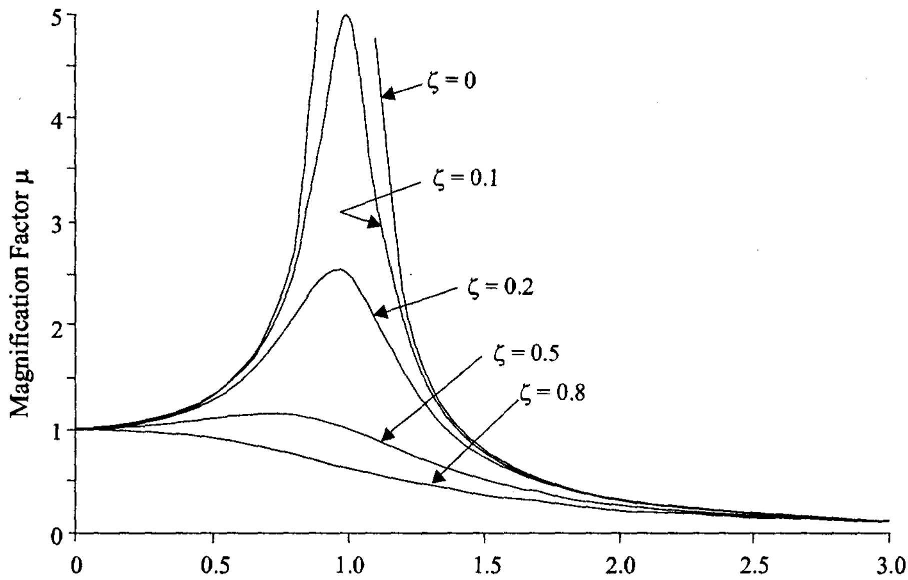


Figure 2.2.2-4 Magnification Factor μ vs. Frequency Ratio β

At this point it is worth mentioning that:

- Since there is no appreciable variation in damped and undamped natural frequency for damping value up to 20 % (see Figure 2.1.2-5), the response has been plotted for time period corresponding to natural frequency of the system instead of its damped natural frequency.

- The transient response dies out in about 3 cycles
- The total response, which is sum of transient and steady state response, therefore remains same as that of steady state response
- For frequency ratio of unity i.e. $\beta_y = 1$, response shoots up significantly (see Figure 2.2.2-4). This condition is termed as **resonance condition**

It is also seen from the Figure 2.2.2-4 that under this resonance condition i.e. $\beta_y = 1$, response rises to infinity for 0 % damping. (This statement is only for academic interest and of least practical significance as every system has some amount of damping and amplitude shall never rise to infinity).

From the point of view of machine foundation design, it is thus desirable to avoid resonance condition to avoid building up of the amplitude of vibration. In other words it is desirable to keep natural frequency sufficiently away from excitation frequency.

From the above, we can summarize as:

- Natural Frequency (undamped) $p_y = \sqrt{k_y / m}$ rad/s
- Steady state response $y_p(t) = \delta_y \dfrac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} \sin(\omega t - \phi)$
- Max. Force transmitted to support $F_T = F_y \, \mu_y \sqrt{1 + (2\beta_y \zeta_y)^2}$

SOLUTION (2.2.2-2)

Rewriting equation (2.2.2-2)

$$\ddot{y} + 2p_y \zeta_y \dot{y} + p_y^2 y = \delta_y p_y^2 \sin \omega t \qquad (a)$$

The solution to the equation (2.2.2-2) has two parts:

i) Complimentary solution

ii) Particular solution

Complimentary solution: *It is the free vibration response of the system.*

For complimentary solution, equation (a) becomes (for free vibration RHS = 0)

$$\ddot{y}_c + 2p_y \zeta_y \dot{y}_c + p_y^2 y_c = 0 \qquad (b)$$

Here subscript 'c' in y_c *refers to complimentary solution*

System vibrates with natural frequency p_y and the response is given as (see equation (2.1.2-19)):

$$y_c(t) = e^{-p_y \zeta_y t}\left(A\cos p_{dy} t + B\sin p_{dy} t\right); \quad \text{where } p_{dy} = p_y\sqrt{\left(1-\zeta_y^2\right)}$$

***Particular solution:** It is the forced vibration response of the system.*

For particular solution, equation (a) is written as

$$\ddot{y}_p + 2p_y\zeta_y\dot{y}_p + p_y^2 y_p = \delta_y p_y^2 \sin\omega t \tag{c}$$

Here subscript ' p ' in y_p refers to particular solution

The system vibrates with the frequency of the excitation force i.e. frequency 'ω '.

Let solution be of the form

$$y_p(t) = C\sin\omega t + D\cos\omega t \tag{d}$$

Differentiating, we get

$$\begin{aligned}\dot{y}_p(t) &= \omega\,(C\cos\omega t - D\sin\omega t)\\ \ddot{y}_p(t) &= -\omega^2\left(C\sin\omega t + D\cos\omega t\right)\end{aligned} \tag{e}$$

Substituting in (a), equation becomes

$$\begin{aligned}&-\omega^2\left(C\sin\omega t + D\cos\omega t\right) + 2p_y\zeta_y\omega\,(C\cos\omega t - D\sin\omega t)\\ &+ p^2(C\sin\omega t + D\cos\omega t) = \delta_y\, p_y^2\sin\omega t\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}&(-C\omega^2 - 2p_y\zeta_y\omega D + p_y^2 C - \delta_y\, p_y^2)\sin\omega t\\ &+(-\omega^2 D + 2p_y\zeta_y\omega C + p_y^2 D)\cos\omega t = 0\end{aligned} \tag{f}$$

For equation (f) to be true for all values of t, coefficients of $\sin\omega t$ & $\cos\omega t$ *should independently be equal to zero. Thus we get*

$$\begin{aligned}&-C\omega^2 - 2p_y\zeta_y\omega D + p_y^2 C - \delta_y\, p_y^2 = 0; \quad \textit{Or}\\ &C\left(p_y^2 - \omega^2\right) - D\left(2p_y\zeta_y\omega\right) - \delta_y\, p_y^2 = 0\end{aligned} \tag{g}$$

$$\begin{aligned}&-\omega^2 D + 2p_y\zeta_y\omega C + p_y^2 D = 0\\ &C\left(2p_y\zeta_y\omega\right) + D\left(p_y^2 - \omega^2\right) = 0\end{aligned} \tag{h}$$

Multiplying equation (g) by $(p_y^2-\omega^2)$ *and equation (h) by* $(2p_y\zeta_y\omega)$ *and rearranging terms, we get*

$$C(p_y^2-\omega^2)^2 - D(p_y^2-\omega^2)(2p_y\zeta_y\omega) = \delta_y\, p_y^2(p_y^2-\omega^2)$$

$$C(2p_y\zeta_y\omega)^2 + D(p_y^2-\omega^2)(2p_y\zeta_y\omega) = 0$$

Solving, we get

$$C = \delta_y\, p_y^2 \frac{(p_y^2-\omega^2)}{(p_y^2-\omega^2)^2+(2p_y\zeta_y\omega)^2} = \delta_y \frac{(1-\beta_y^2)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}$$

$$D = -\delta_y\, p_y^2 \frac{(2p_y\zeta_y\omega)}{(p_y^2-\omega^2)^2+(2p_y\zeta_y\omega)^2} = -\delta_y \frac{(2\zeta_y\beta_y)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2} \qquad (i)$$

Substituting (i) in equation (d), particular solution becomes

$$y_p(t) = \delta_y \frac{(1-\beta_y^2)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\sin\omega\, t - \delta_y \frac{(2\zeta_y\beta_y)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\cos\omega\, t \qquad (j)$$

The complete solution then becomes $\quad y(t) = y_c(t) + y_p(t)$

$$y(t) = \underbrace{e^{-p_y\zeta_y t}(A\cos p_{dy}t + B\sin p_{dy}t)}_{\substack{\text{Complimentary Solution} \\ \text{Transient Response}}}$$

$$\underbrace{+\delta_y \frac{(1-\beta_y^2)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\sin\omega\, t - \delta_y \frac{(2\zeta_y\beta_y)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\cos\omega\, t}_{\substack{\text{Particular Solution} \\ \text{Steady-state response}}} \qquad (k)$$

Constants A, & B, are evaluated based on initial conditions $y = y(0)$ *and* $\dot{y} = \dot{y}(0)$ *at t=0*

Differentiating equation (k), it gives

$$\dot{y}(t) = -p_y\zeta_y e^{-p_y\zeta_y t}\left(A\cos p_{dy}t + B\sin p_{dy}t\right) + e^{-p_y\zeta_y t}\left(-p_d A\sin p_{dy}t + p_d B\cos p_{dy}t\right)$$

$$+\left(\omega\,\delta_y \frac{(1-\beta_y^2)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\cos\omega\, t + \omega\,\delta_y \frac{(2\zeta_y\beta_y)}{(1-\beta_y^2)^2+(2\zeta_y\beta_y)^2}\sin\omega\, t\right) \qquad (l)$$

Substituting initial condition $y = y(0)$ *and* $\dot{y} = \dot{y}(0)$ *in equations (k & l), it gives*

$$A = y(0) + \delta_y \frac{(2\zeta_y \beta_y)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \tag{m}$$

$$B = \frac{1}{p_{dy}} \left\{ \begin{aligned} &\dot{y}(0) + p_y \zeta_y \; y(0) + p_y \zeta_y \; \delta_y \frac{(2\zeta_y \beta_y)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \\ &-\omega \, \delta_y \frac{(1-\beta_y^2)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \end{aligned} \right\} \tag{n}$$

For a specific case of $y(0) = 0$ *and* $\dot{y}(0) = 0$, *we get*

$$A = \delta_y \frac{(2\zeta_y \beta_y)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \tag{o}$$

$$B = \frac{1}{p_{dy}} \left\{ p_y \zeta_y \; \delta_y \frac{(2\zeta_y \beta_y)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} - \omega \; \delta_y \frac{(1-\beta_y^2)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \right\} \tag{p}$$

Complimentary solution thus becomes

$$y_c(t) = \delta_y \, e^{-p_y \zeta_y t} \left\{ \begin{aligned} &\frac{(2\zeta_y \beta_y)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \cos p_{dy} t \\ &+ \frac{1}{\sqrt{(1-\zeta_y^2)}} \frac{\zeta_y (2\zeta_y \beta_y) - \beta_y (1-\beta_y^2)}{(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2} \sin p_{dy} t \end{aligned} \right\} \tag{q}$$

Where $\delta_y = \dfrac{F_y}{k_y}$ *represents static deflection under the influence of force* F_y

Rearranging terms of equation (j), Particular Solution becomes

$$y_p(t) = \delta_y \left\{ \frac{(1-\beta_y^2)}{\left[(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2\right]} \sin \omega t + \frac{-2\zeta_y \beta_y}{\left[(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2\right]} \cos \omega t \right\} \tag{r}$$

By combining terms, the equation becomes

$$y_p(t) = \delta_y \frac{1}{\sqrt{(1-\beta_y^2)^2 + (2\beta_y \zeta_y)^2}} \sin(\omega t - \phi) \tag{s}$$

Where $\quad \phi = \tan^{-1}\left(\dfrac{2\beta_y \zeta_y}{(1-\beta_y^2)} \right)$ *(t)*

Here ϕ is the phase angle, which gives the time by which the steady state response lags behind the excitation force.

Response becomes $y_p(t) = \delta_y \mu_y \sin(\omega t - \phi)$ *(u)*

$$\mu_y = \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}}$$ *is Dynamic Magnification Factor along Y* *(v)*

Total Response thus becomes $y(t) = y_c(t) + y_p(t)$ *(w)*

2.2.3 Damped System - Dynamic Force Internally Generated

In machine foundation design, more often than not, the dynamic force is internally generated by machine itself. Let us address this issue by considering a SDOF System as given below:

Consider a damped SDOF system having mass m, spring stiffness k_y and damping c_y as shown in Figure 2.2.3-1. Machine mass m has a rotating component of mass m_r rotating at speed ω having eccentricity e.

m_r = Rotating mass
e = Eccentricity
ω = Speed of rotation in rad/sec

Dynamic force generated by the system $F(t) = m_r e\omega^2 \sin \omega t$

The system considered is same as that of Figure 2.2.2-1 with the difference that dynamic force applied to the mass is $m_r e\omega^2 \sin \omega t$ instead of $F_y \sin \omega t$.

Equation of motion:

Substituting $F_y = m_r e\omega^2$ in equation 2.2.2-2, we get

$$\ddot{y} + 2p_y \zeta_y \dot{y} + p_y^2 y = \frac{m_r e\omega^2}{k_y} p_y^2 \sin \omega t \tag{2.2.3-1}$$

Steady state response

With $\frac{m_r e\omega^2}{k_y} = \delta_y$, equation (2.2.3-1) becomes identical to equation (2.2.2-2)). Steady-state response thus becomes (See equation (2.2.2-5)

$$
\begin{aligned}
y_p(t) &= \delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} \sin(\omega t - \phi) \\
&= \frac{m_r e\omega^2}{k_y} \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} \sin(\omega t - \phi)
\end{aligned}
\tag{2.2.3-2}
$$

Substituting $p_y^2 = k_y/m$ & $\beta_y = \omega/p_y$ and rewriting, equation becomes

$$
y_p(t) = \frac{em_r}{m} \frac{\beta_y^2}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \sin(\omega t - \phi) \tag{2.2.3-3}
$$

Where $\phi = \tan^{-1}\left(\frac{2\zeta_y \beta_y}{1-\beta_y^2}\right)$ (2.2.3-4)

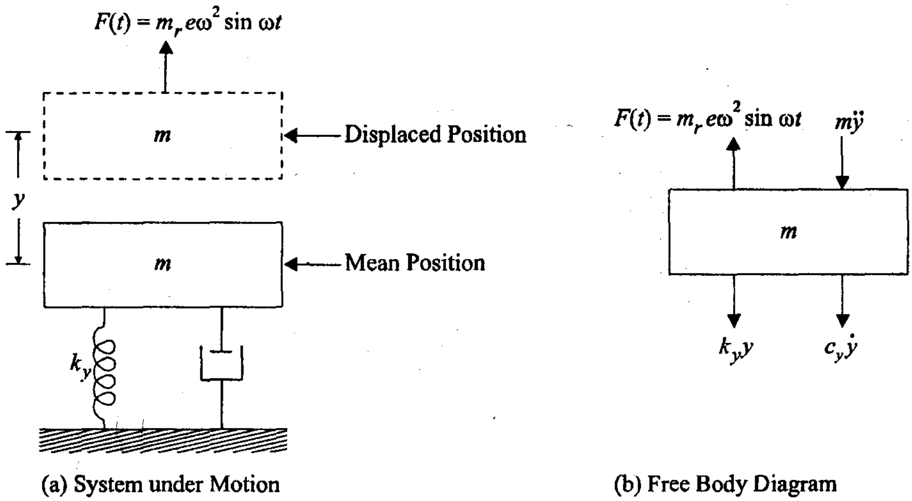


Figure 2.2.3-1 Forced Vibration-Damped SDOF System Dynamic Force Internally Generated

For a given ratio of rotating mass to system mass, equation 2.2.3-3 represents magnification of dynamic amplitude over eccentricity as a function of frequency ratio β_y for various values of damping. Plot of steady state response **(equation (2.2.3-3))** is shown in Figure 2.2.3-2.

Rewriting in non-dimensional form

$$\frac{y/e}{m_r/m} = \frac{\beta_y^2}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \sin(\omega t - \phi) \tag{2.2.3-5}$$

$$\frac{y/e}{m_r/m} = \mu_y \sin(\omega t - \phi) \tag{2.2.3-6}$$

Here $$\mu_y = \frac{\beta_y^2}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \tag{2.2.3-7}$$

Plot of equation (2.2.3-7) for various values of damping is shown in figure 2.2.3-3

Force transmitted to support:

Since the system is damped system, the force $F_T(t)$ transmitted to support is force due to spring reaction + force due to damping. Thus we get

$$F_T(t) = k_y y_p + c_y \dot{y}_p \tag{2.2.3-8}$$

From equation (2.2.3-3), integrating $y_p(t)$, we get

$$\dot{y}_p(t) = \frac{em_r}{m} \omega \frac{\beta_y^2}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \cos(\omega t - \phi) \tag{2.2.3-9}$$

Substituting (2.2.3-3) and (2.2.3-9) in equation (2.2.3-8), we get

$$\begin{aligned} F_T(t) = k_y \frac{em_r}{m} \frac{\beta_y^2}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \sin(\omega t - \phi) \\ + c_y \frac{em_r}{m} \omega \frac{\beta_y^2}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y \beta_y\right)^2}} \cos(\omega t - \phi) \end{aligned} \tag{2.2.3-9a}$$

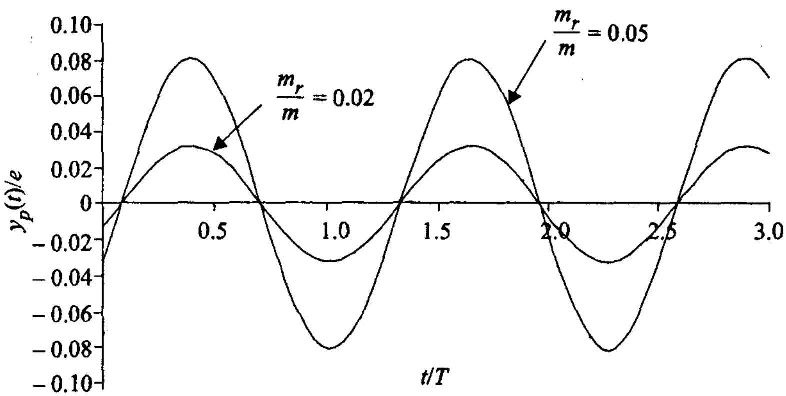


Figure 2.2.3-2 Ratio of Steady-State Response to Rotor Eccentricity-Forced Vibration - Damped System - Dynamic Force Internally Generated

Representing in terms of generated dynamic force $F_y = m_r e\omega^2$ and simplifying RHS of equation (2.2.3-9a), we get

$$k_y e\frac{m_r}{m}\beta_y^2 = \frac{k_y e m_r}{m}\frac{\omega^2}{p_y^2} = F_y \qquad \text{and}$$

$$\frac{c_y e m_r}{m}\beta_y^2\omega = \frac{c_y e m_r}{m}\frac{\omega^2}{p_y^2}\omega = F_0 2\zeta_y\beta_y$$

With this equation (2.2.3-9a) becomes

$$F_T(t) = F_y\left\{\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}}\sin(\omega t-\phi)+\frac{2\beta_y\zeta_y}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}}\cos(\omega t-\phi)\right\}$$

Maximum value of transmitted force thus becomes:

$$F_T = F_y\frac{\sqrt{1+(2\beta_y\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}} = m_r e\omega^2\frac{\sqrt{1+(2\beta_y\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta_y\right)^2}} \qquad (2.2.3\text{-}10)$$

It is seen that this equation is same as equation (2.2.2-11) where the force $F_y \sin\omega t$ is externally applied.

From the above, we can summarize as:

- Natural Frequency (undamped) $p_y = \sqrt{k_y / m}$ rad/s
- Steady state response $y_p(t) = \dfrac{m_r e\omega^2}{k_y} \dfrac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\zeta_y\beta_y\right)^2}} \sin(\omega t - \phi)$
- Max. Force transmitted to support $F_T = m_r e\omega^2 \dfrac{\sqrt{1+(2\beta_y\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta_y\right)^2}}$

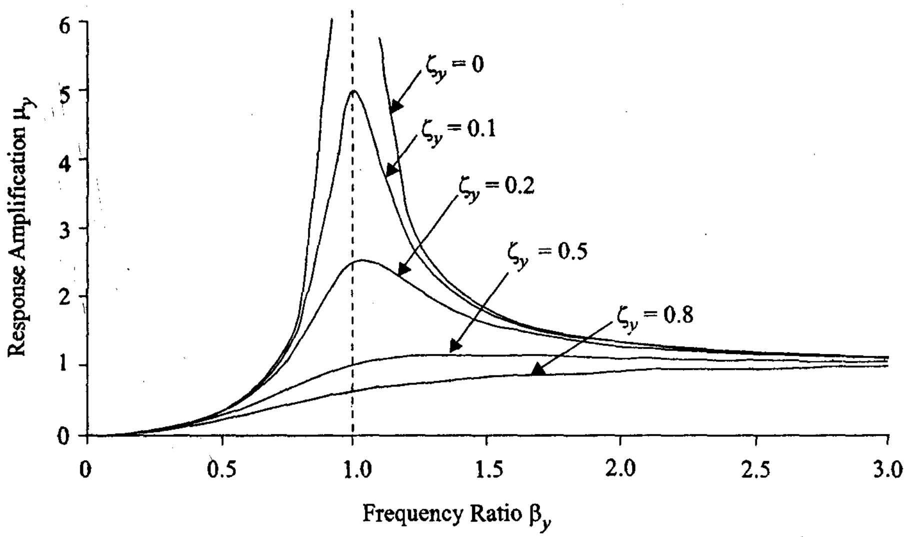


Figure 2.2.3-3 Response Amplification μ_y vs. Frequency Ratio β_y - Forced Vibration-Damped System - Dynamic Force Internally Generated

2.2.4 Damped System - Dynamic Excitation Applied At Base

Consider a damped SDOF system having mass m, spring stiffness k_y and damping c_y as shown in Figure 2.2.4-1. A dynamic excitation in form of ground acceleration $\ddot{y}_g(t)$ is applied at the base of the system.

Equation of motion: Let the displacement of the mass be y_m & that of the base be y_g. The inertia force developed is $m\ddot{y}_m$ & acts opposite to direction of motion.

The spring deformation = $y_m - y_g$

Spring reaction force = $k_y(y_m - y_g)$

Damping force = $c_y(\dot{y}_m - \dot{y}_g)$

Considering equilibrium of forces (see free body diagram), the equation of motion is written as

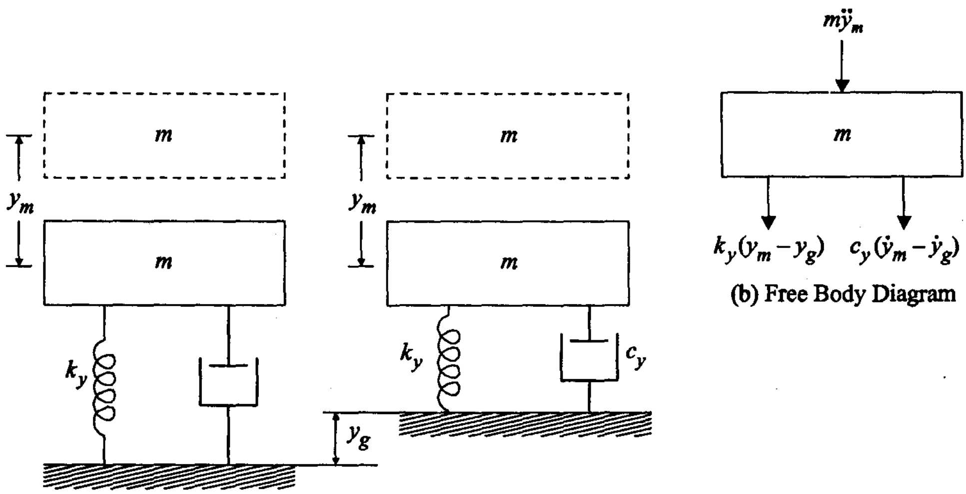
$$m\ddot{y}_m + c_y(\dot{y}_m - \dot{y}_g) + k_y(y_m - y_g) = 0$$

Rewriting equation by subtracting $m\,\ddot{y}_g$ from both LHS & RHS, it gives

$$m\ddot{y}_m - m\ddot{y}_g + c_y(\dot{y} - \dot{y}_g) + k_y(y - y_g) = -m\ddot{y}_g$$

Representing $y_m - y_g = y$; $\dot{y}_m - \dot{y}_g = \dot{y}$; $\ddot{y}_m - \ddot{y}_g = \ddot{y}$ and substituting, it gives

$$m\ddot{y} + c_y\dot{y} + k_y y = -m\ddot{y}_g$$



(a) Forced Vibration-Damped SDOF System - Dynamic Excitation Applied at Base

(b) Free Body Diagram

Figure 2.2.4-1 Forced Vibration-Damped SDOF System - Dynamic Excitation Applied at the Base

Considering ground excitation acceleration as sinusoidal i.e. $\ddot{y}_g(t) = \ddot{y}_g \sin\omega t$, the equation becomes:

$$m\ddot{y} + c_y\dot{y} + k_y y = -m\ddot{y}_g \sin\omega t \qquad (2.2.4\text{-}1)$$

Denoting $(c_y/m) = 2p_y\zeta_y$; $(k_y/m) = p_y^2$ *and* $F_y = -m\ddot{y}_g$ and substituting, the equation becomes

$$\ddot{y} + 2p_y\zeta_y\,\dot{y} + p_y^2\,y = -\frac{m\ddot{y}_g}{m}\sin\omega t = \frac{F_y}{m}\sin\omega t \qquad (2.2.4\text{-}2)$$

Substituting $\delta_y = \frac{F_y}{k_y} = \frac{F_y}{m\,p_y^2}$ the equation becomes

$$\ddot{y} + 2p_y\zeta_y\,\dot{y} + p_y^2 y = p_y^2\delta_y\sin\omega t \qquad (2.2.4\text{-}3)$$

It is noticed that equation (2.2.4-3) is similar to equation (2.2.2-2). Thus we can write the steady-state response on the same lines as equation (2.2.2-5).

$$y_p(t) = \delta_y\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta_y\right)^2}}\sin(\omega t - \phi) \qquad (2.2.4\text{-}4)$$

Here $\delta_y = \frac{m\ddot{y}_g}{k_y}$ (see equation 2.2.4-2)

We can write this in terms of magnification factor as

$$y_p(t) = \delta_y\,\mu_y\left(\sin\omega t - \phi\right) \qquad (2.2.4\text{-}5)$$

Where

$\mu_y = \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta_y\right)^2}}$ is magnification factor &

$\phi = \tan^{-1}\frac{2\zeta_y\beta_y}{\left(1-\beta_y^2\right)}$ is the phase angle

Force transmitted to support:

Since the system is damped system, the force $F_T(t)$ transmitted to support is force due to spring reaction + force due to damping. Thus we get

$$F_T(t) = k_y\,y_p + c_y\,\dot{y}_p \qquad (2.2.4\text{-}6)$$

From equation (2.2.4-4), integrating $y_p(t)$, we get

$$\dot{y}_p(t) = \delta_y\,\omega\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta_y\right)^2}}\cos(\omega t - \phi) \qquad (2.2.4\text{-}7)$$

Substituting (2.2.4-4) and (2.2.4-7) in equation (2.2.4-6), simplifying and rearranging the terms (**on the similar lines as given in § 2.2.2**), we get

$$F_T = k_y\,\delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}}\sin(\omega t-\phi)+c_y\,\delta_y\,\omega\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}}\cos(\omega t-\phi)$$

$$F_T = \delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}}\left(k_y\sin(\omega t-\phi)+c_y\omega\cos(\omega t-\phi)\right)$$

Maximum value of transmitted force as:

$$F_T = \frac{m\ddot{y}_g}{k_y}\frac{\sqrt{k_y^2+(c_y\omega)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}} = \frac{F_y}{k_y}\frac{k_y\sqrt{1+\frac{(c_y\omega)^2}{k_y^2}}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}}$$

$$F_T = F_y\frac{\sqrt{1+\left(\frac{(c_y\omega)}{mp_y^2}\right)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}} = F_y\frac{\sqrt{1+\left(\frac{(2mp_y\zeta\omega)}{mp_y^2}\right)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}}$$

$$F_T = F_y\frac{\sqrt{1+(2\,\beta_y\,\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}} \qquad (2.2.4\text{-}8)$$

$$F_T = -m\,\ddot{y}_g\frac{\sqrt{1+(2\,\beta_y\,\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}} \qquad (2.2.4\text{-}9)$$

From the above, we can summarize as:

- Natural Frequency (undamped) $p_y = \sqrt{k_y/m}$ rad/sec
- Steady state response $y_p(t) = -\frac{m\,\ddot{y}_g}{k_y}\frac{1}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\zeta_y\,\beta_y\right)^2}}(\sin\omega t-\phi)$
- Max. Force transmitted to support $F_T = -m\,\ddot{y}_g\frac{\sqrt{1+(2\,\beta_y\,\zeta_y)^2}}{\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\,\zeta_y\right)^2}}$

2.2.5 Undamped System – Subjected to Impact Loads

Let us consider a freely falling mass ' m_0 ' falling from height ' h ' and striking mass ' m_1 ' of an undamped SDOF spring-mass system, initially at rest, as shown in Figure 2.2.5-1. Let us consider that mass m_1 is at rest before the impact and the impact is central.

Let v_0' & v_1' represent velocity of masses m_0 & m_1 **before impact** and v_0 & v_1 represent velocity of masses m_0 & m_1 **after impact.**

From conservation of momentum, we get:

$$\underbrace{m_1 \times v_1' + m_0 \times v_0'}_{\text{Before Impact}} = \underbrace{m_1 \times v_1 + m_0 \times v_0}_{\text{After Impact}}$$

Since $v_1' = 0$, we get

$$m_0 \times v_0' = m_1 \times v_1 + m_0 \times v_0 \qquad (2.2.5\text{-}1)$$

In order to evaluate v_1, we use Newton's hypothesis, which states that for central impact of the two bodies, the relative velocity of the two bodies after the impact is in constant ratio to their relative velocities before impact and is in opposite direction.

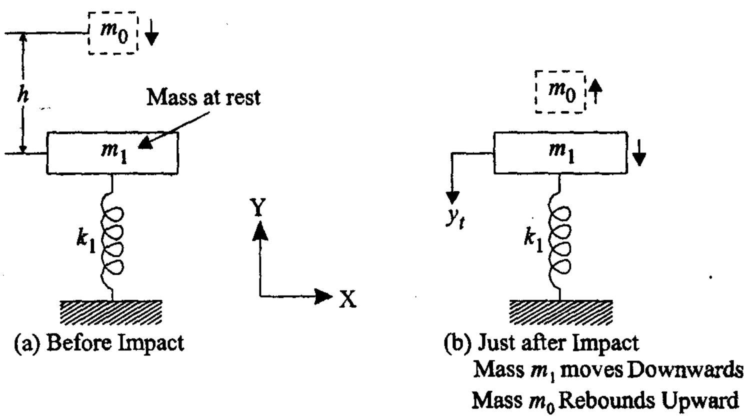


Figure 2.2.5-1 Undamped SDOF Spring Mass System Subjected to Impact Load Mass m_0 Freely Falling over Mass m_1 from Height h

This gives:

$$e = -\frac{v_1 - v_0}{(v_1' - v_0')} = \frac{v_1 - v_0}{v_0'} \quad \text{or} \quad v_0 = v_1 - ev_0' \qquad (2.2.5\text{-}2)$$

Here e is called **Coefficient of Restitution** that depends upon properties of the material of the masses m_0 & m_1. For perfectly plastic central impact, the value of e is zero and for perfectly elastic central impact e is equal to unity. For real bodies in practice, the value lies in the range $0 < e < 1$ and for all practical purposes it's reasonably good to use $e = 0.5$.

Substituting 2.2.5-2 in 2.2.5-1 and simplifying, we get

$$v_1 = v_0' \times \frac{(1+e)}{(1+\lambda_1)} \qquad (2.2.5\text{-}3)$$

Here λ_1 represents ratio of mass m_1 to mass m_0. That is:

$$\lambda_1 = \frac{m_1}{m_0} \qquad (2.2.5\text{-}4)$$

For **freely falling body** of mass m_0 from height h, the velocity just before impact is given as

$$v_0' = \sqrt{2gh} \qquad (2.2.5\text{-}5)$$

Substituting this in 2.2.5-3, we get

$$v_1 = \sqrt{2gh} \times \frac{(1+e)}{(1+\lambda_1)} \qquad (2.2.5\text{-}6)$$

This is the initial velocity imparted by the falling mass to stationary mass m_1 at time $t = 0$.

Hence, solution of a SDOF system subjected to impact load, thus this becomes an initial velocity problem of a SDOF system having mass m_1 and spring stiffness k_1.

Equation of motion (refer equation 2.1.1.2) of the SDOF system, as shown in Figure 2.2.5-1, is written as

$$m_1\ddot{y}_1 + k_1 y_1 = 0 \qquad (2.2.5\text{-}7)$$

This gives natural frequency (refer equation 2.1.1.3) as

$$p_1 = \sqrt{\frac{k_1}{m_1}} \quad \text{rad/s} \qquad (2.2.5\text{-}8)$$

The response of the SDOF system is given as (refer equation 2.1.1.5)

$$y_1 = y_1(0)\cos p_1 t + \frac{\dot{y}_1(0)}{p_1}\sin p_1 t \tag{2.2.5-9}$$

Here $y_1(0)$ & $y_1(0)$ represent initial displacement and initial velocity at time $t = 0$.

Maximum amplitude of the SDOF system is given as (refer equation 2.1.1-6)

$$\rho_1 = \sqrt{\left(y_1(0)^2 + \left(\frac{\dot{y}_1(0)}{p_1} \right)^2 \right)} \tag{2.2.5-10}$$

For the present case, we know that:

$$y_1(0) = 0 \quad \& \quad \dot{y}_1(0) = v_1 \quad at \quad t = 0 \tag{2.2.5-11}$$

Substituting equation (2.2.5-12) in to equations (2.2.5-9) & (2.2.5-10), we obtain response of SDOF system as:

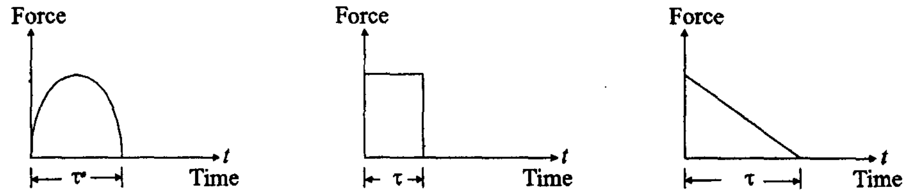
$$y_1 = y_1(0)\cos p_1 t + \frac{\dot{y}_1(0)}{p_1}\sin p_1 t = \frac{\dot{y}_1(0)}{p_1}\sin p_1 t = \frac{v_1}{p_1}\sin p_1 t \tag{2.2.5-12}$$

$$\rho_1 = \sqrt{\left(y_1(0)^2 + \left(\frac{\dot{y}_1(0)}{p_1} \right)^2 \right)} = \frac{\dot{y}_1(0)}{p_1} = \frac{v_1}{p_1} \tag{2.2.5-13}$$

This approach shall be useful for **Design of Foundation for Impact Machines** covered in **Chapter 11**.

2.2.6 Undamped System – Subjected to Impulsive Loads

Impulsive loading is a special class of dynamic loading and generally consists of a single impulse of short duration.



(a) Half Sine Wave Pulse (b) Rectangular Pulse (c) Traingular Pulse

Figure 2.2.6-1 Typical Pulse Loading

Typical example of impulsive loads that could be expressed by simple analytical functions are a half sine wave pulse, a rectangular pulse, a triangular pulse etc. having very short duration. These are shown in Figure 2.2.6-1.

Consider a SDOF system subjected to impulsive loading (applied one at a time) as shown in Figure 2.2.6-2.

The mpulsive load $F(t)$ is applied at the mass i.e. the applied pulse has maximum force amplitude of F_y and pulse duration τ where $\tau = \dfrac{\pi}{\omega}$.

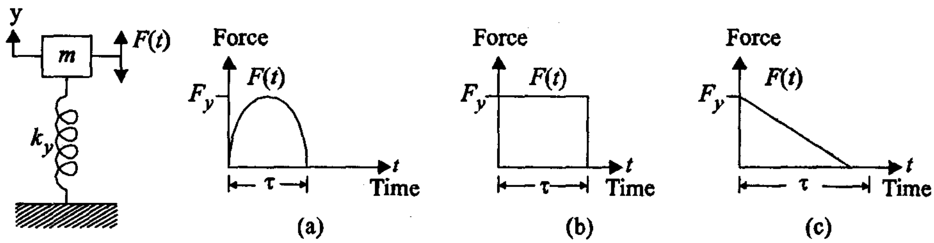


Figure 2.2.6-2 SDOF System Subjected to Impulsive Load

Equation of Motion:

(a) When applied load is a Half Sine Pulse

$$\begin{aligned} m\ddot{y} + k_y y &= F(t) = F_y \sin \omega t \qquad & 0 < t < \tau \\ m\ddot{y} + k_y y &= 0 & t \ge \tau \end{aligned} \tag{2.2.6-1}$$

(b) When applied load is a Rectangular Pulse

$$\begin{aligned} m\ddot{y} + k_y y &= F(t) = F_y \qquad & 0 < t < \tau \\ m\ddot{y} + k_y y &= 0 & t \ge \tau \end{aligned} \tag{2.2.6-2}$$

(c) When applied load is a Triangular Pulse

$$\begin{aligned} m\ddot{y} + k_y y &= F(t) = F_y(\tau - t)/\tau \qquad & 0 < t < \tau \\ m\ddot{y} + k_y y &= 0 & t \ge \tau \end{aligned} \tag{2.2.6-3}$$

For each of the above loading case, the first equation of motion gives response during the pulse i.e. **Phase I - Forced Vibration Response** and the second equation gives response after the pulse i.e.

Phase II - Free Vibration Response. It is also note worthy that the maximum response reaches in a very short time before system damping gets effective.

Natural Time Period of the system T $$T = \frac{2\pi}{p_y}; \quad p_y = \sqrt{k_y / m} \qquad (2.2.6\text{-}4)$$

Pulse duration τ $$\tau = \frac{\pi}{\omega}; \quad \omega \text{ is the excitation frequency} \qquad (2.2.6\text{-}5)$$

Frequency Ratio $$\beta_y = \frac{\omega}{p_y} = \frac{(\pi / \tau)}{(2\pi / T)} = \frac{T}{2\tau} \qquad (2.2.6\text{-}6)$$

We know that

$$\beta_y < 1; \quad \omega < p_y; \quad \frac{\pi}{\tau} < \frac{2\pi}{T}; \quad \frac{\tau}{T} > \frac{1}{2} \quad \&$$

$$\beta_y > 1; \quad \omega > p_y; \quad \frac{\pi}{\tau} > \frac{2\pi}{T}; \quad \frac{\tau}{T} < \frac{1}{2}$$

It can be shown (derivation not given) that for $\beta_y < 1$ *i.e. for* $\frac{\tau}{T} > \frac{1}{2}$ *maximum response occurs during forced vibration phase i.e. Phase I and for* $\beta_y > 1$ *i.e. for* $\frac{\tau}{T} < \frac{1}{2}$, *maximum response occurs during Free Vibration Phase i.e. Phase II. Let us now compute response of the SDOF system subjected to applied impulsive loading.*

(a) When applied load is Half Sine Pulse

Response in Phase I for $t \le \tau$

For undamped SDOF system subjected to harmonic loading $F(t) = F_y \sin \omega t$, the steady state response is given as (Refer equation 2.2.1-7a):

For $t \le \tau$ $$y(t) = \frac{F_y}{k_y} \frac{1}{(1 - \beta_y^2)} (\sin \omega t - \beta_y \sin p_y t) \qquad (2.2.6\text{-}7)$$

Differentiating equation 2.2.6-7 and equating it to zero, we get the time when the response $y(t)$ is maximum. This gives

$$\dot{y}(t) = \frac{F_y}{k_y} \frac{1}{(1 - \beta_y^2)} (\omega \cos \omega t - \beta_y \, p_y \cos p_y t) = \frac{F_y}{k_y} \frac{\omega}{(1 - \beta_y^2)} (\cos \omega t - \cos p_y t) = 0$$

This gives $\left(\cos\omega t - \cos p_y t\right) = 0$ or $\omega t = 2\pi - p_y t$

Simplifying, we get $t = \dfrac{2\pi}{\left(\omega + p_y\right)} = \dfrac{2\pi}{\omega\left(1 + \dfrac{1}{\beta_y}\right)}$

This gives $\omega t = \dfrac{2\pi\,\beta_y}{\left(\beta_y + 1\right)}$ & $p_y t = \dfrac{\omega t}{\beta_y} = \dfrac{2\pi}{\left(\beta_y + 1\right)}$

Substituting this in equation (2.2.6-7), we get maximum response as:

$$y_{\max}(t) = \frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\left(\sin\frac{2\pi\beta_y}{\left(\beta_y+1\right)} - \beta_y \sin\frac{2\pi}{\left(\beta_y+1\right)}\right)$$

$$\text{This gives}\quad \mu_y = \frac{y_{\max}(t)}{\dfrac{F_y}{k_y}} = \frac{1}{\left(1-\beta_y^2\right)}\left(\sin\frac{2\pi\beta}{\left(\beta_y+1\right)} - \beta_y \sin\frac{2\pi}{\left(\beta_y+1\right)}\right) \qquad (2.2.6\text{-}8)$$

This is valid only for $\beta_y < 1$ or $\dfrac{\tau}{T} > 0.5$

Response in Phase II for $t \ge \tau$

The free vibration response (Refer equation 2.1.1-5) depends upon displacement and velocity of the system at $t = \tau$ i.e. $y(\tau)\,\&\,\dot{y}(\tau)$

At time $t = \tau$ $\quad y(\tau) = \dfrac{F_y}{k_y}\dfrac{1}{\left(1-\beta_y^2\right)}\left(\sin\omega\tau - \beta_y \sin p_y\tau\right)$

$$\omega = \frac{\pi}{\tau};\quad \omega\tau = \pi;\quad \beta_y = \frac{\omega}{p_y};\quad p_y\tau = \frac{\pi}{\beta_y}$$

$$y(\tau) = \frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\left(0 - \beta_y \sin\left(\pi/\beta_y\right)\right)$$

$$y(\tau) = -\frac{F_y}{k_y}\frac{\beta_y}{\left(1-\beta_y^2\right)}\sin\left(\pi/\beta_y\right) \qquad (2.2.6\text{-}9)$$

Differentiating equation 2.2.6-7, we get

$$\dot{y}(t) = \frac{F_y}{k_y}\frac{1}{\left(1-\beta_y^2\right)}\left(\omega\cos\omega t - \beta_y\,p_y\cos p_y t\right) = \frac{F_y}{k_y}\frac{\omega}{\left(1-\beta_y^2\right)}\left(\cos\omega t - \cos p_y t\right)$$

At $t = \tau$ $\quad \dot{y}(t) = \frac{F_y}{k_y}\frac{\omega}{\left(1-\beta_y^2\right)}\left(\cos\omega\tau - \dot{\cos} p_y\tau\right)$

$$\dot{y}(t) = \frac{F_y}{k_y}\frac{\omega}{\left(1-\beta_y^2\right)}\left(-1-\cos\frac{\pi}{\beta_y}\right) \qquad (2.2.6\text{-}10)$$

Maximum amplitude is given as (refer equation 2.1.1-6)

$$\rho_y = \sqrt{\left(y(\tau)^2 + \left(\frac{\dot{y}(\tau)}{p_y}\right)^2\right)} = \frac{F_y}{k_y}\frac{\beta_y}{\left(1-\beta_y^2\right)}\sqrt{\left(\sin\frac{\pi}{\beta_y}\right)^2 + \left(1+\cos\frac{\pi}{\beta_y}\right)^2}$$

$$\rho_y = \frac{F_y}{k_y}\frac{\beta_y}{\left(1-\beta_y^2\right)}\sqrt{\left(2+2\cos\frac{\pi}{\beta_y}\right)} = \frac{F_y}{k_y}\frac{2\beta_y}{\left(1-\beta_y^2\right)}\cos\frac{\pi}{2\beta_y} \qquad (2.2.6\text{-}11)$$

From this we get Response Magnification μ_y as:

$$\mu_y = \frac{\rho_y}{\left(F_y/k_y\right)} = \frac{2\,\beta_y}{\left(1-\beta_y^2\right)}\cos\frac{\pi}{2\beta_y} \qquad (2.2.6\text{-}12)$$

This is valid only for $\beta_y > 1$ or $\frac{\tau}{T} < 0.5$

The derivation of Response Magnification Factor μ_y for other shapes of Impulsive Loads is not presented here in the text. The same could easily be evaluated on the similar lines. From the above it can be generalized that for any given SDOF system subjected to impulsive load $F(t)$ having a specific pulse shape, maximum Response of the system thus becomes:

$$y_{\max} = \mu_y \times \frac{F_y}{k_y} \qquad (2.2.6\text{-}13)$$

Here F_y is the peak magnitude of the applied load, k_y is the stiffness and μ_y is the **Response Magnification Factor.**

Table 2.2.6-1 Dynamic Magnification Factor vs. Ratio of Pulse Duration to Natural Time period τ/T

	Ratio of Pulse Duration to Natural Time period τ/T										
Pulse Type	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Rectangular Pulse	0	1.18	1.9	2	2	2	2	2	2	2	2
Half Sine Pulse	0	0.8	1.38	1.7	1.8	1.75	1.67	1.6	1.52	1.42	1.34
Triangular Pulse	0	0.68	1.05	1.3	1.46	1.55	1.61	1.66	1.7	1.72	1.75

Plot of Response Magnification factor μ_y vs. Ratio of Pulse Duration τ to Natural Time Period T of a SDOF system subjected to Impulsive Load is given in Figure 2.2.6-3 for each of the pulse shape. The tabulated values are given in Table 2.2.6-1.

It can also be shown that (derivation not given):

i) **When the Pulse Duration is Very Short compared to time period of the system** i.e. $\frac{\tau}{T} \le 0.5$, the maximum response occurs during its free vibration mode and the shape of the pulse has no influence on the response of the system.

The applied force becomes an impulse and is given by

$$I = \int_0^\tau F(t)\, dt \qquad (2.2.6\text{-}14)$$

Maximum amplitude becomes

$$y_{\max} = \frac{I}{m \times p_y} = \frac{I \times p_y}{m \times p_y^2} = \frac{I \times p_y}{k_y} \qquad (2.2.6\text{-}15)$$

Here product $I \times p_y$ represents equivalent force i.e. $F_{eq} = I \times p_y$ and k_y represents system stiffness.

ii) **When the pulse duration is relatively long compared to time period of the system** i.e. $\frac{\tau}{T} > 0.5$, the maximum response occurs during the pulse duration and the shape of the pulse has significant influence on the response. Overall response maxima depend upon the rate of rise of the loading. In other words this suggests that the maximum amplitude will be higher in case of rectangular pulse compared to half sine wave or triangular pulse because the rate of rise of the loading is higher in case of rectangular pulse.

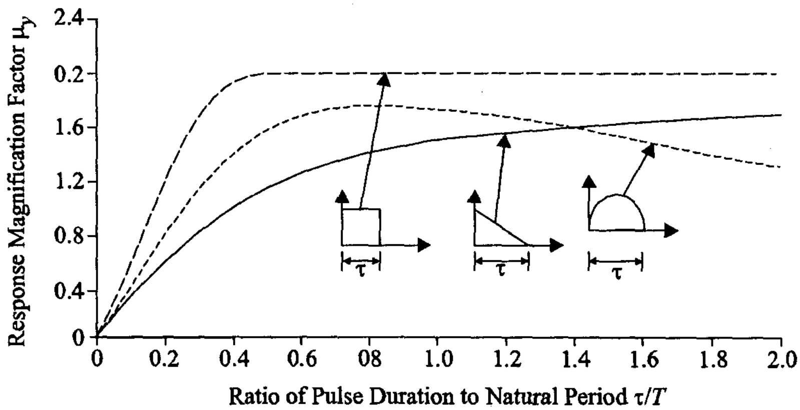


Figure 2.2.6-3 SDOF System - Response Magnification Factor vs. Ratio of Pulse Duration to Natural Period

EXAMPLE PROBLEMS: Free Vibration - SDOF System

Units used throughout the text

Force, Weight	N
Mass	kg
Length	m
Time	s
Gravity g	m/s^2
Elastic Modulus	N/m^2
Pressure	N/m^2
Density (Mass density)	kg/m^3

Note: Units given other than these are converted to these units for computation

P 2.1-1

A machine having mass of $m = 500$ kg is supported by a linear translational spring having stiffness of $k_y = 200$ kN/m along Y-direction as shown in Figure P 2.1-1. Consider the system as undamped. Compute a) natural frequency of the spring mass system; b) the static deflection of the spring and c) compute its natural frequency using the static deflection and compare with (a) above?

Solution:

a) Natural Frequency

$k_y = 200000$ N/m; $m = 500$ kg

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{200000}{500}} = 20 \text{ rad/s}$$

$$f_y = \left(\frac{20}{2\pi}\right) = 3.183 \text{ cycles/sec} = 3.183 \text{ Hz}$$

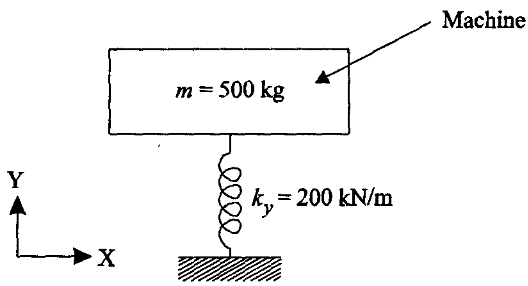


Figure P2.1-1 Machine Supported by Vertical Spring

b) Static deflection of the spring (along Y-direction) δ = Force/stiffness

Weight $mg = 500 \times 9.81 = 4905$ N;

Stiffness $k_y = 200000$ N/m;

δ = Weight/Stiffness = 4905/200,000= .024525 m

c) Natural frequency in terms of δ

$$\delta = \frac{mg}{k_y}; \quad \frac{k_y}{m} = \frac{g}{\delta}; \quad p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{g}{\delta}}$$

$$p_y = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.24525}} = 20 \quad \text{rad/s}, \qquad f_y = \frac{1}{2\pi} p_y = \frac{20}{2\pi} = 3.183 \text{ Hz}$$

The frequency thus computed is the same as that computed in (a) above.

Note: One can also represent δ in other units, say in mm, cm, inch, feet etc. In that case, while computing $p_y = \sqrt{\frac{g}{\delta}}$, value of g should be in units compatible with deflection i.e. if δ is in mm, g should be in mm/sec^2 and so on.

P 2.1-2

A machine of mass $m = 500$ kg is supported at the end of a RCC cantilever beam 100mm wide and 200mm deep having span of 2000 mm as shown in Figure P 2.1-2. Consider the system as undamped. Mass density of beam material is 2500 kg/cu.m. Consider motion of the mass only along Y direction in X-Y plane as shown. Elastic modulus of concrete is E = 3 x 10^7 kN/m^2.

Find Natural frequency of the beam mass system

- **When beam is considered mass-less**
- **When beam mass is considered**

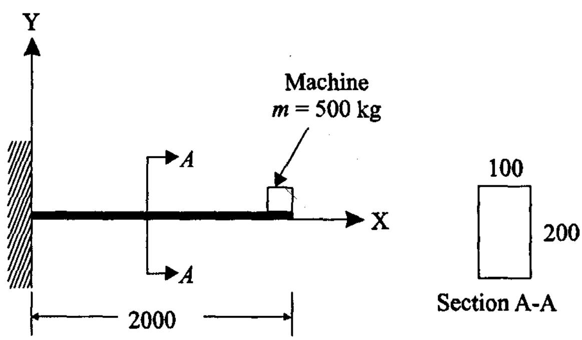


Figure P2.1-2 Machine Supported by Cantilever Beam

Solution:

(i) Beam is considered as mass-less

a) Natural frequency of the beam mass system

Stiffness of Cantilever Beam in Y-direction

Apply unit load (P =1.0 N) at free end of the beam along Y

Deflection at the free end $= \delta = \dfrac{PL^3}{3EI}$

$P = 1.0 \quad \text{N}$

$E = 3 \times 10^7 \ \text{kN/m}^2 = 3 \times 10^{10} \ \text{N/m}^2$

$I = (0.1 \times 0.2^3)/12 = 0.667 \times 10^{-4} \ \text{m}^4$

Stiffness $k_y = \dfrac{P}{\delta} = \dfrac{3EI}{L^3} = \dfrac{3 \times 3 \times 10^{10} \times .667 \times 10^{-4}}{2^3} = 750375 \quad \text{N/m}$

Frequency = $p_y = \sqrt{k_y / m} = \sqrt{750375/500} = 38.73 \quad \text{rad/s}$

$f_y = 6.16 \ \text{Hz}$

Let us also compute natural frequency using static deflection under given loading condition.

(b) Static Deflection

$W = 500 \times 9.81 = 4905 \ \text{N}$ $\qquad L = 2000 \ mm = 2.0 \ m$

$E = 3 \times 10^{10} \quad \text{N/m}^2$ $\qquad I = 0.667 \times 10^4 \ \text{m}^4$

$$\delta = (WL^3 / 3EI) = \frac{4905 \times 2^3}{3 \times 3 \times 10^{10} \times 0.667 \times 10^{-4}} = 0.0065 \ \text{m}$$

(c) Natural frequency using static deflection

$$p_y = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.0065}} = 38.84 \quad \text{rad/s}; \qquad f_y = 6.18 \quad \text{Hz}$$

This is same as obtained above. (The minor difference is only due to rounding off the numbers)

ii) Beam Mass is considered

Mass of the beam $m_b = 0.1 \times 0.2 \times 2.0 \times 2500 = 100$ kg

Generalised mass of beam $m_b^* = 0.23 \times 100 = 23$ kg

Total Equivalent mass $m^* = 500 + 23 = 523$ kg

Natural frequency $p_y = \sqrt{\frac{750375}{523}} = 37.88$ rad/s; $f_y = 6.028$ Hz

(Consideration of beam mass thus causes about 2.5 % reduction in natural frequency in this case.)

P 2.1-3

A Machine of mass 5000 kg is supported at the center of a simply supported RCC beam 200 mm x 500 mm deep and span 4000 mm as shown in Figure P 2.1-3. Consider the system as undamped. Consider motion of the mass only along Y direction in X-Y plane as shown. E_{conc} = 3 x 10^{10} N/m^2; Beam mass density 2500 kg/m^3. Beam mass to be ignored. Consider beam as (a) Simply Supported Beam and (b) Fixed-Fixed Beam. Find Natural frequency when: i) Beam Mass is ignored & ii) Beam mass is considered

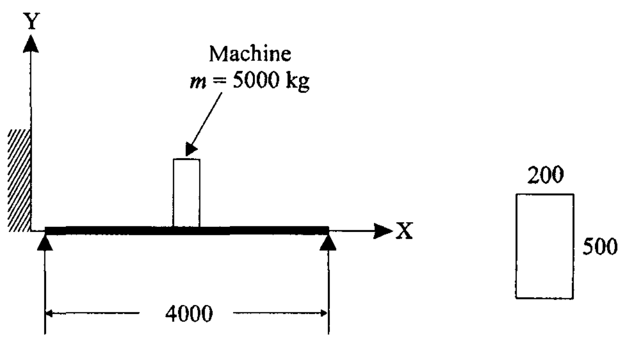


Figure P2.1-3 Machine Supported at Center of Simply Supported Beam

Solution:

i) When Beam Mass is ignored

a) Natural Frequency when Beam is Simply-Supported

Static deflection at beam center $= \delta = \dfrac{WL^3}{48EI} = \dfrac{5000 \times 9.81 \times 4^3}{48 \times 3 \times 10^{10} \times \left(\dfrac{1}{12} 0.2 \times 0.5^3\right)} = 0.001046$ m

Stiffness $k_y = \dfrac{W}{\delta} = \dfrac{5000 \times 9.81}{0.001046} = 4.6875 \times 10^7$ N/m

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{4.6875 \times 10^7}{5000}} = 96.82 \text{ rad/s}; \quad f_y = 15.41 \text{ Hz}$$

We can also compute natural frequency from static deflection δ

$$p_y = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.001046}} = 96.82 \text{ rad/s}$$

b) Natural Frequency when beam is Fixed-Fixed

$$k_y = \frac{192EI}{L^3} = \frac{192 \times 3 \times 10^{10} \times \frac{1}{12}\left(0.2 \times 0.5^3\right)}{4^3} = 18.75 \times 10^7 \text{ N/m};$$

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{18.75 \times 10^7}{5000}} = 193.6 \text{ rad/s}$$

ii) When Beam Mass is considered

Beam Mass $m_b = 0.2 \times 0.5 \times 4.0 \times 2500 = 1000$ kg

Generalised mass of beam $m_b^* = 0.37 \times 1000 = 0.37 \times 1000 = 370$ kg

Total Mass $m^* = 5000 + 370 = 5370$ kg

Natural Frequency

(a) **Beam Simply Supported** $p_y = \sqrt{\frac{k_y}{m^*}} = \sqrt{\frac{4.6875\times10^7}{5370}} = 93.43$ rad/s

(b) **Beam Fixed-Fixed** $p_y = \sqrt{\frac{k_y}{m^*}} = \sqrt{\frac{18.75\times10^7}{5370}} = 186.85$ rad/s

NOTE: In both the case, a reduction of about 3.5 % in Natural frequency is seen compared to when beam mass is ignored.

P 2.1-4

A RCC Block having Length (along Z-axis) $L = 2500$ mm, **Width** $B = 1500$ mm **and Height** $H = 400$ mm **is supported by a rotational spring (Rocking about Z-axis) having stiffness of** $k_\phi = 2\times10^6$ Nm/rad **attached at center point of base of the block, point O as shown as shown in Figure P 2.1-4. Consider the system as undamped. Density of concrete is 2500 kg/m^3.**

Find natural frequency of the system

a) **Considering that the system performs only rocking motion about Z-axis passing through CG of the base area point O**
b) **If the applied spring at point O is in Rocking direction about X-axis and the system performs only rocking motion about X-axis passing through center point O**
c) **If the applied spring at point O is in Torsional direction about Y-axis and the system performs only torsional motion about Y-axis passing through center point O**

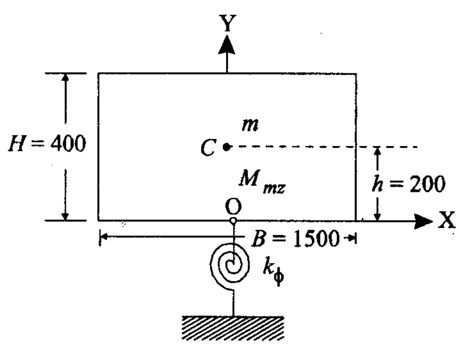


Figure P 2.1-4 Block Supported by Rotational Spring Attached at Point O

Solution:

a) Rocking about Z axis

Mass of the block $m = 2500 \times 2.5 \times 1.5 \times 0.4 = 3750$ kg

Rotational Stiffness of spring (Rocking about Z-axis) $k_{\phi} = 2 \times 10^6$ N m/rad

Height of centroid C above center of base O $h = 0.5 \times H = 0.5 \times 0.4 = 0.2$ m

Mass moment of inertia of the block about Z axis passing through centroid C= M_{mz}

$$M_{mz} = \frac{m}{12}\left[B^2 + H^2\right] = \frac{3750}{12}\left[1.5^2 + 0.4^2\right] = 753.125$$

Mass moment of inertia of the block about Z axis passing through point O = M_{moz}

$$M_{moz} = M_{mz} + m h^2 = 753.125 + 3750 \times (0.2)^2 = 903.125$$

Rocking Natural Frequency (about Z axis)

$$p_{\phi} = \sqrt{\frac{(k_{\phi} - mgh)}{M_{moz}}} = \sqrt{\frac{(2 \times 10^6 - 3750 \times 9.81 \times 0.2)}{903.125}} = 46.97 \quad \text{rad/s}$$

b) Rocking about X axis

Mass of the block $m = 3750$ kg

Rotational Stiffness of spring (Rocking about X-axis) $k_{\theta} = 2 \times 10^6$ Nm/rad

Height of centroid C above center of base O $h = 0.2$ m

Mass moment of inertia of the block about X axis passing through centroid C = M_{mx}

$$M_{mx} = \frac{m}{12}\left[L^2 + H^2\right] = \frac{3750}{12}\left[2.5^2 + 0.4^2\right] = 2003.125$$

Mass moment of inertia of the block about X axis passing through point O = M_{mox}

$$M_{mox} = M_{mx} + m h^2 = 2003.125 + 3750 \times (0.2)^2 = 2153.125$$

Rocking Natural Frequency (about X axis)

$$p_{\theta} = \sqrt{\frac{(k_{\theta} - mgh)}{M_{mox}}} = \sqrt{\frac{(2 \times 10^6 - 3750 \times 9.81 \times 0.2)}{2153.125}} = 30.421 \quad \text{rad/s}$$

c) Torsional motion about Y- axis

Mass of the block $m = 3750$ kg

Torsional Stiffness of spring (about Y-axis) $k_\psi = 2\times10^6$ Nm/rad

Height of centroid C above center of base O $h = 0.5\times H = 0.5\times0.4 = 0.2$ m

Since point C and point O lie on the same vertical line, Mass Moment of Inertia of the block about Y-axis passing through point C i.e. M_{my} is the same as the Mass moment of inertia of the block about Y-axis passing through point O i.e. M_{moy}.

$$M_{moy} = \frac{m}{12}\left[L^2 + B^2\right] = \frac{3750}{12}\left[2.5^2 + 1.5^2\right] = 2656.25$$

Torsional Natural Frequency (about Y axis)

$$p_\psi = \sqrt{\frac{k_\psi}{M_{moy}}} = \sqrt{\frac{2\times10^6}{2656.25}} = 27.44 \quad \text{rad/s}$$

P 2.1-5

For the data given in Problem P 2.1-1, consider that system has 10% damping i.e. $\zeta_y = 0.1$. Initial conditions are $y(0) = 30$ mm and $\dot{y}(0) = 200$ mm/sec. Find

a) **Natural frequency (both undamped and damped natural frequencies)**
b) **Also compute free vibration response history for damping ratio of 10% as well as 5%.**

Solution:

a) Natural frequency

m	500 kg
k_y	200,000 N/m
$y(0)$	30 mm $= 0.03$ m
$\dot{y}(0)$	0.2 m/sec

$$p_y = \sqrt{k_y/m} = \sqrt{\frac{200000}{500}} = 20 \text{ rad/s}$$

This corresponds to Time Period $T = \frac{2\pi}{p_y} = \frac{2\pi}{20} = 0.31416$ s

Damping constant $\zeta_y = 0.10$

Damped Natural Frequency for 10% damping

$$p_{dy} = p_y\sqrt{(1-\zeta_y{}^2)} = 20\sqrt{(1-0.1^2)} = 19.9 \text{ rad/s}$$

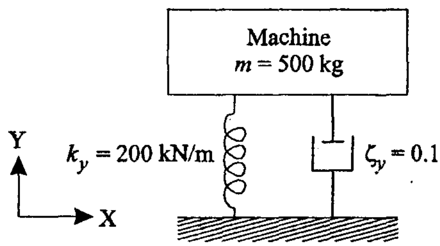


Figure P 2.1-5 Machine Supported by Vertical Spring - System Damping 10%

b) Response time history for damped system for 5% & 10 % damping

Damped Natural Frequency for 5 % damping

$$p_{dy} = p_y\sqrt{(1-\zeta_y^2)} = 20\sqrt{(1-0.05^2)} = 19.975 \text{ rad/s}$$

Rewriting equation (2.1.2-21)

$$y(t) = e^{-p_y \zeta_y t}\left(y(0)\cos p_{dy} t + \frac{(\dot{y}(0) + p_y\zeta_y\, y(0))}{p_{dy}} \sin p_{dy} t \right)$$

Substituting for $p_y, \zeta_y, p_{dy}, y(0) \& \dot{y}(0)$ the equation gives free vibration response for damped system. Response history is shown in Figure P 2.1-5a for $\zeta = 5\,\% \;\&\; \zeta = 10\,\%$

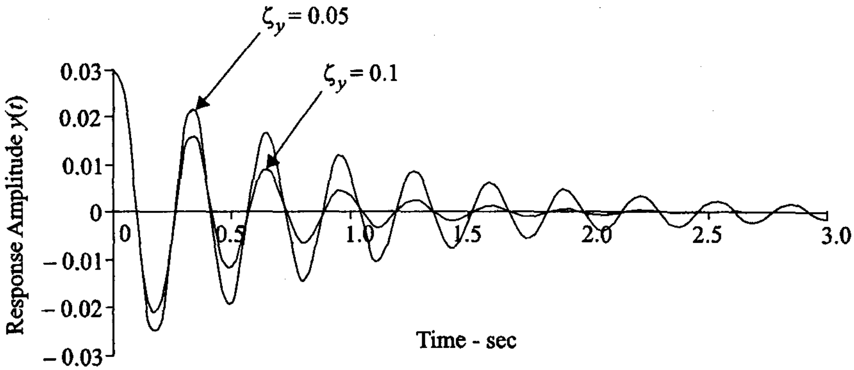


Figure P2.1-5a Free Vibration Response Time History for Damping $\zeta_y = 0.1$ and $\zeta_y = 0.05$

P 2.1-6

A Machine of mass 5000 kg is supported at the center of a RCC portal frame beam as shown in Figure P 2.1 -7. Frame beam is 200 mm x 500 mm deep and column section is 200 X 400 mm. Frame span is 4000 mm (cneter to center) and height of frame is 6000 mm (up to beam center) as shown. Consider the system as undamped. Elastic Modulus of concrete is $E_c = 3\times10^{10}$ N/m^2 and its mass density is $\rho_c = 2500$ kg/m^3. Find Natural frequency for a) Frame motion along Y only & b) Frame motion along X only for the following two conditions:

i) Beam is Elastic
ii) Beam material is considered rigid

Solution:

i) Beam is Elastic

Elastic Modulus of Material	$E_c = 3\times10^7$ kN/m^2
Mass density of the material	$\rho_c = 2.5$ t/m^3
Span of Beam is	$L = 4.0$ m
Height of Frame	$H = 6.0$ m
Area of Beam Crossection	$A_b = 0.2\times0.5 = 0.10$ m^2

Area of Column Crossection $A_c = 0.2 \times 0.4 = 0.08\ \text{m}^2$

Moment of Inertia Beam Crossection $I_b = \frac{1}{12} \times 0.2 \times 0.5^3 = 0.00208\ \text{m}^4$

Moment of Inertia Column Crossection $I_c = \frac{1}{12} \times 0.2 \times 0.4^3 = 0.00107\ \text{m}^4$

Stiffness ratio factor $k = \frac{I_b/L}{I_c/H} = \frac{0.00208/4}{0.00107/6} = 2.916$

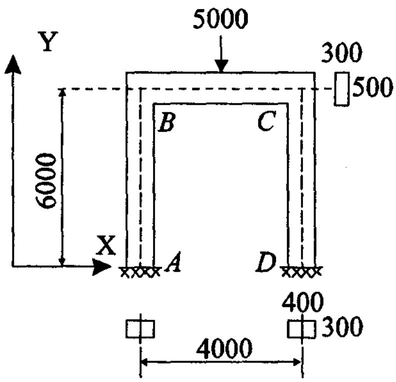


Figure P 2.1-6 Machine mass supported at Frame Beam center

a) Motion along Y (Vertical motion)

- **Deflection δ_{yb} at beam center:**

Machine mass at frame beam center $m = 5.0\ \text{t}$

Beam Mass $m_b = 0.10 \times 4 \times 2.5 = 1.0\ \text{t}$

Generelaised Beam Mass at beam center $m_b^* = 0.45 m_b = 0.45 \times 1 = 0.45\ \text{t}$

Effective Mass at Beam Center $m^* = m + 0.45 m_b = 5.0 + 0.45 = 5.45\ \text{t}$

Deflection at beam center δ_{yb} (see equation 2.1.1-34)

$$\delta_{yb} = \frac{m^* g L^3}{96 E I_b} \times \frac{2k+1}{k+2} = \frac{5.45 \times 9.81 \times 4^3}{96 \times 3 \times 10^7 \times 0.00208} \times \frac{2 \times 2.916 + 1}{2.916 + 2} = 7.938 \times 10^{-4}\ \text{m}$$

- **Deflection δ_{yc} at column top**

Mass of each column $m_c = 0.08 \times 6 \times 2.5 = 1.2$ t

Generalised column mass (each column) $m_c^* = 0.33 \times 1.2 = 0.396$ t

Total Effective mass at column top $m^* = m + m_b + 2 \times m_c^*$

$m^* = 5.0 + 1.0 + 2 \times 0.396 = 6.792$ t

Vertical deflection of columns δ_{yc} (see equation 2.1.1-35)

$$\delta_{yc} = \frac{m^* \times g}{2 \times (E A_c / H)} = \frac{6.792 \times 9.81}{2 \times (3 \times 10^7 \times 0.08/6)} = 0.8329 \times 10^{-4} \text{ m}$$

Total vertical deflection $\delta_y = \delta_{yb} + \delta_{yc}$

$$\delta_y = 7.938 \times 10^{-4} + 0.8329 \times 10^{-4} = 8.771 \times 10^{-4} \text{ m}$$

Natural Frequency $p_y = \sqrt{\frac{g}{\delta_y}} = \sqrt{\frac{9.81}{8.771 \times 10^{-4}}} = 105.75$ rad/s

Just for the academic interest, let us compare the results with that of **Problem P 2.1-3** having same Beam size, Beam Span and Machine Mass. From the results of P 2.1-3, we notice that:

- When Beam is simply supported, Natural Frequency is $p_y = 93.43$ rad/s
- When Beam is fixed – fixed, Natural Frequency is $p_y = 186.85$ rad/s
- When considering as portal frame (the present case), Vertical Natural Frequency is $p_y = 105.75$ rad/s

For the portal frame, it is noticed that stiffness ratio k is close to 3. This indicates that beam is about 3 times stiffer than column. In other words, beam beahaviour is more biased towards simply supported case rather than fixed beam case. This gets confirmed from the present results.

b) Motion along X (Transverse motion)

Machine mass $m = 5.0$ t

Generalised beam mass $m_b^* = m_b = 1.0$ t

Generalised column mass (each column) $m_c^* = 0.23 m_c = 0.23 \times 1.2 = 0.276$ t

Total Effective Mass at Frame Column top $m^* = 5.0 + 1.0 + 2 \times 0.276 = 6.552$ t

Transverse deflection at column top δ_x (see equation 2.1.1-37)

$$\delta_x = \frac{m^* \times g \times H^3}{12 E I_c} \frac{2+3k}{1+6k} = \frac{6.552 \times 9.81 \times 6^3}{12 \times 3 \times 10^7 \times 0.00107} \frac{2+3 \times 2.916}{1+6 \times 2.916} = 0.02 \text{ m}$$

Natural Frequency $p_x = \sqrt{\frac{g}{\delta_x}} = \sqrt{\frac{9.81}{0.02}} = 22.14$ rad/s

ii) Beam is Material is considered Rigid

Stiffness ratio factor (beam Elastic Modulus is considered infinite)

$$k = \frac{(E I_b / L)}{(E I_c / H)} = \infty$$

a) Motion along Y (Vertical motion)

Since beam is rigid, there is no elastic deformation of the beam. Thus $\delta_{yb} = 0$

Only deformation along Y is that due to column.

Total Effective Mass at Column Top
$m^* = 5.0 + 1.0 + 2 \times 0.33 \times 1.2 = 6.792$ t

Vertical deflection of columns δ_{yc} (see equation 2.1.1-35)

$$\delta_{yc} = \frac{m^* \times g}{2 \times (E A_c / H)} = \frac{6.792 \times 9.81}{2 \times (3 \times 10^7 \times 0.08 / 6)} = 0.8329 \times 10^{-4} \text{ m}$$

Total vertical deflection $\delta_y = \delta_{yb} + \delta_{yc} = 0 + 0.8329 \times 10^{-4} = 8.329 \times 10^{-5}$ m

Natural Frequency $p_y = \sqrt{\frac{g}{\delta_y}} = \sqrt{\frac{9.81}{8.329 \times 10^{-5}}} = 343$ rad/s

b) Motion along X (Transverse motion)

Effective Mass at Frame column top

$m^* = 5.0 + 1.0 + 2 \times 0.23 \times 1.2 = 6.552$ t

Transverse deflection at column top δ_x (see equation 2.1.1-37)

$$\delta_x = \frac{m^* \times g \times H^3}{12 E I_c} \frac{2+3k}{1+6k}$$

for $k = \infty$, we get $\delta_x = \dfrac{m^* \times g \times H^3}{24 E I_c}$

$$\delta_x = \frac{m_c^* \times g \times H^3}{24 E I_c} = \frac{6.552 \times 9.81 \times 6^3}{24 \times 3 \times 10^7 \times 0.00107} = 0.018 \text{ m}$$

Natural Frequency $p_x = \sqrt{\dfrac{g}{\delta_x}} = \sqrt{\dfrac{9.81}{0.018}} = 23.34$ rad/s

EXAMPLE PROBLEMS: (SDOF System - Forced Vibration Response)

P 2.2-1

In Example Problem P 2.1-1, consider that the system has 10% damping & the mass is acted upon by external dynamic load (harmonic load) of $F_y = 20$ N with excitation frequency ω of a) 10 rad/s, b) 20 rad/s, c) 30 rad/s, d) 80 rad/s. Compute:

i. **Maximum amplitude of vibration for each of the above loading case**
ii. **Maximum Reaction Force transmitted to support in each case**
iii. **Plot Response history (Transient and steady state response) for case a) & b) for initial conditions of $y(0) = 0$ & $\dot{y}(0) = 0$**

Solution:

Natural Frequency (see solution P 2.1-1)

k_y=200 kN/m ; m= 500 kg ; $\zeta_y = 0.1$

Undamped Natural Frequency ω= 20 rad/s

Damped Natural Frequency ω_d = 19.8997 rad/s

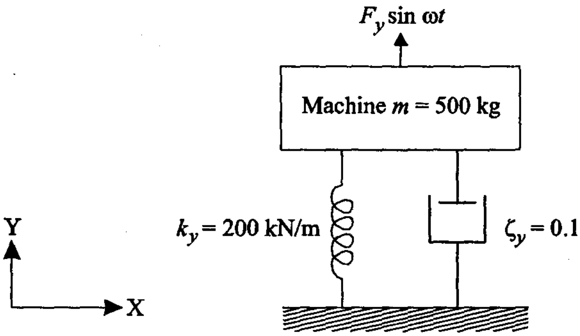


Figure P 2.2-1 Forced Vibration-Damped SDOF System - Damping 10%
Dynamic Force Externally Applied

It is seen that there is hardly any appreciable change in damped frequency. Hence for all practical purposes, it is good enough to compute only undamped natural frequency.

i) **Maximum Amplitude of vibration**

a) **Dynamic Force** = 20 sin 10 t

$F_y = 20$ N; $\omega = 10$ rad/s; $p_y = 20$ rad/s; $\zeta_y = 0.1$; $\beta_y = \dfrac{w}{p_y} = 0.5$

Dynamic Magnification factor $\mu_y = \dfrac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} = 1.32164$

Amplitude = $y(t) = \dfrac{F_y}{k_y} \mu_y \sin(\omega t - \phi)$; for maximum amplitude, $\sin(\omega t - \phi) = 1$

Max. Amplitude $y(t)_{\max} = \dfrac{F_y}{k_y} \mu_y$

$y(t)_{\max} = (20/200000) \times 1.32164 = 0.00013 \text{ m} = 130$ microns

b) **Dynamic Force** = 20 sin 20 t

$F_y = 20N$; $\omega = 20$ rad/s; $p_y = 20$ rad/s; $\zeta_y = 0.1$; $\beta_y = \dfrac{w}{p_y} = 1$

Dynamic Magnification factor $\mu_y = \dfrac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} = 5$

Max. Amplitude $y(t)_{\max} = \frac{F_y}{k_y} \mu_y$

$y(t)_{\max} = (20/200000) \times 5 = 0.0005 \text{ m} = 500 \text{ microns}$

c) **Dynamic Force** = $20 \sin 30t$

$F_y = 20$ N ; $\omega = 30$ rad/s; $p_y = 20$ rad/s; $\zeta_y = 0.1$; $\beta_y = \frac{w}{p_y} = 1.5$

Dynamic Magnification factor $\mu_y = \frac{1}{\sqrt{(1-\beta_y^2)^2 + (2\beta_y \zeta_y)^2}} = 0.778$

Maximum Amplitude $y(t)_{\max} = \frac{F_y}{k_y} \mu_y$

$y(t)_{\max} = (20/200000) \times 0.778 \times 1 \times 10^6 = 77.8 \text{ microns}$

d) **Dynamic Force** = $20 \sin 80t$

$F_y = 20N$; $\omega = 80$ rad/sec; $p_y = 20$ rad/sec; $\zeta_y = 0.1$; $\beta_y = \frac{w}{p_y} = 4$

Dynamic Magnification factor $\mu_y = \frac{1}{\sqrt{(1-\beta_y^2)^2 + (2\beta_y \zeta_y)^2}} = 0.066$

Maximum Amplitude $y(t)_{\max} = \frac{F_y}{k_y} \mu_y$

$y(t)_{\max} = (20/200000) \times 0.066 \times 1 \times 10^6 = 6.6 \text{ microns}$

From these results, following observations are made:

1. ***Magnification factor rises sharply with frequency ratio approaching unity***

2. ***Though stiffness is of the order of 200 kN/m, even a small dynamic force of 20 N is able to cause amplitudes as high as 500 microns at resonance.***

Maximum Reaction Force transmitted to support:

From equation (2.2.2-11), we get max-transmitted force as $F_T = F_y \; \mu_y \sqrt{1 + (2\beta_y \zeta_y)^2}$

Max. Reaction force transmitted

Case a) $F_T = 20 \times 1.32164 \times \sqrt{1 + (2 \times 0.5 \times 0.1)^2} = 26.56$ N

Case b) $F_T = 20 \times 5 \times \sqrt{1 + (2 \times 1 \times 0.1)^2} = 101.98$ N

Case c) $F_T = 20 \times 0.778 \times \sqrt{1 + (2 \times 1.5 \times 0.1)^2} = 16.245$ N

Case d) $F_T = 20 \times 0.066 \times \sqrt{1 + (2 \times 4 \times 0.1)^2} = 1.69$ N

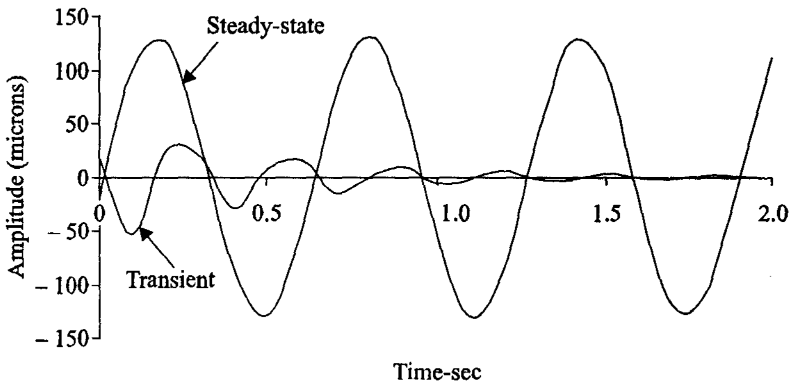


Figure P2.2-1a Respose History for Force $F(t) = 20 \sin 10\,t$

iii) Response history

For initial conditions of $y(0) = 0$ & $\dot{y}(0) = 0$ use equation (2.2.2-4) for Transient Response & use equation (2.2.2-5) for Steady State Response. As mentioned earlier, use natural frequency p_y instead of p_{dy} for all response computations.

Transient Response:

$$y_c(t) = \frac{F_y}{k_y} e^{-p_y \zeta_y t} \left[\frac{2\zeta_y \beta_y}{\left[(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2\right]} \cos p_{dy} t + \frac{1}{\sqrt{(1-\zeta_y^2)}} \left\{ \begin{aligned} &\left(\zeta_y \frac{2\zeta_y \beta_y}{\left[(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2\right]} \right) \\ -&\left(\beta_y \frac{(1-\beta_y^2)}{\left[(1-\beta_y^2)^2 + (2\zeta_y \beta_y)^2\right]} \right) \end{aligned} \right\} \sin p_{dy} t \right]$$

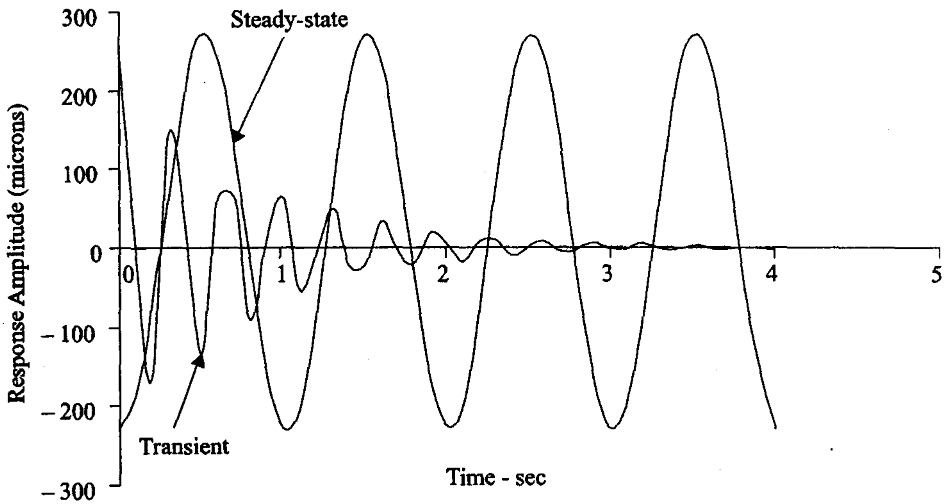


Figure P2.2-1b Response History for Force $F(t) = 20 \sin 20\, t$

Steady State Response: $y_p(t) = \dfrac{F_y}{k_y} \mu_y \sin(\omega t - \phi)$ $\phi = \tan^{-1}\left(\dfrac{2\beta_y \zeta_y}{(1-\beta_y^2)}\right)$

Case a) Force 20 sin 10 t

$\omega = 10, \quad p_y = 20, \quad \beta_y = 0.5, \quad \zeta_y = 0.1, \quad F_y = 20, \quad k_y = 200000$

Response history plot giving transient as well as steady state response is as shown in Figure P 2.2-1a.

Case b) Force 20 sin 20 t

$\omega = 20, \quad p_y = 20, \quad \beta_y = 1, \quad \zeta_y = 0.1, \quad F_y = 20, \quad k_y = 200000$

Response history plot giving transient as well as steady state response is shown in Figure P 2.2-1b

P 2.2-2

In Example Problem P 2.1-2, consider that the system has 5 % damping and the mass is acted upon by external dynamic load of a) $20 \sin 25t$, b) $20 \sin 50t$. Compute Maximum

Amplitude of vibration and Maximum Reaction Force Transmitted to Support. Contribution of Beam Mass is to be considered.

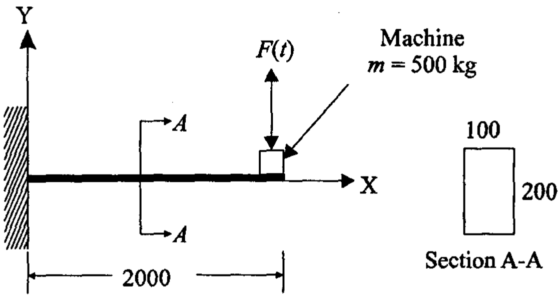


Figure P2.2-2 Machine supported by cantilever beam system damping $z = 5\%$, Dynamic Force $F(t) = F(t)$ externally applied

Solution:

(i) **Beam mass is considered**

Stiffness of Cantilever Beam in Y-direction **(see solution P 2.1-2)**

Stiffness $k_y = 750375$ N/m

Natural frequency $p_y = 37.88$ rad/s

i) Maximum Amplitude of vibration

Case a) Force $20 \sin 25t$

$$\omega = 25 \text{ rad/sec}; \quad F_y = 20 \text{ N}; \quad \zeta_y = 0.05$$

Frequency ratio $\beta_y = \dfrac{\omega}{p_y} = \dfrac{25}{37.88} = 0.66$

Dynamic Magnification factor

$$\mu_y = \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta_y\right)^2}} = \frac{1}{\sqrt{\left(1-0.66^2\right)^2 + \left(2\times0.66\times0.05\right)^2}} = 1.76$$

Amplitude (max) $(F_y / k_y) \times \mu_y \quad = \frac{20}{750375} \times 1.76 \times 1 = 46.9 \times 10^{-6}$ m

$= 46.9$ microns

Case b) Force $20 \sin 50t$

$\omega = 50$ rad/s; $F_y = 20$ N; $\zeta_y = 0.05$

Frequency ratio $\beta_y = \frac{\omega}{p_y} = \frac{50}{37.88} = 1.32$

Dynamic Magnification factor $\mu_y = \frac{1}{\sqrt{\left(1 - 1.32^2\right)^2 + \left(2 \times 1.32 \times 0.05\right)^2}} = 1.326$

Amplitude (max) $= \frac{20}{750375} \times 1.326 = 35.3 \times 10^{-6}$ m $= 35.3$ microns

ii) Maximum Reaction Force Transmitted to Support

Max. Force transmitted to support $F_T = F_y \; \mu_y \sqrt{1 + (2\, \beta_y \, \zeta_y)^2}$

Case a) $F_T = 20 \times 1.76 \times \sqrt{1 + (2 \times 0.66 \times 0.05)^2} = 35.2$ N

Case b) $F_T = 20 \times 1.326 \times \sqrt{1 + (2 \times 1.32 \times 0.05)^2} = 26.75$ N

P 2.2-3

A machine of mass $500\,\text{kg}$ is supported on a RCC Block of size $L = 2500$ mm; $B = 1500$ mm; & $H = 400$ mm. Density of concrete is 2500 kg/m^3. The block in turn is supported by a Lateral Translational spring in X direction having stiffness of $k_x = 2 \times 10^7$ N/m attached at CG of the base area of the block (point O) as shown in Figure P 2.2-3. Height of the CG of machine (point C) above top of the Block is $h_1 = 100$ mm. CG of Block and CG of the machine lie on the same vertical line. Consider that the system is undamped and it is constrained to translate only along X-axis. A dynamic force of $F_x = 200$ N at excitation frequency of $\omega = 50$ rad/s is applied at the machine mass CG along X-axis. Find natural frequency of the system and compute maximum amplitude of vibration at the machine center point C.

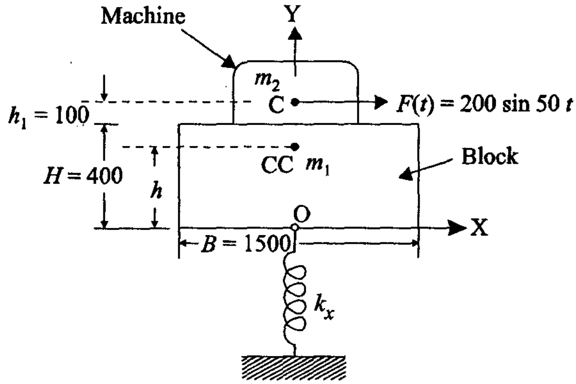


Figure P2.2-3 Machine on Block - Block Supported by Translational Spring in X-Direction attached at Base Center Point O

Solution:

Spring applied in X-direction and Dynamic Force acts in X-direction

Machine mass m_2 $= 500$ kg

Block size $= 2.5 \times 1.5 \times 0.4$ m

Mass of Block m_1 $2.5 \times 1.5 \times 0.4 \times 2500 = 3750$ kg

Total Mass $(m_1 + m_2)$ $= 4250$ kg

System is constrained to translate only along X-axis

Stiffness of supporting spring $k_x = 2 \times 10^7$ N/m

Natural Frequency $p_x = \sqrt{\dfrac{k_x}{m}} = \sqrt{\dfrac{2 \times 10^7}{4250}} = 68.6$ rad/s

Since the system is constrained to translate only along X-axis, the dynamic force at O is same as F_x applied at machine center.

Dynamic Force at O $F_x(t) = 200 \sin 50t$

Frequency ratio $\beta_x = \dfrac{\omega}{p_x} = \dfrac{50}{68.6} = 0.73$

Magnification factor $\mu_x = \left| \dfrac{1}{(1 - \beta_x^2)} \right| = 2.14$

Amplitude at point O (see equation 2.2.1-5a; With appropriate changes made for force in X-direction)

$$x_p(t) = \frac{F_x}{k_x} \frac{1}{\left(1-\beta_x^2\right)} \sin \omega t$$

Maximum amplitude at point O

$$x(\max) = \frac{200}{(2\times 10^7)} \times 2.14 = 0.214\times 10^{-4} \text{ m} = 21.4 \text{ microns}$$

Maximum amplitude at point C: Since the system is constrained to translate only along X-axis, amplitude at C is same as amplitude at O i.e. 21.4 microns.

P 2.2-4

A machine of mass 500 kg is supported on a RCC Block of size $L = 2500\text{ mm}; B = 1500\text{ mm}; \& H = 400\text{ mm}$. Density of concrete is 2500 kg/m^3. The block in turn is supported by a rotational spring having stiffness of $k_\phi = 2\times 10^6\text{ Nm/rad}$ attached at center of the base of the block (point O) as shown in Figure P 2.2-4. Height of the CG of machine (point C) above top of the Block is $h_1 = 100\text{ mm}$. CG of Block and CG of the machine lie on the same vertical line. Consider system to be undamped. System is constrained to perform only rocking motion about Z-axis passing through O. A dynamic force of $F_x = 200\text{ N}$ at excitation frequency of $\omega = 50\text{ rad/s}$ is applied at the machine mass CG along X-axis.

a) **Compute natural frequency and maximum amplitude of vibration at the machine center point C.**
b) **If rotational stiffness acts at O about X-axis and the applied dynamic force at point C acts along Z-axis, compute natural frequency and maximum amplitude of vibration at the machine center point C.**

Solution:

a) Rotational stiffness acts at O about Z-axis and the applied dynamic force at point C acts along X-axis

Machine mass	m_2	$= 500$ kg
Block size		$= 2.5\times 1.5\times 0.4$ m
Mass of Block	m_1	$2.5\times 1.5\times 0.4\times 2500 = 3750$ kg
Total Mass	$(m_1 + m_2)$	$= 4250$ kg

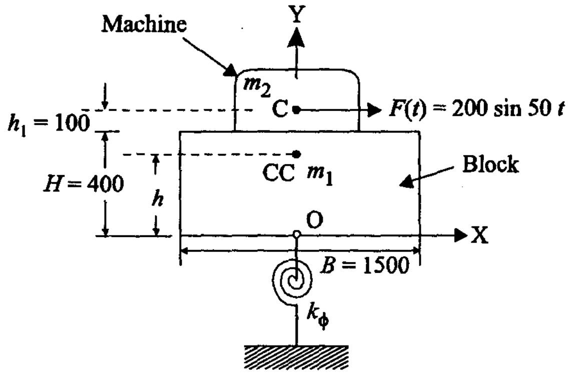


Figure P2.2-4 Machine on Block - Rotational Spring Attached to Base Center Point O about Z-axis - Dynamic Force $F(t) = 200 \sin 50\,t$ applied at Machine Center along X-axis

Let us denote Overall centroid (Block +Machine) as CC

Height of overall centroid CC above base

$$h = \frac{3750 \times 0.2 + 500 \times (0.1 + 0.4)}{4250} = 0.2354 \quad \text{m}$$

System rotates about Z-axis passing through base center point O

Mass Moment of Inertia about base center point O = M_{moz}

$$M_{moz} = \frac{3750}{12} \times \left(1.5^2 + 0.4^2\right) + 3750 \times 0.2^2 + 500 \times (0.1 + 0.4)^2 = 1028.125$$

Stiffness of supporting spring $\quad k_\phi = 2 \times 10^6 \quad$ Nm/rad

Natural Frequency

$$p_\phi = \sqrt{\frac{k_\phi}{M_{moz}}} = \sqrt{\frac{2 \times 10^6}{1028.125}} = 44.10 \quad \text{rad/s}$$

Just for academic interest let us compute natural frequency by including the term 'mgh', we get

$$p_\phi = \sqrt{\frac{(k_\phi - mgh)}{M_{moz}}} = \sqrt{\frac{(2 \times 10^6 - 4250 \times 9.81 \times 0.2354)}{1028.125}} = 44 \quad \text{rad/s}.$$

It is seen that by ignoring the term mgh there is hardly any difference in the result. Thus the simplification of ignoring the term mgh both for frequency and amplitude computations is OK.

The dynamic force F_x at machine center causes dynamic moment M_ϕ about O

Dynamic moment about O $\quad M_\phi = 200\,(0.1+0.4) = 100 \quad$ Nm

Frequency ratio $\quad \beta_\phi = \left(50/_{44.1}\right) = 1.134$

Magnification factor $\quad \mu_\phi = \left|\frac{1}{\left(1-\beta_\phi^2\right)}\right| = 3.5$

Amplitude at point O (see equation 2.2.1-10)

$$\phi_p(t) = \frac{M_\phi}{k_\phi}\frac{1}{\left(1-\beta_\phi^2\right)}\sin\omega\, t = \frac{M_\phi}{k_\phi}\mu_\phi \sin\omega\, t$$

Maximum amplitude $\quad \phi(\max) = \frac{100}{(2\times10^6)}\times 3.5 = 1.75\times10^{-4} \quad$ rad

Amplitude at point C in X-direction

$$(0.1+0.4)\times 0.1.75\times10^{-4} = 8.75\times10^{-5}\ \text{m} \quad = 87.5\ \text{microns}$$

b) Rotational stiffness acts at O about X-axis and the applied dynamic force at point C acts along Z-axis

System rotates about X-axis passing through base center point O

Mass Moment of Inertia about base center point O= M_{mox}

$$M_{mox} = \frac{3750}{12}\times\left(2.5^2+0.4^2\right) + 3750\times0.2^2 + 500\times(0.1+0.4)^2 = 2278.125$$

Stiffness of supporting spring $\quad k_\theta = 2\times10^6$ Nm/rad

Natural Frequency

$$p_\theta = \sqrt{\frac{k_\theta}{M_{mox}}} = \sqrt{\frac{2\times10^6}{2278.125}} = 29.6 \quad \text{rad/s}$$

The dynamic force F_z at machine center causes dynamic moment M_θ about O

Dynamic moment about O $\quad M_\theta = 200\,(0.1+0.4) = 100 \quad \text{Nm}$

Frequency ratio $\quad \beta_\theta = \left(50/29.6\right) = 1.69$

Magnification factor $\quad \mu_\theta = \left|\dfrac{1}{(1-\beta_\theta^2)}\right| = 0.537$

Amplitude at point O (see equation 2.2.1-10)

$$\theta_p(t) = \frac{M_\theta}{k_\theta}\frac{1}{\left(1-\beta_\theta^2\right)}\sin\omega t = \frac{M_\theta}{k_\theta}\mu_\theta \sin\omega t$$

Maximum amplitude $\quad \theta(\max) = \dfrac{100}{(2\times10^6)}\times 3.43 = 0.2685\times10^{-4} \quad \text{rad}$

Amplitude at point C in Z direction

$$(0.1+0.4)\times 0.2685\times10^{-4} = 1.34\times10^{-5}\ \text{m} \quad = 13.4\ \text{microns}$$

P 2.2-5

A machine of mass $500\,\text{kg}$ is supported on a RCC Block of size $L = 2500\,\text{mm};\ B = 1500\,\text{mm};\ \&\ H = 400\,\text{mm}$. Density of concrete is 2500 kg/m^3. The block in turn is supported by a rotational spring having stiffness of $k_\psi = 2\times10^6$ Nm/rad attached at center of the base of the block (point O) as shown in Figure P 2.2-5. Height of the CG of machine (point C) above top of the Block is $h_1 = 100\,\text{mm}$. CG of Block and CG of the machine lie on the same vertical line. Consider that the system only perform rocking motion about Y-axis passing through O. Consider machine radius of gyration $r_y = 300\,\text{mm}$. Find natural frequency of the system? Also compute maximum amplitude of vibration at the corners at the top of the block when a dynamic couple of $M_\psi = 100\,\text{Nm}$ at excitation frequency of $\omega = 50\,\text{rad/s}$ is applied at the machine mass CG about Y-axis. Consider system to be undamped.

Solution:

The entire problem data is same as that for Problem P 2.2-4 except that the applied spring is in ψ direction and the applied dynamic couple is about Y-axis and the block is allowed to rotate about Y-axis passing through O

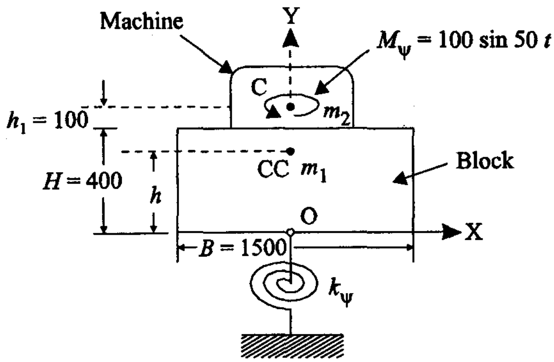


Figure P2.2-5 Machine on Block - Block Supported by Rotational Spring Attached to Base Center Point O - Dynamic Moment M_ψ = 100 sin 50t applied at Machine Center about-Y

Machine mass	m_2	$= 500$ kg
Mass of Block	m_1	$= 3750\ kg$
Total Mass	$(m_1 + m_2)$	$= 4250$ kg

Let us denote Overall centroid (Block +Machine) as CC

Height of overall centroid CC above base $h = 0.2354$ m

System rotates about Y-axis passing through base center point O

Radius of Gyration $r_y = 300$ mm

Mass Moment of Inertia about Y at base center point O = M_{moy}

$$M_{moz} = \frac{3750}{12} \times \left(2.5^2 + 1.5^2\right) + 500 \times (0.3)^2 = 2701.25$$

Stiffness of supporting spring $k_\phi = 2 \times 10^6$ Nm/rad

Natural Frequency

$$p_\psi = \sqrt{\frac{k_\phi}{M_{moy}}} = \sqrt{\frac{2 \times 10^6}{2701.25}} = 27.2 \quad \text{rad/s}$$

Dynamic moment about Y at O $M_\psi = 100 \sin 50t$ Nm

Frequency ratio $\beta_\psi = \left(50/27.2\right) = 1.838$

Magnification factor $\mu_\phi = \left|\dfrac{1}{\left(1-\beta_\phi^2\right)}\right| = 0.42$

Amplitude at point O

$$\psi_p(t) = \frac{M_\psi}{k_\psi}\frac{1}{\left(1-\beta_\phi^2\right)}\sin\omega t = \frac{M_\psi}{k_\psi}\mu_\psi \sin\omega t$$

Maximum amplitude $\psi(\max) = \dfrac{100}{(2\times10^6)}\times0.42 = 0.21\times10^{-4}$ rad

Amplitude at Top corner of the foundation

Since all the four corner points at top of the foundation are equidistant from point O, consider right top corner for purpose of computation.

Distance of corner point from O along X-axis =750 mm

Distance of corner point from O along Z-axis =1250 mm

X amplitude thus becomes $A_x = 0.75\times0.21\times10^{-4} = 1.575\times10^{-5}\ m = 15.75$ microns

We get Z amplitude as $A_z = 1.25\times0.21\times10^{-4} = 2.625\times10^{-5}\ m = 26.25$ microns

Total amplitude of corner point $A = \sqrt{{A_x}^2 + {A_z}^2} = \sqrt{15.75^2 + 26.25^2} = 30.6$ microns

P 2.2-6

A mass $m_0 = 3500$ kg falls freely from a height of $h = 2.0$ m over a foundation having mass $m_1 = 250000$ kg. The foundation is supported by a linear spring having stiffness of $k_1 = 4.2\times10^6$ kN/m. Consider $e = 0.5$. Compute maximum amplitude of the foundation.

Solution:

$m_1 = 250000$ kg; $m_0 = 3500$ kg; $k_1 = 4.2\times10^6$ kN/m

$e = 0.5$; $h = 2.0$ m

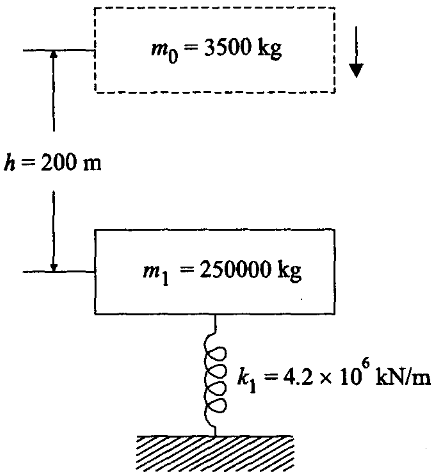


Figure P2.2-6 SDOF System Subjected to Impact Load

Natural Frequency $$p_1 = \sqrt{\frac{4.2\times10^6\times10^3}{250000}} = 129.6 \quad \text{rad/s}$$

Mass Ratio $$\lambda_1 = \frac{250000}{3500} = 71.42$$

Velocity of mass m_0 before impact

$$v_0' = \sqrt{2gh} = \sqrt{2\times9.81\times2} = 6.26 \quad \text{m/s}$$

Velocity of mass m_1 after impact (see equation 2.2.5-3)

$$v_1 = v_0' \times \frac{(1+e)}{(1+\lambda_1)} = 6.26\times\frac{(1+0.5))}{(1+71.42)} = 0.1296 \quad \text{m/s}$$

We get response of the SDOF system as (refer equation 2.2.5-12)

$$y_1 = \frac{v_1}{p_1}\sin p_1 t = \frac{0.1296}{129.6}\sin 129.6t$$

Maximum Amplitude (refer equation 2.2.5-13)

$$\rho_1 = \frac{v_1}{p_1} = \frac{0.1296}{129.6} = 0.001 \quad \text{m} \quad or \quad \rho_1 = 1 \quad \text{mm}$$

P 2.2-7

A SDOF system having mass $m_1 = 2500$ kg and stiffness $k_1 = 4.2\times10^4$ kN/m is subjected to an impulsive load. The pulse shape is rectangular and peak magnitude of the applied force is $F_0 = 50$ kN and pulse duration is (a) 50 milisec and (b) 10 milisec. Compute maximum amplitude of the foundation.

Solution:

$m_1 = 2500$ kg

$k_1 = 4.2\times10^4$ kN/m

Natural Frequency $$p_1 = \sqrt{\frac{4.2\times10^4\times10^3}{2500}} = 129.6 \quad \text{rad/s}$$

Natural Time Period of the system $$T = \frac{2\pi}{p} = \frac{2\pi}{129.6} = 0.048 \quad \text{s}$$

Magnitude of the applied Force $F_0 = 50$ kN

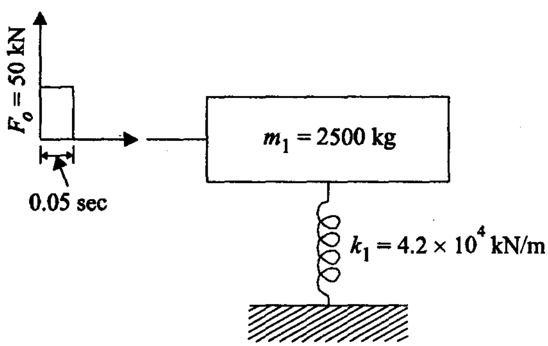


Figure P 2.2-7 SDOF System Subjected to Rectangular Pulse

(a) Pulse Duration $\tau = 50\times10^3 = 0.05$ s

Excitation Frequency $$\omega = \frac{\pi}{\tau} = \frac{\pi}{0.05} = 62.83 \text{ rad/s}$$

Frequency Ratio $$\beta = \frac{\omega}{p} = \frac{62.83}{129.6} = 0.485$$

Ratio $\frac{\tau}{T} = \frac{0.05}{0.048} = 1.031$

Response Magnification from Figure 2.2.6-3 $\mu = 2.0$

Maximum Amplitude

$$y_{max} = \mu \times \frac{F_0}{k_1} = 2.0 \times \frac{50 \times 10^3}{4.2 \times 10^7} = 2.4 \times 10^{-3} \quad \text{m}$$

$$= 2.4 \times 10^{-3} \times 10^3 \quad = 2.4 \quad \text{mm}$$

(b) Pulse Duration $\tau = 10 \times 10^3 = 0.01$ s

Excitation Frequency $\omega = \frac{\pi}{\tau} = \frac{\pi}{0.01} = 314.16$ rad/s

Frequency Ratio $\beta = \frac{\omega}{p} = \frac{314.16}{129.6} = 2.424$

Ratio $\frac{\tau}{T} = \frac{0.01}{0.048} = 0.2083$

Response Magnification from Figure 2.2.6-3 $\mu = 1.22$

Maximum Amplitude

$$y_{max} = \mu \times \frac{F_0}{k_1} = 1.22 \times \frac{50 \times 10^3}{4.2 \times 10^7} = 1.452 \times 10^{-3} \text{m} = 1.452 \quad \text{mm}$$

It is noted that when pulse duration is short, amplitude reduces for the same applied force. Since pulse duration is short compared to natural time period, we can also evaluate maximum amplitude by equation (2.2.6-15).

$$y_{max} = \frac{I \times p}{k}$$

Here I is the Impulse i.e. area of the pulse diagram, p is the natural frequency and k is the stiffness. Substituting values we get:

$$I = F_0 \times \tau = 50 \times 10^3 \times 0.01 = 500 \quad \text{Ns}$$

$$I \times p = 500 \times 129.6 = 64800 \quad \text{N}$$

$$y_{max} = \frac{I \times p}{k} = \frac{64800}{4.2 \times 10^7} = 1.543 \times 10^{-3} \text{ m} = 1.543 \text{ mm}$$

It is noticed that this amplitude is nearly same as computed above using Response Magnification Factor.

P 2.2-8

In problem P 2.1-3, consider that system has 5 % damping and mass is acted upon by a dynamic force 200 N at excitation frequency of 15 Hz. Compute maximum amplitude of vibration of the mass.

Solution: (see solution 2.1-3) Beam mass ignored

a) When beam is Simply- Supported

Stiffness $k_y = 4.6875 \times 10^7$ N/m

$p_y = 96.82$ rad/s

Excitation Frequency $\omega = 15 \times 2 \times \pi = 94.24$ rad/s

Frequency Ratio $\beta = \dfrac{\omega}{p} = \dfrac{94.24}{96.82} = 0.97$

Damping $\zeta = 0.05$

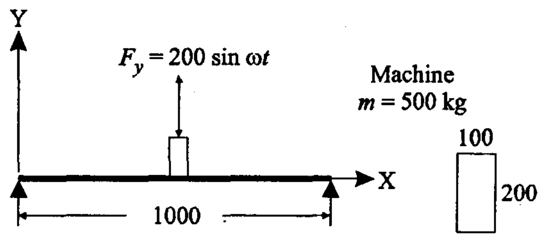


Figure P 2.2-8 Machine Supported at Center of Simply Supported Beam Subjected to Dynamic Force $F_y = 200 \sin \omega t$

Dynamic magnification factor

$$\mu = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}} = \frac{1}{\sqrt{(1-0.97^2)^2 + (2\times 0.97\times 0.05)^2}} = 8.8$$

Excitation Force Magnitude $F_y = 200$ N

Static deflection $\delta_{st} = \dfrac{200}{4.6875\times10^{7}} = 4.267\times10^{-6}$ m

Maximum Amplitude $A_y = 4.267\times10^{-6}\times8.8 = 37.55\times10^{-6}$ m $= 37.55$ microns

b) When beam is Fixed-fixed

$k_y = 18.75\times10^{7}$ N/m ; $p_y = 193.6$ rad/s

Excitation Frequency $\omega = 15\times2\times\pi = 94.24$ rad/s

Frequency Ratio $\beta = \dfrac{\omega}{p} = \dfrac{94.24}{193.6} = 0.48$

Damping $\zeta = 0.05$

Dynamic magnification factor

$$\mu = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\beta\zeta)^2}} = \frac{1}{\sqrt{(1-0.48^2)^2 + (2\times0.48\times0.05)^2}} = 1.3$$

Excitation Force Magnitude $F_y = 200$ N

Static deflection $\delta_{st} = \dfrac{200}{18.75\times10^{7}} = 1.067\times10^{-6}$ m

Maximum Amplitude $A_y = 1.067\times10^{-6}\times1.3 = 1.39\times10^{-6}$ m $= 1.4$ microns

ii) When Beam Mass is considered

a) Beam Simply Supported

$p_y = 93.43$ rad/s; $\beta = 1.008$; $\zeta = 0.05$; $\mu = 9.8$; $A_y = 41.8\times10^{-6}$ m $= 41.8$ microns

b) Beam Fixed-Fixed

$p_y = 186.85$ rad/s; $\beta = 0.504$; $\zeta = 0.05$; $\mu = 1.38$; $A_y = 1.47\times10^{-6}$ m $= 1.5$ microns

P 2.2-9

In Problem P 2.1-6, consider a dynamic force $F(t) = F_0 \sin \omega t$, as given below, is applied at frame beam center vertically along Y as well as horizontally along X (one at a time). Consider damping $\zeta = 5\%$. Compute response of the mass when excitation frequency of the applied force is:

i) **Applied Force** $F(t) = 0.2 \sin 100t$

ii) **Applied Force** $F(t) = 0.2 \sin 15t$

Solution:

a) Motion along Y (Vertical motion)

Effective Mass at Beam Center $m^* = 5.45$ t

Deflection at beam center $\delta_{yb} = 7.938 \times 10^{-4}$ m

Stiffness of frame beam = $k_{beam} = \dfrac{m^* g}{\delta_{yb}} = \dfrac{5.45 \times 9.81}{7.938 \times 10^{-4}} = 67353$ kN/m

Effective mass at column top $m^* = 6.792$ t

Deflection δ_{yc} at column top $\delta_{yc} = 0.8329 \times 10^{-4}$ m

Column Stiffness (both columns) $k_{column} = \dfrac{m^* g}{\delta_{yc}} = \dfrac{6.792 \times 9.81}{0.8329 \times 10^{-4}} = 800000$ kN/m

Beam and column stiffness are in series.

Total vertical stiffness k_y is given as:

$$\frac{1}{k_y} = \frac{1}{k_{beam}} + \frac{1}{k_{column}} = \left(\frac{1}{67353} + \frac{1}{800000} \right) = 1.6097 \times 10^{-5}$$

$k_y = 62123$ kN/m

Natural Frequency $p_y = 105.75$ rad/s

i) **Applied Force** $F(t) = 0.2 \sin 100t$

Excitation Frequency $\omega = 100$ rad/s

Frequency ratio $\beta_y = \dfrac{\omega}{p_y} = \dfrac{100}{105.75} = 0.946$

Damping $\zeta = 0.05$

Magnification factor μ_y

$$\mu_y = \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta_y\right)^2}} = \frac{1}{\sqrt{\left(1-0.946^2\right)^2 + \left(2\times 0.946\times 0.05\right)^2}} = 7.07$$

Static deflection $\delta_{st} = \dfrac{F_0}{k_y} = \dfrac{0.2}{62123} = 3.22\times 10^{-6}$ m

Vertical Amplitude y

$$y = \delta_{st}\times \mu_y = 3.22\times 10^{-6}\times 7.07 = 22.8\times 10^{-6} \text{ m} = 22.8 \text{ microns}$$

Note: Just for academic interest, if system is considered as undamped, we get

$$\mu_y = \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2}} = \frac{1}{\left(1-0.946^2\right)} = 9.52$$

$$y = 3.22\times 10^{-6}\times 9.52 = 30.65\times 10^{-6} \text{ m} = 30.6 \text{ microns}$$

ii) Applied Force $F(t) = 0.2\sin 15t$

Excitation Frequency $\omega = 15$ rad/sec

Frequency ratio $\beta_y = \dfrac{\omega}{p_y} = \dfrac{15}{105.75} = 0.142$

Damping $\zeta = 0.05$

Magnification factor μ_y

$$\mu_y = \frac{1}{\sqrt{\left(1-0.142^2\right)^2 + \left(2\times 0.142\times 0.05\right)^2}} = 1.04$$

Static deflection $\delta_{st} = \dfrac{F_0}{k_y} = \dfrac{0.2}{62123} = 3.22\times 10^{-6}$ m

Vertical Amplitude y

$$y = \delta_{st} \times \mu_y = 3.22 \times 10^{-6} \times 1.04 = 3.4 \times 10^{-6} \text{ m} = 3.4 \text{ microns}$$

b) Motion along X (Transverse motion)

Total Effective Mass at Frame Column top $m^* = 6.552$ t

Transverse deflection at column top $\delta_x = 0.02$ m

Frame stiffness (Transverse) $k_x = \dfrac{6.552 \times 9.81}{0.02} = 3213.7$ kN/m

Natural Frequency $p_x = 22.14$ rad/s

i) Applied Force $F(t) = 0.2 \sin 100t$

Excitation Frequency $\omega = 100$ rad/s

Frequency ratio $\beta_x = \dfrac{\omega}{p_x} = \dfrac{100}{22.14} = 4.515$

Damping $\zeta = 0.05$

Magnification factor: As frequency ratio is more than 2, damping effect on the magnification will be insignificant

$$\mu_x = \frac{1}{\left|\left(1 - 4.515^2\right)\right|} = 0.0516$$

Static deflection $\delta_{st} = \dfrac{F_0}{k_x} = \dfrac{0.2}{3213.7} = 6.22 \times 10^{-5}$ m

Transverse Amplitude x

$$x = \delta_{st} \times \mu_x = 6.22 \times 10^{-5} \times 0.0516 = 3.21 \times 10^{-6} \text{ m} = 3.2 \text{ microns}$$

ii) Applied Force $F(t) = 0.2 \sin 15t$

Excitation Frequency $\omega = 15$ rad/s

Frequency ratio $\beta_x = \dfrac{\omega}{p_x} = \dfrac{15}{22.14} = 0.6775$

Damping $\zeta = 0.05$

Magnification factor μ_x

$$\mu_x = \frac{1}{\sqrt{\left(1-0.6775^2\right)^2 + \left(2\times 0.6775\times 0.05\right)^2}} = 1.834$$

Static deflection $\delta_{st} = 6.22\times 10^{-5}$ m

Transverse Amplitude x

$$x = \delta_{st}\times\mu_x = 6.22\times 10^{-5}\times 1.834 = 1.14\times 10^{-4}\ \text{m} = 114\ \text{microns}$$

3

MULTI- DEGREE OF FREEDOM SYSTEMS

- Free and Forced Vibration
- Two Spring Mass System
- Three Spring Mass System
- Multiple Spring Mass System Connected by a Rigid Bar
- Rigid Block supported by Translational and Rotational Springs
- Portal Frame
- Harmonic Loads
- Impact Loads

Example Problems

MULTI- DEGREE OF FREEDOM SYSTEMS

Every physical system is a complex system and requires mathematical idealization to a fairly good degree of accuracy for good results. Though SDOF System is the simplest way to understand vibration behaviour, its application to physical problems gets restricted, as, in most of the cases, it does not adequately represent behaviour of the prototype. It may be desirable to adopt appropriate mathematical idealization for nearly true representation of the prototype. Mathematical idealization by a higher degree freedom system (2DOF, 3DOF, ---, nDOF) may be the right choice. In the present context and for machine foundation design application, development of analysis is restricted to only two and three degree of freedom system as this is adequate for most of the problems. However, where mathematical modeling calls for idealization with further higher DOF systems, it is advisable to use standard computer packages as it may turn out to be too tedious to perform manual computations.

After **SDOF** system, next step is to understand vibration behaviour of **Two Degrees** Of Freedom System and **Three Degrees** of Freedom System from the point of view of machine foundation design. Two degrees of freedom system is one where two coordinates are required to define displaced position of the structure. Similarly for three degrees of freedom system, three coordinates are required to define displaced position of the structure.

From the study of SDOF System, we note the following:

i) Equation of motion for SDOF y is $m\ddot{y} + k_y y = 0$

ii) Frequency equation is $\left|\left(k_y - mp_y^2\right)\right| = 0$, that gives natural frequency as $p_y = \sqrt{\frac{k_y}{m}}$

iii) It is observed that there is hardly any appreciable change in the damped natural frequency vs. undamped natural frequency for systems having damping less than 20%.

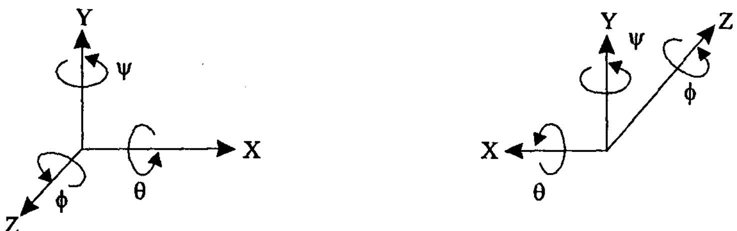
In view of iii) above, for all future cases, analysis is restricted to only undamped system as majority of the structural systems considered for machine foundation design application have damping less than 20%. However, influence of damping is considered for computation of amplitudes only in case of resonance (as will be seen later for forced vibration response).

The analysis is developed for most commonly used systems for machine foundation applications. These are:

- Two Spring Mass System
- Rigid Block supported by Vertical, Translational & Rotational Springs
- Three Spring Mass System

From the above we see that a SDOF system has only one stiffness term and one mass term. For a two DOF system, we shall get stiffness matrix $[k]$ of the order of $[2\times 2]$ and mass matrix $[m]$ also of the order of $[2\times 2]$. Similarly for a 3-DOF system, we shall have stiffness matrix $[k]$ of the order of $[3\times 3]$ and mass matrix $[m]$ also of the order of $[3\times 3]$ and the frequency equation shall be $\left|\left(k-mp^2\right)\right| = 0$.

Coordinate System followed throughout the text is **right hand thumb rule** as shown in Figure 3-1.



(a) Z-Axis towards Reader (b) Z-Axis away from Reader

Displacements x, y & z are +ve along X, Y & Z Axes. Rotations θ, ψ, ϕ about X, Y & Z Axes are +ve considering rotations from Y to Z, Z to X & X to Y

Figure 3-1 Notations for Displacements and Rotations

3.1 TWO DEGREES OF FREEDOM SYSTEM - FREE VIBRATION

Both the systems, i.e. Spring Mass System as well as Block-Foundation System are covered for analysis. The spring mass system has been added only for academic purposes.

3.1.1 Two Spring Mass System- Linear Springs

Consider a two spring mass system, having masses m_1 & m_2, spring stiffness k_1 & k_2 as shown in Figure 3.1.1-1. Two coordinates namely y_1 & y_2 are the two degrees of freedom that define displaced position of the masses m_1 & m_2.

Equation of Motion:

For any given system, there are many ways of writing equation of motion. We use only method Considering equilibrium of forces.

Let us write equations of motion using equilibrium of forces acting on free body diagram. Figure 3.1.1-1(a) shows position of the masses m_1 & m_2 at rest. Masses are disturbed and set free for motion. At any instant of time t, let the displaced positions of masses m_1 & m_2 be y_1 & y_2 respectively as shown in figure 3.1.1-1 (b). Forces acting on the masses are shown in the free body diagram of figure 3.1.1-1(c). Inertia force acts opposite to direction of motion.

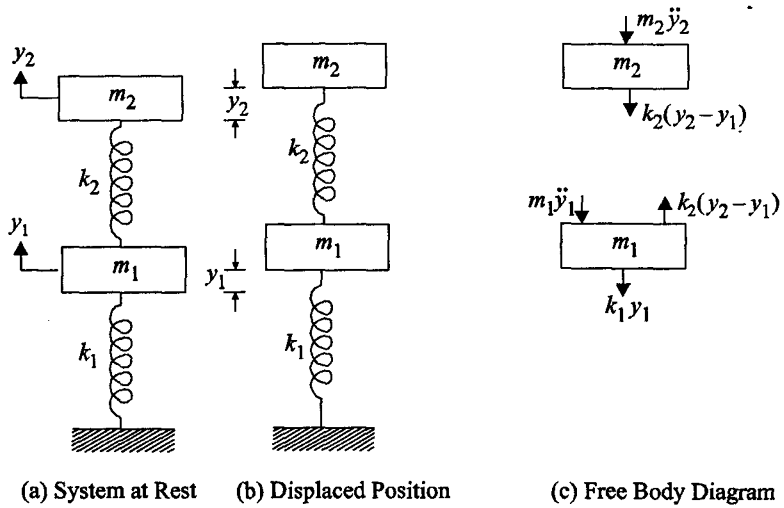


Figure 3.1.1-1 An Undamped Two Spring Mass System

Considering equilibrium of forces on the free body diagram, we get the equation of motion as:

$$m_1\ddot{y}_1 + k_1 y_1 - k_2(y_2 - y_1) = 0$$
$$m_2\ddot{y}_2 + k_2(y_2 - y_1) = 0 \qquad (3.1.1\text{-}1)$$

Rewriting in Matrix form, equation becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.1.1-2}$$

Here $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$ represents mass matrix and $\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ represents stiffness matrix

As seen from equation (3.1.1-2), though there is no coupling in mass matrix but stiffness matrix is coupled through off-diagonal terms. Thus equation of motion is a coupled one.

Solution to Equation of Motion:

Natural Frequency

In chapter 2, for solution to SDOF system, we obtained natural frequency p_y as

$$p_y = \sqrt{\frac{k_y}{m}} \quad \text{i.e.} \quad k_y = mp_y^2 \quad \text{or} \quad k_y - mp_y^2 = 0 \tag{3.1.1-3}$$

$$k_y - mp_y^2 = 0 \quad \text{is termed as Frequency Equation} \tag{3.1.1-4}$$

For two DOF system, (equation of motion equation 3.1.1-2), the frequency equation is given by

$$\text{Determinant} \quad \left| k - mp^2 \right| = 0 \tag{3.1.1-5}$$

$$\text{Or} \quad \begin{vmatrix} (k_1 + k_2 - m_1 p^2) & (-k_2) \\ (-k_2) & (k_2 - m_2 p^2) \end{vmatrix} = 0 \tag{3.1.1-6}$$

Expanding the determinant, we get

$$(k_1 + k_2 - m_1 p^2)(k_2 - m_2 p^2) - (-k_2)(-k_2) = 0 \qquad \text{Or}$$

$$m_1 m_2 \left\{ p^4 - p^2 \left(\frac{k_2}{m_2} + \frac{k_2}{m_1} + \frac{k_1}{m_1} \right) + \left(\frac{k_1}{m_1} \frac{k_2}{m_2} \right) \right\} = m_1 m_2 \, \Delta(p^4) = 0 \tag{3.1.1-7}$$

$$\text{Where} \quad \Delta(p^4) = \left\{ p^4 - p^2 \left(\frac{k_2}{m_2} + \frac{k_2 + k_1}{m_1} \right) + \left(\frac{k_1}{m_1} \frac{k_2}{m_2} \right) \right\} \tag{3.1.1-8}$$

Since masses m_1 & m_2 are non-zero, roots of $\Delta(p^4)=0$ will give two natural frequencies p_1 & p_2 corresponding to 1st mode and 2nd mode of vibration respectively.

$$\Delta(p^4)=\left\{p^4-p^2\left(\frac{k_2}{m_2}+\frac{k_2+k_1}{m_1}\right)+\left(\frac{k_1}{m_1}\frac{k_2}{m_2}\right)\right\}=0 \tag{3.1.1-9}$$

Solution of this equation will give two natural frequencies. Solving, we get

$$p^2=\left(\frac{k_2}{2m_2}+\frac{k_2+k_1}{2m_1}\right)\mp\frac{1}{2}\sqrt{\left(\frac{k_2}{m_2}+\frac{k_2+k_1}{m_1}\right)^2-4\left(\frac{k_1}{m_1}\frac{k_2}{m_2}\right)}$$

Or
$$p^2=\frac{1}{2}\left\{\frac{k_2}{m_2}+\frac{k_2}{m_1}+\frac{k_1}{m_1}\mp\sqrt{\left(\frac{k_2}{m_2}+\frac{k_2}{m_1}+\frac{k_1}{m_1}\right)^2-4\left(\frac{k_1}{m_1}\frac{k_2}{m_2}\right)}\right\} \tag{3.1.1-10}$$

At times it is convenient to express the frequency equation in terms of limiting frequencies. Expressing these in terms of Limiting Frequencies, we get:

Denoting

$$p_{L1}=\sqrt{\frac{k_1}{m_1}} \quad \text{as limiting frequency for mass } m_1$$

$$p_{L2}=\sqrt{\frac{k_2}{m_2}} \quad \text{as limiting frequency for mass } m_2 \tag{3.1.1-10a}$$

$$\&\ \lambda=\frac{m_2}{m_1} \quad \textit{as mass ratio}$$

Equation (3.1.1-10) is re-written in terms of limiting frequencies as

$$p_{1,2}^2=\frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)\mp\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2\,p_{L2}^2\right)}\right\} \tag{3.1.1-11}$$

Two natural frequencies p_1 & p_2 are

$$p_1^2=\frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)-\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2\,p_{L2}^2\right)}\right\} \tag{3.1.1-12}$$

$$p_2^2=\frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)+\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2\,p_{L2}^2\right)}\right\} \tag{3.1.1-13}$$

Since p_1 & p_2 are roots of $\Delta(p^4)=0$, $\Delta(p^4)$ could also be represented in terms of its roots i.e. its natural frequencies p_1 & p_2. Expressing in terms of natural frequencies p_1 & p_2, **(derivation not given, readers may please derive themselves),** we get:

$$\Delta(p^4)=(p^2-p_1^2)(p^2-p_2^2) \tag{3.1.1-14}$$

From equation 3.1.1-12 & 13, it can be seen that

$$p_1 \times p_2 = p_{L1} \times p_{L2} \tag{3.1.1-14a}$$

Mode Shapes:

As we have seen that a SDOF system has only one frequency and vibrates in one mode only. Similarly a 2 Degrees of Freedom (2DOF) System has two natural frequencies and two modes of vibration. Let us first evaluate its mode shapes.

Let the general solution of the equation of motion be of the form

$$\begin{aligned} y_1 &= A_1 \sin(pt+\phi) \\ y_2 &= A_2 \sin(pt+\phi) \end{aligned} \tag{3.1.1-15}$$

In the 1st mode since the system vibrates with frequency p_1, we can consider the solution to be of the form

$$\begin{aligned} y_1' &= A_1' \sin(p_1 t+\phi') \\ y_2' &= A_2' \sin(p_1 t+\phi') \end{aligned} \tag{3.1.1-16}$$

Here quantities with single prime are indicative of the first mode.

Here y_1' & y_2' represent the response of mass m_1 & m_2, A_1' & A_2' are the amplitudes of mass m_1 & m_2 and ϕ' represents the phase angle in the 1st mode. Differentiating, we get the 2nd derivative as

$$\begin{aligned} \ddot{y}_1' &= -{p_1}^2 A_1' \sin(p_1 t+\phi') \\ \ddot{y}_2' &= -{p_1}^2 A_2' \sin(p_1 t+\phi') \end{aligned} \tag{3.1.1-17}$$

Substituting equation (3.1.1-16) and (3.1.1-17) in equation of motion, (equation 3.1.1-2), it gives

$$-p_1^2 \sin(p_1 t+\phi')\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} A_1' \\ A_2' \end{Bmatrix}+\sin(p_1 t+\phi')\begin{bmatrix} (k_1+k_2) & (-k_2) \\ (-k_2) & (k_2) \end{bmatrix}\begin{Bmatrix} A_1' \\ A_2' \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Rearranging terms, we get

$$\begin{bmatrix} (k_1+k_2-m_1 p_1^2) & (-k_2) \\ (-k_2) & (k_2-m_2 p_1^2) \end{bmatrix}\begin{Bmatrix} A_1' \\ A_2' \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.1.1\text{-}18)$$

Solution of this gives

$$\frac{A_1'}{A_2'}=\frac{k_2}{k_1+k_2-m_1 p_1^2}=\frac{k_2-m_2 p_1^2}{k_2}=\frac{1}{\alpha'} \qquad (3.1.1\text{-}19)$$

Here α' represents amplitude ratio of mass m_2 to the mass m_1 in the first mode of vibration. This indicates that for a given system, there exists a constant ratio between amplitudes of masses m_1 & m_2 for the first mode.

Similarly for the 2nd mode the system will vibrate with frequency p_2

We can consider the solution to be of the form

$$\begin{aligned} y_1'' &= A_1'' \sin(p_2 t+\phi'') \\ y_2'' &= A_2'' \sin(p_2 t+\phi'') \end{aligned} \qquad (3.1.1\text{-}20)$$

Here quantities with double prime are indicative of the second mode.

Here y_1'' & y_2'' represent the response of mass m_1 & m_2, A_1'' & A_2'' are the amplitudes of mass m_1 & m_2 and ϕ'' is the phase angle in the 2nd mode.

Substituting equation (3.1.1-20) and its 2nd derivative in equation (3.1.1-2), and solving, we get

$$\frac{A_1''}{A_2''}=\frac{k_2}{k_1+k_2-m_1 p_2^2}=\frac{k_2-m_2 p_2^2}{k_2}=\frac{1}{\alpha''} \qquad (3.1.1\text{-}21)$$

Here α'' represents amplitude ratio of second mass to the first mass in second mode of vibration. This indicates that for a given system, there exists a constant ratio between amplitudes of masses m_1 & m_2 for the second mode too.

Values of constants $A_1', A_2', A_1'' \& A_2''$ are determined based on initial conditions. It can well be proved that equation (3.1.1-19), that represents **Mode I,** is always positive i.e. both the masses move in phase in relation to equilibrium position. Similarly it can be proved that equation 3.1.1-21, that represents **Mode II,** is always negative i.e. both the masses move out of phase in relation to equilibrium position. It is also interesting to note that the first natural frequency (lower natural frequency) is always lower than the lowest limiting frequency (given by equation 3.1.1-10a) and the second natural frequency (higher natural frequency) is always higher than the highest limiting frequency. (***Proof not given- readers may attempt the same on their own***).

The two modes are shown in Figure 3.1.1-2.

The mode when both masses move in phase in relation to equilibrium position is termed as **Fundamental Mode or Principal Mode or Normal Mode of Vibration**. In this mode the system vibrates with lowest frequency called the **fundamental frequency**. In the second mode the system vibrates with next higher frequency or second frequency

Free Vibration Response

Having obtained natural frequencies of vibration, the next step is to evaluate its free vibration response. The general solution (see equation 3.1.1-15) thus becomes:

Response of mass m_1

$$\begin{aligned} y_1 = y_1' + y_1'' &= \underbrace{A_1' \sin(p_1 t + \phi')}_{\text{1st Mode Response}} + \underbrace{A_1'' \sin(p_2 t + \phi'')}_{\text{2nd Mode Response}} \\ &= A_1' \sin(p_1 t + \phi') + A_1'' \sin(p_2 t + \phi'') \end{aligned} \tag{3.1.1-22}$$

Response of mass m_2

$$\begin{aligned} y_2 = y_2' + y_2'' &= \underbrace{A_2' \sin(p_1 t + \phi')}_{\text{1st Mode Response}} + \underbrace{A_2'' \sin(p_2 t + \phi'')}_{\text{2nd Mode Response}} \\ &= A_2' \sin(p_1 t + \phi') + A_2'' \sin(p_2 t + \phi'') \\ &= \alpha' A_1' \sin(p_1 t + \phi') + \alpha'' A_1'' \sin(p_2 t + \phi'') \end{aligned} \tag{3.1.1-23}$$

The values of constants are determined from initial conditions.

Let the initial conditions be

$$\text{At} \quad t = 0 \rightarrow y_1(t) = y_1(0); \dot{y}_1(t) = \dot{y}_1(0); y_2(t) = y_2(0); \dot{y}_2(t) = \dot{y}_2(0) \tag{3.1.1-24}$$

Differentiating equation 3.1.1-22 & 3.1.1-23, we get

$$\dot{y}_1 = p_1 A_1' \cos(p_1 t + \phi') + p_2 A_1'' \cos(p_2 t + \phi'') \tag{3.1.1-25}$$

$$\dot{y}_2 = p_1 A_1' \alpha' \cos(p_1 t + \phi') + p_2 A_1'' \alpha'' \cos(p_2 t + \phi'') \tag{3.1.1-26}$$

Substituting equation 3.1.1-24 into equations 3.1.1-22, 23, 25 & 26, we get the values of the four constants $A_1', A_2', \phi' \& \phi''$. Using equations 3.1.1-19 & 21 we get values of $A_1'' \& A_2''$.

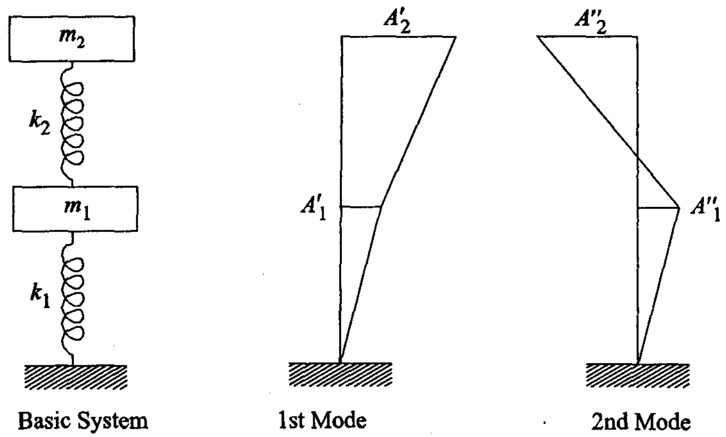


Figure 3.1.1-2 Mode Shapes of an Undamped Two Spring Mass System

Having evaluated these constants, equation 3.1.1-22 & 23 yields the free vibration response of the system. It is well known that every physical system has some inherent damping present in it. Since free vibration response is only transient response it dies out quickly depending upon the value of the damping present in the system. Therefore it is not of much interest from the point of view of machine foundation design. However in specific cases (as we will see later) it may be desirable to compute this transient response too.

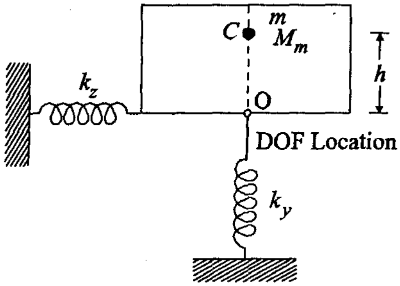
3.1.2 A Rigid Block supported by Vertical and Translational Springs

(This combination of springs does not represent any practical application. The derivation is given for academic purposes only to demonstrate that Vertical and Translational modes are uncoupled)

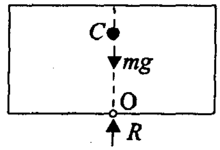
Consider a rigid block of mass m having its centroid at C. The block is supported by vertical spring of stiffness k_y and translational spring of stiffness k_x attached at center of base O. The height of centroid from base is h. The block is constrained to move only in X & Y direction as shown in part (a) of the Figure 3.1.2-1.

Static Equilibrium: The vertical spring k_y supports the self-weight of the block and develops vertical reaction R to counteract the self-weight mg. This position of the block is termed as position at rest and has been shown in part (b) of the Figure 3.1.2-1.

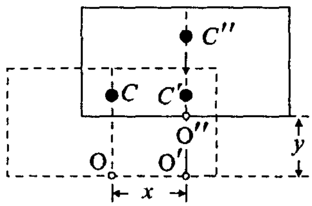
Considering equilibrium at rest position, we get $mg - R = 0$ (3.1.2-1)



(a) Block with Centroid C-Rotation ϕ at O restrained. Translational spring stiffness k_x and Vertical spring stiffness k_y



(b) Block Position at Rest. Self Weight mg Supported by Reaction R at support point O



(c) Displaced Position

Figure 3.1.2-1 A Rigid Block Supported by Translational & Vertical Springs

Equation of motion

The block is displaced and released to oscillate freely. The displaced position of the block at any instant of time t is as shown in part (c) of the Figure 3.1.2-1. It is seen that due to x translation, point O moves to O' and centroid C moves to C'. Similarly due to y displacement, point O' moves to O'' and centroid C' moves to C''.

Let us consider these displacements and corresponding forces developed one by one. Reaction force and inertia force developed are shown in Figure 3.1.2-2.

Consider first x - translation as shown. Inertia force and reaction force act in direction opposite to direction of motion. Reaction forces and inertia forces developed are shown in part (a) of the figure. We get:

Spring reaction force at O' along X- direction $k_x x$

Inertia force at C' along X-direction $m\ddot{x}$

Now consider displacement along Y direction. Reaction force and inertia force developed are shown in part (b) of the figure. We get:

Spring reaction force at O'' along Y- direction $k_y y$

Inertia force at C'' along Y-direction $m\ddot{y}$

Considering equilibrium of forces (at DOF location), we get:

$\sum F_x = 0$ $m\ddot{x} + k_x x = 0$ (3.1.2-2)

$\sum F_y = 0$ $m\ddot{y} + k_y y + mg - R = 0$

Substituting 3.1.2-1, we get $m\ddot{y} + k_y y = 0$ (3.1.2-3)

$\sum M_z = 0$ **as the rotation about O is constrained**

These equations are called equation of motion.

Figure (c) shows total reaction forces developed due to both the displacement as well as due to static equilibrium.

It is seen that **these equations of motion are un-coupled**. In other words these equations are independent. Each of this equation represents SDOF system. Solution to these equations will yield natural frequencies and free vibration response of the block. (For Solution to SDOF System - See Chapter 2)

Solving equation (3.1.2-2), we get:

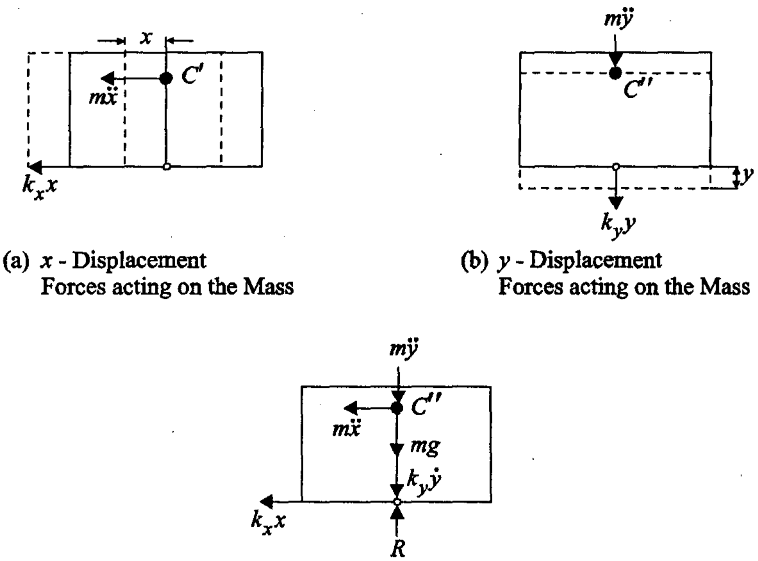
Natural frequency in X translation $p_x = \sqrt{\dfrac{k_x}{m}}$ (3.1.2-4)

Free Vibration Response is given as $x = A\sin p_x t + B\cos p_x t$ (3.1.2-5)

Solution of equation (3.1.2-3) gives

Natural frequency in Y- direction $p_y = \sqrt{\dfrac{k_y}{m}}$ (3.1.2-6)

Free Vibration Response is given as $y = A\sin p_y t + B\cos p_y t$ (3.1.2-7)



(a) x - Displacement Forces acting on the Mass

(b) y - Displacement Forces acting on the Mass

(c) Total Forces acting on the Mass

Figure 3.1.2-2 Forces Acting on the Mass

3.1.3 A Rigid Block Supported by Vertical and Rotational Springs

(This spring combination also does not represent any practical application. The derivation is given for academic purposes only to demonstrate that Vertical and Rotational modes are uncoupled)

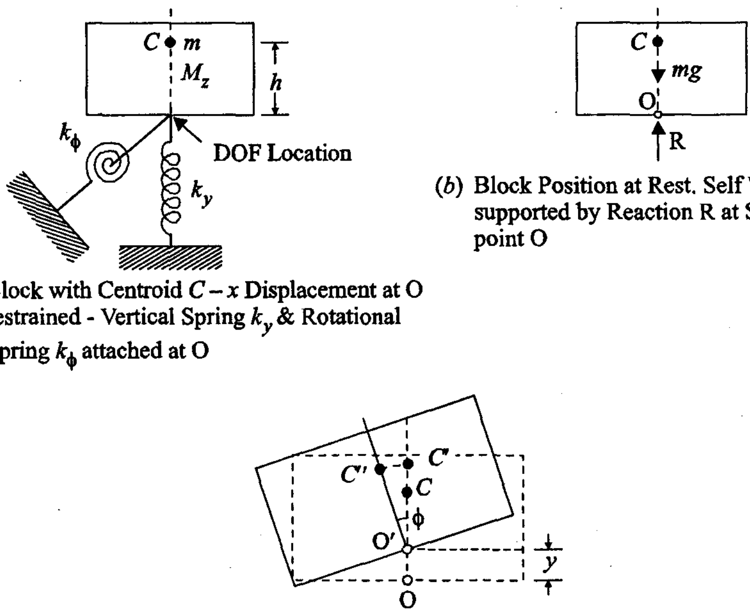
Consider a rigid block supported by vertical and Rotational spring. The block has its center at C, and two springs one vertical and one rotational spring is connected at base center point O. The block has mass m and Mass Moment of Inertia about Z-axis passing through block centroid C is M_{mz}. The height of centroid C above base center O is h. The block is constrained to move only in vertical Y direction and rotate about Z-axis passing through O. The block is as shown in part (a) of the Figure 3.1.3-1.

Static Equilibrium: The vertical spring k_y supports the self-weight of the block and develops vertical reaction R to counteract the self-weight mg. This position of the block is termed as position at rest and has been shown in part (b) of the Figure 3.1.3-1.

Considering equilibrium at rest position, we get $$mg - R = 0 \qquad (3.1.3\text{-}1)$$

Equation of motion

The block is displaced slightly and released to oscillate freely. Consider that due to y displacement, point O moves to O' and centroid C moves to C'. Similarly due to rotation ϕ centroid C' moves to C''. The displaced position of the block at any instant of time t is as shown in part (c) of the Figure 3.1.3-1.



(a) Block with Centroid $C-x$ Displacement at O restrained - Vertical Spring k_y & Rotational Spring k_ϕ attached at O

(b) Block Position at Rest. Self Weight mg supported by Reaction R at Support point O

(c) Displaced Position

Figure 3.1.3-1 Rigid Block Supported by Vertical & Rotational Springs

Let us consider these displacements and corresponding reactions developed one by one. Reaction forces and inertia forces developed are shown in Figure 3.1.3-2.

Consider first displacement along Y direction. Reaction force and inertia force developed are shown in part (a) of the figure. We get,

Spring reaction force at O' along Y- direction $k_y y$

Inertia force at C' along Y-direction $m\ddot{y}$

Now consider rotation ϕ. It is interesting to note that rotation ϕ at O' gives rise to rotational inertia as well as translational inertia at C''. Reaction force and inertia force developed are shown in part (b) of the Figure.

Spring reaction force at O' along ϕ $\quad k_\phi \phi$

Inertia force developed at C'' along direction normal to centerline $\quad mh\ddot{\phi}$

Inertia force developed at C'' along ϕ $\quad M_{mz}\ddot{\phi}$

Figure (c) shows total reaction forces developed due to both the displacement as well as due to static equilibrium.

Considering equilibrium of forces at DOF location (See Figure c), equation of motion is written as

$\sum F_y = 0 \qquad m\ddot{y} + k_y y + mg - R = 0$

Substituting 3.1.3-1, we get $\qquad m\ddot{y} + k_y y = 0 \qquad$ (3.1.3-2)

$\sum M_z = 0 \qquad M_{mz}\ddot{\phi} + (mh\ddot{\phi} \times h) + (k_\phi \times \phi) - (mgh\sin\phi) = 0$

For ϕ to be small, $h\sin\phi = h\phi$. Substituting, we get

$$\left(M_{mz} + mh^2\right)\ddot{\phi} + \left(k_\phi - mgh\right)\phi = 0$$

Or
$$\left(M_{moz}\right)\ddot{\phi} + \left(k_\phi - mgh\right)\phi = 0 \qquad (3.1.3\text{-}3)$$

Here $M_{moz} = M_{mz} + mh^2$ represents Mass Moment of Inertia of the block about O.

It is seen that equation 3.1.3-2 & 3.1.3-3 are uncoupled. Each of this equation, when solved (**see Chapter 2**), gives natural frequency and response.

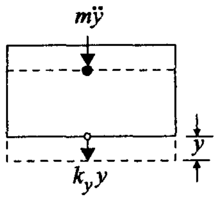
The solution to equation (3.1.3-2) gives

Vertical natural frequency $\qquad p_y = \sqrt{\dfrac{k_y}{m}} \qquad$ (3.1.3-4)

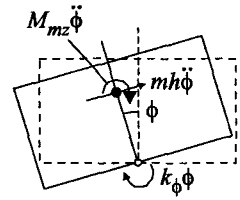
The solution to equation (3.1.3-3) gives

Rotational natural frequency $\qquad p_\phi = \sqrt{\dfrac{k_\phi - mgh}{M_{moz}}} \qquad$ (3.1.3-5)

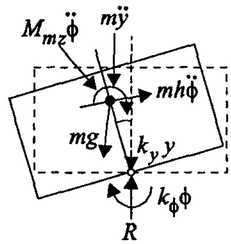
For response for each case, **see § 2.1.1**



(a) Forces acting on the Mass due to y Displacement



(b) Forces acting on the Mass due to ϕ Rotation



(c) Total Forces acting on the Mass

Figure 3.1.3-2 Forces acting on the Mass

3.1.4 A Rigid Block supported by Translational and Rotational Springs

(This spring combination in itself does not represent any practical application. In order to simulate a real practical situation, this combination needs to be clubbed with the vertical support spring. Since vertical mode of vibration is uncoupled with lateral mode or rotational mode (see 3.1.2 & 3.1.3), vertical spring is not included in the formulation. The present derivation is given to demonstrate the coupling between Translational and Rotational modes)

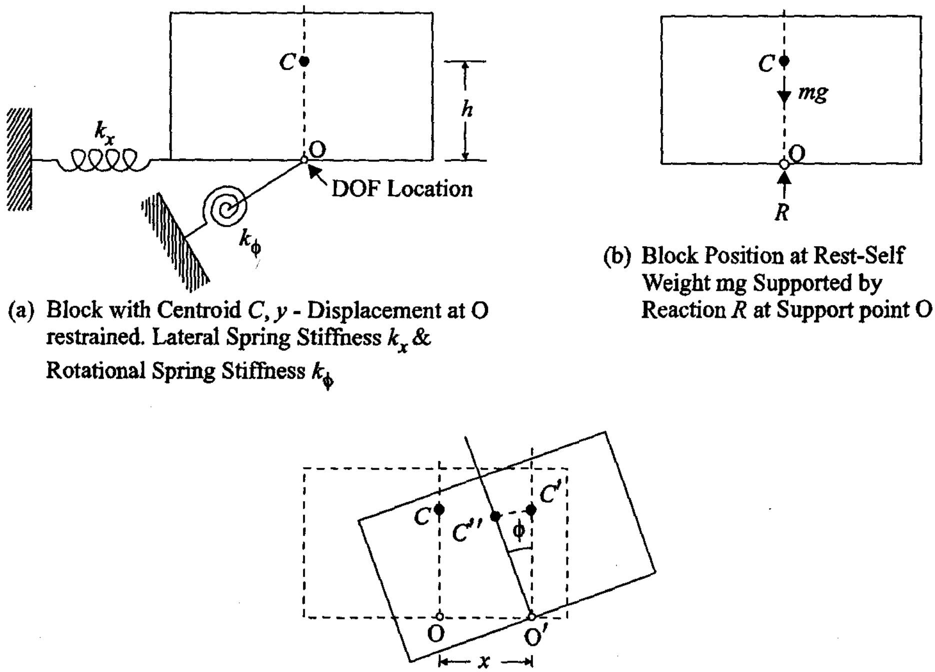
3.1.4.1 Motion in X-Y Plane (Y being vertical axis)

Consider a rigid block supported by a Translational and a Rotational spring. The block has its centroid at C, and two springs, one Translational, having stiffness k_x and the other Rotational, having stiffness k_ϕ are connected at base center point O.

The block has mass m and Mass Moment of Inertia about Z-axis passing through block centroid C as M_{mz}. The height of centroid C above base center O is h. The block is constrained such that it can have only x- translation and ϕ rotation about O and the movement along Y is restrained. The block is as shown in part (a) of the Figure 3.1.4-1.

Static Equilibrium: Let us first consider the position of the mass at rest i.e. mean position of the mass. The gravity force mg is taken care of by reaction R by the restraint at support O. Part (b) of the figure shows position at rest. Considering equilibrium at rest position, we get

$$mg - R = 0 \qquad (3.1.4\text{-}1)$$



(a) Block with Centroid C, y - Displacement at O restrained. Lateral Spring Stiffness k_x & Rotational Spring Stiffness k_ϕ

(b) Block Position at Rest-Self Weight mg Supported by Reaction R at Support point O

(c) Displaced Position

Figure 3.1.4-1 A Rigid Block Supported by Translational & Rotational Springs

Equation of motion:

Consider that at any instant of time t, the block has moved by x and rotated by angle ϕ. Due to x translation, point O moves to O' and centroid C moves to C' and due to rotation ϕ at O' centroid C' moves to C''. Figure 3.1.4-1 Part (c) shows the displaced position of the block.

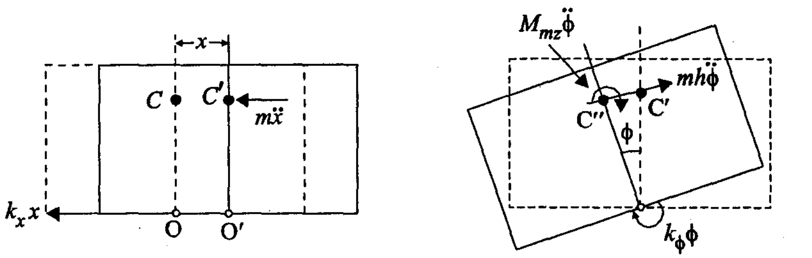
Let us consider these displacements and corresponding reactions one by one. Reaction forces and inertia forces developed are shown in Figure 3.1.4-2.

Consider first x translation. Forces developed are shown in part (a) of the figure. We get,

Spring reaction force at O' along X- direction $k_x x$

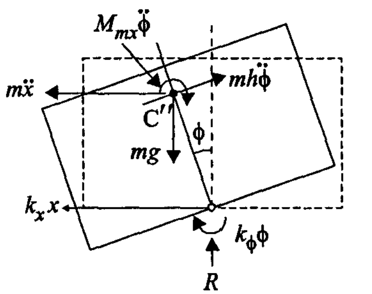
Inertia force at C' along X-direction $m\ddot{x}$

Now consider rotation ϕ at O'. This rotation gives rise to rotational inertia as well as translational inertia at C''. Reaction force and inertia force developed are shown in part (b) of the Figure 3.1.4-2.



(a) x - Displacement
Forces acting on the Mass

(b) ϕ - Rotation
Forces acting on the Mass



(c) Total Forces acting on the Mass

Figure 3.1.4-2 Forces acting on the Mass

Spring reaction force at O' along ϕ $k_\phi \phi$

Translational Inertia force developed at C'' (as shown) = $mh\ddot{\phi}$

Rotational Inertia force developed at C'' (as shown) = $M_{mz}\ddot{\phi}$

Figure (c) shows total reaction forces developed due to both the displacement as well as due to static equilibrium.

Considering equilibrium of the forces (at DOF location point O'), we get

$\sum F_x = 0$ $\quad m\ddot{x} - mh\ddot{\phi}\cos\phi + k_x x = 0$

For ϕ to be small $\quad mh\ddot{\phi}\cos\phi = mh\ddot{\phi}$, we get

$$m\ddot{x} - mh\ddot{\phi} + k_x x = 0 \qquad (3.1.4\text{-}2)$$

$\sum M_z = 0$ $\quad M_{mz}\ddot{\phi} + (mh\ddot{\phi} \times h) - (m\ddot{x}) \times h\cos\phi + (k_\phi \times \phi) - (mgh\sin\phi) = 0$

For ϕ to be small, $h\cos\phi = h$ & $h\sin\phi = h\phi$. Substituting, we get

$$(M_{mz} + mh^2)\ddot{\phi} - mh\ddot{x} + (k_\phi - mgh)\phi = 0$$

$$\text{Or} \qquad M_{moz}\ddot{\phi} - mh\ddot{x} + (k_\phi - mgh)\phi = 0 \qquad (3.1.4\text{-}3)$$

Here $M_{moz} = M_{mz} + mh^2$ represents Mass Moment of Inertia of the block about Z-axis at O.

$\sum F_y = 0$ $\quad mg - R - mh\ddot{\phi}\sin\phi = 0$

For small ϕ, the component $mh\ddot{\phi}\sin\phi$ is very small and can be equated to zero i.e. $mh\ddot{\phi}\sin\phi = 0$. This gives $mg - R = 0$ and that is equation (3.1.4-1).

Writing in matrix form, we get

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & (k_\phi - mgh) \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

It is seen from this equation that there is no coupling in stiffness matrix but mass matrix is coupled through off-diagonal terms. **Thus equations of motion are said to be coupled one.**

It may be noted that for all practical real life problems, the influence of the term mgh is insignificant and hence ignored in the equation of motion. The equation of motion thus becomes:

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3.1.4\text{-}4)$$

Solution to equation of motion 3.1.4-4 gives two natural frequencies and associated mode shapes. Natural Frequencies are given as *(see* equations *(h) & (i) – Solution 3.1.4-4).*

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z p_x^2 p_\phi^2} \quad (3.1.4\text{-}5)$$

$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z p_x^2 p_\phi^2} \quad (3.1.4\text{-}6)$$

Here $\gamma_z = \frac{M_{mz}}{M_{moz}}; \quad p_x^2 = \frac{k_x}{m}; \quad p_\phi^2 = \frac{k_\phi}{M_{moz}}$ (3.1.4-7)

Associated mode shapes are given as *(see equations (l) & (m) - SOLUTION 3.1.4-4).*

$$\frac{A_1}{B_1} = -h\left(\frac{p_1^2}{p_x^2 - p_1^2}\right) = \frac{-M_{moz}}{mh}\left(\frac{(p_\phi^2 - p_1^2)}{p_1^2}\right) \quad (3.1.4\text{-}8)$$

$$\frac{A_2}{B_2} = -h\left(\frac{p_2^2}{p_x^2 - p_2^2}\right) = \frac{-M_{moz}}{mh}\left(\frac{(p_\phi^2 - p_2^2)}{p_2^2}\right) \quad (3.1.4\text{-}9)$$

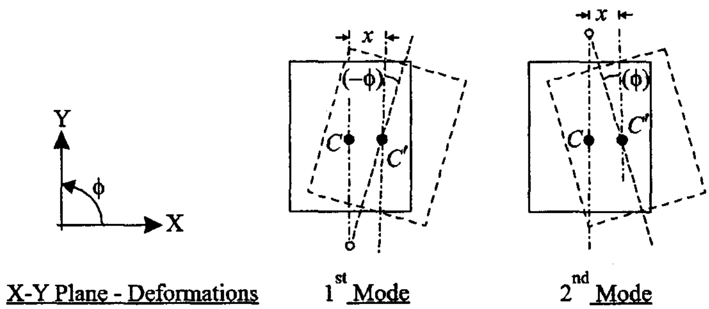


Figure 3.1.4-3 Mode Shapes in X-Y Plane

From equations (3.1.4-8) it is noticed that amplitude **ratio (A_1 /B_1) is always negative** because the value of the quantity in parenthesis is always positive as $p_1 < (p_x \,\&\, p_\phi)$. On the other hand equation (3.1.4-9) indicates that **ratio (A_2 /B_2) is always positive** because the value of the quantity in parenthesis is always negative as $p_2 > (p_x \,\&\, p_\phi)$. This, in other words, indicates that in the 1st mode if the block translates say in positive X-direction, its rotation shall be in negative ϕ direction i.e. clockwise whereas in 2nd mode if the translation is in positive X-direction then rotation shall also be in positive ϕ direction i.e. anticlockwise. These mode shapes are shown in Figure 3.1.4-3.

Free vibration response being **transient response** is not of much interest from the point of view of machine foundation design since it dies out quickly based on the damping present in the system. Only in specific cases (as we will see later) it may be desirable to compute transient response too. Constants A_1, B_1 & A_2, B_2, are evaluated using initial conditions.

SOLUTION 3.1.4-4

A) Natural Frequency

Rewriting equation 3.1.4-4:

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix}+\begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (a)$$

$$\text{Frequency Equation} = \left|\left(k - mp^2\right)\right| = 0 \qquad (b)$$

Substituting equation (a) in equation (b), we get frequency equation as

$$\begin{vmatrix} (k_x - mp^2) & mhp^2 \\ mhp^2 & (k_\phi - M_{moz}p^2) \end{vmatrix} = 0$$

Simplifying, we get

$$\begin{aligned} &(k_x - mp^2)(k_\phi - M_{moz}p^2) - m^2h^2p^4 = 0 \\ &k_x(k_\phi - M_{moz}p^2) - mp^2(k_\phi - M_{moz}p^2) - m^2h^2p^4 = 0 \\ &(k_xk_\phi - M_{moz}k_xp^2) - (mp^2k_\phi - mp^2M_{moz}p^2) - m^2h^2p^4 = 0 \\ &mM_{moz}p^4 - p^2\left(mk_\phi + M_{moz}k_x\right) + k_xk_\phi - m^2h^2p^4 = 0 \end{aligned} \qquad (c)$$

Since $M_{moz} = \left(M_{mz} + mh^2\right)$ *and denoting* $\gamma = \dfrac{M_{mz}}{M_{moz}}$ *and simplifying, we get*

$$m\left(M_{mz} + mh^2\right)p^4 - p^2\left(mk_\phi + M_{moz}k_x\right) + k_xk_\phi - m^2h^2p^4 = 0$$

$$mM_{mz}p^4 - p^2\left(mk_\phi + M_{moz}k_x\right) + k_xk_\phi = 0$$

$$mM_{mz}\left\{p^4 - p^2\frac{1}{\gamma}\left(p_\phi^2 + p_x^2\right) + \frac{1}{\gamma}p_\phi^2p_x^2\right\} = mM_{mz}\Delta\left(p^4\right) = 0 \qquad (d)$$

$$Here\ \gamma_z = \frac{M_{mz}}{M_{moz}};\ p_x^2 = \frac{k_x}{m};\quad p_\phi^2 = \frac{k_\phi}{M_{moz}} \qquad (e)$$

Here p_1 represents lower frequency and p_2 represents higher frequency
Frequency equation thus becomes:

$$\Delta(p^4) = \left[p^4 - p^2\frac{1}{\gamma_z}\left\{p_x^2 + p_\phi^2\right\} + \frac{1}{\gamma_z}p_x^2 p_\phi^2\right] = 0 \qquad (f)$$

Roots of $\Delta(p^4) = 0$ will give two natural frequencies; one corresponding to translational mode and other to rotational mode. Solving we get

$$p^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) \mp \frac{1}{2\gamma_z}\sqrt{\left\{p_x^2 + p_\phi^2\right\}^2 - 4\gamma_z\, p_x^2 p_\phi^2} \qquad (g)$$

This gives:

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left\{p_x^2 + p_\phi^2\right\}^2 - 4\gamma_z\, p_x^2 p_\phi^2} \qquad (h)$$

$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left\{p_x^2 + p_\phi^2\right\}^2 - 4\gamma_z\, p_x^2 p_\phi^2} \qquad (i)$$

*Here p_1 representing 1st natural frequency **(lower natural frequency)** corresponds to 1st mode of vibration & p_2 representing 2nd natural frequency **(higher natural frequency)** corresponds to 2nd mode of vibration.*

From the equations (h) & (i), it can be proved that $p_1^2 \times p_2^2 = \frac{1}{\gamma_z} p_\phi^2 p_x^2$

B) Free Vibration Response

Two natural frequencies have been obtained as given by equations (h) & (i). There are two associated modes of vibration. In the 1st mode the system will vibrate with frequency p_1 & in 2nd mode with frequency p_2. Let us now evaluate the two mode shapes:

Let the solution be represented as

$$x = A\sin(pt+\alpha)$$
$$\phi = B\sin(pt+\alpha) \qquad (j)$$

Equation (a) $$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substituting equation (j) in to equation of motion (a), we get

$$\begin{bmatrix} k_x - mp^2 & mhp^2 \\ mhp^2 & (k_\phi - M_{moz}p^2) \end{bmatrix}\begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (k)$$

Simplifying, we get amplitude ratios in 1st mode as (use frequency p_1 for the first mode):

$$\frac{A_1}{B_1} = -\frac{mhp_1^2}{k_x - mp_1^2} = -\frac{k_\phi - M_{moz}p_1^2}{mhp_1^2}$$

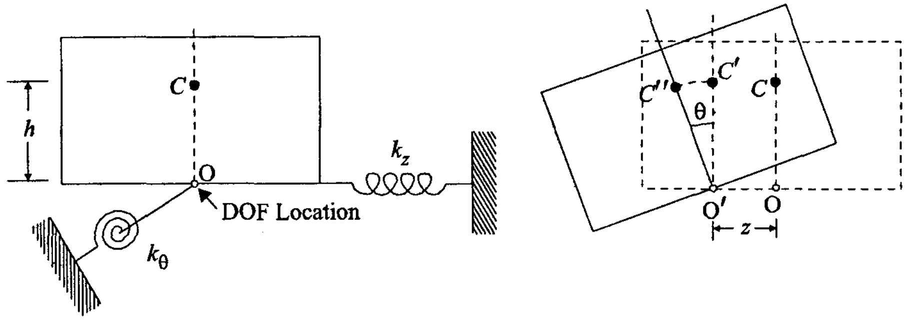
$$\frac{A_1}{B_1} = -h\left(\frac{p_1^2}{p_x^2 - p_1^2}\right) = \frac{-M_{moz}}{mh}\left(\frac{(p_\phi^2 - p_1^2)}{p_1^2}\right) \qquad (l)$$

Solving for amplitude ratios in 2nd mode (use frequency p_2 for the second mode), we get:

$$\frac{A_2}{B_2} = -h\left(\frac{p_2^2}{p_x^2 - p_2^2}\right) = \frac{-M_{moz}}{mh}\left(\frac{(p_\phi^2 - p_2^2)}{p_2^2}\right) \qquad (m)$$

3.1.4.2 Motion in Y - Z plane (Y being vertical axis)

Consider the same rigid block (as in §3.1.4.1) supported by a Translational spring along Z having stiffness k_z and a Rotational spring about X having stiffness k_θ. These springs are connected at base center point O. The block is as shown in Figure 3.1.4-4.



(a) Block with Centroid C, y - Displacement at O restrained. Lateral Spring Stiffness k_z & Rotational Spring Stiffness k_θ

(b) Displaced Position

Figure 3.1.4-4 A Rigid Block Supported by Translational & Rotational Springs in Y-Z Plane

The block has mass m and Mass Moment of Inertia about Z-axis passing through block centroid C as M_{mx} . The height of centroid C above base center O is h . The block is constrained such that it can have only z - translation and θ rotation about O and the movement along Y is restrained. The block is as shown in part (a) of the Figure 3.1.4-1.

Following procedure similar to that for X-Y Plane, we get equation of motion as:

Writing in matrix form, we get

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.1.4-10}$$

Proceeding on the similar lines as for § 3.1.4.1, we get natural Frequencies as:

$$p_1^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) - \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x p_z^2 p_\theta^2} \tag{3.1.4-11}$$

$$p_1^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) + \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x p_z^2 p_\theta^2} \tag{3.1.4-12}$$

$$\text{Here } \gamma_x = \frac{M_{mx}}{M_{mox}}; \; p_z^2 = \frac{k_z}{m}; \quad p_\theta^2 = \frac{k_\theta}{M_{mox}} \tag{3.1.4-13}$$

Associated mode shapes are given as

$$\frac{A_1}{B_1} = h\left(\frac{p_1^2}{p_z^2 - p_1^2}\right) = \frac{M_{mox}}{mh}\left(\frac{(p_\theta^2 - p_1^2)}{p_1^2}\right) \tag{3.1.4-14}$$

$$\frac{A_1}{B_1} = h\left(\frac{p_2^2}{p_z^2 - p_2^2}\right) = \frac{M_{mox}}{mh}\left(\frac{(p_\theta^2 - p_2^2)}{p_2^2}\right) \quad (3.1.4\text{-}15)$$

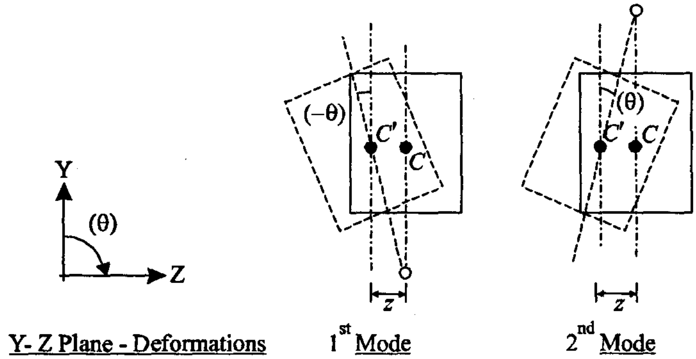


Figure 3.1.4-5 Mode Shapes in Y-Z Plane

From equations (3.1.4-14) it is noticed that amplitude **ratio (A_1 /B_1) is always positive** because the value of the quantity in parenthesis is always positive as $p_1 < p_\theta$. On the other hand equation (3.1.4-15) indicates that **ratio (A_2 /B_2) is always negative** because the value of the quantity in parenthesis is always negative as $p_2 > p_\theta$. This, in other words, indicates that in the 1st mode if the block translates say in positive Z-direction, its rotation shall be in positive θ direction i.e. anti-clockwise whereas in 2nd mode if the translation is in positive Z-direction then rotation shall be in negative θ direction i.e. anticlockwise. These mode shapes are shown in Figure 3.1.4-5.

SOLUTION 3.1.4-10

A) Natural Frequency

Rewriting equation 1:

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix}\begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix}\begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (a)$$

$$\textit{Frequency Equation} \quad = \left|(k - mp^2)\right| = 0 \quad (b)$$

Substituting equation (a) in equation (b), we get frequency equation as

$$\begin{vmatrix} (k_z - mp^2) & -mhp^2 \\ -mhp^2 & (k_\theta - M_{mox}p^2) \end{vmatrix} = 0$$

Simplifying, we get

$$(k_z - mp^2)(k_\theta - M_{mox}p^2) - m^2h^2p^4 = 0$$

$$k_z(k_\theta - M_{mox}p^2) - mp^2(k_\theta - M_{mox}p^2) - m^2h^2p^4 = 0$$

$$k_zk_\theta - M_{mox}k_zp^2 - mp^2k_\theta + mp^2M_{mox}p^2 - m^2h^2p^4 = 0$$

$$mM_{mox}p^4 - p^2(mk_\theta + M_{mox}k_z) + k_zk_\theta - m^2h^2p^4 = 0$$

$$m(M_{mox} - mh^2)p^4 - p^2(mk_\theta + M_{mox}k_z) + k_zk_\theta = 0$$

$$m(M_{mx})p^4 - p^2(mk_\theta + M_{mox}k_z) + k_zk_\theta - m^2h^2p^4 = 0$$

$$mM_{mx}\left\{p^4 - p^2\frac{(mk_\theta + M_{mox}k_z)}{mM_{mx}} + \frac{k_zk_\theta}{mM_{mx}}\right\} = 0$$

$$mM_{mx}\left\{p^4 - p^2\frac{mM_{mox}}{mM_{mx}}\left(\frac{k_\theta}{M_{mox}} + \frac{k_z}{m}\right) + \frac{k_zk_\theta}{mM_{mx}}\right\} = 0$$

$$mM_{mx}\left\{p^4 - p^2\frac{1}{\gamma}\left(p_\theta^2 + p_x^2\right) + \frac{k_zk_\theta}{mM_{mx}}\frac{M_{mox}}{M_{mox}}\right\} = 0$$

$$mM_{mx}\left\{p^4 - p^2\frac{1}{\gamma}\left(p_\theta^2 + p_x^2\right) + p_x^2p_\theta^2\frac{1}{\gamma}\right\} = 0$$

$$mM_{mx}\Delta\left(p^4\right) = 0$$

$$\Delta\left(p^4\right) = \left\{p^4 - p^2\frac{1}{\gamma}\left(p_\theta^2 + p_x^2\right) + \frac{1}{\gamma}p_x^2p_\theta^2\right\} \tag{c}$$

$$mM_{mx}\left\{p^4 - p^2\frac{1}{\gamma_x}\left(p_\theta^2 + p_z^2\right) + \frac{1}{\gamma_x}p_\theta^2p_z^2\right\} = mM_{mx}\Delta\left(p^4\right) = 0$$

Roots of $\Delta\left(p^4\right) = 0$ will yield two natural frequencies

$$p^4 - p^2\frac{1}{\gamma_x}\left(p_\theta^2 + p_z^2\right) + \frac{1}{\gamma_x}p_\theta^2p_z^2 = 0 \tag{d}$$

Solving we get

$$p^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) \mp \frac{1}{2\gamma_x}\sqrt{\left\{p_z^2 + p_\theta^2\right\}^2 - 4\gamma_x\,p_z^2p_\theta^2}$$

$$\text{Here } \gamma_x = \frac{M_{mx}}{M_{mox}};\ p_z^2 = \frac{k_z}{m};\quad p_\theta^2 = \frac{k_\theta}{M_{mox}} \tag{e}$$

Here p_1 represents lower frequency and p_2 represents higher frequency

Frequency equation thus becomes:

$$\Delta\left(p^4\right)=\left[p^4-p^2\frac{1}{\gamma_x}\left\{p_z^2+p_\theta^2\right\}+\frac{1}{\gamma_z}p_z^2p_\theta^2\right]=0 \tag{f}$$

Roots of $\Delta(p^4)=0$ will give two natural frequencies; one corresponding to translational mode and other to rotational mode. Solving we get

$$p^2=\frac{1}{2\gamma_x}\left(p_z^2+p_\theta^2\right)\mp\frac{1}{2\gamma_x}\sqrt{\left\{p_z^2+p_\theta^2\right\}^2-4\gamma_x p_z^2p_\theta^2} \tag{g}$$

This gives:

$$p_1^2=\frac{1}{2\gamma_x}\left(p_z^2+p_\theta^2\right)-\frac{1}{2\gamma_x}\sqrt{\left\{p_z^2+p_\theta^2\right\}^2-4\gamma_x p_z^2p_\theta^2} \tag{h}$$

$$p_2^2=\frac{1}{2\gamma_x}\left(p_z^2+p_\theta^2\right)+\frac{1}{2\gamma_x}\sqrt{\left\{p_z^2+p_\theta^2\right\}^2-4\gamma_x p_z^2p_\theta^2} \tag{i}$$

Here p_1 representing 1st natural frequency (lower natural frequency) corresponds to 1st mode of vibration & p_2 representing 2nd natural frequency (higher natural frequency) corresponds to 2nd mode of vibration.

From the equations (h) & (i), it can be proved that $p_1^2\times p_2^2=\frac{1}{\gamma_x}p_z^2p_\theta^2$

B) Free Vibration Response

Two natural frequencies have been obtained as given by equations (h) & (i). There are two associated modes of vibration. In the 1st mode the system will vibrate with frequency p_1 & in 2nd mode with frequency p_2. Let us now evaluate the two mode shapes:

Let the solution be represented as

$$\begin{aligned} z&=A\sin(pt+\alpha)\\ \theta&=B\sin(pt+\alpha)\end{aligned} \tag{j}$$

Equation (a) $\begin{bmatrix} m & mh \\ mh & M_{mox}\end{bmatrix}\begin{Bmatrix}\ddot{z}\\ \ddot{\theta}\end{Bmatrix}+\begin{bmatrix}k_z & 0\\ 0 & k_\theta\end{bmatrix}\begin{Bmatrix}z\\ \theta\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$

Substituting equation (j) in to equation of motion (a), we get

$$\begin{bmatrix} k_z - mp^2 & -mhp^2 \\ -mhp^2 & (k_\theta - M_{mox}p^2) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (k)$$

Simplifying, we get amplitude ratios in 1st mode as (use frequency p_1 for the first mode):

$$\frac{A_1}{B_1} = h\left(\frac{p_1^2}{p_z^2 - p_1^2}\right) = \frac{M_{mox}}{mh}\left(\frac{(p_\theta^2 - p_1^2)}{p_1^2}\right) \qquad (l)$$

Solving for amplitude ratios in 2nd mode (use frequency p_2 for the second mode), we get:

$$\frac{A_1}{B_1} = h\left(\frac{p_2^2}{p_z^2 - p_2^2}\right) = \frac{M_{mox}}{mh}\left(\frac{(p_\theta^2 - p_2^2)}{p_2^2}\right) \qquad (m)$$

3.1.5 Multiple Spring Mass Systems connected by a massless Rigid Bar

Consider a Multi-Spring Mass System connected by a massless rigid bar as shown in Figure 3.1.5-1. Figure (a) shows distances of each spring from an arbitrary axis. Let C_k & C_m represent Center of Stiffness and of the System respectively. Let $\bar{x}_m$ & $\bar{x}_k$ represent distance of Center of Mass and center of stiffness from arbitrary axis. Let e represent eccentricity between Center of Mass & Center of Stiffness. The system is constrained to move only in X-Y plane.

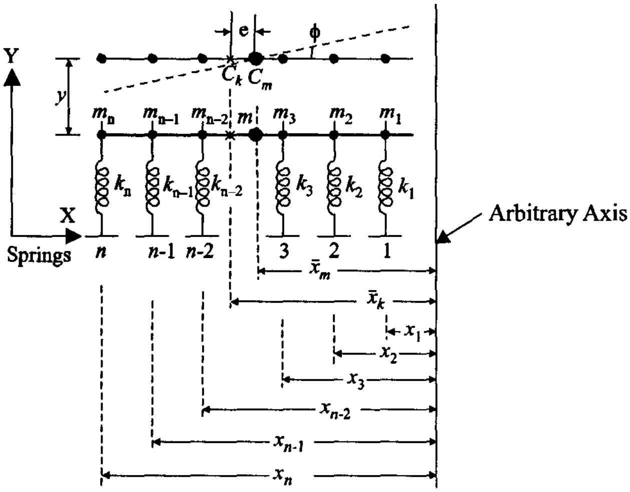
Two coordinates namely translation y (along Y) and rotation ϕ (about Z) represent two degrees of freedom that define displaced position of the system. Let x_i represent the distance of i^{th} frame from any arbitrary axis parallel to Y as shown in part (a) of the figure.

CG overall mass m (point C_m) from that arbitrary axis

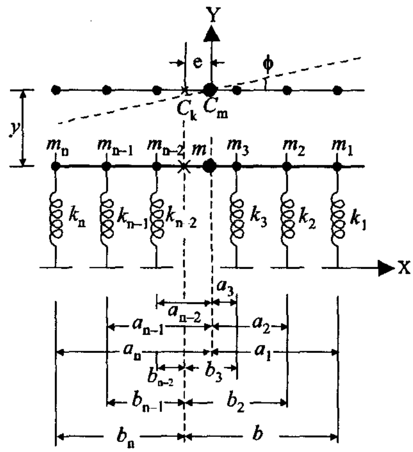
$$m = \sum m_i \; ; \qquad \bar{x}_m = \frac{\sum m_i x_i}{m}$$

CG of Overall stiffness k (point C_k) from that arbitrary axis

$$k_y = \sum k_i \; ; \qquad \bar{x}_k = \frac{\sum k_i x_i}{k}$$



(a) Distances from Arbitrary Axis



(b) Origin @ Center of Mass - Distances from Center of Mass and Center of Stiffness

Figure 3.1.5-1 Multiple Spring Mass Systems Connected by Massless Rigid Bar

Let us consider center of mass C_m as origin.

Let a_i represent distance of i^{th} frame from center of mass point C_m and b_i represent distance of i^{th} frame from center of stiffness point C_k as shown in part (b) of the figure.

Eccentricity e (distance between C_m & C_k) $\qquad e = \bar{x}_m - \bar{x}_k$

We can represent the system as a Two Degree of Freedom System with a simplified model having Mass m, Mass Moment of Inertia M_m, Translational Spring Stiffness k_y (along Y) and Rotational Spring Stiffness k_ϕ (about Z) as shown in Fig. 3.1.5-2.

Equation of motion: Consider the motion in X-Y plane at any instant of time t. Mass moves by a distance y and rotates about Z by ϕ. Let center of mass point C_m represent DOF location for equation of motion. Equation of motion thus becomes:

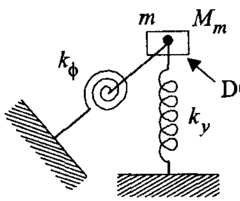


Figure 3.1.5-2 Equivalent Two DOF System Supported by Translational & Rotational Springs

$$m\ddot{y} + k_y\, y = 0 \tag{3.1.5-1}$$

$$M_m\, \ddot{\phi} + k_\phi\, \phi = 0 \tag{3.1.5-2}$$

Here m represents overall Mass, M_m represents Mass moment of Inertia, k_y represents overall linear stiffness and k_ϕ represents overall rotational stiffness of the system. Based on the system parameters, let us evaluate m, M_m, k & k_ϕ

From the figure, we get $y_i = y + a_i\, \phi$ & $a_i = b_i + e$ (3.1.5-3)

a) Total Mass of the system @DOF

Total Inertia force $\sum m_i \ddot{y}_i = \sum m_i \left(\ddot{y} + a_i \ddot{\phi}\right) = \ddot{y}\sum m_i + \ddot{\phi}\sum m_i a_i$

Since a_i is the distance from center of mass $\sum m_i a_i = 0$

$$\sum m_i \ddot{y}_i = \ddot{y}\sum m_i = m\ddot{y}$$

$$m = \sum m_i \tag{3.1.5-4}$$

b) Total Linear Stiffness @DOF

Total resisting force at DOF location (point C_m) $\sum(k_i y_i)$

$$\sum k_i y_i = \sum\{k_i(y + a_i\phi)\} = \sum\{k_i y + k_i(e + b_i)\phi\}$$
$$= \sum(k_i y + k_i e\phi + k_i b_i \phi) = y\sum k_i + e\phi\sum k_i + \phi\sum k_i b_i$$

Since b_i is the distance from center of stiffness, summation $\sum k_i b_i = 0$

$$\sum(k_i y_i) = k_y\, y + k_y\, e\phi$$

$$k_y = \sum k_i \tag{3.1.5-5}$$

c) Total Mass moment of Inertia of the system @ DOF

Total Rotary Inertia Moment

$$\sum(m_i \ddot{y}_i a_i) = \sum\{m_i a_i(\ddot{y} + a_i)\} = \ddot{y}\sum m_i a_i + \ddot{\phi}\sum(m_i a_i^2)$$

Since a_i is the distance from center of mass $\sum m_i a_i = 0$

$$\sum(m_i \ddot{y}_i a_i) = \ddot{\phi}\sum(m_i a_i^2) = M_m \ddot{\phi}$$
$$M_m = \sum m_i a_i^2 \qquad (3.1.5\text{-}6)$$

d) Total Rotational Stiffness of the system @DOF

Total resisting moment at DOF location

$$\begin{aligned}
\sum(k_i y_i a_i) &= \sum\{k_i a_i(y + a_i\phi)\} = \sum(k_i a_i y) + \sum(k_i a_i^2 \phi) \\
&= \phi\sum(k_i a_i^2) + y\sum\{k_i(e + b_i)\} = \phi\sum(k_i a_i^2) + \sum(y k_i e) + \sum(y k_i b_i) \\
&= \phi\sum(k_i a_i^2) + ey\sum k_i + y\sum k_i b_i = \phi\sum(k_i a_i^2) + key \\
&= k_y\, e\, y + \phi\sum(k_i(b_i + e)^2) = key + \phi\sum(k_i(b_i^2 + e^2 + 2eb_i)) \\
&= k_y\, e\, y + \phi\sum k_i b_i^2 + \phi e^2\sum k_i + 2e\phi\sum k_i b_i \\
&= k_y\, e\, y + \phi\sum k_i b_i^2 + ke^2\phi + 2e\phi\sum k_i b_i
\end{aligned}$$

Since b_i is the distance from center of stiffness, summation $\sum k_i b_i = 0$

$$\sum(k_i y_i a_i) = k_y\, e\, y + k_\phi\, \phi + k_y\, e^2 \phi$$
$$k_\phi = \sum k_i b_i^2 \qquad (3.1.5\text{-}7)$$

Substituting equations 3.1.5-4 to 7 into the equations 3.1.5-1&2, we get:

$$m\ddot{y} + k_y(y + e\phi) = 0 \qquad (3.1.5\text{-}8)$$

$$M_m\ddot{\phi} + k_y\, e\, y + k_\phi\, \phi + k_y\, e^2\phi = 0 \qquad (3.1.5\text{-}9)$$

Here $m = \sum m_i$; $k_y = \sum k_i$; $M_m = \sum m_i a_i^2$; $k_\phi = \sum(k_i\, b_i^2)$

Simplifying equations 3.1.5-8, we get

$$\ddot{y} + p_y^2\, y + e\, p_y^2\, \phi = 0; \quad \text{Here } p_y = \sqrt{\frac{k_y}{m}} \tag{3.1.5-10}$$

Simplifying equations 3.1.5-9, we get

$$M_m\, \ddot{\phi} + k_y\, y e + k_y\, e^2\, \phi + k_\phi\, \phi = 0$$

$$\ddot{\phi} + \frac{k_y}{M_m} y e + \frac{k_y}{M_m} e^2\, \phi + \frac{k_\phi}{M_m}\phi = 0 \tag{3.1.5-11}$$

$$\ddot{\phi} + p_y^2\, y \frac{e}{r^2} + p_y^2 \frac{e^2}{r^2}\phi + p_\phi^2\, \phi = 0$$

Terms $p_y = \sqrt{\frac{k_y}{m}}$; $p_\phi = \sqrt{\frac{k_\phi}{M_m}}$ & $r = \sqrt{\frac{M_m}{m}}$ represent limiting translational frequency, limiting rotational frequency and equivalent radius of gyration respectively.

It is also noted that both these equations 3.1.5-10 & 11 are coupled through eccentricity term e. If eccentricity becomes zero, i.e. $e = 0$, both these equations get uncoupled and the limiting frequencies become natural frequencies.

Natural Frequencies:

Rewriting equations 3.1.5-10 & 11 in matrix form, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} p_y^2 & e\, p_y^2 \\ p_y^2 \frac{e}{r^2} & \left(p_y^2 \frac{e^2}{r^2} + p_\phi^2 \right) \end{bmatrix}\begin{Bmatrix} y \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.1.5-12}$$

Frequency equation $\left| k - mp^2 \right| = 0$

$$\begin{vmatrix} p_y^2 - p^2 & e\, p_y^2 \\ p_y^2 \frac{e}{r^2} & \left(p_y^2 \frac{e^2}{r^2} + p_\phi^2 - p^2 \right) \end{vmatrix} = 0 \tag{3.1.5-13}$$

Simplifying, we get

$$p^4 - p^2\left(p_y^2\left(1+\frac{e^2}{r^2}\right)+p_\phi^2\right)+p_y^2 p_\phi^2 = 0$$

$$p^4 - p^2\left(\alpha\, p_y^2 + p_\phi^2\right)+p_y^2 p_\phi^2 = 0 \qquad (3.1.5\text{-}14)$$

Here $\alpha = \left(1+\frac{e^2}{r^2}\right)$

Roots of the equation 3.1.5-14 will yield two natural frequencies.

$$p_{1,2}^2 = \frac{1}{2}\left\{\left(\alpha\, p_y^2 + p_\phi^2\right) \pm \sqrt{\left(\alpha\, p_y^2 + p_\phi^2\right)^2 - 4 p_y^2 p_\phi^2}\right\} \qquad (3.1.5\text{-}15)$$

3.1.6 A Portal Frame supporting mass at Beam Center

Consider a portal frame supporting mass m at beam center as shown in Figure 3.1.6-1. Consider that portal frame is constrained to move only in X-Y plane. Possible motion directions are i) motion along X and ii) motion along Y.

Single Degree of Freedom representation of the system for motion along X as well as along Y is described in Chapter 2 (§2.1.1.4.5).

Representing motion along Y as Two Degrees of Freedom system, consider the portal frame with the properties (same as those used in (§2.1.1.4.5) as under:

Elastic Modulus of Material (Both column & Beam)	E
Mass density of the material	ρ
Span of Beam is	L
Height of Frame	H
Area of Beam Cross-section	A_b
Area of Column Cross-section	A_c
Moment of Inertia Beam Cross-section	I_b
Moment of Inertia Column Cross-section	I_c

Mathematical model is shown in Figure (d).

Equations of motion are (see equation 3.1.1-1)

$$m_1\ddot{y}_1 + k_1 y_1 - k_2(y_2 - y_1) = 0$$
$$m_2\ddot{y}_2 + k_2(y_2 - y_1) = 0 \qquad (3.1.6\text{-}1)$$

Let us evaluate m_1, m_2, k_1 & k_2.

Degrees of Freedom y_1 & y_2

i) **Mass** m_1 & m_2

a) Mass m_2

Machine Mass on the frame beam	m
Mass of Beam	$m_b = \rho \times A_b \times L$
Generalised mass of frame beam	$m_b^* = 0.45\, m_b$

Note: For simply supported beam the factor for equivalent mass is 0.485 (see equation 2.1.1-26) and for fixed-fixed beam this is close to 0.37 (see equation 2.1.1-33). For a frame this value is taken as 0.45 (close to average).

Mass m_2 $\qquad m_2 = m + 0.45\, m_b \qquad$ (3.1.6-2)

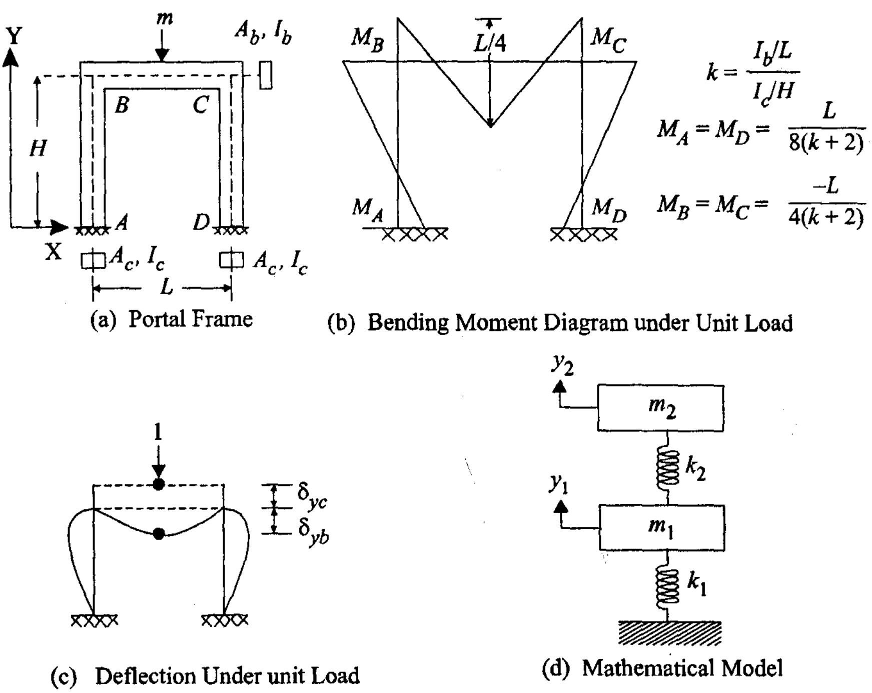


Figure 3.1.6.-1 Portal Frame with Machine Mass *m* at Beam Center - Deflection and Bending Moments - Vibration in Vertical Mode

b) Mass m_1

Mass of each column $m_c = \rho \times A_c \times H$

Generalised mass of each column (see equation 2.1.1-15) $m_c^* = 0.33\, m_c$

Mass m_1 =Total mass on column top – Mass m_2

$$\begin{aligned} m_1 &= \{(m + m_b + 2\times 0.33\times m_c) - m_2\} \\ m_1 &= \{(m + m_b + 2\times 0.33\times m_c) - (m + 0.45\, m_b)\} \\ m_1 &= 0.55\, m_b + 2\times 0.33\times m_c \end{aligned} \tag{3.1.6-3}$$

ii) Stiffness k_1 & k_2

It is seen from the mathematical model that DOF y_2 pertains to beam deflection δ_{yb} and DOF y_1 pertains to column deflection δ_{yc}.

a) Beam Deformation under unit Load δ_{yb} :

Stiffness Ratio Factor $k = \dfrac{(E I_b/L)}{(E I_c/H)} = \dfrac{(I_b/L)}{(I_c/H)}$

From bending moment diagram of beam alone (as shown in the Figure), we get:

Deflection due to span moment $\dfrac{L}{4}$ $\delta_{yb1} = \dfrac{L^3}{48 E I_b}$

Due to support moments $\dfrac{L}{4(k+2)}$ $\delta_{yb2} = \dfrac{L}{4(k+2)} \times \dfrac{L^2}{8 E I_b}$

Net beam deflection at center $\delta_{yb} = \delta_{yb1} - \delta_{yb2}$

$$\delta_{yb} = \frac{L^3}{48 E I_b} - \frac{L}{4(k+2)} \times \frac{L^2}{8 E I_b} = \frac{L^3}{96 E I_b} \times \frac{2k+1}{k+2}$$

Stiffness k_2
$$k_2 = \frac{1}{\delta_{yb}} = \frac{96 E I_b}{L^3} \times \frac{k+2}{2k+1} \tag{3.1.6-4}$$

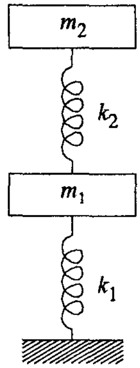
b) Column Deformation under unit Load δ_{yc}:

Vertical deflection of columns $\delta_{yc} = \dfrac{1}{2 \times (EA_c/H)}$

Stiffness k_1 $k_1 = \dfrac{2EA_c}{H}$ (3.1.6-5)

iii) Natural Frequencies (see § 3.1.1)

We can represent the system as shown with $m_1, m_2, k_1 \,\&\, k_2$ as given below



$$m_2 = m + 0.45\, m_b$$

$$m_1 = 0.55\, m_b + 2 \times 0.33 \times m_c$$

$$k_1 = \frac{2EA_c}{H}$$

$$k_2 = \frac{1}{\delta_{yb}} = \frac{96EI_b}{L^3} \times \frac{k+2}{2k+1}$$

$$p_{L1} = \sqrt{\frac{k_1}{m_1}};\; p_{L2} = \sqrt{\frac{k_2}{m_2}};\; \lambda = \frac{m_2}{m_1}$$

Natural frequencies: Rewriting equation 3.1.1-11

$$p_{1,2}^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) \mp \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\} \quad (3.1.6\text{-}6)$$

Roots of this equation give two natural frequencies $p_1 \,\&\, p_2$.

3.2 TWO DEGREES OF FREEDOM SYSTEM - FORCED VIBRATION

3.2.1 Un-damped Two Spring Mass System Subjected to Harmonic Loads

Consider a two spring mass system having dynamic excitation force as shown in Figure 3.2.1-1. In any practical system, the excitation force can be considered in four ways:

a) Dynamic force on mass m_2
b) Dynamic force on mass m_1
c) Dynamic force on masses $m_2 \,\&\, m_1$
d) Dynamic force applied at the base

Let us consider these loading cases one by one.

a) Dynamic force on mass m_2: System is shown in Figure 3.2.1-1 (a)

Equation of motion is written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \sin \omega t \end{Bmatrix} \tag{3.2.1-1}$$

i) **Complimentary Solution:** For complimentary solution **(see § 3.1.1)**
ii) **Particular Solution:**

Under the influence of excitation force, the system will vibrate with frequency of excitation force ω. Consider the solution to be of the form

$$\begin{aligned} y_1 &= C_1 \sin \omega t \\ y_2 &= C_2 \sin \omega t \end{aligned} \tag{3.2.1-2}$$

Differentiating, we get 2nd derivative as

$$\begin{aligned} \ddot{y}_1 &= -\omega^2 C_1 \sin \omega t \\ \ddot{y}_2 &= -\omega^2 C_2 \sin \omega t \end{aligned} \tag{3.2.1-3}$$

Substituting equation (3.2.1-2) & (3.2.1-3) in equation (3.2.1-1), it gives

$$\begin{bmatrix} (k_1 + k_2 - \omega^2 m_1) & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} \tag{3.2.1-4}$$

Solution of this gives **(see SOLUTION 3.2.1-4)** dynamic response y_1 & y_2 of the masses m_1 & m_2 (see equation *(h) & (i)*):

$$y_1 = \frac{F_2}{k_1} \frac{1}{(1-\beta_1^2)(1-\beta_2^2)} \sin \omega t \tag{3.2.1-5}$$

$$y_2 = \frac{F_2}{k_2} \frac{\left(1 + \lambda \frac{\beta_{L1}^2}{\beta_{L2}^2} - \beta_{L1}^2\right)}{(1-\beta_1^2)(1-\beta_2^2)} \sin \omega t \tag{3.2.1-6}$$

Here $p_{L1} = \sqrt{\frac{k_1}{m_1}}$; $p_{L2} = \sqrt{\frac{k_2}{m_2}}$; $\lambda = \frac{m_2}{m_1}$; $\beta_1 = \frac{\omega}{p_1}$; $\beta_2 = \frac{\omega}{p_2}$; $\beta_{L1} = \frac{\omega}{p_{L1}}$; $\beta_{L2} = \frac{\omega}{p_{L2}}$

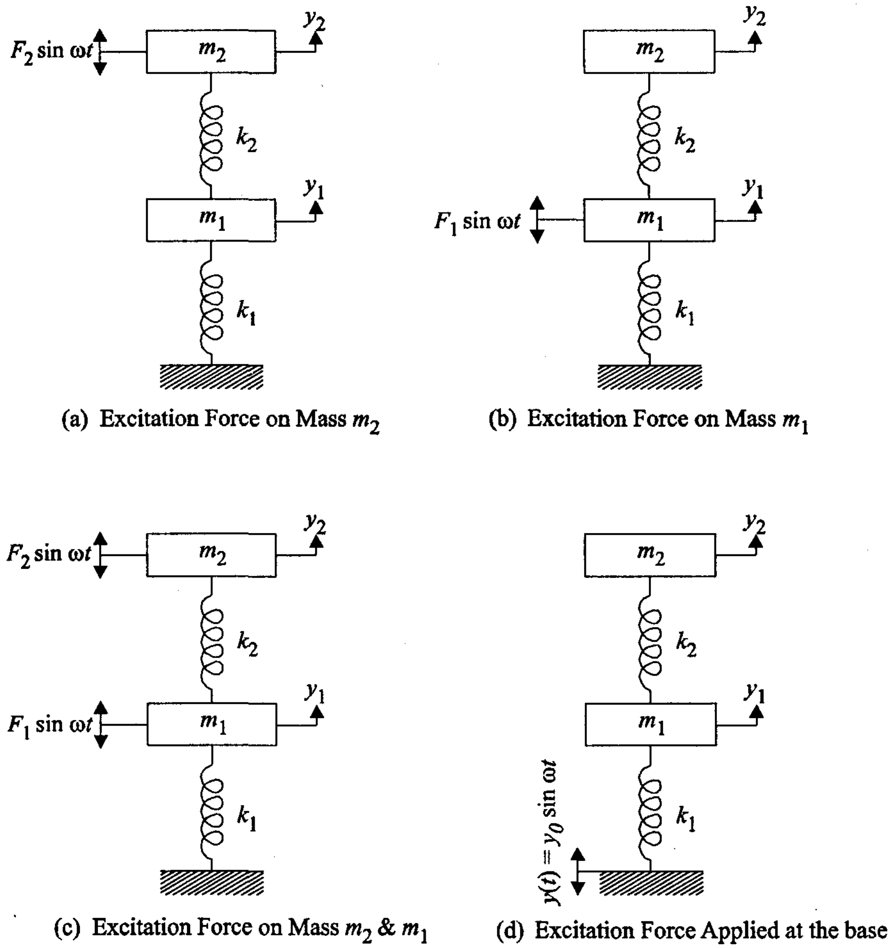


Figure 3.2.1-1 An Undamped Two Spring Mass System with Excitation Force (a) on Mass m_2 (b) on Mass m_1 (c) on Both Masses m_2 & m_1 & (d) at base $y(t) = y_0 \sin \omega t$

Resonance condition

It is seen from the equations (3.2.1-5& 6) that amplitude rises to infinity when either β_1 *or* β_2 becomes unity. Such a condition is termed as **resonance condition** at first or second natural frequency. Thus for a 2 DOF undamped system there exist 2 resonance conditions where amplitude rises to infinity. As every physical system has inherent damping present in the system, this damping plays a predominant role in reducing the amplitude of motion at resonance.

Taking advantage of the derivation done for damped SDOF system, it can be said that in case of resonance with first natural frequency, the response of the system at resonance is obtained by replacing the term $\left(1-\beta_1^2\right)$ in these equations by $\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}$. Similarly in case of resonance with second natural frequency, the response to the system at resonance is obtained by replacing the term $\left(1-\beta_2^2\right)$ by $\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}$.

b) Dynamic force on mass m_1:

System is as shown in Figure 3.2.1-1 (b). Equation of motion is written as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}=\begin{Bmatrix} F_1\sin\omega t \\ 0 \end{Bmatrix} \tag{3.2.1-7}$$

Complimentary solution to the equation of motion remains the same as for case (a) above. Since there is a change only on the right hand side of the equation of motion, only particular solution will get affected. Using the same procedure as in (a) above, the response becomes:

$$y_1=\frac{F_1}{k_1}\frac{\left(1-\beta_{L2}^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \tag{3.2.1-8}$$

$$y_2=\frac{F_1}{k_1}\frac{1}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \tag{3.2.1-9}$$

For resonance amplitudes see explanation given in case (a).

c) Dynamic force applied at mass m_1 & m_2

System is as shown in Figure 3.2.1-1 (c). Equation of motion is written as:

Equation of motion is written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}=\begin{Bmatrix} F_1\sin\omega t \\ F_2\sin\omega t \end{Bmatrix} \tag{3.2.1-10}$$

Since there is a change only on the right hand side of the equation, using the same procedure as in (a) above, the response becomes:

$$y_1 = \frac{\frac{F_1}{k_1}\left(1-\beta_{L2}^2\right)+\frac{F_2}{k_1}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (3.2.1\text{-}11)$$

$$y_2 = \frac{\frac{F_1}{k_1}+\frac{F_2}{k_2}\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (3.2.1\text{-}12)$$

It is noticed that equation (3.2.1-11) is summation of equations (3.2.1-5) & (3.2.1-8). Similarly equation (3.2.1-12) is summation of equations (3.2.1-6) & (3.2.1-9). Thus one can evaluate response for separate load cases and perform linear summation for overall response.

For resonance amplitudes see explanation given in case (a).

d) Dynamic Force applied at the base

For machine foundation application in industrial environment, it is the dynamic displacement transmitted through base of the system that is generally encountered rather than applied force. Hence the case of **Dynamic displacement** $y(t) = y_0 \sin\omega t$ **applied at the base is considered.**

System as given in Figure 3.2.1-1 (d) is shown again in Figure 3.2.1-2 along with its free body diagram.

Equation of motion:

When the base exhibits a dynamic displacement $y(t) = y_0 \sin\omega t$, under the displaced condition, the net displacement of the spring k_1 becomes $y_1 - y_0 \sin\omega t$.

Considering equilibrium of forces on the free body diagram, we get

$$\begin{aligned} m_1\ddot{y}_1 + k_1(y_1 - y_0\sin\omega t) - k_2(y_2 - y_1) &= 0 \\ m_2\ddot{y}_2 + k_2(y_2 - y_1) &= 0 \end{aligned} \qquad (3.2.1\text{-}13)$$

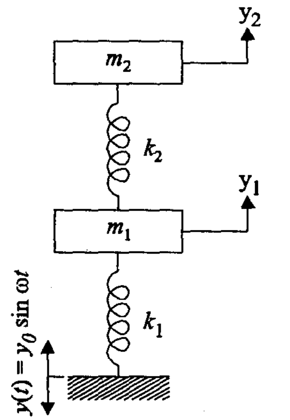
Simplifying we can write

$$\begin{aligned} m_1\ddot{y}_1 + k_1y_1 - k_2(y_2 - y_1) &= k_1y_0\sin\omega t \\ m_2\ddot{y}_2 + k_2(y_2 - y_1) &= 0 \end{aligned} \qquad (3.2.1\text{-}14)$$

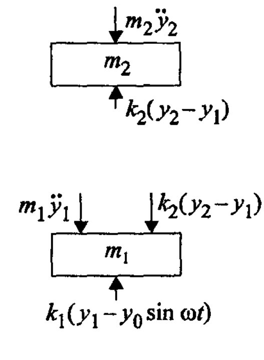
Writing in matrix form, Equation of motion is written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} k_1y_0\sin\omega t \\ 0 \end{Bmatrix} \qquad (3.2.1\text{-}15)$$

Solution to equation (3.2.1-15): Complimentary solution remains the same as for case (a) above. Since there is a change only on the right hand side of the equation of motion, only particular solution will get affected.



Base Excitation $y(t) = y_0 \sin \omega t$



Free Body Diagram

Figure 3.2.1-2 Two DOF System with Base Excitation $y(t) = y_0 \sin \omega t$

Particular solution: Comparing equation of motion (3.2.1-15) with equation (3.2.1-7), it is seen that for $F_1 = k_1 y_0$, these equations are identical

Thus replacing $F_1 = k_1 y_0$ in equations (3.2.1-11) & (3.2.1-12), the solution to equation (3.2.1-21) becomes:

$$y_1 = y_0 \frac{\left(1 - \beta_{L2}^2\right)}{\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)} \sin \omega t \tag{3.2.1-16}$$

$$y_2 = y_0 \frac{1}{\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)} \sin \omega t \tag{3.2.1-17}$$

For resonance amplitudes see explanation given in case (a).

SOLUTION 3.2.1-4

Rewriting equation 3.2.1-4

$$\begin{bmatrix} (k_1+k_2-\omega^2 m_1) & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} \qquad (a)$$

Pre-multiplying by $\begin{bmatrix} (k_1+k_2-\omega^2 m_1) & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix}^{-1}$ *on both LHS & RHS, it gives:*

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{bmatrix} (k_1+k_2-\omega^2 m_1) & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix}$$

(For inverse of matrix refer any relevant book on Matrices)

Simplifying we get,

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{bmatrix} (k_1+k_2-\omega^2 m_1) & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} = \frac{\begin{bmatrix} k_2-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix} \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix}}{\begin{vmatrix} (k_1+k_2-\omega^2 m_1) & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{vmatrix}}$$

Simplifying the denominator, we get

$$\begin{vmatrix} (k_1+k_2-\omega^2 m_1) & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{vmatrix} = m_1 m_2 \left\{ \omega^4 - \omega^2 \left(\frac{k_2}{m_2} + \frac{k_2}{m_1} + \frac{k_1}{m_1} \right) + \left(\frac{k_1}{m_1} \frac{k_2}{m_2} \right) \right\} = m_1 m_2 \, \Delta(\omega^4)$$

Substituting, we get

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \frac{1}{m_1 m_2 \, \Delta(\omega^4)} \begin{bmatrix} k_2-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix} \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} \qquad (b)$$

Since p_1 *&* p_2 *are the two natural frequencies of the system,* $\Delta(\omega^4)$ *is written as*

$$\Delta(\omega^4) = \left\{ (\omega^2 - p_1^2)(\omega^2 - p_2^2) \right\} = \left\{ (p_1^2 - \omega^2)(p_2^2 - \omega^2) \right\}$$

Substituting, the equation (b) becomes:

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \frac{1}{m_1 m_2 (p_1{}^2 - \omega^2)(p_2{}^2 - \omega^2)} \begin{bmatrix} k_2-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix} \begin{Bmatrix} 0 \\ F_2 \end{Bmatrix} \qquad (c)$$

Solving for C_1 & C_2, it gives

$$C_1 = \frac{k_2 F_2}{m_1 m_2 \left(p_1^{\,2} - \omega^2\right)\left(p_2^{\,2} - \omega^2\right)} \qquad (d)$$

$$C_2 = \frac{\left(k_1 + k_2 - \omega^2 m_1\right)F_2}{m_1 m_2 \left(p_1^{\,2} - \omega^2\right)\left(p_2^{\,2} - \omega^2\right)} \qquad (e)$$

We can also compute amplitudes using Crammer's rule (for Crammer's rule refer any book on engineering mathematics). This simplifies the computation process. From equation (a), we get:

$$C_1 = \frac{\begin{vmatrix} 0 & -k_2 \\ F_2 & k_2 - \omega^2 m_2 \end{vmatrix}}{\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix}} = \frac{F_2 k_2}{m_1 m_2 \Delta(\omega^4)}$$

$$\Delta(\omega^4) = \left\{\left(\omega^2 - p_1^2\right)\left(\omega^2 - p_2^2\right)\right\} = \left\{\left(p_1^2 - \omega^2\right)\left(p_2^2 - \omega^2\right)\right\}$$

$$C_1 = \frac{F_2 k_2}{m_1 m_2 \left(p_1^{\,2} - \omega^2\right)\left(p_2^{\,2} - \omega^2\right)} \qquad (f)$$

$$C_2 = \frac{\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & 0 \\ -k_2 & F_2 \end{vmatrix}}{\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix}} = \frac{F_2\left(k_1 + k_2 - \omega^2 m_1\right)}{\begin{vmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{vmatrix}} = \frac{F_2\left(k_1 + k_2 - \omega^2 m_1\right)}{m_1 m_2 \left(p_1^{\,2} - \omega^2\right)\left(p_2^{\,2} - \omega^2\right)} \qquad (g)$$

It is noticed that C_1 & C_2 as given by equations (f) & (g) are same as those in equation (d) & (e).Thus it may be convenient and simple to use Crammer's rule any subsequent derivation.

Simplifying further, we get

$$C_1 = \frac{F_2 k_2}{m_1 m_2 p_1^{\,2} p_2^{\,2}\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)} = \frac{F_2 k_2}{m_1 m_2 p_{L1}^{\,2} p_{L2}^{\,2}\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)} = \frac{F_2}{k_1}\frac{1}{\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)}$$

$$C_2 = \frac{F_2 m_1\left(\frac{k_1}{m_1} + \frac{k_2}{m_1} - \omega^2\right)}{m_1 m_2 p_1^{\,2} p_2^{\,2}\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)} = \frac{F_2 m_1\left(\frac{k_1}{m_1} + \frac{k_2}{m_1} - \omega^2\right)}{m_1 m_2 p_{L1}^{\,2} p_{L2}^{\,2}\left(1 - \beta_1^2\right)\left(1 - \beta_2^2\right)}$$

$$C_2 = \frac{F_2 m_1\left(p_{L1}{}^2 + p_{L2}{}^2\lambda - \omega^2\right)}{k_1 k_2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} = \frac{F_2}{k_2}\frac{\left(1+\frac{p_{L2}{}^2}{p_{L1}{}^2}\lambda - \beta_{L1}^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

Substituting for C_1 & C_2 in equation (3.2.1-2), particular solution becomes

$$y_1 = \frac{F_2}{k_1}\frac{1}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (h)$$

$$y_2 = \frac{F_2}{k_2}\frac{\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2} - \beta_{L1}^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (i)$$

Here $p_{L1} = \sqrt{\frac{k_1}{m_1}};\ p_{L2} = \sqrt{\frac{k_2}{m_2}};\ \lambda = \frac{m_2}{m_1};\ \beta_1 = \frac{\omega}{p_1};\ \beta_2 = \frac{\omega}{p_2};\ \beta_{L1} = \frac{\omega}{p_{L1}};\ \beta_{L2} = \frac{\omega}{p_{L2}}$ (j)

3.2.2 Un-damped Two Spring Mass System- Subjected to Impact Load

Consider mass m_0 freely falling from height h on a two spring mass system as shown in Figure 3.2.1-3. Let us consider that mass m_2 is at rest before the impact and the impact is central. The problem is initial velocity problem and its treatment is similar to one discussed in **Chapter 2 - § 2.2.5.**

Let v_0' & v_2' represent velocity of masses m_0 & m_2 **before impact** and v_0 & v_2 represent velocity of masses m_0 & m_2 **after impact.**

From conservation of momentum, we get:

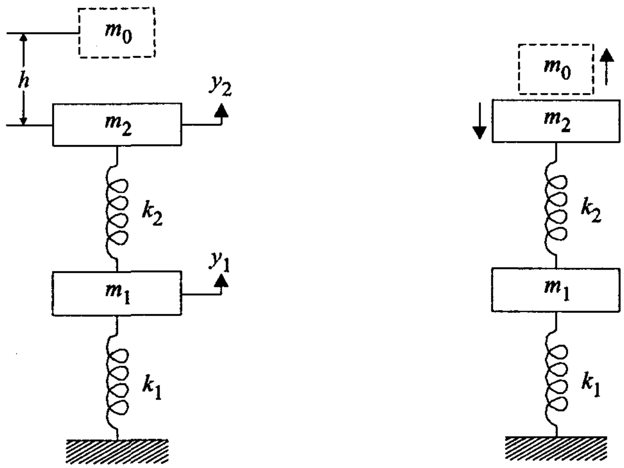
$$\underbrace{m_2 \times v_2' + m_0 \times v_0'}_{\text{Before Impact}} = \underbrace{m_2 \times v_2 + m_0 \times v_0}_{\text{After Impact}}$$

Since mass m_2 is at rest before the impact, i.e. $v_2' = 0$. Substituting, we get

$$m_0 \times v_0' = m_2 \times v_2 + m_0 \times v_0 \qquad (3.2.1\text{-}18)$$

To evaluate v_2 (see equation 2.2.5-2), using **Coefficient of Restitution** e , we get

$$e = -\frac{v_2 - v_0}{(v_2' - v_0')} = \frac{v_2 - v_0}{v_0'} \quad \text{Or} \quad v_0 = v_2 - ev_0' \tag{3.2.1-19}$$



(a) Before Impact
Mass m_0 freely falling from height h over Mass m_2

(b) Just after Impact
Mass m_2 moves downwards
Mass m_0 rebounds upward

Figure 3.2.1-3 An Undamped Two DOF System-Subjected to an Impact Load Mass m_0 Freely Falling over Mass m_2 from Height h

Coefficient of Restitution e depends upon properties of the material of the masses m_0 & m_2. For perfectly plastic central impact, the value of e is zero and for perfectly elastic central impact e is equal to unity. For real bodies in practice, the value lies in the range $0 < e < 1$ and for all practical purposes it's reasonably good to use $e = 0.5$.

Substituting 3.2.1-19 in 3.2.1-18 and simplifying, we get

$$v_2 = v_0' \times \frac{(1+e)}{(1+\lambda_2)} \tag{3.2.1-20}$$

Here $\lambda_2 = \frac{m_2}{m_0}$ λ_2 represents ratio of mass m_2 to mass m_0 (3.2.1-21)

For **freely falling body** of mass m_0 from height h, the velocity just before impact is given as

$$v_0' = \sqrt{2gh} \qquad (3.2.1\text{-}22)$$

Substituting this in 3.2.1-20, we get

$$v_2 = \sqrt{2gh} \times \frac{(1+e)}{(1+\lambda_2)} \qquad (3.2.1\text{-}22a)$$

This is the initial velocity imparted by the falling mass to stationary mass m_2 at time $t = 0$. Thus, solution to Two Spring Mass System subjected to Impact Loads becomes an **Initial Velocity problem.**

Equation of motion of the Two Mass System (refer equation 3.1.1-1)

$$\begin{aligned} m_1\ddot{y}_1 + k_1 y_1 - k_2(y_2 - y_1) &= 0 \\ m_2\ddot{y}_2 + k_2(y_2 - y_1) &= 0 \end{aligned} \qquad (3.2.1\text{-}23)$$

Let the general solution of the equation of motion be of the form (see 3.1.1-15)

$$\begin{aligned} y_1 &= A_1 \sin(pt + \phi) \\ y_2 &= A_2 \sin(pt + \phi) \end{aligned} \qquad (3.2.1\text{-}23a)$$

The two natural frequencies are (see equation 3.1.1-12 & 3.1.1-13):

$$p_1^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) - \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\} \qquad (3.2.1\text{-}24)$$

$$p_2^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) + \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\} \qquad (3.2.1\text{-}25)$$

Here, $p_{L1} = \sqrt{\dfrac{k_1}{m_1}}$; $p_{L2} = \sqrt{\dfrac{k_2}{m_2}}$ & $\lambda = \dfrac{m_2}{m_1}$ (3.2.1-26)

Free Vibration Response (see equations 3.1.1-22 & 23):

In 1st mode the system vibrates with frequency p_1. The solution becomes:

$$\begin{aligned} y_1' &= A_1' \sin(p_1 t + \phi') \\ y_2' &= A_2' \sin(p_1 t + \phi') \end{aligned}$$

In 2nd mode the system vibrates with frequency p_2. The solution becomes:

$$y_1'' = A_1'' \sin(p_2 t + \phi'')$$
$$y_2'' = A_2'' \sin(p_2 t + \phi'')$$

Here, quantities with single prime are indicative of the 1st mode and quantities with double prime are indicative of the 2nd mode.

Total solution (combining both the modes) thus becomes:

$$\begin{aligned} y_1 &= A_1' \sin(p_1 t + \phi') + A_1'' \sin(p_2 t + \phi'') \\ y_2 &= A_2' \sin(p_1 t + \phi') + A_2'' \sin(p_2 t + \phi'') \end{aligned} \qquad (3.2.1\text{-}27)$$

Amplitude Response in 1st mode is given as (see equations 3.1.1-19):

$$\begin{aligned} \frac{A_1'}{A_2'} &= \frac{k_2}{k_1 + k_2 - m_1 p_1^2} = \frac{k_2 - m_2 p_1^2}{k_2} = \frac{1}{\alpha'} \\ \alpha' &= \frac{k_2}{k_2 - m_2 p_1^2} = \frac{p_{L2}^2}{p_{L2}^2 - p_1^2}; \quad A_2' = \alpha' A_1' \end{aligned} \qquad (3.2.1\text{-}28)$$

Amplitude Response in 2nd mode is given as (see equations 3.1.1-21):

$$\begin{aligned} \frac{A_1''}{A_2''} &= \frac{k_2}{k_1 + k_2 - m_1 p_2^2} = \frac{k_2 - m_2 p_2^2}{k_2} = \frac{1}{\alpha''} \\ \alpha'' &= \frac{k_2}{k_2 - m_2 p_2^2} = \frac{p_{L2}^2}{p_{L2}^2 - p_2^2}; \quad A_2'' = \alpha'' A_1'' \end{aligned} \qquad (3.2.1\text{-}29)$$

By rearranging terms, we get

$$\begin{aligned} \frac{1}{(\alpha' - \alpha'')} &= \frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)} \\ \frac{\alpha'}{(\alpha' - \alpha'')} &= \frac{\left(p_{L2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)}; \quad \frac{\alpha''}{(\alpha' - \alpha'')} = \frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)} \end{aligned} \qquad (3.2.1\text{-}30)$$

Substituting $A_2' = \alpha' A_1'$ & $A_2'' = \alpha'' A_1''$ in to equation (3.2.1-27), the equation becomes:

$$y_1 = A_1' \sin(p_1 t + \phi') + A_1'' \sin(p_2 t + \phi'')$$
$$y_2 = \alpha' A_1' \sin(p_1 t + \phi') + \alpha'' A_1'' \sin(p_2 t + \phi'') \qquad (3.2.1\text{-}31)$$

The values of constants A_1', A_2', A_1'', A_2'' are determined from initial conditions.

The initial conditions are:

$$\text{At} \quad t = 0 \rightarrow \quad y_1(t) = 0; \quad \dot{y}_1(t) = 0; \quad y_2(t) = 0; \quad \dot{y}_2(t) = v_2 \qquad (3.2.1\text{-}32)$$

Differentiating equation 3.2.1-31, we get

$$\dot{y}_1 = p_1 A_1' \cos(p_1 t + \phi') + p_2 A_1'' \cos(p_2 t + \phi'')$$
$$\dot{y}_2 = p_1 A_1' \alpha' \cos(p_1 t + \phi') + p_2 A_1'' \alpha'' \cos(p_2 t + \phi'') \qquad (3.2.1\text{-}33)$$

Substituting equation 3.2.1-32 into equations 3.2.1-31 & 33, we get:

$$0 = A_1' \sin(\phi') + A_1'' \sin(\phi'') \qquad \text{(i)}$$
$$0 = \alpha' A_1' \sin(\phi') + \alpha'' A_1'' \sin(\phi'') \qquad \text{(ii)}$$
$$0 = p_1 A_1' \cos(\phi') + p_2 A_1'' \cos(\phi'') \qquad \text{(iii)}$$
$$v_2 = \alpha' p_1 A_1' \cos(\phi') + \alpha'' p_2 A_1'' \cos(\phi'') \qquad \text{(iv)}$$

Solving equations (i) & (ii) gives

$$\sin \phi' (\alpha' A_1' - \alpha'' A_1') = 0$$

Since $A_1', A_1'', \alpha' \,\&\, \alpha''$ are non-zero quantities, it yields that

$$\phi' = 0 \;\&\; \phi'' = 0;$$
$$\sin \phi' = \sin \phi'' = 0 \quad \& \quad \cos \phi' = \cos \phi'' = 1$$

Equations (iii) & (iv) thus become

$$0 = p_1 A_1' + p_2 A_1'' \qquad \text{(v)}$$
$$v_2 = \alpha' p_1 A_1' + \alpha'' p_2 A_1'' \qquad \text{(vi)}$$

Solving (v) & (vi), we get,

$$A_1' p_1(\alpha' - \alpha'') = v_2; \qquad A_1' = \frac{1}{(\alpha' - \alpha'')} \frac{v_2}{p_1} \tag{3.2.1-34}$$

Substituting for $(\alpha' - \alpha'')$ from 3.2.1-30, we get

$$A_1' = \frac{v_2}{p_1} \frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)} \tag{3.2.1-35}$$

Since $A_2' = \alpha' A_1'$ & $\alpha' = \dfrac{p_{L2}^2}{\left(p_{L2}^2 - p_1^2\right)}$

Substituting for $\alpha' A_1'$ we get

$$A_2' = \frac{v_2}{p_1} \times \frac{\left(p_{L2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)} \tag{3.2.1-36}$$

Equation (v) gives $A_1'' = -\dfrac{p_1}{p_2} A_1'$

$$A_1'' = -\frac{v_2}{p_2} \frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)} \tag{3.2.1-37}$$

Since $A_2'' = \alpha'' A_1''$ & $\alpha'' = \dfrac{p_{L2}^2}{\left(p_{L2}^2 - p_2^2\right)}$ we get

$$A_2'' = \alpha'' A_1'' = -\frac{v_2}{p_2} \frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)} \tag{3.2.1-38}$$

Rewriting solution y_1 from equation 3.2.1-27

$$y_1 = A_1' \sin(p_1 t) + A_1'' \sin(p_2 t)$$

Substituting (3.2.1-35& 37), we get:

$$y_1 = v_2 \frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)} \left\{ \frac{\sin(p_1 t)}{p_1} - \frac{\sin(p_2 t)}{p_2} \right\} \tag{3.2.1-39}$$

Rewriting solution y_2 from equation 3.2.1-27

$$y_2 = A_2' \sin(p_1 t) + A_2'' \sin(p_2 t)$$

Substituting (3.2.1-36& 38), we get:

$$y_2 = v_2 \times \frac{\left(p_{L2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)} \frac{\sin(p_1 t)}{p_1} - v_2 \times \frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)} \frac{\sin(p_2 t)}{p_2} \qquad (3.2.1\text{-}40)$$

Coefficients of $\sin p_1 t$ represent amplitudes in 1st mode of vibration and that of $\sin p_2 t$ represent amplitudes in 2nd mode of vibration.

This approach shall be useful for Design of Foundation for Impact Machines covered in Chapter 11.

3.2.3 A Rigid Block supported by Translational & Rotational Springs

3.2.3.1 A Rigid Block supported by Translational & Rotational Springs in X-Y Plane (Y being vertical axis)

Consider the motion in X-Y plane having two DOF i.e. translation along X and rotation ϕ about z-axis. Since the system has only two DOF i.e. $x \,\&\, \phi$, only two types of forces could be applied, one along x and the other along ϕ. It is to be noted that these forces could be applied at any point but these have to be transferred to the DOF location as equation of motion is at DOF location. Let us consider these dynamic forces one by one.

3.2.3.1.1 Dynamic Force along X-axis applied at DOF location

Consider the rigid block (supported by Translational and Rotational spring as in § **3.1.4.1**) subjected to dynamic forces $F_x \sin\omega t$ in x direction acting at point O as shown in Figure 3.2.2-1.

Two springs, **one translational & one rotational,** are connected to the block at base center point O. The block has its centroid at C & height of centroid C above base center O is h. The block has mass m and mass moment of Inertia M_{mz} about Z-axis passing through block centroid C. The block is constrained to move only in lateral X direction and rotate about Z-axis passing through O.

Degrees of freedom: DOF 1 – Translation along X-axis at point O ; and

DOF 2 – Rotation about Z-axis passing through O.

Dynamic Force: Force $F_x \sin\omega t$ acting at point O as shown.

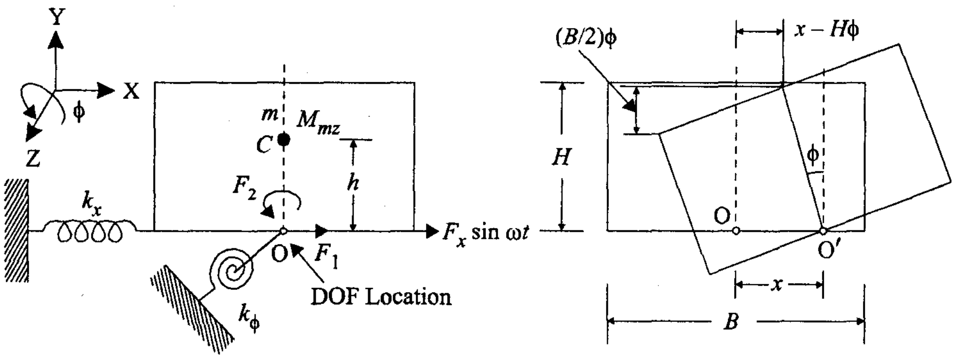
Transfer the dynamic forces to DOF Locations: Let the equivalent dynamic forces at O be F_1 along DOF 1 & F_2 along DOF 2.

Since the applied dynamic force is applied at DOF location, it does not need to be transferred. Considering equilibrium of forces, by statics, we get

$F_1 = F_x \sin \omega t$ & $F_2 = 0$

Hence Dynamic Force at DOF location:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ 0 \end{Bmatrix} \quad (3.2.2\text{-}1)$$



(a) Dynamic Forces @ DOF (b) Amplitudes @ DOF & @ Foundation Top

Figure 3.2.2-1 Block with Centroid C - y - Displacement at O Restrained
Dynamic Force $F_x \sin \omega t$ applied at Point O

Equation of Motion:

Rewriting equation (3.1.4-4) for free vibration:

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.2.2\text{-}2)$$

Substituting equation (3.2.2-1) on RHS for forcing function, the equation of motion for forced vibration becomes:

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ 0 \end{Bmatrix} \quad (3.2.2\text{-}3)$$

Solution to this equation of motion has two parts viz.

i) Complimentary Solution
ii) Particular solution

Complimentary solution: **(See solution 3.1.4-4)**

Natural frequencies are given as (see equations 3.1.4-5 & 6):

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$
$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2} \qquad (3.2.2\text{-}4)$$

Here $\gamma_z = \dfrac{M_{mz}}{M_{moz}}; \; p_x^2 = \dfrac{k_x}{m}; \quad p_\phi^2 = \dfrac{k_\phi}{M_{moz}}$

Particular solution:

Solving the equation **(see solution 3.2.2-3)**, we get the steady state response equations (*g*) & *(h)*.

The steady state response at point O is given as

$$x = \{\delta_x\}\frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t$$

$$\phi = -\delta_x \frac{mh}{M_{moz}}\frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t$$

Here $\delta_x = \dfrac{F_x}{k_x}$ represents static deflection & x & ϕ are the amplitudes at DOF Locations i.e. point O.

For maximum response @ point O, substituting $\sin\omega t = 1$, we get

$$x_o = \{\delta_x\}\frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.2\text{-}5)$$

$$\phi_o = -\delta_x \frac{mh}{M_{moz}}\frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.2\text{-}6)$$

Here x_o & ϕ_o represent maximum response @ O along X & about Z axes respectively

SOLUTION 3.2.2-3

Rewriting the equation (3.2.2-3)

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin\omega t \\ 0 \end{Bmatrix} \tag{a}$$

Let solution be of the form

$$x = C\sin\omega t; \quad \ddot{x} = -\omega^2 C\sin\omega t \tag{b}$$

$$\phi = D\sin\omega t; \quad \ddot{\phi} = -\omega^2 D\sin\omega t \tag{c}$$

Substituting in (a) it gives

$$\begin{bmatrix} k_x - m\omega^2 & mh\omega^2 \\ mh\omega^2 & k_\phi - \omega^2 M_{moz} \end{bmatrix}\begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} F_x \\ 0 \end{Bmatrix} \tag{d}$$

Using Crammer's rule we get

$$C = \frac{\begin{vmatrix} F_x & mh\omega^2 \\ 0 & k_\phi - \omega^2 M_{moz} \end{vmatrix}}{mM_{mz}\Delta(\omega^4)} = \frac{F_x\left(k_\phi - \omega^2 M_{moz}\right)}{mM_{mz}p_1^2 p_2^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} = \frac{F_x}{k_x}\frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \tag{e}$$

$$D = \frac{\begin{vmatrix} k_x - m\omega^2 & F_x \\ mh\omega^2 & 0 \end{vmatrix}}{mM_{mz}\Delta(\omega^4)} = -\frac{F_x mh\omega^2}{k_x k_\phi\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} = -\frac{F_x}{k_x}\frac{mh}{M_{moz}}\frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \tag{f}$$

Substituting in (b) & (c), we get response amplitude as:

$$x = \{\delta_x\}\left\{\frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right\}\sin\omega t \tag{g}$$

$$\phi = \left\{-\delta_x\frac{mh}{M_{moz}}\right\}\left\{\frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right\}\sin\omega t \tag{h}$$

Here $\delta_x = \dfrac{F_x}{k_x}$

3.2.3.1.2 Dynamic Moment about Z–axis applied at DOF location

Consider that the block is subjected to dynamic moment $M_\phi \sin \omega t$ along ϕ direction acting at point O as shown in Figure 3.2.2-2.

$$F_1 = 0 \quad \& \quad F_2 = M_\phi \sin \omega t$$

Hence Dynamic Force at DOF location:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\phi \sin \omega t \end{Bmatrix} \tag{3.2.2-7}$$

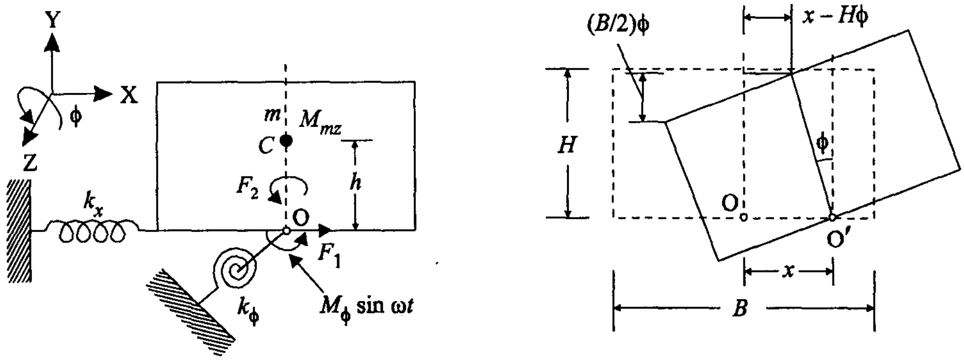
Equation of Motion: The equation of motion for forced vibration thus becomes:

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\phi \sin \omega t \end{Bmatrix} \tag{3.2.2-8}$$

Solving the equation **(see equations *(e)* & *(f)* solution 3.2.2-8)**, we get the steady state response as:

$$x = -h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \sin \omega t; \quad \phi = \delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \sin \omega t$$

Here $\delta_\phi = \dfrac{M_\phi}{k_\phi}$



(a) Dynamic Moment @ DOF (b) Amplitude @ DOF & @ Foundation Top

Figure 3.2.2-2 Block having Centroid $C-y$ Displacement at O Restrained Dynamic Moment M_ϕ sin ωt applied at point O

Maximum response @ point O is written as:

$$x_o = -h\delta_\phi \frac{\beta_x^2}{(1-\beta_1^2)(1-\beta_2^2)} \tag{3.2.2-9}$$

$$\phi_o = \delta_\phi \frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)} \tag{3.2.2-10}$$

Here x_o & ϕ_o represent amplitudes at DOF Locations point O along X & about Z respectively.

SOLUTION 3.2.2-8

Rewriting the equation (3.2.2-8)

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & k_\phi \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\phi \sin \omega t \end{Bmatrix} \tag{a}$$

Let solution be of the form

$$x = C \sin \omega t; \quad \ddot{x} = -\omega^2 C \sin \omega t \tag{b}$$

$$\phi = D \sin \omega t; \quad \ddot{\phi} = -\omega^2 D \sin \omega t \tag{c}$$

Substituting in (a) it gives

$$\begin{bmatrix} k_x - m\omega^2 & mh\omega^2 \\ mh\omega^2 & k_\phi - \omega^2 M_{moz} \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\phi \end{Bmatrix} \tag{d}$$

Using Crammer's rule, we get

$$C = \frac{\begin{vmatrix} 0 & mh\omega^2 \\ M_\phi & k_\phi - \omega^2 M_{moz} \end{vmatrix}}{mM_{mz}\Delta(\omega^4)} = -\frac{M_\phi mh\omega^2}{mM_{mz} p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)} = -\frac{M_\phi}{k_\phi} \frac{h\beta_x^2}{(1-\beta_1^2)(1-\beta_2^2)}$$

$$x = -h\delta_\phi \frac{\beta_x^2}{(1-\beta_1^2)(1-\beta_2^2)} \sin \omega t \tag{e}$$

$$D = \frac{\begin{vmatrix} k_x - m\omega^2 & 0 \\ mh\omega^2 & M_\phi \end{vmatrix}}{mM_{mz}\Delta(\omega^4)} = \frac{M_\phi(k_x - m\omega^2)}{k_x k_\phi (1-\beta_1^2)(1-\beta_2^2)} = \frac{M_\phi}{k_\phi} \frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)}$$

$$\phi = \delta_\phi \frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)} \sin \omega t \qquad (f)$$

Here $\delta_\phi = \frac{M_\phi}{k_\phi}$

3.2.3.1.3 Dynamic Force acting at a point above the block along X - Direction

Now consider that the block is subjected to dynamic forces $F_x \sin \omega t$ in x direction acting at point T at a distance s above base of the block as shown in Figure 3.2.2-3.

Transfer the dynamic forces to DOF Locations:

Let the equivalent dynamic forces at O be F_1 along DOF 1 & F_2 along DOF 2. Considering equilibrium of forces, by statics, we get

$$F_1 = F_x \sin \omega t \qquad \& \qquad F_2 = -s\, F_x \sin \omega t$$

(Anticlockwise moment in X-Y plane is positive –for notations see Figure 3-1)

Hence Dynamic Force at DOF location:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ -s\, F_x \sin \omega t \end{Bmatrix} \qquad (3.2.2\text{-}11)$$

It is noted that the dynamic force F_x acting at a distance s above the DOF location point O results in a dynamic force equal to F_x and a dynamic moment equal to $M_\phi = -F_x \times s$ applied at O.

Equation of Motion: Equation of motion for forced vibration becomes:

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ M_\phi \sin \omega t \end{Bmatrix} \qquad (3.2.2\text{-}12)$$

Where $M_\phi = -F_x \times s$

Forced Vibration Response to this equation is nothing but summation of responses given by equations (3.2.2-5), (3.2.2-6), (3.2.2-9) & (3.2.2-10).

Response x in X-direction thus becomes summation of x-response given by equation (3.2.2-5) & equation (3.2.2-9) and so is the case with ϕ-response which is summation of equation (3.2.2-6) & equation (3.2.2-10). Maximum response @ point O is thus written as:

$$x_o = \left\{ \{\delta_x\} \frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \right\} \tag{3.2.2-13}$$

$$\phi_o = \delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - \delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \tag{3.2.2-14}$$

Here $\delta_x = \dfrac{F_x}{k_x}$, $\delta_\phi = \dfrac{(M_\phi)}{k_\phi}$ and $x_o \& \phi_o$ represent amplitudes at DOF Location point O along X & about Z respectively.

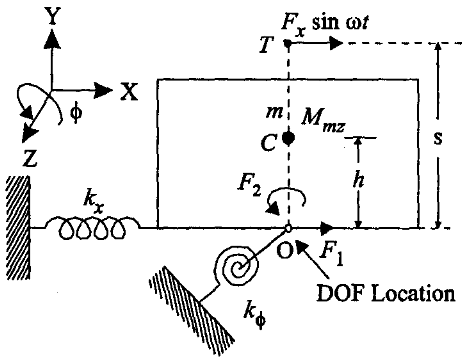


Figure 3.2.2-3 Block with Centroid C - y - Displacement at O Restrained
Dynamic Force F_x sin ωt applied at Point T

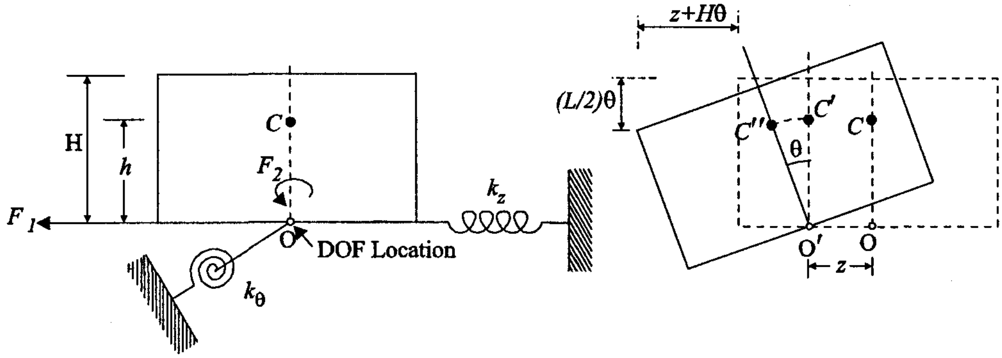
3.2.3.2 A Rigid Block supported by Translational & Rotational Springs in Y-Z Plane (Y being vertical axis)

Consider the motion in Y-Z plane (Y being vertical axis) having two DOF i.e. translation along Z and rotation θ about X-axis. Since the system has only two DOF i.e. $z \& \theta$, only two types of forces could be applied, one along z and the other along θ. It is to be noted that these forces could be applied at any point but these have to be transferred to the DOF location as equation of motion is at DOF location.

Let us consider these dynamic forces one by one.

3.2.3.2.1 Dynamic Force along Z-axis applied at DOF location

Consider the rigid block subjected to dynamic forces $F_z \sin \omega t$ in z direction acting at point 'O' as shown in Figure 3.2.2-4.



(a) Block with Centroid C, y - Displacement at O restrained. Lateral Spring Stiffness k_z & Rotational Spring Stiffness k_θ

(b) Displaced Position

Figure 3.2.2-4 A Rigid Block Supported by Translational & Rotational Springs in Y-Z Plane Dynamic Force F_1 &F_2 applied at DOF Location Point O

Two springs, **one translational & one rotational,** are connected to the block at base center point O. The block has its centroid at C & height of centroid C above base center O is h. The block has mass m and mass moment of Inertia M_{mx} about X-axis passing through block centroid C. The block is constrained to move only in lateral Z direction and rotate about X-axis passing through O.

Degrees of freedom: DOF 1 - Translation along Z-axis at point O & DOF 2 - Rotation about X-axis passing through O.

Dynamic Force: Consider only Force $F_z \sin \omega t$ acting at point O as shown.

Transfer the dynamic forces to DOF Locations: Let the equivalent dynamic forces at O be F_1 along DOF 1 & F_2 along DOF 2.

Since the applied dynamic force is applied at DOF location, it does not need to be transferred. Considering equilibrium of forces, by statics, we get

$$F_1 = F_z \sin \omega t \quad \& \quad F_2 = 0$$

Hence Dynamic Force at DOF location:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} F_z \sin \omega t \\ 0 \end{Bmatrix} \tag{3.2.2-15}$$

Equation of Motion:

Rewriting equation (3.1.4-4) for free vibration:

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.2.2-16}$$

Substituting equation (3.2.2-1) on RHS for forcing function, the equation of motion for forced vibration becomes:

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} F_z \sin \omega t \\ 0 \end{Bmatrix} \tag{3.2.2-17}$$

Solution to this equation of motion has two parts viz.

i) Complimentary Solution
ii) Particular solution.

Complimentary solution: (See solution 3.1.4-4)

Natural frequencies are given as (see equations 3.1.4-5 & 6):

$$p_1^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) - \frac{1}{2\gamma_x}\sqrt{\left\{p_z^2 + p_\theta^2\right\}^2 - 4\gamma_x\, p_z^2 p_\theta^2}$$

$$p_2^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) + \frac{1}{2\gamma_x}\sqrt{\left\{p_z^2 + p_\theta^2\right\}^2 - 4\gamma_x\, p_z^2 p_\theta^2} \tag{3.2.2-18}$$

Here Here $\gamma_x = \dfrac{M_{mx}}{M_{mox}}$; $p_z^2 = \dfrac{k_z}{m}$; $p_\theta^2 = \dfrac{k_\theta}{M_{mox}}$

Particular solution:

Solving the equation **3.2.2-17 (see solution 3.2.2-17)**, we get the steady state response. The steady state response at point O is given as

$$z_o = \{\delta_z\}\left\{\frac{\left(1-\beta_\theta^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right\}\sin\omega t; \quad \theta_o = \left\{\delta_z \frac{mh}{M_{mox}}\right\}\left\{\frac{\beta_\theta^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right\}\sin\omega t$$

$$Here\ \ \delta_z = \frac{F_z}{k_z}$$

These (z & θ) are the amplitudes at DOF Locations i.e. point O.

For maximum response @ point O, substituting $\sin\omega t = 1$, we get

$$z_o = \{\delta_z\}\left\{\frac{\left(1-\beta_\theta^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right\} \quad (3.2.2\text{-}19)$$

$$\theta_o = \left\{\delta_z \frac{mh}{M_{mox}}\right\}\left\{\frac{\beta_\theta^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right\} \quad (3.2.2\text{-}20)$$

Here z_o & θ_o represent maximum response @ O along X & about Z axes respectively

SOLUTION **3.2.2-17**

Rewriting the equation (3.2.2-17)

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix}\begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix}\begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} F_z \sin\omega t \\ 0 \end{Bmatrix} \quad (a)$$

Let solution be of the form

$$z = C\sin\omega t; \quad \ddot{z} = -\omega^2 C\sin\omega t \quad (b)$$

$$\theta = D\sin\omega t; \quad \ddot{\theta} = -\omega^2 D\sin\omega t \quad (c)$$

Substituting in (a) it gives

$$\begin{bmatrix} k_z - m\omega^2 & -mh\omega^2 \\ -mh\omega^2 & k_\theta - \omega^2 M_{mox} \end{bmatrix}\begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} F_z \\ 0 \end{Bmatrix} \quad (d)$$

Using Crammer's rule we get

$$C = \frac{\begin{vmatrix} F_z & -mh\omega^2 \\ 0 & k_\theta - \omega^2 M_{mox} \end{vmatrix}}{mM_{mx}\Delta\left(\omega^4\right)}$$

Since p_1 & p_2 are the roots of the frequency equation, we can represent $\Delta(\omega^4)$ as

$$\Delta(\omega^4) = (\omega^2 - p_1^2)(\omega^2 - p_2^2) = p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)$$

$$C = \frac{F_z(k_\theta - \omega^2 M_{mox})}{mM_{mx}\Delta(\omega^4)} = \frac{F_z(k_\theta - \omega^2 M_{mox})}{mM_{mx} p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)}$$

$$C = \frac{F_z M_{mox}\left(\dfrac{k_\theta}{M_{mox}} - \omega^2\right)}{mM_{mx} p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)} = \frac{F_z M_{mox}(p_\theta^2 - \omega^2)}{mM_{mx} p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)}$$

$$C = \frac{F_z M_{mox} p_\theta^2 (1-\beta_\theta^2)}{mM_{mx} p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)} = \frac{F_z}{m}\frac{M_{mox}}{M_{mx}}\frac{p_\theta^2(1-\beta_\theta^2)}{p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)}$$

$$p_1^2 \times p_2^2 = \frac{1}{\gamma_x} p_\theta^2 p_z^2; \quad \frac{M_{mox}}{M_{mx}} = \frac{1}{\gamma_x}$$

$$C = \frac{F_z}{m}\frac{1}{\gamma_x p_1^2 p_2^2}\frac{p_\theta^2(1-\beta_\theta^2)}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{F_z}{m}\frac{1}{p_\theta^2 p_z^2}\frac{p_\theta^2(1-\beta_\theta^2)}{(1-\beta_1^2)(1-\beta_2^2)}$$

$$C = \frac{F_z}{kz}\frac{(1-\beta_\theta^2)}{(1-\beta_1^2)(1-\beta_2^2)} \qquad (e)$$

Similarly we get

$$D = \frac{\begin{vmatrix} k_z - m\omega^2 & F_z \\ -mh\omega^2 & 0 \end{vmatrix}}{mM_{mx}\Delta(\omega^4)} = -\frac{-F_z mh\omega^2}{k_z k_\theta (1-\beta_1^2)(1-\beta_2^2)} \qquad (f)$$

$$D = \frac{F_z}{k_z}\frac{mh\omega^2}{M_{mox} p_\theta^2 (1-\beta_1^2)(1-\beta_2^2)} = \frac{F_z}{k_z}\frac{mh}{M_{mox}}\frac{\beta_\theta^2}{(1-\beta_1^2)(1-\beta_2^2)}$$

Substituting in (b) & (c), we get response amplitude as:

$$z = \{\delta_z\}\left\{\frac{(1-\beta_\theta^2)}{(1-\beta_1^2)(1-\beta_2^2)}\right\}\sin\omega t \qquad (g)$$

$$\theta = \left\{\delta_z \frac{mh}{M_{mox}}\right\}\left\{\frac{\beta_\theta^2}{(1-\beta_1^2)(1-\beta_2^2)}\right\}\sin\omega t \qquad (h)$$

Here $\delta_z = \dfrac{F_z}{k_z}$

3.2.3.2.2 Dynamic Moment about X –axis applied at DOF location

Consider that the block is subjected to dynamic moment $M_\theta \sin\omega t$ along θ direction acting at point O as shown in Figure 3.2.2-2.

$$F_1 = 0 \quad \& \qquad F_2 = M_\theta \sin\omega t$$

Hence Dynamic Force at DOF location:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\theta \sin\omega t \end{Bmatrix} \tag{3.2.2-21}$$

Equation of Motion: The equation of motion for forced vibration thus becomes:

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\theta \sin\omega t \end{Bmatrix} \tag{3.2.2-22}$$

Solving the equation **(see solution 3.2.2-22)**, we get the steady state response as

$$z = h\delta_\theta \frac{\beta_z^2}{(1-\beta_1^2)(1-\beta_2^2)} \sin\omega t$$

$$\theta = \delta_\theta \frac{(1-\beta_z^2)}{(1-\beta_1^2)(1-\beta_2^2)} \sin\omega t$$

Here $\delta_\theta = \dfrac{M_\theta}{k_\theta}$

Maximum response @ point O is written as:

$$z_o = h\delta_\theta \frac{\beta_z^2}{(1-\beta_1^2)(1-\beta_2^2)} \tag{3.2.2-23}$$

$$\theta_o = \delta_\theta \frac{(1-\beta_z^2)}{(1-\beta_1^2)(1-\beta_2^2)} \tag{3.2.2-24}$$

Here z_o & θ_o represent amplitudes at DOF Locations point O along Z & about X respectively.

***SOLUTION* 3.2.2-22**

Rewriting the equation (3.2.2-22)

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\theta \sin \omega t \end{Bmatrix} \qquad (a)$$

Let solution be of the form

$$z = C \sin \omega t; \quad \ddot{z} = -\omega^2 C \sin \omega t \qquad (b)$$

$$\theta = D \sin \omega t; \quad \ddot{\theta} = -\omega^2 D \sin \omega t \qquad (c)$$

Substituting in (a) it gives

$$\begin{bmatrix} k_z - m\omega^2 & -mh\omega^2 \\ -mh\omega^2 & k_\theta - \omega^2 M_{mox} \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_\theta \end{Bmatrix} \qquad (d)$$

Using Crammer's rule, we get

$$C = \frac{\begin{vmatrix} 0 & -mh\omega^2 \\ M_\theta & k_\theta - \omega^2 M_{mox} \end{vmatrix}}{mM_{mx}\Delta(\omega^4)} = -\frac{-M_\theta mh\omega^2}{mM_{mx}p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)}$$

$$C = \frac{M_\theta}{k_\theta} \frac{h\beta_z^2}{(1-\beta_1^2)(1-\beta_2^2)}$$

$$z = h\delta_\theta \frac{\beta_z^2}{(1-\beta_1^2)(1-\beta_2^2)} \sin \omega t \qquad (e)$$

$$D = \frac{\begin{vmatrix} k_z - m\omega^2 & 0 \\ -mh\omega^2 & M_\theta \end{vmatrix}}{mM_{mx}\Delta(\omega^4)} = \frac{1}{mM_{mx}} \frac{M_\theta (k_z - m\omega^2)}{p_1^2 p_2^2 (1-\beta_1^2)(1-\beta_2^2)}$$

$$p_1^2 p_2^2 = \frac{1}{\gamma_x} p_z^2 p_\theta^2; \qquad \frac{M_{mx}}{M_{mox}} = \gamma_x; \quad M_{mx} p_1^2 p_2^2 = M_{mox} p_z^2 p_\theta^2$$

$$D = \frac{1}{m}\frac{M_\theta\left(k_z - m\omega^2\right)}{M_{mox}p_z^2 p_\theta^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} = \frac{1}{m}\frac{mM_\theta\left(p_z^2 - \omega^2\right)}{p_z^2 M_{mox} p_\theta^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

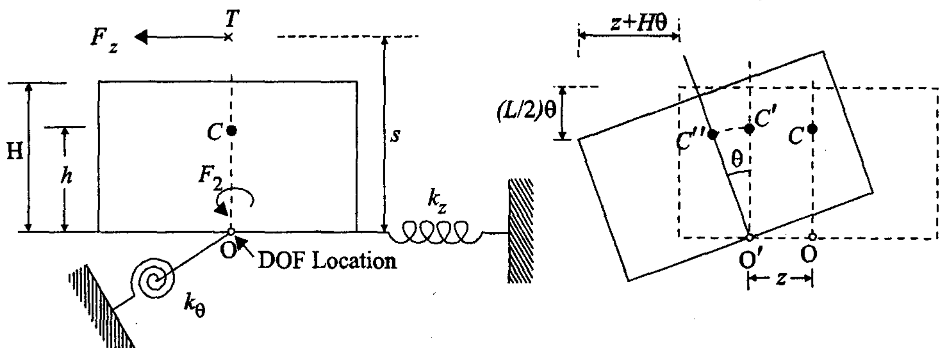
$$D = \frac{1}{mp_z^2}\frac{mM_\theta}{k_\theta}\frac{p_z^2\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} = \frac{M_\theta}{k_\theta}\frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$\theta = \delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (f)$$

Here $\delta_\theta = \dfrac{M_\theta}{k_\theta}$

3.2.3.2.3 Dynamic Force acting at a point above the block along Z - Direction

Now consider that the block is subjected to dynamic forces $F_z \sin\omega t$ in z direction acting at point T at a distance s above base of the block as shown in Figure 3.2.2-5.



(a) Block with Centroid C, y - Displacement at O restrained. Lateral Spring Stiffness k_z & Rotational Spring Stiffness k_θ

(b) Displaced Position

Figure 3.2.2-5 A Rigid Block Supported by Translational & Rotational Springs in Y-Z Plane Dynamic Force F_z applied at Point T above the block

Transfer the dynamic forces to DOF Locations:

Let the equivalent dynamic forces at O be F_1 along DOF 1 & F_2 along DOF 2.

Considering equilibrium of forces, by statics, we get

$$F_1 = F_z \sin \omega t \qquad \& \qquad F_2 = s\, F_z \sin \omega t$$

Hence Dynamic Force at DOF location:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} F_z \sin \omega t \\ s\, F_z \sin \omega t \end{Bmatrix} \qquad (3.2.2\text{-}25)$$

It is noted that the dynamic force F_z acting at a distance s above the DOF location point O results in a dynamic force equal to F_z and a dynamic moment equal to $M_\theta = F_z \times s$ applied at O.

Equation of Motion: Equation of motion for forced vibration becomes:

$$\begin{bmatrix} m & mh \\ mh & M_{mox} \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_z & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{Bmatrix} F_z \sin \omega t \\ M_\theta \sin \omega t \end{Bmatrix} \qquad (3.2.2\text{-}26)$$

Where $M_\theta = F_z \times s$

Forced Vibration Response to this equation is nothing but summation of responses given by equations (3.2.2-19), (3.2.2-20), (3.2.2-23) & (3.2.2-24).

Maximum response @ point O is thus written as:

$$z_o = \{\delta_z\} \left\{ \frac{\left(1-\beta_\theta^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \right\} + h\delta_\theta \frac{\beta_z^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.2\text{-}27)$$

$$\theta_o = \left\{ \delta_z \frac{mh}{M_{mox}} \right\} \left\{ \frac{\beta_\theta^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \right\} + \delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.2\text{-}28)$$

Here $\delta_z = \dfrac{F_z}{k_z}$, $\delta_\theta = \dfrac{(M_\theta)}{k_\theta}$ and z_o & θ_o represent amplitudes at DOF Location point O along Z & about X respectively.

3.2.3.3 Amplitude at resonance

Response given by the equations 3.2.2-5 & 6, 9 &10, 13 &14 are for undamped conditions and do not hold well at or near to resonance. For evaluating amplitudes in the proximity of resonance, refer to § **3.4.2.1.1.**

3.2.4 Multiple Spring Mass Systems connected by a massless Rigid Bar

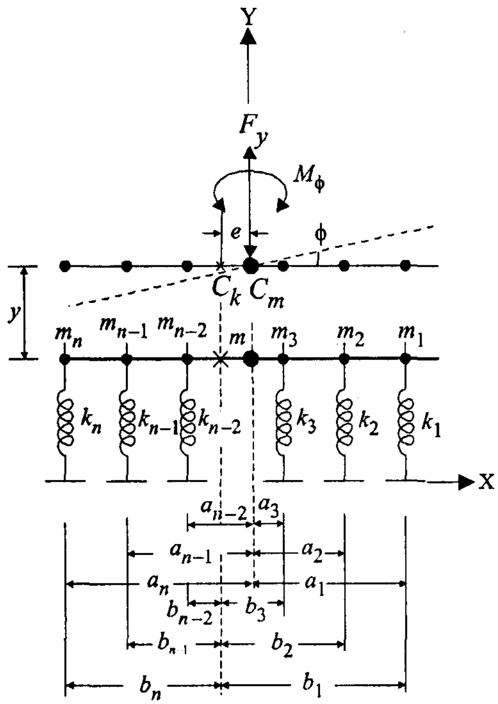


Figure 3.2.3-1 Multiple Spring Mass Systems Connected by massless Rigid Bar Subjected to Dynamic Force & Dynamic Moment applied at Center of Mass

Consider a multi spring mass system (as considered in § 3.1.5) connected by a massless rigid bar subjected to dynamic force $F(t) = F_0 \sin \omega t$ and dynamic moment $M(t) = M_0 \sin \omega t$ applied at center of Mass C_m , as shown in Figure 3.2.3-1.

Rewriting equations of motion for free vibration (equations 3.1.5-8 &9), we get

$$m\ddot{y}+k_y(y+e\phi)=0$$
$$M_m\ddot{\phi}+k_y\,e\,y+k_\phi\,\phi+k_y\,e^2\phi=0 \qquad (3.2.3\text{-}1)$$

Substituting Dynamic Forces on RHS of the equations, we get equations of motion for forced vibration as:

$$m\ddot{y}+k_y(y+e\phi)=F_0\sin\omega t$$
$$M_m\ddot{\phi}+k_y\,e\,y+k_\phi\,\phi+k_y\,e^2\phi=M_0\sin\omega t \qquad (3.2.3\text{-}2)$$

Solving the equation **(see solution 3.2.3-2** equations *(f) & (g)***)**, we get the steady state response as:

$$y=\frac{\delta_{y(static)}\left(1+\dfrac{\beta_\phi^2}{\beta_y^2}\times\dfrac{e^2}{r^2}-\beta_\phi^2\right)-\delta_{\phi(static)}\times e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (3.2.3\text{-}3)$$

$$\phi=\frac{\delta_{\phi(static)}\left(1-\beta_y^2\right)-\delta_{y(static)}\dfrac{\beta_\phi^2}{\beta_y^2}\times\dfrac{e}{r^2}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (3.2.3\text{-}4)$$

For maximum amplitude, $\sin\omega t=1$. We get maximum amplitudes as:

$$y_{(\max)}=\frac{\delta_{y(static)}\left(1+\dfrac{\beta_\phi^2}{\beta_y^2}\dfrac{e^2}{r^2}-\beta_\phi^2\right)-\delta_{\phi(static)}\times e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.3\text{-}5)$$

$$\phi_{(\max)}=\frac{\delta_{\phi(static)}\left(1-\beta_y^2\right)-\delta_{y(static)}\dfrac{\beta_\phi^2}{\beta_y^2}\dfrac{e}{r^2}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.3\text{-}6)$$

Here

$$\delta_{y(static)}=\frac{F_0}{k_y};\ \delta_{\phi(static)}=\frac{M_0}{k_\phi};\quad \beta_1=\frac{\omega}{p_1}\ \&\ \beta_2=\frac{\omega}{p_2};\quad \beta_y=\frac{\omega}{p_y}\ \&\ \beta_\phi=\frac{\omega}{p_\phi};\quad r^2=\frac{M_m}{m}$$

Amplitude at Resonance

For computing response at resonance, let us consider damping coefficient as ζ. In case excitation frequency is in resonance with frequency p_1, replace $\left(1-\beta_1^2\right)$ with $\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}$ in the denominator and in case excitation frequency is in resonance with frequency p_2, replace $\left(1-\beta_2^2\right)$ with $\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}$ in the denominator in the equations 3.2.3-5 & 6.

Solution 3.2.3-2

Rewriting equation 3.2.3-2

$$\begin{aligned} m\ddot{y}+k_y(y+e\phi) &= F_0 \sin\omega t \\ M_m\ddot{\phi}+k_y e y+k_\phi \phi+k_y e^2\phi &= M_0\sin\omega t \end{aligned} \tag{a}$$

Let the solution be of the form $y = C\sin\omega t;\quad \phi = D\sin\omega t$ (b)

Differentiating, we get $\ddot{y}=-\omega^2 C\sin\omega t;\quad \ddot{\phi}=-\omega^2 D\sin\omega t$

Substituting in equation (a), we get

$$\begin{aligned} -\omega^2 mC+k_y(C+eD) &= F_0 \\ -\omega^2 DM_m+k_y eC+k_\phi D+k_y e^2 D &= M_0 \end{aligned}$$

Simplifying and writing in matrix form, we get

$$\begin{bmatrix} k_y-\omega^2 m & k_y e \\ k_y e & k_\phi+k_y e^2-\omega^2 M_m \end{bmatrix}\begin{Bmatrix} C \\ D \end{Bmatrix}=\begin{Bmatrix} F_0 \\ M_0 \end{Bmatrix} \tag{c}$$

Applying Crammer's rule, we get

$$C=\frac{\begin{vmatrix} F_0 & k_y e \\ M_0 & \left(k_\phi+k_y e^2-\omega^2 M_m\right) \end{vmatrix}}{\begin{vmatrix} k_y-\omega^2 m & k_y e \\ k_y e & k_\phi+k_y e^2-\omega^2 M_m \end{vmatrix}}=\frac{\begin{vmatrix} F_0 & k_y e \\ M_0 & \left(k_\phi+k_y e^2-\omega^2 M_m\right) \end{vmatrix}}{mM_m\Delta\left(\omega^4\right)}$$

Simplifying, we get

$$C=\frac{F_0\left(k_\phi+k_y e^2-\omega^2 M_m\right)-M_0 k_y e}{mM_m p_1^2 p_2^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}=\frac{F_0\left(k_\phi+k_y e^2-\omega^2 M_m\right)-M_0 k_y e}{mM_m p_y^2 p_\phi^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$C=\frac{F_0\left(k_\phi+k_y e^2-\omega^2 M_m\right)-M_0 k_y e}{k_y k_\phi\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}=\frac{\frac{F_0}{k_y}\left(1+\frac{k_y}{k_\phi}e^2-\frac{\omega^2 M_m}{k_\phi}\right)-\frac{M_0}{k_\phi}e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$C=\frac{\frac{F_0}{k_y}\left(1+\frac{k_y}{k_\phi}e^2-\beta_\phi^2\right)-\frac{M_0}{k_\phi}e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}=\frac{\frac{F_0}{k_y}\left(1+\frac{mp_y^2}{M_m p_\phi^2}e^2-\beta_\phi^2\right)-\frac{M_0}{k_\phi}e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$C=\frac{\delta_{y(static)}\left(1+\frac{\beta_\phi^2}{\beta_y^2}\frac{e^2}{r^2}-\beta_\phi^2\right)-\delta_{\phi(static)}\times e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (d)$$

$$D=\frac{\begin{vmatrix}k_y-\omega^2 m & F_0\\ k_y e & M_0\end{vmatrix}}{\begin{vmatrix}k_y-\omega^2 m & k_y e\\ k_y e & k_\phi+k_y e^2-\omega^2 M_m\end{vmatrix}}=\frac{\begin{vmatrix}k_y-\omega^2 m & F_0\\ k_y e & M_0\end{vmatrix}}{mM_m\Delta\left(\omega^4\right)}$$

$$D=\frac{M_0\left(k_y-\omega^2 m\right)-F_0 k_y e}{mM_m p_1^2 p_2^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}=\frac{M_0\left(k_y-\omega^2 m\right)-F_0 k_y e}{mM_m p_y^2 p_\phi^2\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$D=\frac{M_0\left(k_y-\omega^2 m\right)-F_0 k_y e}{k_y k_\phi\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}=\frac{\frac{M_0}{k_\phi}\left(1-\frac{\omega^2 m}{k_y}\right)-\frac{F_0}{k_y}\frac{k_y}{k_\phi}e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}=\frac{\frac{M_0}{k_\phi}\left(1-\beta_y^2\right)-\frac{F_0}{k_y}\frac{\beta_\phi^2}{\beta_y^2}\frac{e}{r^2}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$D=\frac{\delta_{\phi(static)}\left(1-\beta_y^2\right)-\delta_{y(static)}\frac{\beta_\phi^2}{\beta_y^2}\frac{e}{r^2}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (e)$$

Substituting in equation (b), we get

$$y=\frac{\delta_{y(static)}\left(1+\frac{\beta_\phi^2}{\beta_y^2}\frac{e^2}{r^2}-\beta_\phi^2\right)-\delta_{\phi(static)}\times e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (f)$$

$$\phi=\frac{\delta_{\phi(static)}\left(1-\beta_y^2\right)-\delta_{y(static)}\frac{\beta_\phi^2}{\beta_y^2}\frac{e}{r^2}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (g)$$

3.2.5 A Portal Frame supporting mass at Beam Center

Consider a portal frame (as considered in § 3.1.6) subjected to dynamic force $F(t) = F_0 \sin \omega t$ applied to the Mass, as shown in Figure 3.2.4-1.

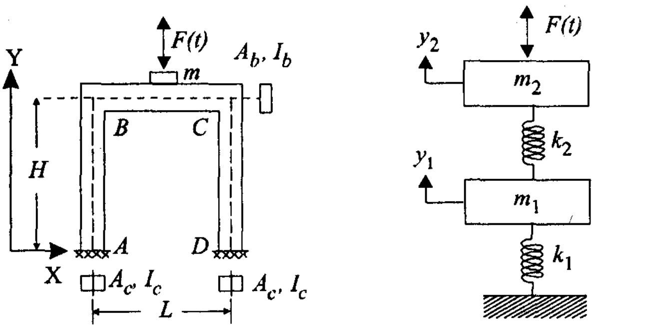


Figure 3.2.4-1 Portal Frame with Machine Mass m at Beam Center - Subjected to Dynamic Force applied at mass location - Vibration in Vertical Mode

Equation of motion:

Equation of motion for the system as shown thus becomes (see equation 3.2.1-4)

$$\begin{bmatrix} (k_1 + k_2 - \omega^2 m_1) & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_0 \end{Bmatrix} \tag{3.2.4-1}$$

For portal frame mass and its stiffness, see equations 3.1.6-2, 3, 4 &5

$$m_2 = m + 0.45\, m_b\,; \quad m_1 = 0.55\, m_b + 2 \times 0.33 \times m_c \tag{3.2.4-2}$$

$$k_1 = \frac{2EA_c}{H}; \quad k_2 = \frac{1}{\delta_{yb}} = \frac{96 E\, I_b}{L^3} \times \frac{k+2}{2k+1} \tag{3.2.4-3}$$

Natural frequencies:

Frequency equation (see equation 3.1.6-6)

$$p_{1,2}^2 = \frac{1}{2}\left\{\left(p_{l,2}^2(1+\lambda) + p_{l,1}^2\right) \mp \sqrt{\left(p_{l,2}^2(1+\lambda) + p_{l,1}^2\right)^2 - 4\left(p_{l,1}^2 p_{l,2}^2\right)}\right\} \tag{3.2.4-4}$$

Here $p_{L1} = \sqrt{\frac{k_1}{m_1}}$; $p_{L2} = \sqrt{\frac{k_2}{m_2}}$; $\lambda = \frac{m_2}{m_1}$

Roots of this equation give two natural frequencies p_1 & p_2.

Steady-State response:

Steady-State response is given as (see equations 3.2.1-5 & 6)

$$y_1 = \frac{F_0}{k_1}\frac{1}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (3.2.4\text{-}5)$$

$$y_2 = \frac{F_0}{k_2}\frac{\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\sin\omega t \qquad (3.2.4\text{-}6)$$

Here $\beta_1 = \frac{\omega}{p_1}$; $\beta_2 = \frac{\omega}{p_2}$; $\beta_{L1} = \frac{\omega}{p_{L1}}$; $\beta_{L2} = \frac{\omega}{p_{L2}}$

Maximum Response:

$$y_{1\,(\max)} = \frac{F_0}{k_1}\frac{1}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.4\text{-}7)$$

$$y_{2\,(\max)} = \frac{F_0}{k_2}\frac{\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (3.2.4\text{-}8)$$

Amplitude at Resonance: (3.2.4-9)

In case of resonance, taking advantage of the derivation done for damped SDOF system, it can be said that in case of resonance with vertical natural frequency p_1, the response to the system at resonance is obtained by replacing the term $\left(1-\beta_1^2\right)$ in denominator by $\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}$ and In case of resonance with vertical natural frequency p_2, the response to the system at resonance is obtained by replacing the term $\left(1-\beta_2^2\right)$ in denominator by $\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}$ in equations 3.2.4-7 & 8.

3.3 THREE DEGREES OF FREEDOM SYSTEM – FREE VIBRATION

From the point of view of application to machine foundation design, development of analysis, in this section, is limited to only i) A Three Spring Mass System Undamped and ii) A Rigid Block supported by Vertical, Translational and Rotational Springs. Again, the spring mass system has been added only for academic purposes.

3.3.1 Three Spring Mass System

Consider a three spring mass system, having masses $m_1, m_2 \& m_3$, spring stiffness $k_1, k_2 \& k_3$ as shown in Figure 3.3.1-1. Three DOF are $y_1, y_2 \& y_3$ associated with masses $m_1, m_2 \& m_3$ respectively. Following steps similar to that for two spring mass system (see 3.1.1), equation of motion is developed. Forces acting on the masses are shown in the free body diagram.

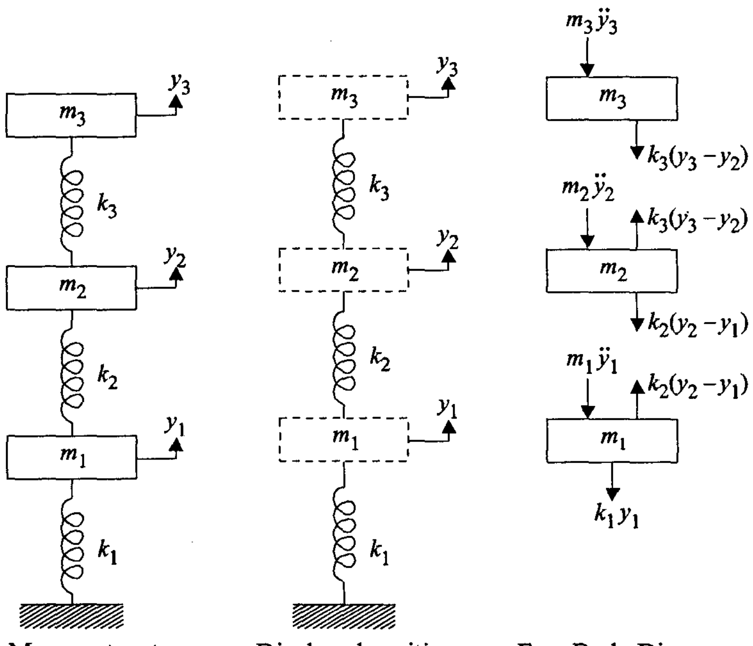


Figure 3.3.1-1 Three Spring Mass System-Undamped

Considering equilibrium of forces (as shown on the free body diagram), we get the equation of motion as:

$$m_1\ddot{y}_1 + k_1y_1 - k_2(y_2 - y_1) = 0$$
$$m_2\ddot{y}_2 + k_2(y_2 - y_1) - k_3(y_3 - y_2) = 0 \qquad (3.3.1\text{-}1)$$
$$m_3\ddot{y}_3 + k_3(y_3 - y_2) = 0$$

Rewriting in Matrix form, equation becomes

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (3.3.1\text{-}2)$$

Here $\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$ represents mass matrix

& $\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$ represents stiffness matrix of the system

It is seen from equation (3.3.1-2) that there is no coupling in mass matrix but stiffness matrix is coupled through off-diagonal terms. Thus equations of motion are coupled.

Frequency equation is written as $\left|k - mp^2\right| = 0$

$$\begin{vmatrix} (k_1 + k_2 - m_1p^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2p^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3p^2) \end{vmatrix} = 0 \qquad (3.3.1\text{-}3)$$

Expanding the determinant, we get

$$p^6 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2} + \frac{k_3}{m_3}\right]p^4 + \left[\frac{(k_1k_2 + k_2k_3 + k_3k_1)}{m_1m_2} + \frac{(k_2k_3)}{m_2m_3} + \frac{(k_1 + k_2)k_3}{m_3m_1}\right]p^2 - \frac{k_1k_2k_3}{m_1m_2m_3} = 0 \qquad (3.3.1\text{-}4)$$

Solution to this equation gives three natural frequencies p_1, p_2 & p_3 corresponding to 1st, 2nd &3rd mode of vibration respectively.

Let the amplitudes of the mass m_1, m_2 & m_3 be represented as A_1, B_1 & C_1 in 1st mode, A_2, B_2 & C_2 in 2nd mode & A_3, B_3 & C_3 in 3rd mode respectively. Following the procedure similar to that developed for two spring mass system, we get equation for the free vibration amplitudes as:

Equations for 1st mode amplitudes:

$$\begin{bmatrix} (k_1 + k_2 - m_1 p_1^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2 p_1^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3 p_1^2) \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \end{Bmatrix} = 0 \qquad (3.3.1\text{-}5)$$

This gives $\dfrac{A_1}{B_1} = \dfrac{k_2}{(k_1 + k_2 - m_1 p_1^2)}$; $\dfrac{B_1}{C_1} = \dfrac{(k_3 - m_3 p_1^2)}{k_3}$

Equations for 2nd mode amplitudes:

$$\begin{bmatrix} (k_1 + k_2 - m_1 p_2^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2 p_2^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3 p_2^2) \end{bmatrix} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \end{Bmatrix} = 0 \qquad (3.3.1\text{-}6)$$

This gives $\dfrac{A_2}{B_2} = \dfrac{k_2}{(k_1 + k_2 - m_1 p_2^2)}$; $\dfrac{B_2}{C_2} = \dfrac{(k_3 - m_3 p_2^2)}{k_3}$

Equations for 3rd mode amplitudes:

$$\begin{bmatrix} (k_1 + k_2 - m_1 p_3^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2 p_3^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3 p_3^2) \end{bmatrix} \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \end{Bmatrix} = 0 \qquad (3.3.1\text{-}7)$$

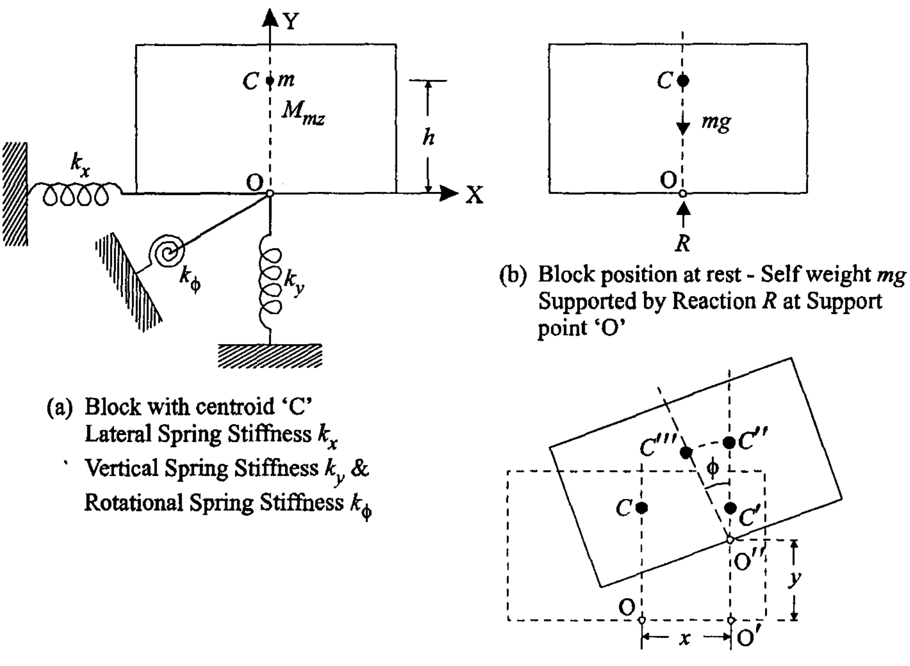
This gives $\dfrac{A_3}{B_3} = \dfrac{k_2}{(k_1 + k_2 - m_1 p_3^2)}$; $\dfrac{B_3}{C_3} = \dfrac{(k_3 - m_3 p_3^2)}{k_3}$

Values of constants $(A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3$ & $C_3)$ are determined based on initial conditions.

Since free vibration response is only transient response, which dies out quickly due to damping present in the system, it is not of much interest from the point of view of machine foundation design. However in specific cases (as we will see later) it may be desirable to compute transient response also.

3.3.2 A Rigid Block supported by Vertical, Translational & Rotational Springs

3.3.2.1 Center of Mass lies vertically above Center of Stiffness



(a) Block with centroid 'C' Lateral Spring Stiffness k_x Vertical Spring Stiffness k_y & Rotational Spring Stiffness k_ϕ

(b) Block position at rest - Self weight mg Supported by Reaction R at Support point 'O'

(c) Displaced position

Figure 3.3.2-1 A Rigid Block Supported by Lateral, Vertical and Rotational Springs

Consider a rigid block supported by Vertical, Translational and Rotational spring. The block considered in X-Y plane is constrained to move only in Vertical Y & Lateral X direction and rotate about Z-axis passing through O.

The block has its centroid at C, and has two translational and one rotational springs i.e. one translational spring in X direction having stiffness k_x, one in vertical Y direction having stiffness

k_y and one rotational spring about Z-axis having stiffness k_ϕ. All the three springs are connected at base center point O. The block has mass m and Mass Moment of Inertia about Z-axis passing through block centroid C is M_{mz}.

The block is considered such that the **centroid** C **lies vertically above base center point** O and the height of centroid C above base center O is h. The block is shown in Figure 3.3.2-1 (a).

Static Equilibrium: The vertical spring k_y supports the self-weight of the block and offers vertical reaction R to counteract the self-weight mg. This position of the block at rest has been shown in part (b) of the Figure 3.3.2-1.

Considering equilibrium at rest position, we get $mg - R = 0$ (3.3.2-1)

Equation of motion: Consider that the block is displaced slightly and released. The block is set into motion i.e. free vibration. At any instant of time t, the block has moved by x along X direction, y along Y direction and rotated by angle ϕ about Z-axis passing through O. The displaced position of the block is shown in part (c) of Figure 3.3.2-1. It is seen that centroid C moves to position C' due to x-translation (see part a), C' moves to C'' due to y-translation & C'' moves to C''' location due to ϕ rotation.

Let us consider these displacements and corresponding reactions one by one. Reaction forces and inertia forces developed are shown in Figure 3.3.2-2.

For better understanding, let us visualize the displaced position in stages viz. x- displacement, y-displacement and rotation about Z-axis and their corresponding reaction and inertia forces.

Part (a) of Figure 3.3.2-2 shows x displacement and the corresponding forces, part (b) shows y displacement and the corresponding forces & part (c) shows ϕ- rotation and the corresponding forces.

Let us consider each movement and the corresponding forces developed. Consider first x translation. Forces developed are (as shown in part (a) of the Figure):

Inertia force along X-axis = $m\ddot{x}$

Spring Reaction force along X-axis = $k_x \times x = x\,k_x$

Now consider y translation. Forces developed are (as shown in part (b) of the Figure):

Inertia force along Y-axis = $m\ddot{y}$

Spring Reaction force along Y-axis = $k_y \times y = y\,k_y$

Now consider ϕ - rotation. Forces developed are shown in part (c) of the figure.

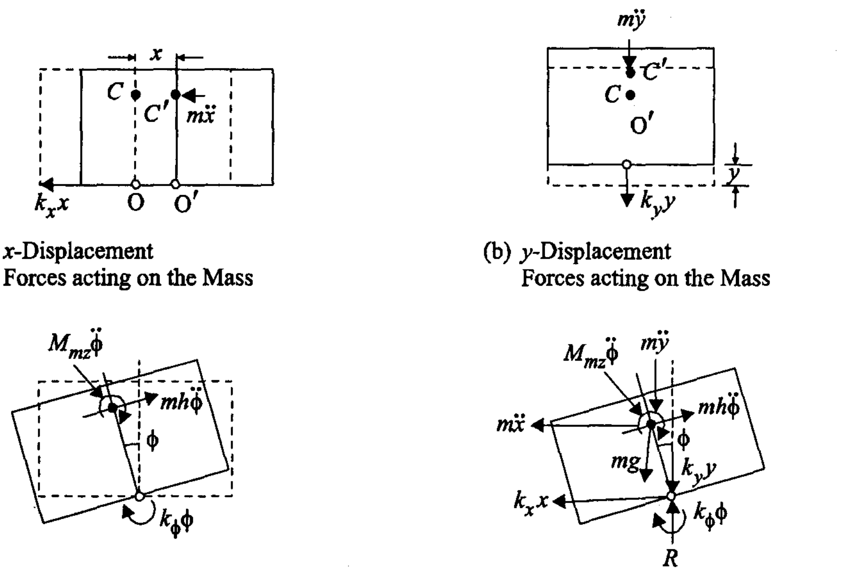
We get,

Rotational Inertia force along ϕ (about Z-axis) = $M_{mz}\ddot{\phi}$

Translational Inertia force (along normal to center line)= $mh\ddot{\phi}$

Spring Reaction force along ϕ (about Z-axis) = $k_\phi \times \phi = \phi\,k_\phi$

This translational Inertia Force has both X & Y components. For ϕ to be small, X-component $mh\ddot{\phi}\cos\phi = mh\ddot{\phi}$ and Y-component $mh\ddot{\phi}\sin\phi = 0$.



(a) x-Displacement
Forces acting on the Mass

(b) y-Displacement
Forces acting on the Mass

(c) ϕ-Rotation
Forces acting on the Mass

(d) Total Forces acting on the Mass

Figure 3.3.2-2 A Rigid Block Supported by Lateral, Vertical and Rotational Springs - Forces Acting on the Mass

Total forces acting on the mass including gravity force and corresponding reactions are shown in part (d) of the figure.

Considering equilibrium at DOF location O, we get

$$\sum F_x = 0 \quad m\ddot{x} - mh\ddot{\phi} + x\,k_x = 0 \tag{3.3.2-2}$$

$$\sum F_y = 0 \quad m\ddot{y} + y\,k_y + mg - R = 0 \tag{3.3.2-3}$$

$$\sum M_z = 0 \quad M_{mz}\ddot{\phi} + (mh\ddot{\phi} \times h) - (m\,\ddot{x}) \times h\cos\phi + (k_\phi \times \phi) - (mgh\sin\phi) = 0 \tag{3.3.2-4}$$

For small ϕ, $h\cos\phi = h$ & $h\sin\phi = h\phi$. Substituting equation 3.3.2-1 in equations 3.3.2-2, 3.3.2-3 & 3.3.2-4, we get

$$m\ddot{x} - mh\ddot{\phi} + k_x x = 0 \tag{3.3.2-5}$$

$$m\ddot{y} + k_y y = 0 \tag{3.3.2-6}$$

$$M_{moz}\,\ddot{\phi} - mh\,\ddot{x} + (k_\phi - mgh)\,\phi = 0 \tag{3.3.2-7}$$

Here $M_{moz} = M_{mz} + mh^2$ represents Mass Moment of Inertia of the block about Z-axis at DOF location Point O.

It is seen from equations (3.3.2-5, 3.3.2-6 & 3.3.2-7) that 2nd equation i.e. equaιion 3.3.2-6 representing motion in Y direction, is totally uncoupled whereas 1st and 3rd equations are coupled.

Writing these equations in matrix form, we get

$$\begin{bmatrix} m & 0 & -mh \\ 0 & m & 0 \\ -mh & 0 & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ y \\ \phi \end{Bmatrix} = 0 \tag{3.3.2-8}$$

It is noticed that the equation 3.3.2-6 is same as equation 3.1.2-3 and **Natural Frequency & Free Vibration Response equations are given by equations** (3.1.2-6) & (3.1.2-7). These equations are reproduced as under:

Natural frequency in Y- direction $p_y = \sqrt{\dfrac{k_y}{m}}$ (3.3.2-8a)

Response is given as $y = A\sin p_y t + B\cos p_y t$ (3.3.2-8b)

The other two equations representing translation in x and rotation ϕ are coupled. Representing these in Matrix form, we get

$$\begin{bmatrix} m & -mh \\ -mh & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & (k_\phi - mgh) \end{bmatrix}\begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

This equation is same as equation (3.1.4-4) and Natural Frequency & Free Vibration Response equations are given by equations 3.1.4-5 to 3.1.4-8 (also see SOLUTION 3.1.4-4). These equations are reproduced as under:

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z p_x^2 p_\phi^2} \quad (3.3.2\text{-}8c)$$

$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z p_x^2 p_\phi^2} \quad (3.3.2\text{-}8d)$$

Here $\gamma_z = \dfrac{M_{mz}}{M_{moz}}$; $p_x^2 = \dfrac{k_x}{m}$; $p_\phi^2 = \dfrac{k_\phi - mgh}{M_{moz}}$ (3.3.2-8e)

Associated mode shapes are given by equations (3.1.4-8) & (3.1.4-9) and are reproduced here for convenience.

$$\frac{A_1}{B_1} = h\frac{-p_1^2}{\left(p_x^2 - p_1^2\right)} \quad (3.3.2\text{-}8f)$$

$$\frac{A_2}{B_2} = h\frac{-p_2^2}{\left(p_x^2 - p_2^2\right)} \quad (3.3.2\text{-}8g)$$

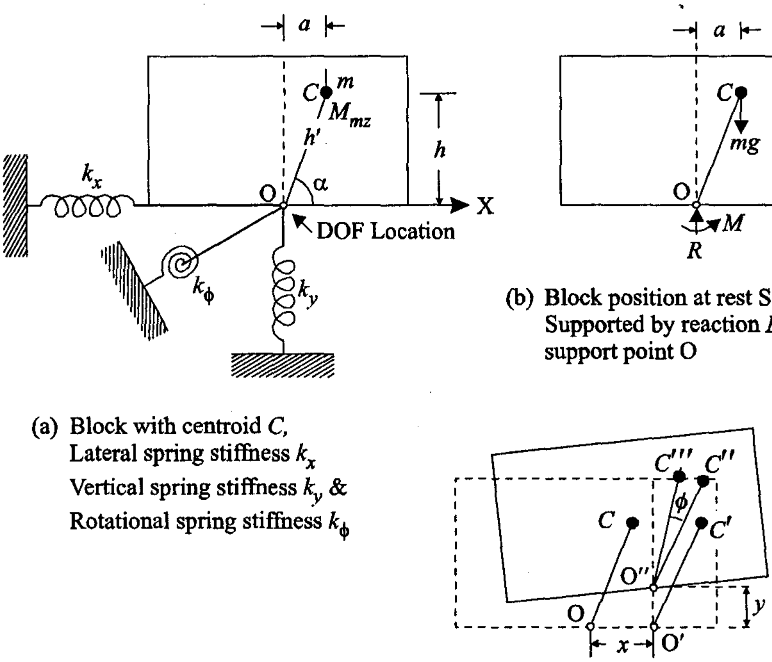
It can be noted that the given 3-DOF system provides:

a) One SDOF System which can be solved independently (solution given in § **2.1.1**)
b) One two degree of freedom system (solution given in § **3.1.4**)

It is thus confirmed that for a rigid block supported by vertical, translational and rotational spring, the vertical mode of vibration is uncoupled from the rest of the two modes of vibration **subject to the condition that there is no eccentricity i.e. the common centroid C lies on the same**

vertical line as base center point O. The system can be analysed as a SDOF system for vertical mode and as a Two Degree of Freedom System for Translational and Rotational Modes (which are coupled).

3.3.2.2 Center of Mass is not in line with Center of Stiffness



(a) Block with centroid C, Lateral spring stiffness k_x Vertical spring stiffness k_y & Rotational spring stiffness k_ϕ

(b) Block position at rest Self weight mg Supported by reaction R at support point O

(c) Displaced position

Figure 3.3.2-3 A Rigid Block Supported by Translational, Vertical and Rotation Springs - Center of Mass eccentric to Center of Stiffness

Now consider the same block as of Figure 3.3.2-1 **but having eccentricity** a between Center of Mass C and Center of Stiffness O. The system is as shown in part (a) of Figure 3.3.2-3.

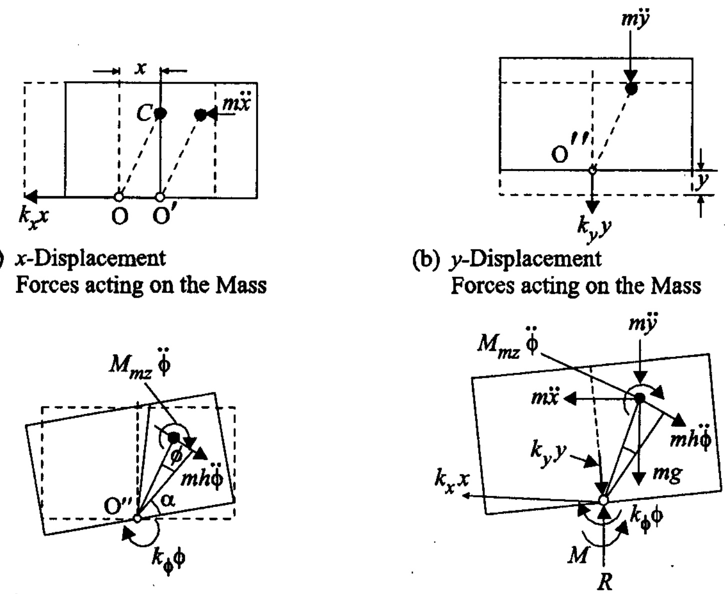
Let the distance of centroid C from O be h' and OC make an angle α with X-axis such that:

$$a = h'\cos\alpha \quad \& \quad h = h'\sin\alpha \tag{3.3.2-9}$$

Static Equilibrium: Since center of mass is eccentric to center of stiffness, the block exerts vertical force and a moment at the static equilibrium position. The vertical spring k_y offers vertical reaction R to counteract the self-weight mg, whereas rotational spring k_ϕ offers rotational moment reaction M to counteract the moment caused by eccentric location of mass mg. This position at rest is shown in part (b) of Figure 3.3.2-3.

Considering equilibrium at rest position, we get

$$\begin{aligned} mg - R &= 0 \\ mg \times a - M &= 0 \end{aligned} \qquad (3.3.2\text{-}10)$$



(a) x-Displacement Forces acting on the Mass

(b) y-Displacement Forces acting on the Mass

(c) ϕ-Rotation Forces acting on the Mass

(d) Total Forces acting on the Mass

Figure 3.3.2-4 A Rigid Block Supported by Translational, Vertical and Rotation Springs - Center of Mass eccentric to Center of Stiffness - Forces acting on the Mass

Equation of motion: Consider that the block is displaced slightly and released. The block is set into motion i.e. free vibration. At any instant of time t, the block has moved by x along X direction, y along Y direction and rotated by angle ϕ about Z-axis passing through 'O'. The displaced position of the block is shown in part (c) of Figure 3.3.2-3. It is seen that centroid C moves to position C' due to x-translation, C' moves to C'' due to y-translation & C'' moves to C''' location due to ϕ rotation.

Reaction forces and inertia forces corresponding to x- displacement, y-displacement and rotation ϕ about Z-axis are shown in part (a), (b) & (c) of Figure 3.3.2-4 respectively. Part (d) shows overall reaction & inertia forces acting on the mass.

Considering equilibrium at DOF location i.e. $\sum F_x = 0\,;\sum F_y = 0 \,\&\, \sum M_z = 0$, we get

$$\sum F_x = 0 \qquad m\ddot{x} + k_x x - mh'\ddot{\phi}\sin\alpha = 0 \qquad (3.3.2\text{-}11)$$

$$\sum F_y = 0 \qquad m\ddot{y} + k_y y + mg - R + mh'\ddot{\phi}\cos\alpha = 0 \qquad (3.3.2\text{-}12)$$

$$\sum M_z = 0$$

$$M_{mz}\ddot{\phi} + mh'\ddot{\phi}\times h' + k_\phi \phi - M + mg\times(a - h'\phi\sin\alpha) - m\ddot{x}\times h + m\ddot{y}\,a = 0 \qquad (3.3.2\text{-}13)$$

Substituting equation 3.3.2-9 & 3.3.2-10 in equations 3.3.2-11, 3.3.2-12 & 3.3.2-13, we get

$$m\ddot{x} - mh\ddot{\phi} + k_x x = 0 \qquad (3.3.2\text{-}14)$$

$$m\ddot{y} + ma\ddot{\phi} + k_y y = 0 \qquad (3.3.2\text{-}15)$$

$$M_{moz}\ddot{\phi} - mh\ddot{x} + ma\ddot{y} + (k_\phi - mg\,h)\phi = 0 \qquad (3.3.2\text{-}16)$$

Here $M_{moz} = M_{mz} + m(h')^2$ is Mass Moment of Inertia of the block about Z-axis at DOF location point O.

Writing in matrix form, the equation of motion becomes:

$$\begin{bmatrix} m & 0 & -mh \\ 0 & m & ma \\ -mh & ma & M_{moz} \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & (k_\phi - mgh) \end{bmatrix}\begin{Bmatrix} x \\ y \\ \phi \end{Bmatrix} = 0 \qquad (3.3.2\text{-}17)$$

It is seen from equation (3.3.2-17) that all the three motions are coupled. It is this aspect that dictates the limits on eccentricity. **It is therefore desirable that the eccentricity (distance between center of mass and center of stiffness) should as far as possible, be close to zero otherwise the Vertical motion shall also get coupled with Translational and Rotational motion.**

For $a = 0$, the equation (3.3.2-17) reduces to same equation as equation (3.3.2-8).

Free Vibration Response: It is relatively difficult to write closed form solution for equation (3.3.2-17). Contrary to earlier cases, the solution to this equation would be relatively complex. Use of computer is therefore recommended for computation of natural frequencies and response.

3.4 THREE DOF SYSTEM – FORCED VIBRATION

In this section we consider only undamped systems. Here also the analysis is limited to i) A Three Spring Mass Undamped System and ii) A Rigid Block supported by Vertical, Translational and Rotational Springs. Again, the spring mass system has been added only for academic purposes.

3.4.1 Three Spring Mass System subjected to Harmonic Excitation

Consider the system as shown in Figure 3.3.3-1. Consider that a dynamic force $F_3 \sin \omega t$ is applied to mass m_3 only.

Under the influence of forcing function, equation (3.3.3-1) gets modified. Equation of motion thus becomes:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3 \end{Bmatrix} \qquad (3.4.1\text{-}1)$$

Solution to this equation of motion has two parts viz.

i) Complimentary Solution
ii) Particular solution.

For complimentary solution, RHS of equation (3.4.1-1) is zero. This gives

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For solution to this equation, **see § 3.3.1**. Solution gives natural frequencies given by equation (3.3.1-4) and free vibration response given by equations (3.3.1-5) to (3.3.1-7).

Particular solution:

System will vibrate with forcing frequency ω. Thus we can represent

$$\begin{gathered} y_1 = D_1 \sin \omega t ; \quad y_2 = D_2 \sin \omega t ; \quad y_3 = D_3 \sin \omega t \\ \ddot{y}_1 = -\omega^2 D_1 \sin \omega t ; \quad \ddot{y}_2 = -\omega^2 D_2 \sin \omega t ; \quad \ddot{y}_3 = -\omega^2 D_3 \sin \omega t \end{gathered} \qquad (3.4.1\text{-}2)$$

Substituting and rearranging terms, we get

$$\begin{bmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2\omega^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3\omega^2) \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3 \end{Bmatrix} \tag{3.4.1-3}$$

Using Crammer's rule, we get

$$D_1 = \frac{\begin{vmatrix} 0 & -k_2 & 0 \\ 0 & (k_2 + k_3 - m_2\omega^2) & -k_3 \\ F_3 & -k_3 & (k_3 - m_3\omega^2) \end{vmatrix}}{\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2\omega^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3\omega^2) \end{vmatrix}}$$

$$D_2 = \frac{\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & 0 & 0 \\ -k_2 & 0 & -k_3 \\ 0 & F_3 & (k_3 - m_3\omega^2) \end{vmatrix}}{\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2\omega^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3\omega^2) \end{vmatrix}}$$

$$D_3 = \frac{\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2\omega^2) & 0 \\ 0 & -k_3 & F_3 \end{vmatrix}}{\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2\omega^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3\omega^2) \end{vmatrix}} \tag{3.4.1-4}$$

It may be noted that the denominator in equation 3.4.1-4 could also be represented in terms of natural frequencies $p_1, p_2 \,\&\, p_3$.

$$\begin{vmatrix} (k_1 + k_2 - m_1\omega^2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3 - m_2\omega^2) & -k_3 \\ 0 & -k_3 & (k_3 - m_3\omega^2) \end{vmatrix} \tag{3.4.1-5}$$

$$= m_1 m_2 m_3 (\omega^2 - p_1{}^2)(\omega^2 - p_2{}^2)(\omega^2 - p_3{}^2)$$

Substituting the values of constants D_1, D_2 & D_3 in equation (3.4.1-2), we get amplitudes y_1, y_2 & y_3 whereas constants D_1, D_2 & D_3 represent maximum value of the amplitude.

3.4.2 A Rigid Block supported by Vertical, Translational & Rotational Springs subjected to Harmonic Excitation

3.4.2.1 Center of Mass is in line with Center of Stiffness. Dynamic Forces applied at a point above the block.

Now consider the same block as of Figure 3.3.2-1 (in X-Y Plane) subjected to Dynamic Forces $F_x \sin \omega t$ & $F_y \sin \omega t$ applied at point T as shown in Figure 3.4.2-1.

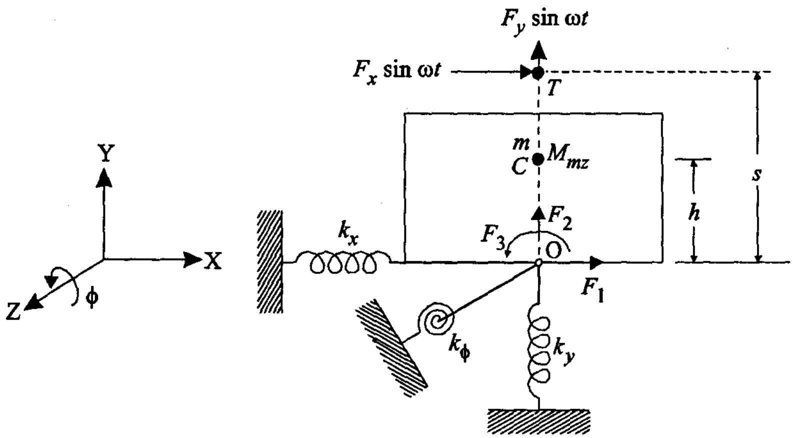


Figure 3.4.2-1 Dynamic Force $F_x \sin \omega t$ & $F_y \sin \omega t$ applied at point T

Transfer the dynamic forces to DOF Locations: Let the equivalent dynamic forces at O be F_1 along DOF 1, F_2 along DOF 2 & F_3 along DOF 3.

Considering equilibrium of forces, by statics, we get

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ F_y \sin \omega t \\ M_\phi \sin \omega t \end{Bmatrix} \qquad \text{where} \quad M_\phi = -s\,F_x \qquad (3.4.2\text{-}1)$$

Adding equation (3.4.2-1) to RHS of equation of motion for free vibration i.e. equation (3.3.2-8), we get equation of motion for forced vibration as:

$$\begin{bmatrix} m & 0 & -mh \\ 0 & m & 0 \\ -mh & 0 & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ y \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ F_y \sin \omega t \\ M_\phi \sin \omega t \end{Bmatrix} \quad (3.4.2\text{-}2)$$

Since vertical motion is uncoupled, separating from the equation, we get

$$m\ddot{y} + k_y y = F_y \sin \omega t \quad (3.4.2\text{-}3)$$

$$\begin{bmatrix} m & 0 \\ 0 & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 \\ 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ M_\phi \sin \omega t \end{Bmatrix} \quad \text{where } M_\phi = \text{-}sF_x \quad (3.4.2\text{-}4)$$

Equation 3.4.2-3 represents vertical motion of the system. It is seen that this equation is same as equation (2.2.1-1). The solution to equation gives response at point O. Maximum response @ O thus becomes (see equation 2.2.1-5a):

$$y_o = \delta_y \frac{1}{\left(1-\beta_y^2\right)}; \qquad \delta_y = \frac{F_y}{k_y} \quad (3.4.2\text{-}5)$$

Equation 3.4.2-4 represents motion of the system in X-Y plane having coupling in $X \,\&\, \phi$. Substituting $(M_\phi = -sF_x)$, it is seen that this equation becomes same as equation (3.2.2-12). Following solution on the similar lines as that for equation (3.2.2-12), we get $x \,\&\, \phi$ response at point O as:

$$x_o = \left[\delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right] \quad (3.4.2\text{-}6)$$

$$\phi_o = \left[\delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - \delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right] \quad (3.4.2\text{-}7)$$

Here $\quad \delta_x = \dfrac{F_x}{k_x} \;\&\; \delta_\phi = \dfrac{M_\phi}{k_\phi}$

3.4.2.1.1 Amplitudes at resonance

For motion in X-Y plane, response amplitudes $x_o, y_o \,\&\, \phi_o$, as given by equations 3.4.2-5,6 &7 represent undamped response that holds good for conditions away from resonance.

When natural frequencies are in proximity to operating speed i.e. conditions of near resonance, the responses given by the above equations do not hold good. For evaluating response at resonance, the equations 3.4.2-5,6 &7 are modified as under:

In case of resonance, taking advantage of the derivation done for damped SDOF system, it can be said that:

i) In case of resonance with vertical natural frequency p_y, the response to the system at resonance is obtained by replacing the term $\left(1-\beta_y^2\right)$ in denominator by $\sqrt{\left(1-\beta_y^2\right)^2+\left(2\beta_y\zeta\right)^2}$ in equation 3.4.2-5.

ii) For coupled motion $x \,\&\, \phi$, in case of resonance with first natural frequency the response at resonance is obtained by replacing the term $\left(1-\beta_1^2\right)$ in denominator by $\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}$ in equations 3.4.2-6 &7, keeping the sign of the term $\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}$ same as that for $\left(1-\beta_1^2\right)$.

iii) In case of resonance with second natural frequency for coupled motion $x \,\&\, \phi$, the response at resonance is obtained by replacing the term $\left(1-\beta_2^2\right)$ by $\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}$ in equations 3.4.2-6 &7, keeping the sign of the term $\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}$ same as that for $\left(1-\beta_2^2\right)$.

Similar modifications must be made while considering motion in Y-Z plane involving responses $y, z \;\&\; \theta$.

Further, since torsional motion in X-Z plane (about Y) involving response ψ is uncoupled, response at resonance is obtained by equations:

$$\psi_o = \delta_\psi \frac{1}{\left(1-\beta_\psi^2\right)} \quad \text{(undamped response)}$$

$$\psi_o = \delta_\psi \frac{1}{\sqrt{\left(1-\beta_\psi^2\right)^2+\left(2\beta_\psi\zeta\right)^2}} \quad \text{(damped response at resonance)} \qquad (3.4.2\text{-}8)$$

$$\delta_\psi = \frac{M_\psi}{k_\psi}$$

It may be noted that responses x_o, y_o & ϕ_o (whether undamped or damped) represent amplitudes at DOF Locations point O in X Y & ϕ directions respectively. For response at any other location viz. top of the foundation, bearing location etc, computations need to be modified accordingly.

3.4.2.2 Center of Mass is not in line with Center of Stiffness - Dynamic Forces applied at a point above the block

Now consider the same block as of Figure 3.3.2-2 (in X-Y Plane). Center of stiffness is point O and center of Mass point C is at distance 'a' from point O. The block is subjected to Dynamic Forces $F_x \sin\omega t$ & $F_y \sin\omega t$ applied at point T as shown in Figure 3.4.2-2. These dynamic forces are to be transferred at DOF location.

Transfer the dynamic forces to DOF Locations: Let the equivalent dynamic forces at O be F_1 along DOF 1, F_2 along DOF 2 & F_3 along DOF 3. Considering equilibrium of forces, by statics, we get

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} F_x \sin\omega t \\ F_y \sin\omega t \\ (-s\,F_x)\sin\omega t \end{Bmatrix} \tag{3.4.2-9}$$

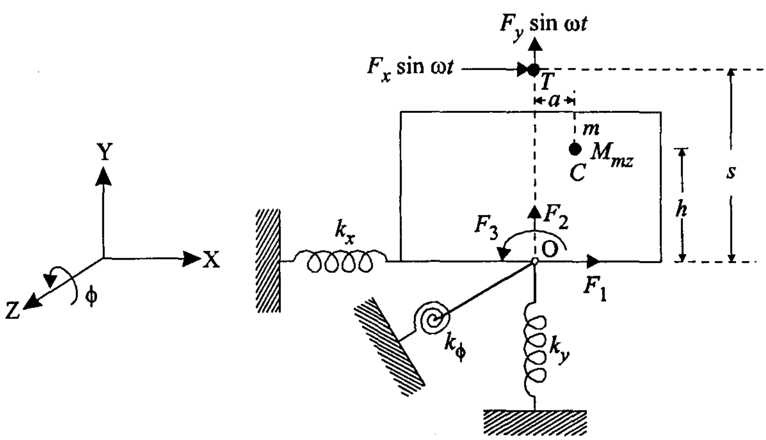


Figure 3.4.2-2 Dynamic Force F_x sin ωt & F_y sin ωt applied at point T
Overall Centroid is offset by distance 'a' from center of Stiffness

Equation of Motion

Adding equation (3.4.2-9) to RHS of equation of motion for free vibration i.e. equation (3.3.2-17), we get

$$\begin{bmatrix} m & 0 & -mh \\ 0 & m & ma \\ -mh & ma & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ y \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin \omega t \\ F_y \sin \omega t \\ (-s F_x) \sin \omega t \end{Bmatrix} \qquad (3.4.2\text{-}10)$$

For free vibration response, **see § 3.3.2.2.**

Forced Response:

Solution of equation 3.4.2-10, using closed form solution techniques, is not only difficult but complex too. We leave it at this stage itself. Should a situation arise, it may be desirable to resort to advanced computational tools/packages for solution to the problem.

EXAMPLE PROBLEMS (§3.1)

(Free Vibration 2-DOF System - Natural Frequency Computation)

P 3.1-1

A 2DOF spring mass system, as shown in Figure P 3.1-1a, has mass $m_1 = 1000\,kg$, mass $m_2 = 500\,kg$, spring stiffness $k_1 = 25\,kN/m$ and $k_2 = 2\,kN/m$. Compute natural frequency of the above spring mass system.

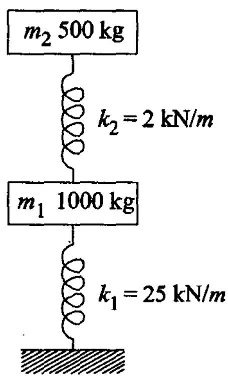


Figure P3.1-1 2 DOF Spring Mass System

Solution

Limiting frequencies

$m_1 = 1000$ kg

$m_2 = 500$ kg

$k_1 = 25000$ N/m

$k_2 = 2000$ N/m

$$\lambda = \frac{m_2}{m_1} = \frac{500}{1000} = 0.5$$

$$p_{L1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{25000}{1000}} = 5 \text{ rad/s}$$

$$p_{L2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{2000}{500}} = 2 \text{ rad/s}$$

Natural frequencies are:

$$p_1^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) - \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\times\left(p_{L1}^2 p_{L2}^2\right)}\right\}$$

$$= \frac{1}{2}\left\{(4\times(1.5)+25) - \sqrt{(4\times(1.5)+25)^2 - 4(25\times 4)}\right\}$$

$p_1 = 1.91$ rad/s

$$p_2^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) + \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\}$$

$$= \frac{1}{2}\left\{(4\times(1.5)+25) + \sqrt{(4\times(1.5)+25)^2 - 4(25\times 4)}\right\}$$

$p_2 = 5.23$ rad/s

It is noticed that the lower natural frequency of the system p_1 is lower than the lowest limiting frequency and the higher natural frequency p_2 is higher than the highest limiting frequency.

P 3.1-2

In problem P 3.1-1, k_1 & m_1 is interchanged with k_2 & m_2. Thus, the spring mass system has masses $m_1 = 500$ kg, $m_2 = 1000$ kg, spring stiffness $k_1 = 2$ kN/m and $k_2 = 25$ kN/m as shown in Figure P 3.1-2. Compute natural frequency of the spring mass system.

Solution

Limiting frequencies

$m_1 = 500$
$m_2 = 1000$
$k_1 = 2000$
$k_2 = 25000$

$$\lambda = \frac{m_2}{m_1} = \frac{1000}{500} = 2.0$$

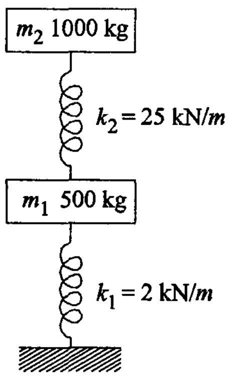


Figure P3.1-2 2 DOF Spring Mass System

$$p_{L1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{2000}{500}} = 2 \text{ rad/s}$$

$$p_{L2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{25000}{1000}} = 5 \text{ rad/s}$$

Natural frequencies are:

$$p_1^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) - \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}{}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\}$$

$$= \frac{1}{2}\left\{(25 \times (1+2) + 4) - \sqrt{(25 \times (1+2) + 4)^2 - 4(25 \times 4)}\right\}$$

$p_1 = 1.134$ rad/s

$$p_2^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)+\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2 p_{L2}^2\right)}\right\}$$

$$= \frac{1}{2}\left\{(25\times(1+2)+4)+\sqrt{(25\times(1+2)+4)^2-4(25\times 4)}\right\}$$

$$p_2 = 8.82 \quad \text{rad/s}$$

It is noticed that the lower natural frequency of the system p_1 is lower than the lowest limiting frequency and the higher natural frequency p_2 is higher than the highest limiting frequency.

P 3.1-3

A machine of mass 500 kg is supported on a RCC Block of size L= 2500 mm, B=1500 mm & H=400 mm deep. Density of concrete is 2500 kg/m^3. The block in turn is supported by a rotational spring having stiffness of $k_\phi = 2\times10^6$ Nm/rad and Translational spring having stiffness of $k_x = 2\times10^7$ N/m attached at base center of the block (point O) as shown. The height of the machine mass above top of Block is 100 mm. Block Centroid C and CG of machine lie on the same vertical line. The system is constrained such that it can move only in translational X direction rock about Z-axis passing through O. Find natural frequency of the system?

$k_\phi = 2\times10^6$ Nm/rad

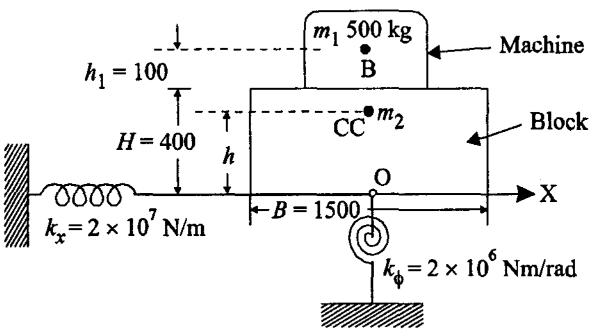


Figure P3.1-3 Machine on RCC Block Supported by Rotational Spring attached to Base Center Point O

Solution:

Machine mass m_1 $= 500$ kg

Mass of Block m_2 $2.5\times1.5\times0.4\times2500 = 3750$ kg

Total Mass $(m = m_1 + m_2)$ $= 4250$ kg

Spring Stiffness in X direction $k_x = 2\times10^7$ N/m

Spring Stiffness in ϕ direction

Let us denote Overall centroid (Block +Machine) as CC

Height of CC from center of base of the block point O

$$h = \frac{500\times(0.1+0.4)+3750\times(0.5\times0.4)}{4250} = 0.2354 \text{ m}$$

Mass Moment of Inertia about Z-axis at centroid CC= M_{mz}

$$M_{mz} = \frac{3750}{12}\times\left(1.5^2+0.4^2\right)+3750\times(0.2354-0.2)^2+500\times(0.1+0.4-0.2354)^2 = 792.83$$

Mass Moment of Inertia about base center point O= M_{moz}

$$M_{moz} = \frac{3750}{12}\times\left(1.5^2+0.4^2\right)+3750\times0.2^2+500\times(0.1+0.4)^2 = 1028.125$$

$$\gamma_z = \frac{M_{mz}}{M_{moz}} = \frac{792.83}{1028.125} = 0.77$$

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{2\times10^7}{4250}} = 68.6 \text{ rad/s}$$

$$p_\phi = \sqrt{\frac{(k_\phi)}{M_{moz}}} = \sqrt{\frac{(2\times10^6)}{1028.125}} = 44.10 \quad \text{rad/s}$$

As mentioned earlier in the text, the influence of term mgh for all practical real life problems is practically insignificant. This can be checked here it self. Considering effect of term mgh, we get:

$$p_\phi = \sqrt{\frac{(k_\phi - mgh)}{M_{moz}}} = \sqrt{\frac{(2\times10^6 - 4250\times9.81\times0.2354)}{1028.125}} = 44 \quad \text{rad/s}$$

It is seen that this value is practically same as obtained above. Hence one can conveniently and safely ignore term mgh for all practical purposes.

$$p^2 = \frac{1}{2\gamma}\left\{\left(p_x^2 + p_\phi^2\right) \mp \sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\,\gamma_z\, p_x^2 p_\phi^2}\right\}$$

Substituting for p_x, p_ϕ & γ_z, we get

$$p^2 = \frac{1}{2\times 0.77}\left\{\left(68.66^2 + 44.1^2\right) \mp \sqrt{\left(68.66^2 + 44.1^2\right)^2 - 4\times 0.77\times 68.66^2\times 44.1^2}\right\}$$
$$= 4318 \mp 2606.7$$

Two positive Roots of p^2 give two natural frequencies as

$$p_1 = \sqrt{4318 - 2606.7} = 41.36 \text{ rad/s}$$
$$p_2 = \sqrt{4318 + 2606.7} = 83.21 \text{ rad/s}$$

EXAMPLE PROBLEMS (§3.2)

(Forced Vibration –2-DOF System - Response Computations)

P 3.2-1

A machine of mass $5000\,\text{kg}$ is supported on a RCC Block of size $L = 4\,\text{m}, B = 2\,\text{m}$ & $H = 3\,\text{m}$ deep. Density of concrete is $2500\,\text{kg/m}^3$. The block in turn is supported by a rotational spring having stiffness of $k_\phi = 2.1\times 10^8\,\text{Nm/rad}$ and Translational spring having stiffness of $k_x = 1.6\times 10^8\,\text{N/m}$ attached at base center of the block (point O) as shown. The height of the machine mass above base of the Block is $3500\,\text{mm}$. Overall Centroid C and center of Stiffness point O lie on the same vertical line. The system is constrained so as to translate only along X & rotate about Z-axis passing through O. A dynamic force of $F_x = 5000\ \ \text{N}@\ 15\,\text{Hz}$ is applied at the machine mass center along X-axis. Find the undamped amplitudes of vibration at foundation base (point O).

Solution:

Mass of Machine	$5000\,\text{kg}$
Mass of foundation Block	$2500\times(4\times 2\times 3) = 60000\,\text{kg}$
Total Mass	$m = 5000 + 60000 = 65000\,\text{kg}$
Spring Stiffness in X direction	$k_x = 1.6\times 10^8\ \ \text{N/m}$

Spring Stiffness in ϕ direction $k_{\phi} = 2.1 \times 10^8$ Nm/rad

Excitation Frequency $\omega = 15 \text{ Hz} = 15 \times 2\pi = 94.24$ rad/s

Applied Dynamic Force:

Magnitude of Dynamic Force: $F_x = 5000\,\text{N}$

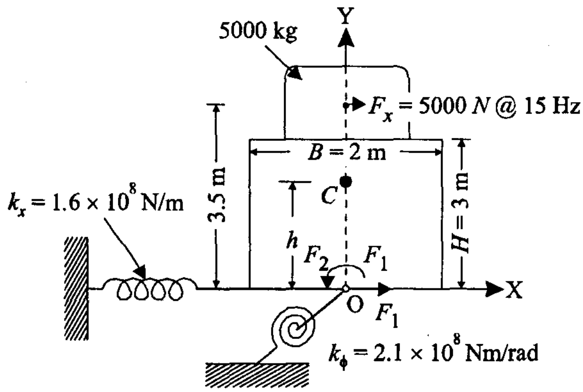


Figure P3.2-1 Machine on RCC Block Supported by Rotational Spring and Translational Spring attached at Base Center Point O

Equivalent Dynamic Forces transferred @ point O:

Transferring forces @ DOF location point O, we get

$$F_1 = 5000\,\text{N}, \qquad F_2 = -5000 \times 3.5 = -17500 \text{ Nm}$$

Let us denote Overall centroid (Block +Machine) as C

Height of overall centroid C from center of base of the block point O

$$h = (5000 \times 3.5 + 60000 \times 3.0/2)/65000 = 1.654 \text{ m}$$

Mass Moment of Inertia (Machine + Block) about Z-axis at base Overall centroid C= M_{mz}

$$M_{mz} = \frac{60000}{12} \times \left(3^2 + 2^2\right) + 60000 \times (1.654 - 1.5)^2 + 5000 \times (3.5 - 1.654)^2 = 83461 \text{ kg m}^2$$

Mass Moment of Inertia about Z-axis at base center point O = M_{moz}

$$M_{moz} = 83461 + 65000 \times (1.654)^2 = 261283 \quad \text{kg m}^2$$

$$\gamma_z = \frac{M_{mz}}{M_{moz}} = \frac{83461}{261283} = 0.319$$

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{1.6 \times 10^8}{65000}} = 49.6\, rad/s$$

$$p_\phi = \sqrt{\frac{k_\phi - mgh}{M_{moz}}} = \sqrt{\frac{2.1 \times 10^8 - 65000 \times 9.81 \times 1.654}{261283}} = 28.27 \quad \text{rad/s}$$

Response in $x \& \phi$ direction:

Natural Frequencies:

Frequency equation is $$p^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) \mp \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$

Substituting for p_x, p_ϕ & γ_z, we get natural frequencies as:

$$p_1 = 25.38 \quad \text{rad/s}; \quad \& \quad p_2 = 97.84 \quad \text{rad/s}$$

$$\beta_1 = \frac{\omega}{p_1} = \frac{94.24}{25.38} = 3.71; \quad \beta_2 = \frac{\omega}{p_2} = \frac{94.24}{97.84} = 0.963$$

$$\beta_x = \frac{\omega}{p_x} = \frac{94.24}{49.6} = 1.899; \quad \beta_\phi = \frac{\omega}{p_\phi} = \frac{94.24}{28.35} = 3.32$$

$$\beta_x^2 = 3.6, \ \beta_\phi^2 = 11.02, \ \beta_1^2 = 13.69, \ \beta_2^2 = 0.925$$

Amplitudes at DOF location O

$$x_o = \left[\delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right]$$

$$\phi_o = \left[\delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - \delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right]$$

Substituting for $\beta_x, \beta_\phi, \beta_1, \beta_2, \delta_x, \delta_\phi$, we get:

$(1-\beta_x^2) = -2.6$, $(1-\beta_\phi^2) = -10.02$, $(1-\beta_1^2) = -12.76$, $(1-\beta_2^2) = 0.073$

$$(1-\beta_1^2)(1-\beta_2^2) = -12.76\times 0.073 = -0.93$$

$$\frac{(1-\beta_\phi^2)}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{-10.02}{-0.93} = 10.77$$

$$\frac{\beta_x^2}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{3.6}{-0.93} = -3.87$$

$$\frac{\beta_\phi^2}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{11.02}{-0.93} = -11.85$$

$$\frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{-2.6}{-0.93} = 2.79$$

$$h = 1.654\text{ m}; \qquad \frac{mh}{M_{moz}} = \frac{65000\times 1.654}{261283} = 0.4115$$

$$\delta_x = \frac{F_x}{k_x} = \frac{5000}{1.6\times 10^8} = 3.125\times 10^{-5}\text{ m}$$

$$\delta_\phi = \frac{M_\phi}{k_\phi} = \frac{-17500}{2.1\times 10^8} = -8.33\times 10^{-5}\text{ rad}$$

x - Amplitude at DOF Location (Point O)

$$x_o = 3.125\times 10^{-5}\times(10.77) - 1.654\times(-8.33\times 10^{-5})\times(-3.87)$$
$$= -19.4\times 10^{-5}\text{ m}$$

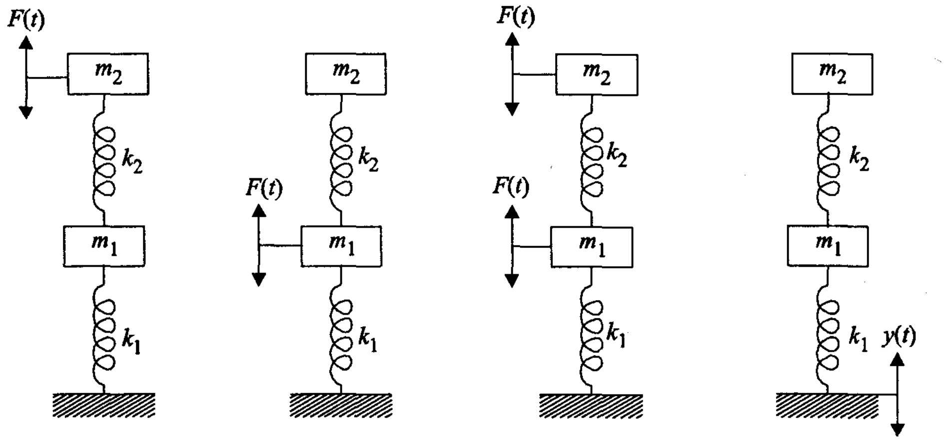
ϕ - Amplitude at DOF Location (Point O)

$$\phi_o = (-8.33)\times 10^{-5}\times(2.79) - 3.125\times 10^{-5}\times 0.4115\times(-11.85)$$
$$= -8.0\times 10^{-5}\text{ rad}$$

P 3.2-2

A 2DOF spring mass system, as shown in Figure P 3.2-2, has mass m_1 =1000 kg, mass m_2 =500 kg, spring stiffness $k_1 = 25$ kN/m and $k_2 = 2$ kN/m. Compute maximum amplitudes of vibration for:

Excitation force $F_2(t) = 100 \sin 10t$ **on Mass** m_2
Excitation force $F_1(t) = 100 \sin 10t$ **on Mass** m_1
Excitation force $F_1(t) = 100 \sin 10t$ **on Mass** m_1 **&** $F_2(t) = 100 \sin 10t$ **on Mass** m_2
Base Excitation displacement $y(t) = 0.001 \sin 10t$



(a) Excitation Force $F(t)$ on Mass m_2 (b) Excitation Force $F(t)$ on Mass m_1 (c) Excitation Force $F(t)$ on Mass m_1 & m_2 (c) Base Excitation

Figure P3.2-2 Two Mass System-Forced Vibration

Solution:

System Data

$m_1 = 1000$ kg ; $m_2 = 500$ kg ; $k_1 = 25000$ N/m ; $k_2 = 2000$ N/m

$F_1 = 100$ N; $F_2 = 100$ N ; $x_0 = 0.001$ m ; $\omega = 10$ rad/sec

$$\lambda = \frac{m_2}{m_1} = 0.5$$

Limiting Frequency $p_{L1} = \sqrt{25000/1000} = 5$ rad/sec

Limiting Frequency $p_{L2} = \sqrt{2000/500} = 2$ rad/sec

Using equations 3.1.1-12 & 13, and substituting for P_{L1}, P_{L2} **&** λ**, we get natural frequencies**

$$p_1^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)-\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2\,p_{L2}^2\right)}\right\} \qquad (3.1.1\text{-}12)$$

$$p_2^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)+\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2\,p_{L2}^2\right)}\right\} \qquad (3.1.1\text{-}13)$$

Natural Frequency $p_1 = 1.912$ rad/s ; $p_2 = 5.229$ rad/s

Amplitudes of vibration

Excitation Frequency $\omega = 10$ rad/s;

Frequency Ratios $\beta_1 = \frac{\omega}{p_1} = \frac{10}{1.912} = 5.23$; $\beta_2 = \frac{\omega}{p_2} = \frac{10}{5.229} = 1.912$

(a) Excitation force $F_2(t) = 100\sin\ 10t$ on Mass m_2

For amplitude of mass m_1 & m_2, refer equations (3.2.1-5) & (3.2.1-6).

$$y_1 = \frac{F_2}{k_2}\frac{1}{\left\{\left(1-\beta_1{}^2\right)\left(1-\beta_2{}^2\right)\right\}}\frac{k_2}{k_1}\sin\omega t$$

$$y_2 = \frac{F_2}{k_2}\frac{1}{\left(1-\beta_1{}^2\right)\left(1-\beta_2{}^2\right)}\left(\frac{k_1+k_2-\omega^2 m_1}{k_1}\right)\sin\omega t$$

For maximum amplitude $\sin\omega t = 1.0$, substituting the values, we get

Amplitude of mass m_1

$$y_1 = \frac{100}{2000}\frac{1}{\left\{\left(1-5.23^2\right)\left(1-1.912^2\right)\right\}}\frac{2000}{25000}$$

$$y_1 = 5.715\times10^{-5}\ \text{m} = 57.15\ \text{microns}$$

Amplitude of mass m_2

$$y_2 = \frac{100}{2000}\frac{1}{\left\{\left(1-5.23^2\right)\left(1-1.912^2\right)\right\}}\left(\frac{25000+2000-10^2\times1000}{25000}\right)$$

$$y_2 = -2.086\times10^{-3}\ \text{m} = -2086\ \text{microns}$$

(b) Excitation force $F_1(t) = 100\ \sin\ 10\,t$ on Mass m_1

Refer equations (3.2.1-8) & (3.2.1-9)

$$y_1 = \frac{F_1}{k_1}\frac{1}{\{(1-\beta_1^2)(1-\beta_2^2)\}}\left\{\frac{(k_2-\omega^2 m_2)}{k_2}\right\}\sin\omega t$$

$$y_2 = \frac{F_1}{k_1}\frac{1}{\{(1-\beta_1^2)(1-\beta_2^2)\}}\sin\omega t$$

For maximum amplitude $\sin\omega t = 1.0$, substituting the values, we get

Amplitude of mass m_1

$$y_1 = \frac{100}{25000}\frac{1}{\{(1-5.23^2)(1-1.912^2)\}}\left\{\frac{(2000-10^2\times 500)}{2000}\right\}$$

$y_1 = -1.371\times 10^{-3}$ m $= -1371$ microns

Amplitude of mass m_2

$$y_2 = \frac{100}{25000}\frac{1}{\{(1-5.23^2)(1-1.912^2)\}}$$

$y_2 = 5.715\times 10^{-5}$ m $= 57.15$ microns

Excitation force

$F_1(t) = 100\ \sin\ 10\,t$ **on Mass** m_1 **&** $F_2(t) = 100\sin 10\,t$ **on Mass** m_2

Refer equations (3.2.1-11) & (3.2.1-12)

$$y_1 = \frac{1}{\{(1-\beta_1^2)(1-\beta_2^2)\}}\left\{\frac{F_1}{k_1}\frac{(k_2-\omega^2 m_2)}{k_2}+\frac{F_2}{k_2}\frac{k_2}{k_1}\right\}\sin\omega t$$

$$y_2 = \frac{1}{\{(1-\beta_1^2)(1-\beta_2^2)\}}\left\{\frac{F_2}{k_2}\frac{(k_1+k_2-\omega^2 m_1)}{k_1}+\frac{F_1}{k_1}\right\}\sin\omega t$$

For maximum amplitude $\sin\omega t = 1.0$, substituting the values, we get

Amplitude of mass m_1

$$y_1 = \frac{1}{\left\{(1-5.23^2)(1-1.912^2)\right\}}\left\{\frac{100}{25000}\frac{(2000-10^2\times500)}{2000}+\frac{100}{2000}\frac{2000}{25000}\right\}$$

$y_1 = -1.314\times10^{-3}$ m = –1314 microns

Amplitude of mass m_2

$$y_2 = \frac{1}{\left\{(1-5.23^2)(1-1.912^2)\right\}}\left\{\frac{100}{2000}\frac{(25000+2000-10^2\times1000)}{25000}+\frac{100}{25000}\right\}$$

$y_2 = -2.029\times10^{-3}$ m = –2029 microns

(c) Base Excitation $y(t) = 0.001 \sin 10t$

Refer equations (3.2.1-16) & (3.2.1-17)

$$y_1 = y_0 \frac{1}{\left\{(1-\beta_1^2)(1-\beta_2^2)\right\}}\left\{\frac{(k_2-\omega^2 m_2)}{k_2}\right\}\sin\omega t$$

$$y_2 = y_0 \frac{1}{\left\{(1-\beta_1^2)(1-\beta_2^2)\right\}}\sin\omega t$$

For maximum amplitude $\sin\omega t = 1.0$, substituting the values, we get

Amplitude of mass m_1

$$y_1 = 0.001\frac{1}{\left\{(1-5.23^2)(1-1.912^2)\right\}}\left\{\frac{(2000-10^2\times500)}{2000}\right\}$$

$y_1 = -3.429\times10^{-4}$ m = –343 microns

Amplitude of mass m_2

$$y_2 = 0.001\frac{1}{\left\{(1-5.23^2)(1-1.912^2)\right\}} = 1.43\times10^{-5} \text{ m} = 14.3 \text{ microns}$$

P 3.2-3

A 2DOF spring mass system, as shown in Figure P 3.2-3, has mass $m_1 = 170$ t, mass $m_2 = 34$ t, spring stiffness $k_1 = 8.01\times10^5$ kN/m and $k_2 = 2.35\times10^6$ kN/m. A mass $m_0 = 1.38$ t freely falls on mass m_2 from a height of $h = 1.70$ m. Considering coefficient of restitution for the impact $e = 0.6$, compute the amplitudes of masses m_1 & m_2.

Solution:

$m_1 = 170\ \text{t};\qquad m_2 = 34\ \text{t};\qquad m_0 = 1.38\ \text{t}$

$k_1 = 8.01\times10^5\ \text{kN/m};\qquad k_2 = 2.35\times10^6\ \text{kN/m};\qquad h = 1.7\ m;\qquad e = 0.6$

Limiting Frequencies:

$$p_{L1}^2 = \frac{k_1}{m_1} = \frac{8.012\times10^6}{170} = 4697;\qquad p_{L2}^2 = \frac{k_2}{m_2} = \frac{2.345\times10^6}{34} = 68978$$

Mass Ratio $\qquad \lambda = \frac{m_2}{m_1} = \frac{34}{170} = 0.2$

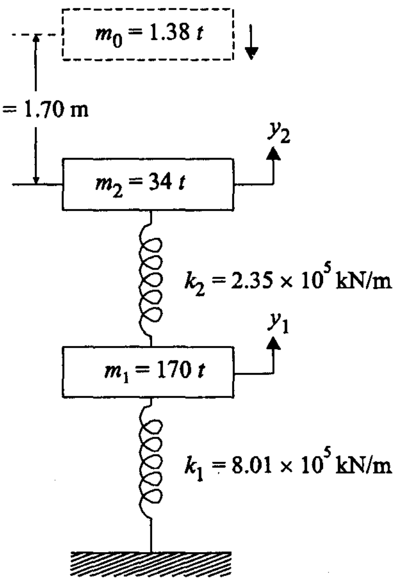


Figure P3.2-3 (a) Two Spring Mass System Subjected to Impact Load

Natural Frequencies:

$$p_1^2 = \frac{1}{2}\left\{(68978\times(1+0.2)+4697) - \sqrt{(68978\times(1+0.2)+4697)^2 - 4\times(4697\times68978)}\right\} = 3876$$

$p_1 = 62.26\ \text{rad/s}$

$$p_2^2 = \frac{1}{2}\left\{(68978\times(1+0.2)+4697)+\sqrt{(68978\times(1+0.2)+4697)^2-4\times(4697\times 68978)}\right\} = 8.355\times 10^4$$

$p_2 = 289$ rad/s

Initial Velocity of mass m_2

Velocity of mass m_0 before impact

$$v_0' = \sqrt{2gh} = \sqrt{2\times 9.81\times 1.7} = 5.775 \quad \text{m/s}$$

$$\lambda_2 = \frac{m_2}{m_0} = \frac{34}{1.38} = 24.63$$

Velocity of mass m_2 after impact

$$v_2 = v_0' \times \frac{(1+e)}{(1+\lambda_2)} = 5.775\times\frac{(1+0.6))}{(1+24.63)} = 0.36 \quad \text{m/s}$$

We get response of the Two Spring Mass System as under (refer equation 3.2.1-39 & 40):

Amplitude of mass m_1 (see equation 3.2.1-39)

$$y_1 = v_2\frac{(p_{L2}^2-p_1^2)(p_{L2}^2-p_2^2)}{p_{L2}^2(p_1^2-p_2^2)}\left\{\frac{\sin(p_1t)}{p_1}-\frac{\sin(p_2t)}{p_2}\right\}$$

Amplitude of mass m_2 (see equation 3.2.1-40)

$$y_2 = v_2\times\frac{(p_{L2}{}^2-p_2{}^2)}{(p_1{}^2-p_2{}^2)}\frac{\sin(p_1t)}{p_1} - v_2\times\frac{(p_{L2}{}^2-p_1{}^2)}{(p_1{}^2-p_2{}^2)}\frac{\sin(p_2t)}{p_2}$$

Amplitude of mass m_1 in 1st mode

$$y_1' = v_2\frac{(p_{L2}^2-p_1^2)(p_{L2}^2-p_2^2)}{p_{L2}^2(p_1^2-p_2^2)}\frac{\sin(p_1t)}{p_1}$$

$$y_1' = 0.36\times\frac{(68978-3876)(68978-83550)}{68978(3876-83550)}\frac{\sin 62.26t}{62.26}\times 10^3 = 0.99918\sin 62.26t \quad \text{mm}$$

Amplitude of mass m_1 in 2nd mode

$$y_1'' = -v_2\frac{(p_{L2}{}^2-p_1{}^2)(p_{L2}{}^2-p_2{}^2)}{p_{L2}{}^2(p_1{}^2-p_2{}^2)}\frac{\sin(p_2t)}{p_2}$$

$$y_1'' = -0.36 \times \frac{(68978-3876)(68978-83550)}{68978(3876-83550)} \frac{\sin 289t}{289} \times 10^3 = -0.215 \sin 289t \quad \text{mm}$$

Amplitude of mass 2

$$y_2' = v_2 \times \frac{\left(p_{L2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)} \frac{\sin(p_1 t)}{p_1}$$

Amplitude of mass m_2 in 1st mode

$$y_2' = \frac{0.36}{62.26} \times \frac{(68978-83550)}{(3876-83550)} \times 10^3 \sin 62.26t = -1.05 \sin 62.26t \quad \text{mm}$$

Amplitude of mass m_2 in 2nd mode

$$y_2'' = -\frac{v_2}{p_2} \times \frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)} \sin p_2 t$$

$$y_2'' = -\frac{0.36}{289} \times \frac{(68978-3876)}{(3876-83550)} \times 10^3 \sin 289t = 1.0 \sin 289t \quad \text{mm}$$

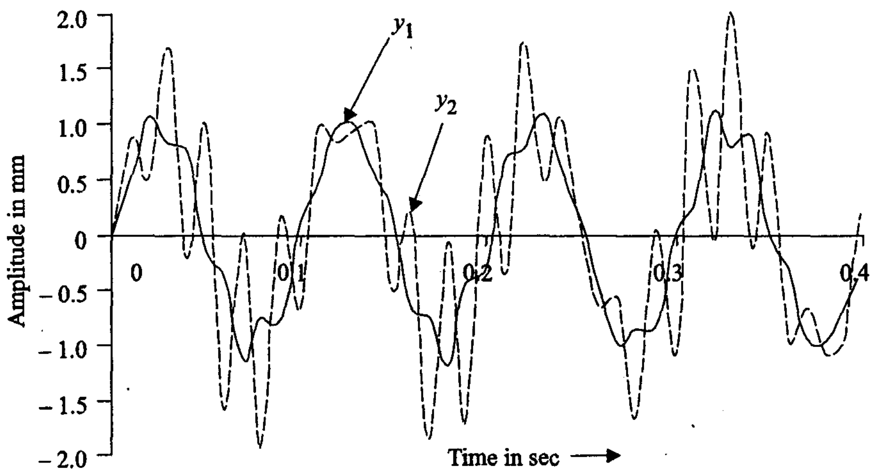


Figure P 3.2-3 (b) Response of Masses m_1 & m_2 Subjected to Impact Loading

Amplitude vs. time plots of mass m_1 and mass m_2 are shown in Figure 3.2-3 (b). From the figure we get

Overall Amplitude of mass m_1 $\quad y_1 = 1.213$ mm

Overall Amplitude of mass m_2 $\quad y_2 = 2.06$ mm

P 3.2-4

A 6 spring mass system connected by a rigid bar, as shown in Figure P 3.2-4, is subjected to dynamic force of $F(t) = 50\sin 100t$ N applied at each mass point. Find response of the system.

Solution (See §3.1.5 & §3.2.3)

Mass Data:
$m_1 = 100$ kg $= 0.1$ t
$m_2 = 0.15$ t; $m_3 = 0.2$ t
$m_4 = 0.3$ t; $m_5 = 0.2$ t; $m_6 = 0.1$ t

Spring Data:
$k_1 = 1000$ kN/m
$k_2 = 1500$ kN/m; $k_3 = 2000$ kN/m
$k_4 = 1000$ kN/m; $k_5 = 2000$ kN/m; $k_6 = 2500$ kN/m

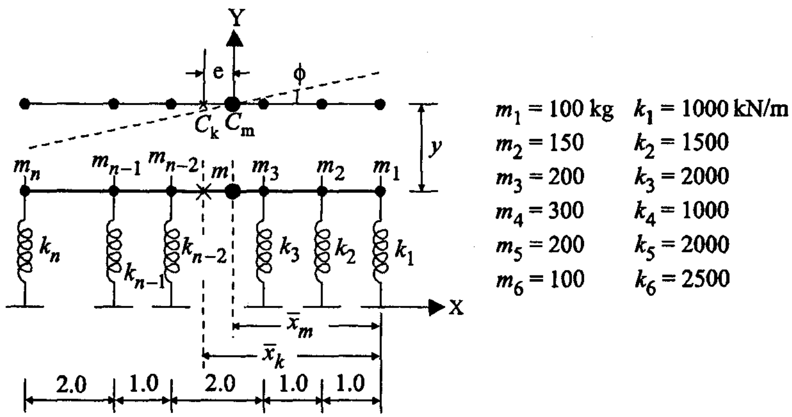


Figure P 3.2-4 Six Spring Mass Systems Connected by Massless Rigid Bar

Center of Mass $\bar{x}_m$

Total Mass $m = 0.10 + 0.15 + 0.20 + 0.3 + 0.2 + 0.1 = 1.05 \text{ t}$

Taking moment about axis passing through spring 1, we get

$$\bar{x}_m = \frac{0.15\times1 + 0.2\times2 + 0.3\times4 + 0.2\times5 + 0.1\times7}{1.05} = \frac{3.45}{1.05} = 3.286 \text{ m}$$

Center of Stiffness

Total Spring Stiffness $k_y = (1000 + 1500 + 2000 + 1000 + 2000 + 2500) = 10000 \text{ kN/m}$

Taking moment about axis passing through spring 1, we get

$$\bar{x}_k = \frac{(1500\times1 + 2000\times2 + 1000\times4 + 2000\times5 + 2500\times7)}{10000} = \frac{37000}{10000} = 3.7 \text{ m}$$

Eccentricity: $e = \bar{x}_m - \bar{x}_k = 3.286 - 3.7 = -0.414 \text{ m}$

Distances of springs from center of mass:

$a_1 = 3.286 \text{ m}; \ a_2 = 3.286 - 1 = 2.286 \text{ m}; \ a_3 = 3.286 - 2 = 1.286 \text{ m}$
$a_4 = 3.286 - 4 = -0.714 \text{ m}; \ a_5 = 3.286 - 5 = -1.714 \text{ m}; \ a_6 = 3.286 - 7 = -3.714 \text{ m}$

Distances of springs from center of stiffness:

$b_1 = 3.7 \text{ m}; \ b_2 = 3.7 - 1 = 2.7 \text{ m}; \ b_3 = 3.7 - 2 = 1.7 \text{ m}$
$b_4 = 3.7 - 4 = -0.3 \text{ m}; \ b_5 = 3.7 - 5 = -1.3 \text{ m}; \ b_6 = 3.7 - 7 = -3.3 \text{ m}$

It is seen that these values of a_i, b_i & e satisfy equation $a_i = b_i + e$

Mass Moment of Inertia M_m: It is the second moment of mass about center of mass

$$M_m = \sum(m_i a_i^2) = 0.1\times3.286^2 + 0.15\times2.286^2 + 0.2\times1.286^2$$
$$+ 0.3\times(-0.714)^2 + 0.2\times(-1.714)^2 + 0.1\times(-3.714)^2 = 4.314 \text{ t m}^2$$

Rotational stiffness about center of stiffness

$$k_\phi = \sum\left(k_i b_i^2\right) = 1000\times 3.7^2 + 1500\times 2.7^2 + 2000\times 1.7^2$$
$$+1000\times(-0.3)^2 + 2000\times(-1.3)^2 + 2500\times(-3.3)^2 = 61100 \text{ kNm}$$

Limiting Translational Frequencies $p_y = \sqrt{\dfrac{k_y}{m}} = \sqrt{\dfrac{10000}{1.05}} = 97.6 \text{ rad/s}$

Limiting Rotational Frequencies $p_\phi = \sqrt{\dfrac{k_\phi}{M_m}} = \sqrt{\dfrac{61100}{4.314}} = 119 \text{ rad/s}$

Equivalent radius of gyration $r = \sqrt{\dfrac{M_m}{m}} = \sqrt{\dfrac{4.314}{1.05}} = 2.027 \text{ m}$

$$\alpha = 1 + \frac{e^2}{r^2} = 1 + \frac{(-0.414)^2}{2.027^2} = 1.042$$

Natural Frequencies: (see equation 3.1.5-15)

$$p_{1,2}^2 = 0.5\times\left\{\left(1.042\times 97.6^2 + 119^2\right) \mp \sqrt{\left(1.042\times 97.6^2 + 119^2\right)^2 - 4\times 97.6^2\times 119^2}\right\}$$

$p_1 = 94.1 \text{ rad/s}; \quad p_2 = 123.4 \text{ rad/s}$

Response:

Dynamic force and moment transferred at center of mass point C_m

Total dynamic force @ center of mass

$F(t) = 50\sin 100t\times 6 = 300\sin 100t \text{ N} = 0.30\sin 100t \text{ kN}$

Point of application of dynamic force $\bar{x}_F = \dfrac{50\times(1+2+4+5+7)}{50\times 6} = \dfrac{350}{300} = 3.167 \text{ m}$

Centroid of dynamic force from center of mass $\bar{x}_m - \bar{x}_F = 3.286 - 3.167 = 0.12 \text{ m}$

Dynamic Moment @ center of mass

$M(t) = (0.30\times 0.12)\sin 100t = 0.036\sin 100t \text{ kNm}$

Maximum Amplitude: (See equation 3.2.3-5 & 6)

Since system has no damping, let us compute response for undamped condition.

$$y_{(\max)} = \frac{\delta_{y(static)}\left(1 + \frac{\beta_\phi^2}{\beta_y^2}\frac{e^2}{r^2} - \beta_\phi^2\right) - \delta_{\phi(static)} \times e}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)};$$

$$\phi_{(\max)} = \frac{\delta_{\phi(static)}\left(1-\beta_y^2\right) - \delta_{y(static)}\frac{\beta_\phi^2}{\beta_y^2}\frac{e}{r^2}}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

$$\delta_{y(static)} = \frac{F_0}{k_y} = \frac{0.3}{10000} = 3.0\times10^{-5} \text{ m}; \quad \delta_{\phi(static)} = \frac{M_0}{k_\phi} = \frac{0.036}{61100} = 5.89\times10^{-7} \text{ rad}$$

$\omega = 97.6$ rad/sec; $p_y = 97.6$ rad/sec; $p_\phi = 119$ rad/sec; $p_1 = 94.1$ rad/sec; $p_2 = 123.4$ rad/sec;

$$\beta_y = \frac{\omega}{p_y} = \frac{100}{97.6} = 1.0246; \quad \beta_\phi = 0.84; \quad \beta_1 = 1.0627; \quad \beta_2 = 0.8104;$$

$$r = 2.027 \text{ m}; \quad e = \text{-}0.414 \text{ m}; \quad \alpha = 1 + \frac{e^2}{r^2} = 1.042$$

$$\left(1 + \frac{\beta_\phi^2}{\beta_y^2}\frac{e^2}{r^2} - \beta_\phi^2\right) = 1 + \frac{0.84^2}{1.0246^2}\times 0.042 - 0.84^2 = 0.3226$$

$$\left(1-\beta_1^2\right)\left(1-\beta_2^2\right) = \left(1-1.0627^2\right)\left(1-0.8104^2\right) = -0.214$$

$$y_{(\max)} = \frac{3.0\times10^{-5}\times0.3226 - 5.89\times10^{-7}\times(\text{-}0.414)}{-0.214} = -4.64\times10^{-5} \text{ m}$$

$$\phi_{(\max)} = \frac{5.89\times10^{-7}\left(1-1.0246^2\right) - 3.0\times10^{-5}\left(\frac{0.84}{1.0246}\right)^2\left(\frac{-0.414}{2.027^2}\right)}{-0.214} = -9.6\times10^{-6} \text{ rad}$$

Maximum Translational Amplitude shall occur at the extreme end of the bar

Max Amplitude

$$A_y = y_{\max} + \phi_{(\max)}\times\left|a_i(\max)\right|$$
$$\left|a_i(\max)\right| = a_6 = 3.714 \text{ m}$$

$$A_y = -4.64\times10^{-5} - 9.6\times10^{-6}\times3.714 = -82.15\times10^{-6} \text{ m}$$
$$\left(A_y\right)_{\max} = 82.15 \text{ microns}$$

P 3.2-5

A Machine of mass 5000 kg is supported at the center of a RCC portal frame beam as shown in Figure P 3.2 -5. Frame beam is 200 mm x 500 mm deep and column section is 200 X 400 mm. Frame span is 4000 mm (center to center) and height of frame is 6000 mm (up to beam center) as shown. Consider that the mass is subjected to a dynamic force $F(t) = 0.2\sin 100t$ applied along Y. Elastic Modulus of concrete is $E_c = 3\times10^{10}$ N/m^2 and its mass density is $\rho_c = 2500$ kg/m^3. Consider damping $\zeta = 5\%$. Compute the response.

Solution:

i) Beam is Elastic

Material properties of Concrete	$E_c = 3\times10^7$ kN/m^2; $\rho_c = 2.5$ t/m^3
Span & Height of Frame	$L = 4.0$ m ; $H = 6.0$ m ;
Area of Beam Cross-section	$A_b = 0.2\times0.5 = 0.10$ m^2
Area of Column Cross-section	$A_c = 0.2\times0.4 = 0.08$ m^2
Mass of Beam	$m_b = 0.1\times4.0\times2.5 = 1.0$ t
Mass of each column	$m_c = 0.08\times6\times2.5 = 1.2$ t
Moment of Inertia Beam Cross-section	$I_b = \frac{1}{12}\times0.2\times0.5^3 = 0.00208$ m^4
Moment of Inertia Column Cross-section	$I_c = \frac{1}{12}\times0.2\times0.4^3 = 0.00107$ m^4
Stiffness ratio factor	$k = \frac{I_b/L}{I_c/H} = \frac{0.00208/4}{0.00107/6} = 2.916$

Motion along Y (Vertical motion): Mathematical model is as shown in the Figure

Mass & Stiffness Properties: (see equations 3.2.4-2 & 3)

$$m_2 = m + 0.45m_b; \quad m_1 = 0.55m_b + 2\times0.33\times m_c$$

$$k_1 = \frac{2EA_c}{H}; \quad k_2 = \frac{1}{\delta_{yb}} = \frac{96EI_b}{L^3}\times\frac{k+2}{2k+1}$$

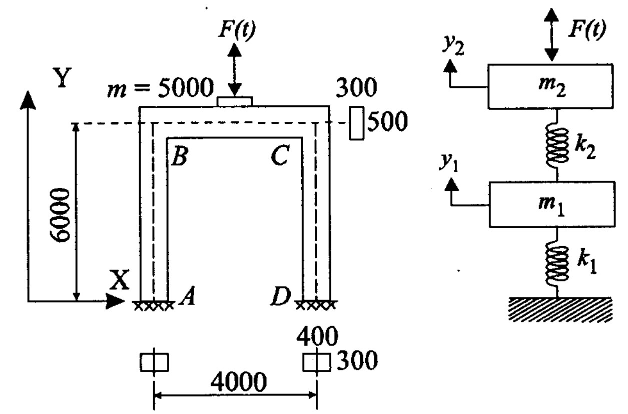


Figure P 3.2-5 Machine mass supported at Portal Frame Beam center Subjected to Dynamic Force *F(t)*

Mass

$$m_1 = 0.55\,m_b + 2\times 0.33\times m_c = 0.55\times 1.0 + 2\times 0.33\times 1.2 = 1.342 \quad \text{t}$$

$$m_2 = m + 0.45\,m_b = 5.0 + 0.45 = 5.45 \ \ \text{t}$$

Stiffness

$$k_1 = \frac{2EA_c}{H} = \frac{2\times 3\times 10^7 \times 0.08}{6.0} = 8\times 10^5 \quad \text{kN/m}$$

$$k_2 = \frac{96E\,I_b}{L^3}\times\frac{k+2}{2k+1} = \frac{96\times 3\times 10^7\times 0.00208}{4^3}\times\frac{2.916+2}{2\times 2.916+1} = 6.735\times 10^4 \quad \text{kN/m}$$

Natural Frequency (see equation 3.2.4-4)

$$p_{L1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{8\times 10^5}{1.342}} = 772.1 \ \text{rad/s}$$

$$p_{L2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{6.735\times 10^4}{5.45}} = 111.16 \quad \text{rad/s}$$

$$\lambda = \frac{m_2}{m_1} = \frac{5.45}{1.342} = 4.06$$

Frequency Equation

$$p_{1,2}^2=\frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)\mp\sqrt{\left(p_{L2}^2(1+\lambda)+p_{L1}^2\right)^2-4\left(p_{L1}^2p_{L2}^2\right)}\right\}$$

Substituting values, we get

$$p_{1,2}^2=\frac{1}{2}\left\{\left(111.16^2\times(1+4.06)+772.1^2\right)\mp\sqrt{\left(111.16^2\times(1+4.06)+772.1^2\right)^2-4\left(772.1^2\times111.16^2\right)}\right\}$$

$p_1=107.23$ rad/sec; $p_2=804.67$ rad/s

Steady-State response:

i) Maximum Undamped Response: (see equations 3.2.4-7 & 8)

$$y_{1\,(\max)}=\frac{F_0}{k_1}\frac{1}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|};\qquad y_{2\,(\max)}=\frac{F_0}{k_2}\frac{\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|}$$

Applied Force $F(t)=0.2\sin 100t$

$$\beta_1=\frac{\omega}{p_1}=\frac{100}{107.23}=0.93;\ \beta_2=\frac{100}{804.67}=0.124;\ \beta_{L1}=\frac{100}{772.1}=0.13;\ \beta_{L2}=\frac{100}{111.16}=0.9$$

$$\frac{1}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|}=\frac{1}{\left|\left(1-0.93^2\right)\right|\left|\left(1-0.124^2\right)\right|}=7.517$$

$$\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)=\left(1+4.06\left(\frac{0.13}{0.9}\right)^2-0.13^2\right)=1.068$$

$$y_{1\,(\max)}=\frac{0.2}{8\times10^5}\times7.517=1.88\times10^{-6}\ \text{m}=1.88\ \text{microns}$$

$$y_{2\,(\max)}=\frac{0.2}{6.735\times10^4}\times1.068\times7.517=2.38\times10^{-5}\ \text{m}=23.8\ \text{microns}$$

ii) Maximum Damped Response:

Since frequency ratio β_1 is in resonance range $(\pm 20\%)$, Equations 3.2.4-7 & 8 get modified as

$$y_{1\,(\max)} = \frac{F_0}{k_1}\frac{1}{\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}\times\left|\left(1-\beta_2^2\right)\right|}\;; \qquad y_{2\,(\max)} = \frac{F_0}{k_2}\frac{\left(1+\lambda\dfrac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)}{\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}\times\left|\left(1-\beta_2^2\right)\right|}$$

Substituting, we get

$$\frac{1}{\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}\times\left|\left(1-\beta_2^2\right)\right|} = \frac{1}{\sqrt{\left(1-0.93^2\right)^2+\left(2\times0.93\times0.05\right)^2}\times\left|\left(1-0.124^2\right)\right|} = 6.19$$

$$y_{1\,(\max)} = \frac{0.2}{8\times10^5}\times6.19 = 1.54\times10^{-6}\ \text{ m} = 1.54\ \text{ microns}$$

$$y_{2\,(\max)} = \frac{0.2}{6.735\times10^4}\times1.068\times6.19 = 1.963\times10^{-5}\ \text{ m} = 19.63\ \text{ microns}$$

EXAMPLE PROBLEMS (§3.4)

P 3.4-1

A machine of mass $5000\,\text{kg}$ is supported on a RCC Block of size $L=4\,\text{m}, B=2\,\text{m}$ & $H=3\,\text{m}$ deep. Density of concrete is $2500\,\text{kg/m}^3$. The block in turn is supported by a rotational spring having stiffness of $k_\phi = 2.1\times10^8\,\text{Nm/rad}$ and Translational springs having stiffness of $k_x = 1.6\times10^8$ N/m & $k_y = 3.2\times10^8$ N/m attached at base center of the block (point O) as shown in Figure P 3.4-1. Height of CG of the machine mass above foundation base of the Block is 3500 mm. Overall Centroid CC and center of Stiffness point O lie on the same vertical line. The system is constrained to move only in X-Y plane i.e. it can translate along X & Y directions & rotate about Z-axis passing through O. Dynamic force $F_x = 5000$ N @15 Hz along X-axis, $F_y = 10000$ N @15 Hz along Y-axis and dynamic moment $M_\phi = 20000$ Nm about Z- axis are applied at the machine mass center. Find the amplitudes of vibration at foundation base (point O). In case of resonance, use damping constant $\zeta = 10\%$.

Solution:

Mass of Machine	5000 kg
Mass of foundation Block	$2500\times(4\times2\times3) = 60000$ kg
Total Mass	$m = 5000+60000 = 65000$ kg

Spring Stiffness in X direction $k_x = 1.6 \times 10^8$ N/m

Spring Stiffness in Y direction $k_y = 3.2 \times 10^8$ N/m

Spring Stiffness in ϕ direction $k_\phi = 2.1 \times 10^8$ Nm/rad

Excitation Frequency $\omega = 15$ Hz $= 15 \times 2\pi = 94.24$ rad/s

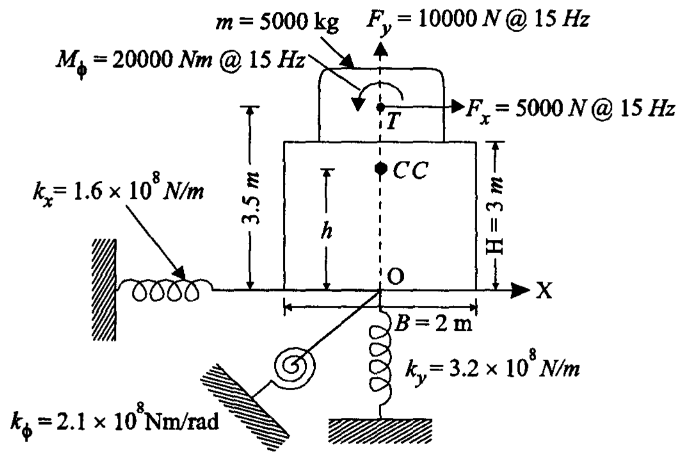


Figure P 3.4-1 Machine on RCC Block Supported by vertical, Translational and Rotational Springs Subjected to Dynamic Forces F_x, F_y, & M_ϕ at Machine center Location

Magnitude of Dynamic Force:

$$F_x = 5000 \text{ N}, \quad F_y = 10000 \text{ N}, \quad M_\phi = 20000 \text{ Nm}$$

Transferring forces @ DOF location point O, we get

$$F_1 = 5000 \text{ N}, \quad F_2 = 10000 \text{ N} \quad \& \quad F_3 = 20000 - 5000 \times 3.5 = 2500 \text{ Nm}$$

Let us denote Overall centroid (Block +Machine) as CC

Height of CC from center of base of the block point O

$h = (5000 \times 3.5 + 60000 \times 3.0/2)/65000 = 1.654$ m

Mass Moment of Inertia (Machine + Block) about Z-axis at base Overall centroid CC $= M_{mz}$

$$M_{mz} = \frac{60000}{12} \times \left(3^2 + 2^2\right) + 60000 \times (1.654 - 1.5)^2 + 5000 \times (3.5 - 1.654)^2 = 83461 \ \ \text{kg m}^2$$

Mass Moment of Inertia about Z-axis at base center point O = M_{moz}

$$M_{moz} = 83461 + 65000 \times (1.654)^2 = 261283 \ \ \text{kg m}^2$$

$$\gamma_z = \frac{M_{mz}}{M_{moz}} = \frac{83461}{261283} = 0.319$$

Limiting Frequencies

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{1.6 \times 10^8}{65000}} = 49.6 \ \ \text{rad/s}$$

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{3.2 \times 10^8}{65000}} = 70.16 \ \ \text{rad/s}$$

$$p_\phi = \sqrt{\frac{k_\phi}{M_{moz}}} = \sqrt{\frac{2.1 \times 10^8}{261283}} = 28.35 \ \ \text{rad/s}$$

Note: For real life problem it is customary to ignore the effect of term mgh while computing limiting frequency, as its influence is negligible. Considering effect of mgh we get frequency of 28.27 rad/s (see below)

$$p_\phi = \sqrt{\frac{k_\phi - mgh}{M_{moz}}} = \sqrt{\frac{2.1 \times 10^8 - 65000 \times 9.81 \times 1.654}{261283}} = 28.27 \ \ \text{rad/s}$$

Response in vertical Y – direction (Undamped):

$$\delta_y = \frac{F_y}{k_y} = \frac{10000}{3.2 \times 10^8} = 3.125 \times 10^{-5} \ \ \text{m}$$

$$\beta_y = \frac{\omega}{p_y} = \frac{94.24}{70.16} = 1.34$$

Vertical Amplitude @ O

$$y_o = \delta_y \times \frac{1}{\left|\left(1-\beta_y^2\right)\right|} = 3.125\times10^{-5} \times \frac{1}{\left|\left(1-1.34^2\right)\right|} = 39.2\times10^{-6} \text{ m}$$

$$= 39.2 \quad \text{microns}$$

Response in $x \,\&\, \phi$ direction: (Undamped)

Natural Frequencies:

Frequency equation is $$p^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) \mp \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$

Substituting for p_x, p_ϕ & γ_z, we get natural frequencies as:

$$p_1 = 25.43 \quad \text{rad/s;} \quad \& \quad p_2 = 97.90 \quad \text{rad/s}$$

$$\beta_1 = \frac{\omega}{p_1} = \frac{94.24}{25.43} = 3.70\,; \quad \beta_2 = \frac{\omega}{p_2} = \frac{94.24}{97.90} = 0.962$$

$$\beta_x = \frac{\omega}{p_x} = \frac{94.24}{49.6} = 1.899\,; \quad \beta_\phi = \frac{\omega}{p_\phi} = \frac{94.24}{28.35} = 3.32$$

$$\beta_x^2 = 3.6\,,\ \beta_\phi^2 = 11.02\,,\ \beta_1^2 = 13.69\,,\ \beta_2^2 = 0.925$$

Amplitudes: Response at DOF location O

$$x_o = \left[\delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right]$$

$$\phi_o = \left[\delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} - \delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right]$$

$$\delta_x = \frac{F_x}{k_x} = \frac{5000}{1.6\times10^8} = 3.125\times10^{-5} \text{ m}$$

$$\delta_\phi = \frac{M_\phi}{k_\phi} = \frac{2500}{2.1\times10^8} = 1.19\times10^{-5} \text{ rad}$$

Substituting for $\beta_x, \beta_\phi, \beta_1, \beta_2, \delta_x, \delta_\phi$, we get:

$(1-\beta_x^2) = -2.6$, $(1-\beta_\phi^2) = -10.02$, $(1-\beta_1^2) = -12.69$, $(1-\beta_2^2) = 0.075$

$(1-\beta_1^2)(1-\beta_2^2) = -12.69 \times 0.075 = -0.95$

$$\frac{(1-\beta_\phi^2)}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{-10.02}{-0.95} = 10.53$$

$$\frac{\beta_x^2}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{3.6}{-0.95} = -3.78$$

$$\frac{\beta_\phi^2}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{11.02}{-0.95} = -11.58$$

$$\frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)} = \frac{-2.6}{-0.95} = 2.73$$

$$h = 1.654 \text{ m}; \qquad \frac{mh}{M_{moz}} = \frac{65000 \times 1.654}{261283} = 0.4115$$

x - Amplitude at DOF Location (Point O)

$$x_o = 3.125 \times 10^{-5} \times (10.53) - 1.654 \times (1.19 \times 10^{-5}) \times (-3.78)$$
$$= 40.34 \times 10^{-5} \text{ m}$$

ϕ - Amplitude at DOF Location (Point O)

$$\phi_o = (1.19) \times 10^{-5} \times (2.754) - 3.125 \times 10^{-5} \times 0.4115 \times (-11.58)$$
$$= 18.25 \times 10^{-5} \text{ rad}$$

Response in $x \& \phi$ direction: (Damped)

It is seen that frequency ratio β_2 is close to unity hence evaluate response using damping $\zeta = 0.1$.

Step 1 $\quad (1-\beta_2^2) = 0.075$

Step 2 $\quad \sqrt{(1-\beta_2^2)^2 + (2 \times \beta_2 \times \zeta)^2} = \sqrt{0.075^2 + (2 \times 0.962 \times 0.1)^2} = 0.206$

Replacing the term $(1-\beta_2^2)$ by $\sqrt{(1-\beta_2^2)^2+(2\times\beta_2\times\zeta)^2}$, keeping the sign of the term under radical same as that of term $(1-\beta_2^2)$, the denominator becomes:

$$(1-\beta_1^2)\sqrt{(1-\beta_2^2)^2+(2\times\beta_2\times\zeta)^2}=(-12.69)\times 0.206=-2.618$$

Substituting this we get

$$\frac{(1-\beta_\phi^2)}{(1-\beta_1^2)\sqrt{(1-\beta_2^2)^2+(2\times\beta_2\times\zeta)^2}}=\frac{-10.02}{-2.618}=3.827$$

$$\frac{\beta_x^2}{(1-\beta_1^2)\sqrt{(1-\beta_2^2)^2+(2\times\beta_2\times\zeta)^2}}=\frac{3.6}{-2.618}=-1.375$$

$$\frac{\beta_\phi^2}{(1-\beta_1^2)\sqrt{(1-\beta_2^2)^2+(2\times\beta_2\times\zeta)^2}}=\frac{11.02}{-2.618}=-4.21$$

$$\frac{(1-\beta_x^2)}{(1-\beta_1^2)\sqrt{(1-\beta_2^2)^2+(2\times\beta_2\times\zeta)^2}}=\frac{-2.6}{-2.618}=1$$

Substituting these, we get amplitudes as:

x - Amplitude at DOF Location (Point O) - Damped

$$x_o=3.125\times10^{-5}\times(3.827)-1.654\times(1.19\times10^{-5})\times(-1.375)$$
$$=14.66\times10^{-5}\text{ m}$$

ϕ - Amplitude at DOF Location (Point O)

$$\phi_o=(1.19)\times10^{-5}\times(1)-3.125\times10^{-5}\times0.4115\times(-4.21)$$
$$=6.6\times10^{-5}\text{ rad}$$

4

VIBRATION ISOLATION

- Principle of Isolation
- Transmissibility Ratio
- Isolation Efficiency
- Isolation Requirements
- Selection of Isolators

Example Problems

VIBRATION ISOLATION

In reference to machine foundation design, the term **ISOLATION** means reduction in the transmission of vibration from machine to the foundation and vice-versa. In other words it means control of transmission of dynamic forces from machine to the foundation and thereby to the adjoining structures and equipment or from the adjoining structures and equipment to the machine through its foundation.

Here we discuss theory of Vibration Isolation covering Principle of Isolation and Isolation Requirements. A brief description of Selection of Isolators has also been included.

4.1 PRINCIPLE OF ISOLATION

Let us consider a damped spring mass system, having mass m, stiffness k and damping c, subjected to dynamic excitation. Consider following two cases:

a) Dynamic Excitation Force $F_E(t)$ is applied at the mass and the Transmitted Force at the base (foundation) is $F_T(t)$ as shown in Figure 4.1-1 (a).

b) Dynamic Excitation Force $F_E(t)$ is applied at the base (foundation) and the Transmitted Force at the mass is $F_T(t)$ as shown in Figure 4.1-1 (b).

In either case, the interest is that the transmitted force from the mass to the foundation (as in case (a)) or from the foundation to the mass (as in case (b)) should be least.

4.2 TRANSMISSIBILITY RATIO

Let us denote the **Transmissibility Ratio** as TR that is defined as the ratio of transmitted force to excitation force.

$$TR = \frac{F_T(t)}{F_E(t)} \tag{4.1-1}$$

Consider the two systems as shown in case (a) and case (b) in Figure 4.1-1. For vibration isolation the interest is that the transmitted force should be minimum in either case i.e. TR should be minimum and it is also true that TR depends upon the dynamic response of SDOF system.

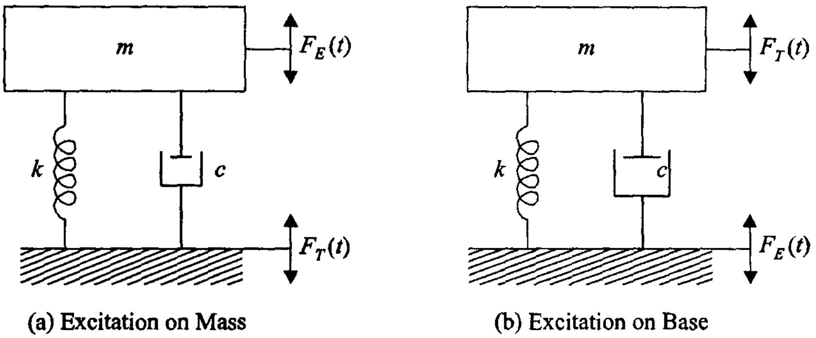


Figure 4.1-1 SDOF Spring Mass System
(a) Excitation on mass
(b) Excitation at base

Let us consider dynamic response of each case:

Case (a) Dynamic Excitation Force $F_E(t)$ is applied at the mass and the Transmitted Force at the base (foundation) is $F_T(t)$. The dynamic force could either be externally applied or internally generated by the machine itself.

i) Let us first consider that the **dynamic force is externally applied**

Let this excitation force be $F_E(t) = F_0 \sin \omega t$ (4.1-2)

Maximum transmitted force $F_T(t)$ to the support is (**see equation 2.2.2-11**) given as;

$$F_T = F_0 \frac{\sqrt{1+(2\beta\zeta)^2}}{\sqrt{(1-\beta^2)^2+(2\beta\zeta)^2}} \quad (4.1\text{-}3)$$

Where ζ is the damping constant & $\beta = \omega/p$ is the frequency ratio

Thus, we get **Transmissibility Ratio** TR as

$$TR = \frac{F_T}{F_E} = \frac{F_T}{F_0} = \frac{\sqrt{1+(2\beta\zeta)^2}}{\sqrt{(1-\beta^2)^2+(2\beta\zeta)^2}} \quad (4.1\text{-}4)$$

ii) Let us now consider that the **dynamic force is internally generated**

Maximum value of transmitted force is given by **(see equation 2.2.3-10)**

$$F_T = F_0 \frac{\sqrt{1+(2\beta\zeta)^2}}{\sqrt{\left(1-\beta^2\right)^2+(2\beta\zeta)^2}} = m_r\, e\,\omega^2 \frac{\sqrt{1+(2\beta\zeta)^2}}{\sqrt{\left(1-\beta^2\right)^2+(2\beta\zeta)^2}} \tag{4.1-5}$$

This equation also gives the Transmissibility ratio TR same as equation 4.1-4 for $F_E = F_0 = m_r\, e\,\omega^2$

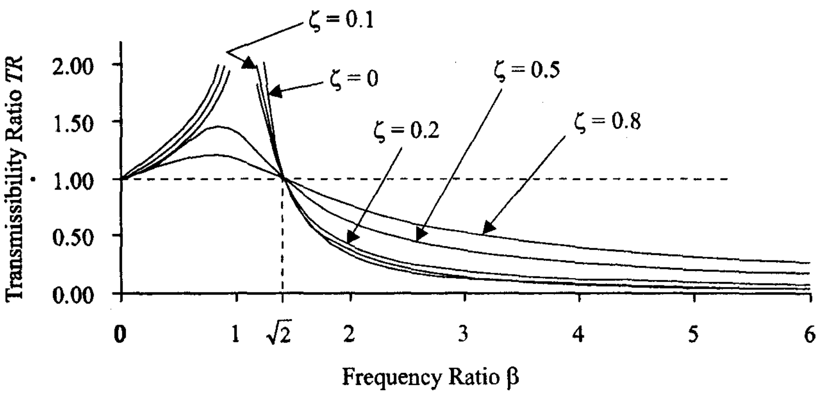


Figure 4.1-2 Transmissibility Ratio *TR* vs. Frequency Ratio β

Case (b) Dynamic Excitation Force $F_E(t)$ is applied at the base (foundation).

For this case, maximum value of transmitted force is given by **(see equation 2.2.4-9)**

$$F_T = -m\,\ddot{y}_g \frac{\sqrt{1+(2\beta\zeta)^2}}{\sqrt{\left(1-\beta^2\right)^2+(2\beta\zeta)^2}} \tag{4.1-6}$$

This equation also gives the Transmissibility ratio TR same as equation 4.1-4 for $F_E = F_0 = -m\,\ddot{y}_g$.

Thus it is clear that irrespective of whether the dynamic force is applied on the mass or applied at the base, the transmitted force remains the same for same system characteristics of SDOF system.

Plot of equation 4.1-4 giving **Transmissibility Ratio** TR vs. Frequency ratio is shown in Figure 4.1-2.

4.3 ISOLATION EFFICIENCY

Let us denote **Isolation Efficiency** as η. Isolation efficiency is thus defined as:

$$\eta = (1 - TR) \tag{4.1-7}$$

(Generally it is convenient to represent isolation efficiency $(\eta \times 100)$ *in percentage.)*

It is clear from this equation that lesser the **Transmissibility Ratio** TR better is the **Isolation Efficiency** η. Having determined Transmissibility Ratio TR (equation 4.1-4), we work out the **Isolation Efficiency** η.

Table 4.1-1 Isolation Efficiency η vs. Frequency Ratio β for different Damping Ratios ζ

Isolation Efficiency

Frequency ratio	Damping			Frequency ratio	Damping		
β	ζ=0.0	ζ=0.1	ζ=0.2	β	ζ=0.0	ζ=0.1	ζ=0.2
2	0.67	0.64	0.59	4.2	0.94	0.92	0.88
2.2	0.74	0.72	0.66	4.4	0.95	0.93	0.89
2.4	0.79	0.77	0.71	4.6	0.95	0.93	0.9
2.6	0.83	0.81	0.75	4.8	0.95	0.94	0.9
2.8	0.85	0.83	0.78	5	0.96	0.94	0.91
3	0.88	0.85	0.81	5.2	0.96	0.94	0.91
3.2	0.89	0.87	0.83	5.4	0.96	0.95	0.92
3.4	0.91	0.89	0.84	5.6	0.97	0.95	0.92
3.6	0.92	0.9	0.85	5.8	0.97	0.95	0.92
3.8	0.93	0.91	0.87	6	0.97	0.96	0.93
4	0.93	0.91	0.87				

Substituting equation 4.1-4 in equation 4.1-7, we get:

$$\eta = (1 - TR) = \left(1 - \frac{\sqrt{1 + (2\beta\zeta)^2}}{\sqrt{\left(1 - \beta^2\right)^2 + (2\beta\zeta)^2}} \right) \tag{4.1-8}$$

Plot of equation (4.1-8) is given in Figure 4.1-3.

Isolation efficiency values for frequency ratio $\beta \geq 2$ are tabulated and given in Table 4.1-1 for different values of isolator damping.

From the Figure 4.1-2 and Figure 4.1-3, following observations are made:

i) Transmissibility Ratio TR is less than unity i.e. $TR < 1$ only for frequency ratio greater than $\sqrt{2}$ i.e. $\beta > \sqrt{2}$

ii) For frequency ratio greater than $\sqrt{2}$, TR decreases with decrease in damping value. In other words, TR is lower for zero damping compared to 10% damping i.e. **damping is not desirable for Isolation.**

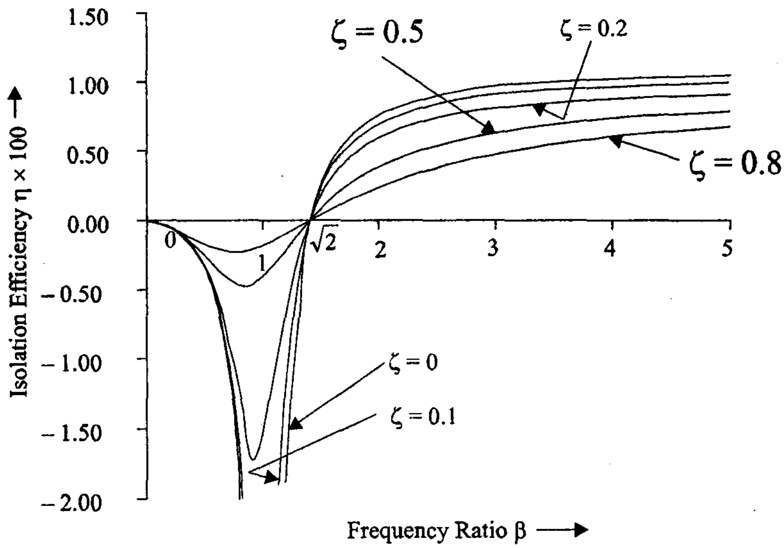


Figure 4.1-3 Isolation Efficiency η vs. Frequency Ratio β for different Damping Values of ζ

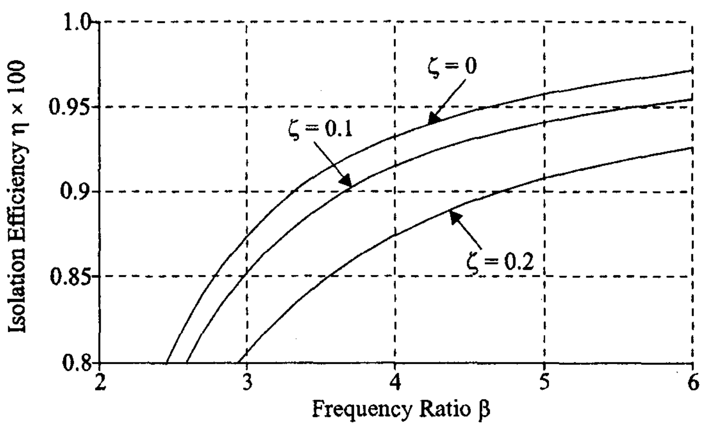


Figure 4.1-4 Isolation Efficiency η > 80% vs. Frequency Ratio β > 2

4.4 ISOLATION REQUIREMENTS

Generally speaking, for machine foundation applications, one would be interested in isolation above 85 % otherwise the very purpose of isolation gets defeated. In view of this, let us view the isolation plot for η > 80 %, which obviously means $\beta > 2$ as shown in Figure 4.1-4. It is noticed from the plot that even for zero damping it requires $\beta = 3$ for $\eta = 88\%$ and $\beta = 5$ for $\eta = 96\%$. It gives an impression that one can achieve as high isolation as desired by just increasing frequency ratio. In reality, this impression, however, does not hold any ground. It is evident from Figure 4.1-3 that there is hardly any appreciable gain in η for $\beta > 6$ which corresponds to $\eta = 97\%$.

This implies that one can at best target for isolation efficiency of about $\eta = 97\%$ knowing that presence of damping in isolators, if any, shall reflect in reduction of η.

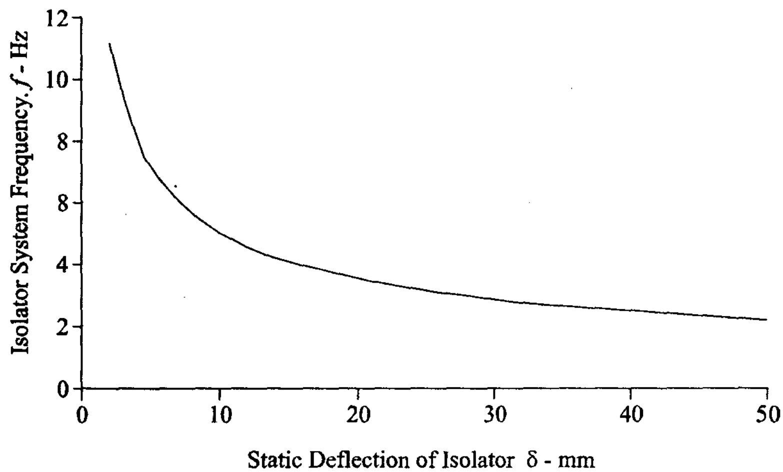


Figure 4.1-5 Isolator system frequency *f* vs. static deflection of isolator unit δ

Let us examine a few more aspects related to this issue

- It is obvious that higher the η, higher shall be β and lower shall be the frequency of isolation system p ($p = \omega/\beta$).
- It is also known that lower the p, lower shall be stiffness of the isolation system k and this lower stiffness would result in higher static deflection δ under self-weight of the system.

We know that $k = mp^2$ & $\delta = mg/k$;

This gives $$p = \sqrt{\frac{g}{\delta}} \text{ rad/s; } or \quad f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \text{ Hz} \qquad (4.1\text{-}9)$$

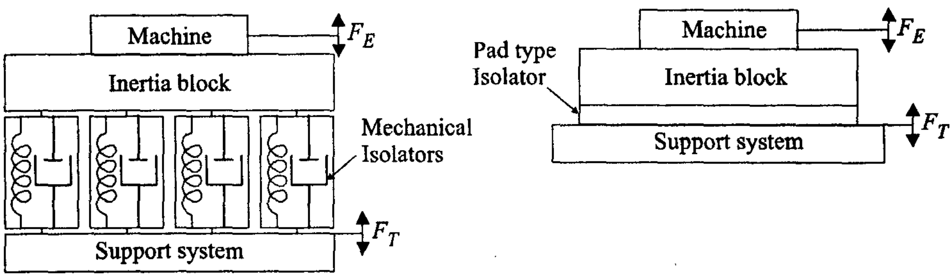
Plot of equation 4.1-9 is given in Figure 4.1-5.

4.5 SELECTION OF ISOLATORS

Consider a machine foundation system on isolators as shown in Figure 4.1-6. Figure shows Machine housed on Inertia Block supported by Isolators, which in turn rests on Support System (Support Structure/Ground). Let us briefly describe inertia block and isolators.

Inertia Block: Inertia block, generally made of RCC, is provided to support the machine. It is made heavy enough (mass 2 to 3 times that of the machine) so as to keep the overall Centroid in stable position. It should be rigid enough so as to have its natural frequencies much above machine speed and its harmonics.

Isolators: These are commercially available devices (as per required specifications) to be installed between inertia block and support system. There are many types of isolators available commercially. We limit our discussions to only two types a) Mechanical Isolators (spring type with or without damping) and b) Sheet/Pad type isolators (Cork, Rubber sheets etc).



(a) Mechanical Isolators (b) Sheet/pad type Isolators

Figure 4.1-6 Machine Foundation Isolation System

Selection of Isolator: It is totally dependent on machine excitation frequency, target isolation efficiency and overall mass of machine + mass of inertia block. There are many ways one can arrive at the specification for required isolators. One of the selection criteria for a) **Mechanical isolators** and b) **Sheet/Pad type isolators** is given as under.

Let us consider machine and isolation block parameters as under:

Machine mass	m_1	kg
Isolator Mass	m_2	kg
Machine speed (rpm)	N	rpm
Excitation Frequency ($\omega = 2\pi N/60$) rad/sec	ω	rad/sec
Target Transmissibility Ratio	TR	

Target Isolation Efficiency (see equation 4.1-7)	η	
For this η, required frequency ratio (From Table 4.1-1)	β	
Thus, the required frequency of isolation system ($p=\omega/\beta$ & $f=p/2\pi$)	f	Hz
For this f, required deflection of isolator (from Figure 4.1-5)	δ	mm

Select the isolator to match this static deflection

a) Mechanical isolators (spring damper unit)

Let the isolator capacity (single isolator) be	R	N
Total mass of machine + isolator ($m = m_1 + m_2$)	m	kg
Number of isolators required ($q = mg/R$)	q	
Vertical stiffness of each isolator ($k_y = R/\delta$)	k_y	N/mm
Lateral stiffness (as specified by manufacturer)	k_x	N/mm
Damping of isolator (as specified by manufacturer)	ζ	%

b) Sheet/ Pad Type isolators (Cork sheets, Rubber pads etc)

Elastic modulus of sheet isolator	E_s	N/m^2
Area of isolation block in contact with sheet/pad isolator	A_b	m^2
Required Thickness of Sheet/Pad isolator ($t = \frac{E_s A_b}{m g}\delta$)	t	mm

Here g is in m/s^2 & δ is in mm

Design of Vibration Isolation System for Real Life Machines is covered in Chapter 12

EXAMPLE PROBLEMS

P 4.1-1

A Machine having mass of 1000 kg operating at 600 rpm is supported on the Foundation resting on the soil. Consider that the machine generates only vertical unbalance force of 500 N at operating speed, design the isolation system with 90 % isolation efficiency such that dynamic force of only 50 N (max) gets transferred to the soil.

Solution:

Provide inertia block to support the machine.

Mass of machine	1000	kg
Assume Mass of inertia block be (twice machine mass)	2000	kg

Total Mass 3000 kg

Operating speed N 600 rpm

Excitation frequency $\omega = \dfrac{2\pi N}{60} = 62.83$ rad/s

Required Isolation efficiency 90 % $\eta = 0.9$

Assume Isolator damping as 10 %

From Figure 4.1-4 or Table 4.1-1(for $\zeta = 0.1$) $\beta = 3.6$

Isolator frequency $p = \dfrac{62.83}{3.6} = 17.45$ rad/s

Or $f = \dfrac{17.45}{2\pi} = 2.78$ Hz

Required isolator deflection ($p = \sqrt{\dfrac{g}{\delta}}$) $\delta = 9810/(17.45)^2 = 32.21$ mm

Provide 4 isolators one at each corner underneath inertia block $q = 4$

Required capacity of each isolator $R = \dfrac{mg}{q} = \dfrac{3000 \times 9.81}{4} = 7357$ N

Let us assume that nearest available isolator (From manufacturers catalogue) gives isolator of capacity of 8000 N and deflection value of 35 mm $\pm 10\%$.

Minimum Deflection $\delta_{min} = 35 - 0.1 \times 35 = 31.5$ mm

Maximum Deflection $\delta_{min} = 35 + 0.1 \times 35 = 38.5$ mm

Vertical stiffness of each isolator

$$k_{y(max)} = (8000/31.5) = 253.97 \text{ N/mm} = 2.54 \times 10^5 \text{ N/m}$$

$$k_{y(min)} = (8000/38.5) = 207.79 \text{ N/mm} = 2.078 \times 10^5 \text{ N/m}$$

With this let us analyse the system. Let us assume that common centroid of machine and inertia block lies at center of inertia block. Let us also consider that isolators are placed symmetrically around inertia block such that center of stiffness matches well with the center of mass (common centroid).

Let us first consider maximum isolator stiffness:

Total vertical stiffness (max) $4 \times 2.54 \times 10^5 = 1.016 \times 10^6$ N/m

Undamped frequency $p=\sqrt{\dfrac{1.016\times10^{6}}{3000}}=18.4\quad \text{rad/s}$

Frequency ratio $\beta=\dfrac{\omega}{p}=\dfrac{62.83}{18.4}=3.41$

For this β we get isolation efficiency (Figure 4.1-4 or Table 4.1-1 $\zeta=0.1$)

$\eta=89\,\%$

$TR=(1-\eta)=(1-.89)=0.11$

Thus maximum force transmitted to support structure is = $0.11\times500=55\;\;\text{N}$

Vertical amplitude at machine location

$$A_y=\frac{500}{1.016\times10^{6}}\frac{1}{\sqrt{(1-(3.41)^2)^2+(2\times3.41\times0.1)^2}}=0.4615\times10^{-4}\quad \text{m}\quad =46\quad \text{microns}$$

Let us first consider minimum isolator stiffness:

Total vertical stiffness (min) $4\times2.078\times10^{5}=8.312\times10^{5}\quad \text{N/m}$

Undamped frequency $p=\sqrt{\dfrac{8.312\times10^{5}}{3000}}=16.64\quad \text{rad/s}$

Frequency ratio $\beta=\dfrac{\omega}{p}=\dfrac{62.83}{16.64}=3.776$

For this β we get isolation efficiency (Figure 4.1-4 or Table 4.1-1 $\zeta=0.1$) $\eta=91\,\%$

$TR=(1-\eta)=(1-.91)=0.09$

Thus maximum force transmitted to support structure is = $0.09\times500=45\quad N$

Vertical amplitude at machine location

$$A_y=\frac{500}{8.312\times10^{5}}\frac{1}{\sqrt{(1-(3.776)^2)^2+(2\times3.776\times0.1)^2}}=4.53\times10^{-5}\quad \text{m}\quad =45.3\quad \text{microns}$$

It is clear that isolation efficiency is approximately 90 % whether we consider higher or lower stiffness value. Force transmitted is also close to the requirement. Hence OK.

PART - II

DESIGN PARAMETERS

5. Design Sub-grade Parameters
6. Design Machine Parameters
7. Design Foundation Parameters

5

DESIGN SUBGRADE PARAMETERS

- Soil Mass participation
- Embedment Effect
- Soil damping
- Dynamic Soil Modulus
- Coefficient of Subgrade Reaction
- Design Soil Parameters
- Equivalent Soil Springs
- Foundation Supported over Elastic Pad
- Foundation Supported over a set of springs
- Foundation Supported over Piles- Equivalent Pile Springs

Example Problems

DESIGN SUBGRADE PARAMETERS

5.1 INTRODUCTION

Following Foundation Support Systems are commonly employed in practice for supporting machines:

i) Foundation Supported directly over soil
ii) Foundation Supported over an Elastic Pad
iii) Foundation Supported over a Set of Springs
iv) Foundation Supported over Piles

Equivalent Springs for all the four systems are covered in § 5.5. The first system i.e. Foundation Supported directly over soil, is discussed in relatively more detail whereas the rest of the systems are discussed briefly.

Soil system is a complex entity in itself and there are many uncertainties associated with it. Only application oriented aspects related to machine foundation design are discussed in this Chapter.

5.2 SOIL ASPECTS INFLUENCING SOIL STRUCTURE INTERACTION

There are many uncertainties associated with site soil exploration, evaluation of dynamic properties and its modeling. Even for static case, modeling of soil, at times, becomes a difficult task. Its representation becomes still more difficult for dynamic case, especially for machine foundation design as soil structure interaction significantly influences the response of the machine foundation system.

Soil investigation of a site is an essential part of the project. For any site, the dynamic soil data is never a unique value. There are various factors that do affect the dynamic soil properties but quantification of their influence is rather difficult. At times the dynamic soil parameters of a site evaluated by one test agency may be in variance with that of other. Such variations could be on account of the method of the test, the quality and level of automation of test equipment, interpretation of the results, etc.

The process of evaluating critical soil properties that influence soil structure interaction is probably the most difficult part of the machine foundation design. The significant aspects of soil properties, which influence soil structure interaction, are:

- Energy Transfer Mechanism
- Soil Mass Participation in Vibration of Foundations
- Effect of Embedment of Foundation
- Applicability of Hook's Law to Soil
- Reduction in Permissible Soil Stress
- Dynamic Soil Parameters

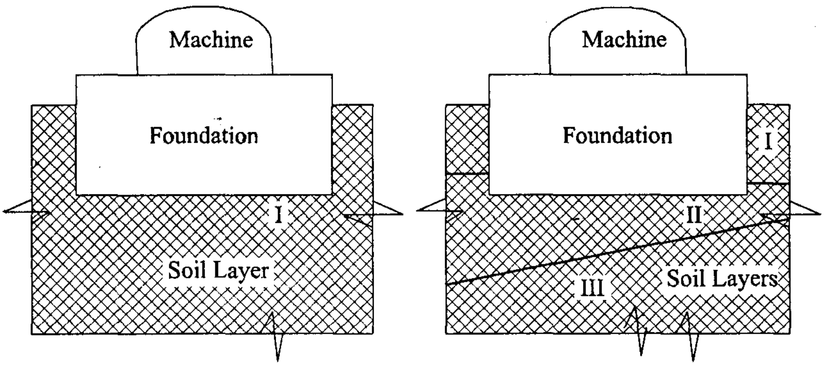
It is well founded that for mathematical modeling of any system, assumptions and approximations are often made in order to simplify the level of complexity resulting in reduced size of the problem. Whereas most of the assumptions made for foundation are generally quantifiable, it is not so with those made for the soil. It requires enormous computational effort to quantify these assumptions and approximations. In majority of the cases, quantification is not attempted at all. A careful investigation of soil characteristics that primarily influence soil structure interaction, therefore, becomes essential. These variations themselves occur in a broad band and so is their influence on the dynamic response.

5.2.1 Energy Transfer Mechanism

The **basic principle** underlying machine foundation design is that the dynamic forces of machine are transmitted to the soil through the foundation in such a way that all kinds of harmful effects are eliminated. In other words, the energy content of the dynamic forces is transmitted to the soil through the foundation. The energy travels in form of waves in all direction in the soil and gets absorbed in the soil itself. If the soil underneath the foundation is not a single layer (but constitutes of several layers), part of the energy from the lower layer will reflect back into the upper layer and thereby into the system.

A typical machine foundation system would mean a machine supported by a foundation block, which in turn rests on the soil. The foundation block is generally embedded to a certain depth below free surface of the soil. A realistic soil representation may contain some variation in the soil strata along the depth. A schematic representation of such a system is shown in Figure 5.2-1. Part (a) of the figure shows single infinite layer of the soil whereas part (b) shows soil as layered media having number of layers (three layers chosen arbitrarily- one layer is considered horizontal and the other inclined).

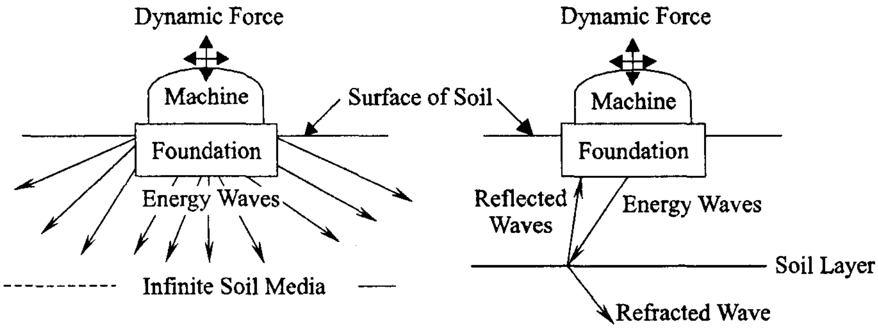
Under static condition, combined machine and foundation mass exerts pressure on the soil and soil in turn deforms. Under dynamic conditions, machine exerts dynamic forces to the soil through the foundation and under the influence of these dynamic forces, the foundation interacts with the soil activating dynamic soil structure interaction, which significantly influences the dynamic response of machine foundation system. The influence is very predominant for block type of foundations whereas it is not so for frame type of foundations.



(a) One Soil Layer only (b) Three Soil Layers - Horizontal & Inclined

Figure 5.2-1 A Schematic Representation of Machine - Foundation on Layered Soil Media - I, II, III Represent Soil Layers

In reality, there may be many more layers below the foundation that influence the process of energy transfer. Such energy transmission takes place by three types of waves namely, Primary Wave or Compression wave (P-Wave), Secondary Wave or Shear Waves (S- Wave) and Surface Wave or Rayleigh Wave (R-Wave).



(a) Soil as Single layer (b) Soil as Layered Media

Figure 5.2-2 Typical Representation of Energy Transmission from Foundation to Soil

Out of these, Rayleigh waves carry a much larger proportion of the total input energy (say about 60 % or more) compared to Shear waves and Primary waves. Hence, from the point of view of machine foundation design, it is the Rayleigh wave that bears more importance.

Computation of such energy transfers from the foundation to the soil through layered media involving various refractions & reflections is a complex task and **its evaluation in true sense is not only difficult but at times becomes impossible.**

In summary, this aspect of soil is not quantifiable from the point of view of machine foundation design.

5.2.2 Soil Mass Participation in Vibration of Foundations

It is a reality that part of the soil mass vibrates along with foundation vibration. The issues that need to be addressed are:

- What is the extent of soil that vibrates with the foundation?
- Does the vibrating soil mass depend upon mode of vibration?
- Does it have any influence on the soil stiffness and damping?
- Can these aspects be quantified? .....and so on.

Just for the sake of understanding, a schematic representation of soil mass participation along with the foundation is shown in Figure 5.2-3. All the shapes shown are arbitrary and carry no relevance to any particular type of soil. These are shown only to present the idea. It can be said qualitatively that quantum of participating soil mass depends not only on various soil parameters but could also depend upon type of dynamic force generated by machine.

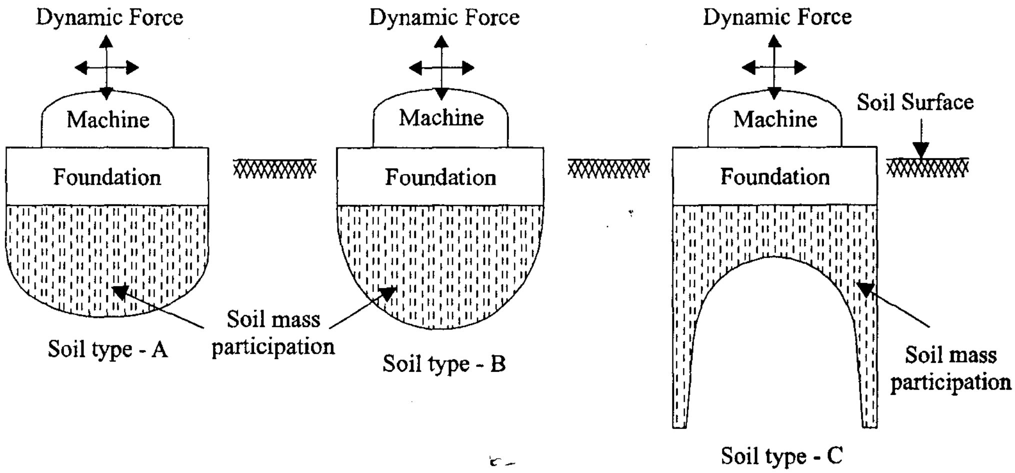


Figure 5.2-3 Typical Representation of Soil Mass Participation with Foundation Vibration for different soil types

There are various opinions expressed by different authors regarding the soil mass participation. According to some, the mass of soil moving with the foundation varies with the dead load, exciting force, base contact area, mode of vibration and the type of soil. As per some other authors, the size of soil participating mass is related to bulb shaped stress distribution curve under the effect of uniformly distributed load.

Till date no concrete formulation is available giving quantification of soil mass participation for different types of soils and what is lacking is perhaps the validation of the results? It is generally the view that soil mass participation will increase the overall effective mass of the machine foundation system and thereby tend to reduce the natural frequency.

Here again, this aspect of soil is also not quantifiable from the point of view of machine foundation design.

For the design purposes, author therefore recommends:

a) For under-tuned foundations, soil mass participation to be ignored
b) For over-tuned foundations, the frequency margin to be increased by additional 5% i.e. natural frequencies to be kept away from operating speed by 25% instead of normal 20 %

5.2.3 Effect of Embedment of Foundation

All machine foundations are invariably embedded partly in to the ground. Many authors have studied this effect and have made varying observations. Some have reported that effect of embedment causes increase in natural frequency and some have reported that it causes reduction in amplitudes. **By and large, it has been generally agreed that embedment tends to reduce the dynamic amplitudes.** The reduction in the amplitudes could either be on account of change in stiffness, change in damping, change in soil mass participation or their combination. This aspect **has also not been quantified** for all types of soils.

Here again, this aspect of soil is also not quantifiable from the point of view of machine foundation design.

For design purposes, author recommends that it will be on the safe side to ignore the embedment effect while computing dynamic response.

5.2.4 Applicability of Hook's Law to Soil

From theory of elasticity, we know that all homogeneous and isotropic materials follow Hook's law i.e.

Young's modulus of elasticity $$E = \frac{stress}{strain} = \frac{\sigma}{\varepsilon} \qquad (5.2\text{-}1)$$

Where, σ is the direct stress (tension/compression)

& ε is the associated linear strain

Let us first examine whether soil behaves like an elastic body? In other words, if the soil is to be represented as an elastic material, it must obey Hook's law. If it does, it becomes convenient to mathematically represent the soil as any other elastic material.

In this context, it may be noted that **Modulus of Elasticity** as well as **Poisson's Ratio** have been found to change with normal pressure. Further, certain soil types exhibit large settlements of foundations under vibratory loads and in certain cases soils even lose their resistance to shear and behave more like liquids. In view of such characteristics, it is difficult to straight away accept the soil as an elastic body for all type of soils.

As a general guideline, it is considered good enough to assume that the foundation undergoes elastic vibrations as long as the total pressure (including static & dynamic pressure) on the soil is lower than its elastic limit.

5.2.5 Reduction in Permissible Soil Stress

For the soil to behave as an elastic material, it is necessary that the total pressure (static + dynamic) exerted by the foundation on the soil remains within elastic limits. A reasonable margin therefore should be kept while assigning bearing capacity to the soil intended to be used for machine foundation application. The dynamic pressure produced by machines not only affects the foundation directly under the machine but to other foundations too, which are away from machine, as the energy gets transmitted through soil in all directions.

It is therefore desirable to keep intended margins for even static foundations i.e. foundations for static equipment including foundations of the building housing machines etc.

Generally **recommended guidelines** for permissible soil pressures for machine foundations and buildings/structures housing machine foundations are:

- For low rpm machines, no reduction of soil stress is needed i.e. one can go up to 100% of the bearing capacity.
- For medium rpm machines, reduction factor should be 10% i.e. permissible bearing pressure should be limited to 90% of the allowable bearing capacity.
- For high rpm machineries, reduction factor should be 20% i.e. permissible bearing pressure should be limited to 80% of the allowable bearing capacity.
- For machines like crushers & hammers producing impact loads, the reduction factor should be 30 % to 50 % i.e. permissible bearing pressure shall be 70% to 50% of the bearing capacity.

5.2.6 Damping in Soil

Damping is an inherent property of soil and its influence on **Forced Vibration Response** is significant but during resonance or near resonance conditions. Different soils exhibit different damping properties depending upon their soil composition and other characteristic parameters. In case of embedded foundations, the depth of embedment also influences damping properties.

Soil Damping comprises of a) Geometrical Damping and b) Material Damping. Whereas Geometrical Damping represents energy radiated away from the foundation, the Material Damping represents the energy lost within the soil due to Hysteresis effects. In the context of application to machine foundation design, the contribution of geometrical damping to Rocking modes of vibration has been reported to be of low order compared to Translational and Torsional modes of vibration.

Damping in the soil has been observed to be both strain and frequency dependent. Same soil exhibits different damping characteristics at different strain levels and similar is the variation for the frequency of excitation. In other words, soil damping not only depends upon stress/strain/contact pressure distribution but also on frequency of vibration. Representation of **frequency dependant soil damping** has not found appropriate place in **Design Industry** for **Real Life Design Problems**. On the other hand, representation in form of **Equivalent Viscous Damping** has found larger acceptability.

It is to be remembered that damping plays role only during resonance. If one is able to avoid resonance of foundation with the machine excitation frequencies at the design stage itself, the significance of damping could be felt only during **Transient Resonance**.

In author's opinion, considering strain and frequency dependent geometrical/radiation damping, as design office practice, is not only difficult but inconvenient too. **The commonly available mathematical tools with industry in general** are not geared to accommodate this type of damping. Use of high-end analytical tools, however, is not recommended for design purposes in view of tight project schedules.

In the absence of any specified data for damping value of a site, the damping coefficient equal to 8 to 10% i.e. $\zeta = 0.08\ to\ 0.1$ could safely be considered for computing response at resonance.

5.3 DYNAMIC SOIL PARAMETERS

The Basic Dynamic Soil Properties (**Dynamic Soil Modulus**) that are required for machine foundation design are Dynamic Shear Modulus G /Elastic Modulus E , Poisson's Ratio ν , Damping Constant ζ and Mass Density ρ .

In addition, evaluation of **Coefficients of Subgrade Reaction** viz. Coefficient of **Uniform Compression** C_u, Coefficient of **Uniform Shear** C_τ, Coefficient of **Non-Uniform Compression** C_ϕ & Coefficient of **Non-Uniform Shear** C_ψ for each site is also recommended.

5.3.1 Dynamic Soil Modulus

From theory of elasticity, we know that the ratio of stress to strain for any elastic material is called its modulus. Ratio of normal stress to normal strain is termed as Elastic Modulus ' E ' and ratio of shear stress to shear strain is termed as Shear Modulus G.

We can write this as:

$$E = \frac{Nomal\ Stress}{Normal\ Strain} = \frac{\sigma}{\varepsilon} \qquad (5.3\text{-}1)$$

$$G = \frac{Shear\ Stress}{ShearStrain} = \frac{\tau}{\gamma} \qquad (5.3\text{-}2)$$

We also know from theory of elasticity that:

$$G = \frac{E}{2(1+\nu)} \qquad (5.3\text{-}3)$$

Here ν is the Poisson's Ratio

For a given soil strata, these properties are determined using either laboratory or field tests.

From theory of wave propagation, we know that

$$V_s^2 = \frac{G}{\rho}\ ,\ V_R^2 = \frac{G}{\rho} \quad \&\ V_c^2 = \frac{E}{\rho} \qquad (5.3\text{-}4)$$

Here V_s represents Shear Wave Velocity, V_R represents Rayleigh Wave Velocity & V_c represents Compression Wave Velocity and ρ represents Mass density. These velocities, for a site, are evaluated using laboratory or field test methods. The commonly employed laboratory and field test methods are:

Laboratory methods:

- Resonant Column Test
- Cyclic Simple Shear Test

- Cyclic Torsional simple Shear
- Cyclic Triaxial Compression Test

Field methods:

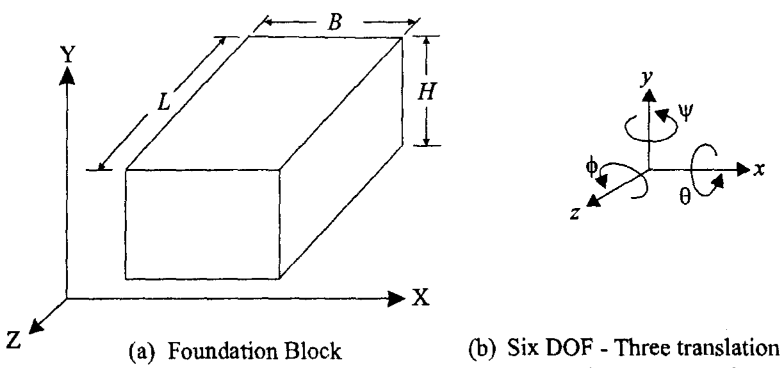
- Cross- Borehole Wave Propagation Test
- Up-Hole or Down-Hole Wave Propagation Test
- Surface Wave Propagation Test

Only application oriented aspects of dynamic soil modulus are discussed here. For details of the test methods, readers are advised to refer applicable codes of practices and standard text books/reference books on this subject. Having determined these velocities with the help of one or more of these tests, E & G are computed as per above equation 5.3-4 and Poisson's Ratio ν is computed using equation 5.3-3.

It may be noted that these values of E & G are applicable at the **Overburden-Pressure** and **Shear Strain Levels** corresponding to respective test methods. For the design purposes, these values are modified for **Overburden Pressure** and **Shear Strain Level** corresponding to the actual foundation **(See § 5.4)**.

5.3.2 Coefficients of Subgrade Reaction

Coefficient of Subgrade Reaction, in a specified deformation mode, is defined as the ratio of the applied pressure to the induced deformation in that deformation mode.



(a) Foundation Block

(b) Six DOF - Three translation x, y, z along X, Y & Z axes & three rotations θ, ψ, ϕ about X, Y & Z axes

Figure 5.3-1 Degree of freedom (DOF's) of a Typical Foundation Block

Consider a foundation block resting on the soil as shown in Figure 5.3-1 (a). The six deformation modes of the block represent six degrees of freedom of the foundation i.e., the block exhibit six deformations namely three translations x, y, z and three rotations $\theta, \psi \& \phi$ as shown in Figure 5.3-1 (b).

Here, vertical deformation y represents **uniform compression** of the soil along Y-axis, lateral translation x & z represent **uniform shear** of the soil along X & Z axes respectively, rotation $\theta \& \phi$ represent rocking about X & Z axes respectively causing **non-uniform compression** of the soil and rotation ψ represents rotation about Y-axis causing **non-uniform shear** of the soil.

Thus in effect, there are only **Four Independent Soil Deformation Modes** namely:

i) **Uniform Compression**
ii) **Uniform Shear**
iii) **Non-uniform Compression**
iv) **Non-uniform Shear**

Thus there will be 4 **Coefficients of Subgrade Reaction** each related to the deformation mode as given above. These **Coefficients** are termed as:

- C_u **Coefficient of Uniform Compression**
- C_τ **Coefficient of Uniform Shear**
- C_ϕ **Coefficient of Non-Uniform Compression &**
- C_ψ **Coefficient of Non-Uniform Shear**

Let us first understand evaluation of **Coefficient of Uniform Compression** and thereafter we shall discuss its correlation with other Coefficients as well as with Dynamic Soil Modulus.

5.3.2.1 Coefficient of Uniform Compression C_u

For a specified deformation mode, **Coefficient of Subgrade Reaction** is defined as the ratio of the pressure to the deformation. Thus Coefficient of Uniform Compression C_u becomes ratio of pressure (vertical pressure causing compression) to the corresponding vertical (compressive) deformation.

Consider a foundation block, as shown in Figure 5.3-2, having base contact area A in X-Z Plane. Consider that the block rests over the soil and a Compressive Force F_y is applied to the block along Y-direction.

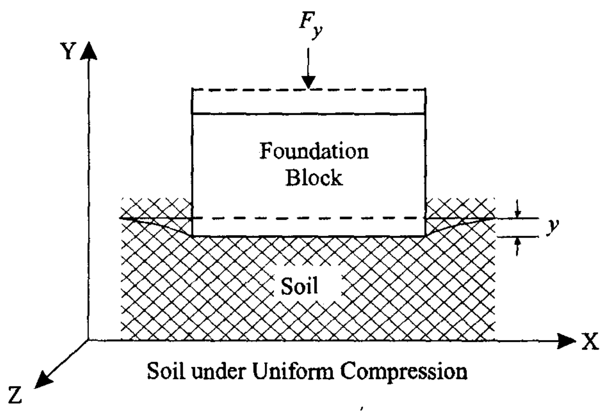


Figure 5.3-2 Force F_y Applied to the Foundation Block Resting over the Soil

Uniform Pressure p_y developed in the soil is given as $p_y = \frac{F_y}{A}$

Consider that this pressure causes uniform compression y to the soil. The **Coefficient of Uniform Compression** C_u is therefore given as ratio of pressure p_y to deformation y. This gives:

$$C_u = \frac{p_y}{y} = \frac{\frac{F_y}{A}}{y} = \frac{F_y}{A\,y} \tag{5.3-5}$$

This is how we understand **Coefficient of Uniform Compression** C_u.

Methods of Evaluation: Various field and laboratory test methods are available for evaluation of C_u. Designers may choose any of the method for evaluation of C_u but author, however, prefers use of field tests. Following field tests are normally recommended for evaluation of Coefficient of Uniform Compression:

5.3.2.1.1 Cyclic Plate Load Test

This test is based on the elastic settlement of a test plate under the influence of uniform loading intensity. For details of test set-up and test method, readers are requested to refer to relevant books on soil dynamics and applicable codes. The basic steps involved (listed for reference) are as under:

i. A pit is excavated at the desired location up to the depth at which the soil properties are to be evaluated. The pit dimensions are not less than 5 times the width of the plate.

ii. A plate (of specified dimension and thickness) is placed in the pit.
iii. First incremental static load is applied to the plate. The load intensity is maintained constant till the rate of settlement becomes negligible. The associated settlement for the load intensity is recorded.
iv. The static load is totally removed (after deformation is stabilized).
v. The next incremental load is applied, load intensity is maintained constant till the rate of settlement becomes negligible and the settlement recorded.
vi. The entire procedure (loading, unloading and incremental re-loading) is repeated till estimated ultimate load is reached.
vii. Ratio of load intensity (pressure) to elastic settlement (for each load) gives Coefficient of Uniform Compression.

Computation of Coefficient of Elastic Uniform Compression

a.	Depth of pit	D	m
b.	Size of the plate	A_p	m^2
c.	Loading intensity (pressure)	p_r	kN/m^2
d.	Total settlement for the loading intensity	S_t	m
e.	Settlement after removal of load	S_p	m
f.	Elastic Settlement	$S_e = S_t - S_p$	m
g.	Coefficient of Elastic Uniform Compression of soil C_u		

$$C_u = \frac{p_r}{S_e} \qquad kN/m^3 \qquad (5.3\text{-}6)$$

The value of C_u thus evaluated is applicable for Area of the foundation equal to Area of the Plate A_p and the corresponding Overburden Pressure. The overburden pressure at the level of the test plate is taken as the pressure corresponding to the depth equal to depth of the test plate+ half the width of the plate. ***This is as recommended by the books giving test procedures for the Cyclic Plate Load Test.***

For a given foundation, the value of C_u is to be modified for actual Area of the Foundation and corresponding Overburden Pressure **(See § 5.4)**.

5.3.2.1.2 Vertical Resonance Test on the Foundation Test Block

This test is based on the resonance of the test block excited by an oscillator producing only vertical dynamic force. For details of test set-up and test method, readers are requested to refer to relevant books on soil dynamics and applicable codes. The basic steps involved (listed for reference) are as under:

i. A pit is excavated at the desired location up to the depth at which the soil properties are to be evaluated
ii. A RCC block is cast in the pit. Pit dimensions should be such that there is a clear gap of minimum 1 m all around the block. Desired anchor bolts, to hold the oscillator on top of the block, are placed in position during casting of the block.
iii. The oscillator is mounted over the block and held down with the help of anchor bolts. (The Oscillator must be placed centrally over the block i.e. CG of the oscillator must lie on the vertical line passing through CG of the block). The oscillator should be such that it produces only vertical excitations.
iv. Two transducers (acceleration/ displacement) pick-ups are mounted on the top of the block to record vertical oscillations of the block.
v. A known eccentricity is set for the eccentric masses of the oscillator so as to produce a known vertical dynamic force as function of frequency.
vi. The oscillator frequency is increased in steps from the initial value and swept through a range (from minimum of 1 Hz. to maximum operating frequency of the oscillator).
vii. The vertical amplitude is measured at each speed of operation.
viii. The entire procedure is repeated for another set of eccentricity/forces.

Computation of Coefficient of Elastic Uniform Compression

a.	Mass of the test block	m_b	kg
b.	Mass of oscillator	m_o	kg
c.	Total Mass	$m = m_b + m_o$	kg
d.	Base contact area of the block with soil	A_b	m^2
e.	Resonant frequency observed	f_b	H_z
f.	Coeff. of Elastic Uniform Compression of soil	C_u	N/m^3
g.	Stiffness of soil	$k_s = C_u \times A_b$	N/m
h.	Natural Frequency of the block	$f_n = \frac{1}{2\pi} \times \sqrt{\frac{k_s}{m}} = \sqrt{\frac{(C_u \times A_b)}{m}}$	H_z

This is nothing but the resonant frequency observed i.e. $f_b = f_n$

This gives $$f_b = \frac{1}{2\pi} \times \sqrt{\frac{k_s}{m}} = \sqrt{\frac{(C_u \times A_b)}{m}} \quad H_z$$

Rearranging terms, we get Coefficient of Elastic Uniform Compression C_u

$$C_u = 4\pi^2 \times f_b^2 \times \frac{m}{A_b} \qquad N/m^3$$

Or $$C_u = 4\pi^2 \times f_b^2 \times \frac{m}{A_b} \times 10^{-3} \quad \text{kN/m}^3 \tag{5.3-7}$$

The value of C_u thus evaluated is applicable for Area of the foundation equal to 'A_b' and the corresponding Overburden Pressure which is taken as equal to the depth of the test block + half the width of the block.

For a given foundation, the value of C_u is to be modified for actual Area of the Foundation and corresponding Overburden Pressure **(See § 5.4)**.

(Just for information, reference is made to Indian standard code of practice IS 5249 that gives test procedure for evaluation of C_u)

5.3.2.1.3 Correlation with Soil Modulus $E, G \,\&\, \nu$

Qualitative assessments of these variations have been reported in the literature by many authors but the quantification has been restricted to **Empirical Relations** only. The **empirical relationship** presented by Barkan (1962) is considered most appropriate one and practically every other author refers to it till date. The expression (**Barkan -1962**) giving relationship of C_u *with* $E, G \,\&\, \nu$ is given as:

$$C_u = 1.13 \frac{E}{\left(1-\nu^2\right)} \frac{1}{\sqrt{A}} \tag{5.3-8}$$

Here A is the area of the foundation, E is the Elastic Modulus of the soil and ν is the Poisson's Ratio of the soil.

Substituting $A = \pi\, r_0^{\,2}$, where r_0 represents the radius of equivalent circular plate and $G = \frac{E}{2(1+\nu)}$ (see equation 5.3-3), equation 5.3-8 becomes:

$$C_u = \frac{4\,G\,r_0}{1-\nu} \frac{1}{A} \tag{5.3-9}$$

Though equations 5.3-8 & 5.3-9 are for circular footing, these have been considered (as reported) applicable for rectangular footings as well.

It is seen from equations 5.3-8 & 5.3-9 that C_u not only depends upon **Soil Modulus** $E \& \nu$ (or $G \& \nu$) but also depends upon **Base Contact Area** of the foundation. In other words, **Foundation Shape and Size** has significant effect on the **Coefficient of Uniform Compression** C_u.

5.3.2.2 Coefficient of Uniform Shear of the Soil - C_τ

For understanding C_τ, consider that a shear force F_x is applied to the block (see Figure 5.3-3) along X-direction.

Uniform Shear stress p_x developed in the soil is given as $$p_x = \frac{F_x}{A}$$

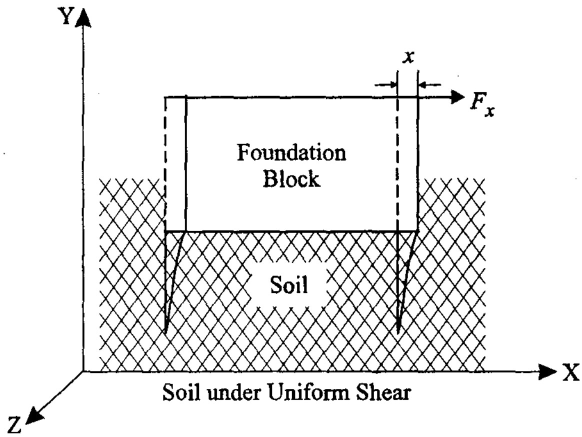


Figure 5.3-3 Force F_x Applied to the Foundation Block Resting over the Soil

Consider that this shear stress produces uniform shear deformation x of the soil. The ratio of the pressure (shear stress) to the deformation therefore gives Coefficient of Subgrade Reaction in Uniform Shear. This is also termed as **Coefficient of Uniform Shear** C_τ. Thus we can write:

$$C_\tau = \frac{p_x}{x} = \frac{F_x}{A}\frac{1}{x} \tag{5.3-10}$$

For coefficient of uniform shear along Z, replacing x by z in equation 5.3-10, we get

$$C_\tau = \frac{p_z}{z} = \frac{F_z}{A}\frac{1}{z} \qquad (5.3\text{-}10a)$$

Evaluation of C_τ may be done using field test methods. The literature review indicates that the work presented by Barkan (1962) giving correlation of C_τ, C_ϕ, and C_ψ with C_u is generally accepted by majority of the authors. Correlation of C_τ with C_u is given as:

$$\frac{C_\tau}{C_u} = 0.5 \qquad (5.3\text{-}11)$$

5.3.2.3 Coefficient of Non-Uniform Compression of Soil - C_ϕ

For understanding C_ϕ, consider that a Moment M_ϕ is applied to the block about Z-direction (see Figure 5.3-4).

Let the Moment of Inertia of the base area of the block about Z-axis be I_{zz}. This moment causes rotation ϕ of the block about Z-axis. Due to this rotation soil experiences non-uniform vertical pressure underneath the base of the block. This rotation also causes soil to undergo non-uniform vertical deformation (termed as non-uniform compression) under the base of the block.

Vertical Pressure p_y developed at a distance x from center is given as

$$p_y = \frac{M_\phi}{I_{zz}} x$$

Vertical deformation y (compression/tension) at distance x from center is $y = x\phi$

The ratio of the pressure to the deformation therefore gives Coefficient of Subgrade Reaction in non-uniform compression. This is also termed as **Coefficient of Non-Uniform Compression** C_ϕ.

Thus we can write:

$$C_\phi = \frac{p_y}{y} = \frac{\frac{M_\phi}{I_{zz}} x}{x\phi} = \frac{M_\phi}{I_{zz}\,\phi} \qquad (5.3\text{-}12)$$

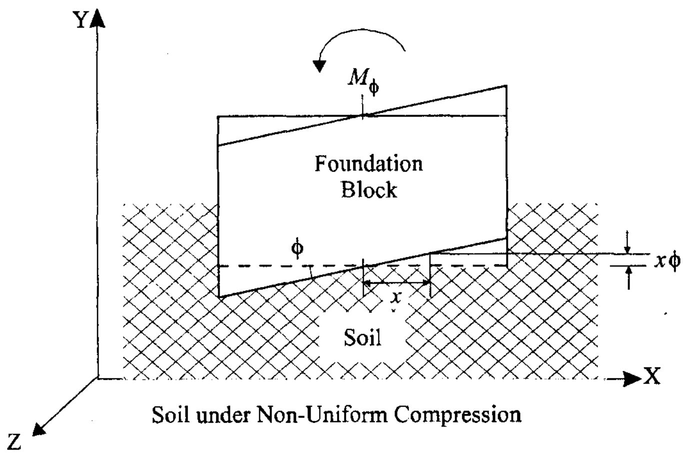


Figure 5.3-4 Moment M_ϕ Applied about Z axis to the Foundation Block Resting over the Soil

For rocking mode about –X, substitute θ in place of ϕ and I_{zz} by I_{xx} in equation 5.3-12, we get

$$C_\theta = \frac{p_y}{y} = \frac{\frac{M_\theta}{I_{xx}} z}{z\theta} = \frac{M_\theta}{I_{xx}\theta} \qquad (5.3\text{-}12a)$$

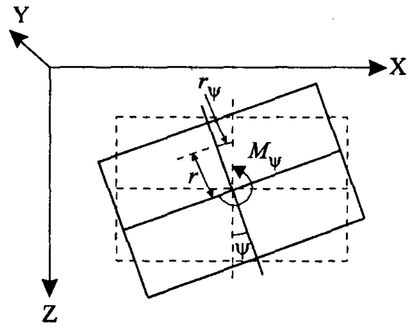
Though desirable, the evaluation of C_ϕ using dynamic test setup has been found to be difficult. In the absence of the test, it may be evaluated based on the C_u using Barkan's correlation:

$$\frac{C_\phi}{C_u} = \frac{C_\theta}{C_u} = 2.0 \qquad (5.3\text{-}13)$$

5.3.2.4 Coefficient of Non-Uniform Shear of the Soil - C_ψ

For understanding C_ψ, consider that a Moment M_ψ is applied to the block (see Figure 5.3-5) in Z-X Plane about Y-axis.

This moment causes rotation ψ of the block about Y-axis. This rotation generates non-uniform shear in the soil underneath the base of the block and causes soil to undergo non-uniform shear deformation underneath the base of the block.



Soil under Non-Uniform Shear

Figure 5.3-5 Moment M_ψ Applied about Y axis to the Foundation Block Resting over the Soil

Let the Moment of Inertia of the base area of the block about Y-axis be I_{yy}. Shear pressure (Shear Stress) $p_{\psi r}$ developed at a radius r from center is given as

$$p_{\psi r} = \frac{M_\psi}{I_{yy}} r$$

Shear deformation at distance r from center is $r\psi$

The ratio of the shear stress to the shear deformation therefore gives Coefficient of Subgrade Reaction in non-uniform shear. This is also termed as Coefficient of Non-Uniform Shear C_ψ. Thus we can write:

$$C_\psi = \frac{p_{\psi r}}{r\psi} = \frac{\frac{M_\psi}{I_{yy}} r}{r\psi} = \frac{M_\psi}{I_{yy}} \frac{1}{\psi} \tag{5.3-14}$$

Evaluation of C_ψ using dynamic test setup is considered difficult. It is recommended that to use Barkan's correlation for its determination, as given below:

$$\frac{C_\psi}{C_\tau} = 1.5 \tag{5.3-15}$$

5.4 DESIGN SOIL PARAMETERS

So far wè have discussed evaluation of Dynamic Soil Parameters. For a given machine foundation, these Dynamic Soil Parameters need to be converted to **Design Soil Parameters.**

For a project of reasonable magnitude, it is common that machines are located spread over the entire area of the project and their foundation depth may also vary depending upon machine type and size. Considering same value of $E, G \& \nu$ and $C_u, C_\phi, C_\tau, C_\psi$ for all foundation sizes and all depths obviously is not the right approach. Design Soil Parameters thus are different from Dynamic Soil Parameters and must account for such variations namely foundation size, foundation depth, etc.

Consider **two similar machines** founded **at two different levels** at the same site. The question arises whether it would be appropriate to use same dynamic soil properties for both the machines? **Obvious answer would be 'no'**. Similarly if the two different machines founded at the same level at the same site have different base contact area, it may also not be appropriate to use same dynamic soil properties for both these machines. Similar is the case with Machines exerting different pressures on the soil i.e. static stress or overburden pressure. Thus the dynamic soil properties need to be modified for each such effect.

For **design office practices**, these effects are generalized as:

i) Effect due to Static stress level or **Overburden Pressure**
ii) Effect due to **Base Contact Area**

The evaluated site soil properties, therefore, need to be suitably modified for a particular machine foundation. Thus, for each foundation, the evaluated **site soil parameters** are to be converted to **design soil parameters** accounting for these effects. These modified soil parameters are termed as **Design Soil Parameters.**

Though the basic dynamic soil parameters required for design of machine foundation are E, G, $C_u, C_\phi, C_\tau, C_\psi$, the discussion here is restricted to only two parameters namely (1) Dynamic shear modulus G and (2) Coefficient of Uniform Compression C_u as other parameters are inter-related as given above in **§ 5.3**.

The recommended Effective depth **for Computing Static Stress** (Overburden Pressure) for different test method is as under:

i) **For Cyclic Plate Load Test Method:** Effective depth for computing static stress is considered equal to founding depth of the plate + half the width of the plate
ii) **For Vertical Resonance Test Method:** Effective depth is considered equal to founding depth of the test block + half the width of the test block

iii) **For Wave Propagation Test Method:** Effective depth is considered equal to half the distance between geo-phones

For the sake of clarity, let us use **suffix '01'** for the **Site Evaluated Parameters** and **suffix '02'** for the **Design Parameters.** Thus **Site and Design Parameters** are referred as:

Site Parameters:

G_{01} Represents **Site Evaluated Dynamic Shear Modulus** of the soil

E_{01} Represents **Site Evaluated Dynamic Elastic Modulus** of the soil

C_{u01} Represents **Site Evaluated Coefficient of Uniform Compression of the soil**

$\overline{\sigma}_{01}$ Represents **Static Stress or Overburden Pressure for site test conditions**

γ_{01} Represents **Shear Strain Value corresponding to site test method**

A_{01} Represents **Base Contact Area corresponding to site test method**

Design Parameters:

E_{02} Represents **Design Dynamic Elastic Modulus of the soil**

G_{02} Represents **Design Dynamic Shear Modulus** of the soil

C_{u02} Represents **Design Coefficient of Uniform Compression of the soil**

$\overline{\sigma}_{02}$ Represents **Design Static Stress or Overburden Pressure for the foundation**

γ_{02} Represents **Design Shear Strain Value for the foundation**

A_{02} Represents **Design Base Contact Area for the foundation**

Many authors have discussed influence of the above effects on the design soil parameters and suggested expressions for accounting for these effects. The commonly adopted and **recommended** correlations between **Site Parameters** and **Design Parameters** (applicable to machine foundation) are listed in **§ 5.4.1, § 5.4.2 & §5.4.3.**

5.4.1 Variation with respect to Static Stress or Overburden Pressure

Guidelines available in the literature are for variation of Dynamic Shear Modulus with respect to Mean Effective Confining Pressure. Author feels that for machine foundation application, it is considered good enough to use **Static Stress (Overburden Pressure)** instead of **Mean Effective Confining Pressure** for determining variation of **Dynamic Shear Modulus.** Variation with respect to static stress may be computed using the same correlation as for Mean Effective Confining Pressure. The relationship is given as under:

$$G_{02} = G_{01}\left(\frac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}\right)^{0.5} \tag{5.4-1}$$

Same correlation could be used for evaluating influence of Static Stress (Overburden Pressure) on **Coefficient of Uniform Compression**. In authors opinion it would not lead to any appreciable error and the converted data could be used with good level of confidence.

$$C_{u02} = C_{u01}\left(\frac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}\right)^{0.5} \tag{5.4-1a}$$

5.4.2 Variation with respect to Base Contact Area of Foundation

The only acceptable guideline available in the literature for computing variation of **Coefficient of Uniform Compression** with respect to **Base Contact Area of Foundation** is that given by Barkan. It is recommended that variation of C_{u02} be considered for base area up to 10 m^2 and no variation be considered for area greater than 10 m^2. The correlation (**After Barkan -1962**) is given as:

$$C_{u02} = C_{u01}\left(\frac{A_{01}}{A_{02}}\right)^{0.5} \tag{5.4-2}$$

$$\text{For } A_{02} > 10\,m^2 \qquad C_{u02} = C_{u02} \text{ for } 10\,m^2 \tag{5.4-3}$$

5.5 EQUIVALENT SPRINGS

In practice, following three types of sub-grade systems are commonly employed for supporting machine foundations. It is necessary to represent these sub-grade systems in terms of Equivalent Springs:

i. Foundation Supported directly over soil
ii. Foundation Supported over an Elastic Pad
iii. Foundation Supported over Piles

5.5.1 Foundation Supported Directly over Soil

For analysis and design of Machine Foundation, which is a 3-D system, soil is represented as **Equivalent Soil Springs** attached to the foundation in each of the six DOF. These equivalent soil springs are evaluated using **Design Soil Parameters** viz. **Design Shear Modulus, Design Coefficient of Uniform Compression**, **Poisson's Ratio etc. (as in § 5.4).**

For evaluation of equivalent soil springs, various soil models have been proposed by various authors but the two models, (i) **Elastic Half Space model** and (ii) **Coefficients of Sub-grade Reaction** are considered **generally acceptable** in the industry. Evaluation of Equivalent Soil Springs using these two models is given as under:

5.5.1.1 Equivalent Soil Springs using Elastic Half Space Model

The model is based on the Dynamic Response of an Isolated Rigid Circular Disk resting on the Surface of the Infinite Soil Medium. The infinite soil medium, termed as Elastic Half Space, is considered as Elastic, Homogeneous and Isotropic, whose elastic properties are defined by Shear Modulus G and Poisson's Ratio ν. Equivalent Soil Springs and Damping Constants (in each of the six DOF) are evaluated using Design Soil Parameters.

Rigid Rectangular Footings: The **mathematical expressions** for evaluation of Equivalent Soil Springs and associated Damping Constants **have been taken** from the following references and are reproduced here.

(i) Whitman R.V., and Richart F.E., Jr., "Design Procedures for Dynamically Loaded Foundations," Journal of the Soil Mechanics and Foundation Division, ASCE Vol. 93, No. SM 6, November 1967

(ii) Richart F.E., Jr., and Whitman R.V., "Comparison of Footing Vibration Tests with Theory," Journal of the Soil Mechanics and Foundation Division, ASCE Vol. 93, No. SM 6, November 1967

(iii) "Vibrations of Soils and Foundations" by Richart, Hall and Woods, Prentice Hall Incorporated, Englewood Cliffs, New Jersey

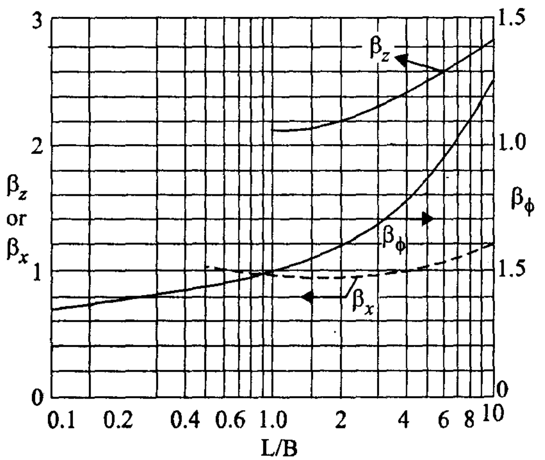


Figure 5.5-1 Coefficients β_x, β_z & β_ϕ for Rectangular Footings (After Whitman & Richart, Jr.)

Representing L as the length of the foundation along Z-axis (axis of rotation), B as width of the foundation along X-axis (perpendicular to axis of rotation) and H as the height of the foundation (along Y-direction i.e. Vertical direction), the expressions for **Equivalent Spring Constants and Damping Constants** in each DOF (vibration mode) are given as under. For B/L or L/B ratio of the foundation, **coefficients $\beta_x, \beta_y, \beta_z, \beta_\theta$ & β_ϕ are as given in Figure 5.5-1.** The symbols and notations have appropriately been changed to be in line with those given in this Book.

Equivalent Soil Springs and Damping Constants:

i) Translational Mode (along X)

Equivalent Spring Constant $$k_x = \beta_x \times 4 \times (1+\nu) \times G \times \sqrt{\frac{L \times B}{4}} \qquad (5.5\text{-}1a)$$

Equivalent Radius $r_0 = \sqrt{\frac{(L \times B)}{\pi}}$; Mass Ratio $b_x = \frac{(1-\nu)}{4} \frac{m}{\rho_s r_0^3}$ (5.5-1b)

Damping Constant $$\zeta_x = \frac{0.288}{\sqrt{b_x}} \qquad (5.5\text{-}1c)$$

ii) **Vertical Mode of Vibration (along Y)**

Equivalent Spring Constant $$k_y = \beta_y \times \frac{G}{(1-\nu)} \times \sqrt{L \times B} \qquad (5.5\text{-}2a)$$

Equivalent Radius $r_0 = \sqrt{\frac{(L \times B)}{\pi}}$; Mass Ratio $b_y = \frac{(1-\nu)}{4} \frac{m}{\rho_s r_0^3}$ (5.5-2b)

Damping Constant $$\zeta_y = \frac{0.425}{\sqrt{b_y}} \qquad (5.5\text{-}2c)$$

iii) Translational Mode (along Z)

Equivalent Spring Constant $$k_z = \beta_z \times 4 \times (1+\nu) \times G \times \sqrt{\frac{L \times B}{4}} \qquad (5.5\text{-}3a)$$

Equivalent Radius $r_0 = \sqrt{\frac{(L \times B)}{\pi}}$; Mass Ratio $b_z = \frac{(1-\nu)}{4} \frac{m}{\rho_s r_0^3}$ (5.5-3b)

Damping Constant $$\zeta_z = \frac{0.288}{\sqrt{b_z}} \quad (5.5\text{-}3c)$$

iv) Rocking Mode (about X)

Equivalent Spring Constant $$k_\theta = \beta_\theta \times \frac{G}{(1-\nu)} \times B \times L^2 \quad (5.5\text{-}4a)$$

Equivalent Radius $r_0 = \left\{\frac{B L^3}{3\pi}\right\}^{1/4}$; Mass Ratio $b_\theta = \frac{3(1-\nu))}{8}\frac{M_{mox}}{\rho_s r_0^5}$ (5.5-4b)

Damping Constant $$\zeta_\theta = \frac{0.15}{(1+b_\theta)\sqrt{b_\theta}} \quad (5.5\text{-}4c)$$

v) Torsional mode about Y axis

Equivalent Spring Constant $$k_\psi = \frac{16G}{3}\left\{\left(\frac{L B(L^2+B^2)}{6\pi}\right)^{1/4}\right\}^3 \quad (5.5\text{-}5a)$$

Equivalent Radius $r_0 = \left\{\frac{L B(L^2+B^2)}{6\pi}\right\}^{1/4}$; Mass Ratio $b_\psi = \frac{3(1-\nu))}{8}\frac{M_{moy}}{\rho_s r_0^5}$. (5.5-5b)

Damping Constant $$\zeta_\psi = \frac{0.5}{(1+2b_\psi)} \quad (5.5\text{-}5c)$$

vi) Rocking Mode (about Z)

Equivalent Spring Constant $$k_\phi = \beta_\phi \times \frac{G}{(1-\nu)} \times B^2 \times L \quad (5.5\text{-}6a)$$

Equivalent Radius $r_0 = \left\{\frac{L B^3}{3\pi}\right\}^{1/4}$; Mass Ratio $b_\phi = \frac{3(1-\nu))}{8}\frac{M_{moz}}{\rho_s r_0^5}$ (5.5-6b)

Damping Constant $$\zeta_\phi = \frac{0.15}{(1+b_\phi)\sqrt{b_\phi}} \quad (5.5\text{-}6c)$$

5.5.1.2 Equivalent Soil Springs using Coefficients of Subgrade Reaction

Soil is represented as **Equivalent Springs and Dashpots in all six DOFs**. The Equivalent Soil Springs are represented as function of **a) Coefficient of Subgrade Reaction of the soil** and **b) Foundation Geometric Parameters.**

It may be noted that only the corrected values of **Coefficients of Subgrade Reaction (Design Soil Parameters as in § 5.4)** should be used for computing **Equivalent Soil Springs.** Mathematical expressions (After Barkan) for these equivalent springs are **given as under:**

a) Soil spring in Lateral X - directions

Rewriting equation 5.3-10 and rearranging terms, we get

$$k_x = \frac{F_x}{x} = C_\tau \times A \qquad (5.5\text{-}7)$$

b) Soil Spring in Vertical Y-Direction

Rewriting equation 5.3-5 and rearranging terms, we get

$$k_y = \frac{F_y}{y} = C_u \times A \qquad (5.5\text{-}8)$$

c) Soil spring in Lateral Z- direction

Rewriting equation 5.3-10a and rearranging terms, we get:

$$k_z = \frac{F_z}{z} = C_\tau \times A \qquad (5.5\text{-}9)$$

d) Soil Spring in rocking θ mode (rocking about X-axis)

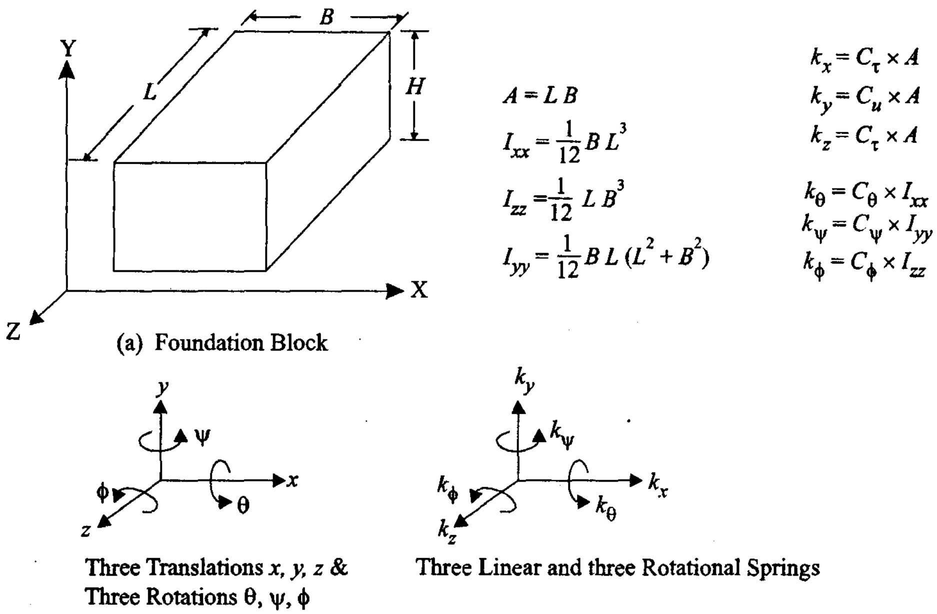
Rewriting equation 5.3-12a and rearranging terms, we get

$$k_\theta = \frac{M_\theta}{\theta} = C_\theta \times I_{xx} \qquad (5.5\text{-}10)$$

e) Soil Spring in Torsional ψ mode (Rotation about vertical Y-axis)

Rewriting equation 5.3-14, and rearranging terms, we get

$$k_\psi = \frac{M_\psi}{\psi} = C_\psi \times I_{yy} \qquad (5.5\text{-}11)$$



(a) Foundation Block

(b) Deformations along X, Y, Z and rotations about X, Y, Z axes

(c) Springs along DOF's

Figure 5.5-2 Spring Constants for a Typical Foundation Block in all Six DOF's

f) Soil Spring in rocking ϕ mode (rocking about Z-axis)

Rewriting equation 5.3-12 and rearranging terms, we get

$$k_\phi = \frac{M_\phi}{\phi} = C_\phi \times I_{zz} \tag{5.5-12}$$

For a typical foundation block, these Equivalent Soil Springs are shown in Figure 5.5-2.

5.5.2 Foundation Supported over an Elastic Pad

This derivation may be found useful incase the foundation rests directly over elastic pads (Rubber pads, Cork Slabs, Isolation pads etc). The elastic pad is mathematically represented as Equivalent springs in all six DOFs i.e. Three Translational Springs and Three Rotational springs.

Consider a **Rigid Plate** of area A resting on an **Elastic Pad** of area A and thickness t as shown in Figure 5.5.3a and DOFs are as shown in Figure 5.5-3b The elastic pad is mathematically represented as Equivalent springs in all six DOFs i.e. Three Translational Springs along X, Y & Z and Three Rotational springs about X, Y & Z.

Equivalent Spring along Vertical Y direction (In Compression/Tension):

Consider a vertical force F_y applied on the rigid plate of area A producing uniform vertical deformation y in the pad (Figure 5.5-4a).

Compressive Stress in the elastic pad $\sigma_y = \frac{F_y}{A}$

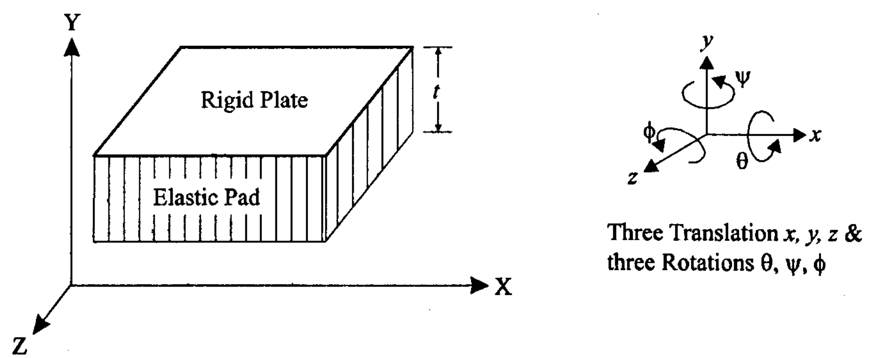
Strain in the elastic pad $\varepsilon_y = \frac{y}{t}$

Elastic Modulus of Pad $E_y = \frac{Stress}{Strain} = \frac{\sigma_y}{\varepsilon_y} = \frac{\frac{F_y}{A}}{\frac{y}{t}} = \frac{F_y t}{A y}$

$$k_y = \frac{F_y}{y} = \frac{E_y A}{t} \quad (5.5\text{-}13)$$

a) **Equivalent Spring along Lateral X & Z directions (In Shear):**

Consider a horizontal shear force F_x applied along X direction on the plate of area A as shown in Figure 5.5-4b. The force produces shear deformation x along X direction.



(a) Rigid Plate of Area A over Elastic Pad of Area A and thickness t

(b) Deformations and Rotations

Figure 5.5-3 A Rigid Plate resting over an Elastic Pad

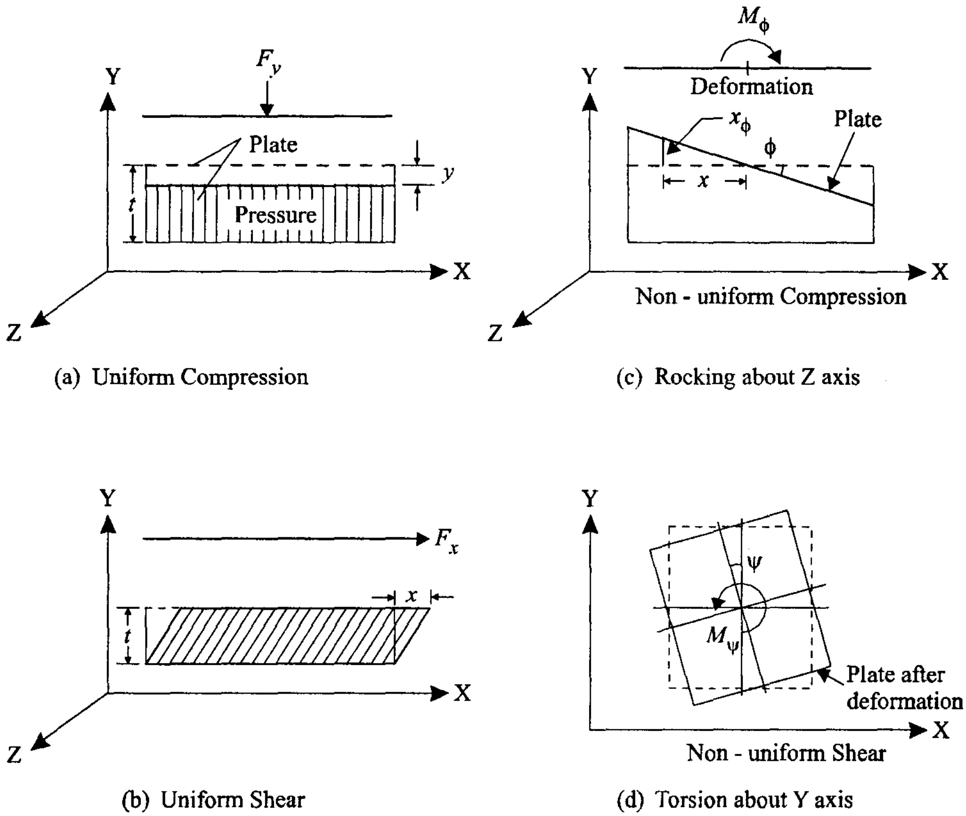


Figure 5.5-4 Rigid Plate Over Elastic Pad (a) Uniform Compression (b) Uniform Shear (c) Rocking about Z axis & (d) Torsion about Y axis

Shearing Stress developed in the pad $\tau_x = \dfrac{F_x}{A}$

Here τ_x represents shear stress on the XZ plane in X direction.

Shear strain $\gamma_x = \dfrac{x}{t}$

Shear Modulus G is the ratio of shear stress to shear strain

$$G_x = \frac{\tau_x}{\gamma_x} = \frac{\frac{F_x}{A}}{\left(\frac{x}{t}\right)} = \frac{F_x t}{A x}$$

Substituting for $G_x = \dfrac{E}{2(1+\nu)}$ and rearranging, we get

Lateral Stiffness k_x of Elastic Pad as: $$k_x = \frac{F_x}{x} = \frac{E\,A}{2(1+\nu)t} \qquad (5.5\text{-}14)$$

Similarly, replacing x with z, we get $$k_z = \frac{F_z}{z} = \frac{E\,A}{2(1+\nu)t} \qquad (5.5\text{-}15)$$

(b) Equivalent Spring rocking about $X \& Z$ axes (Rotational Stiffness):

Now consider a moment M_ϕ applied at the center of the rigid plate about Z-axis. This causes the plate to rotate by angle ϕ about Z-axis passing through center of plate as shown in Figure 5.5-4c. Moment generates non-uniform vertical pressure in the pad that varies from zero at the center to its maximum value at the ends as shown.

Vertical stress developed in the pad at a distance 'x' from the center is $$\sigma_y = \frac{M_\phi}{I_{zz}}x$$

Here I_{zz} is the moment of inertia of the plate about Z-axis

Vertical deformation at the same point (i.e. at distance x) is $$y = x\,\phi$$

Strain at the same point $$\varepsilon_y = \frac{y}{t} = \frac{x\phi}{t}$$

Elastic Modulus $$E = E_y = \frac{\sigma_y}{\varepsilon_y} = \frac{\dfrac{M_\phi}{I_{zz}}x}{\left(\dfrac{x\phi}{t}\right)}$$

We can represent Rotational Stiffness k_ϕ (Rocking about Z-axis) of Elastic Pad as:

$$k_\phi = \frac{M_\phi}{\phi} = \frac{E\,I_{zz}}{t} \qquad (5.5\text{-}16)$$

On the similar lines, we can represent Rotational Stiffness k_θ (Rocking about X-axis) of Elastic Pad as:

$$k_\theta = \frac{M_\theta}{\theta} = \frac{E_y\,I_{xx}}{t} \qquad (5.5\text{-}17)$$

Here I_{xx} is the moment of inertia of the plate about X-axis

(c) Equivalent Spring rotating about Y axis (Torsional Stiffness):

Now consider a moment M_ψ applied at the center of the rigid plate about Y-axis. This causes the plate to rotate by an angle ψ about Y-axis passing through center of plate as shown in Fig. 5.5-4d.

Shear stress developed (in X-Z Plane) at a point at a distance r from the center on the pad due to the moment M_ψ is:

$$\tau_r = \frac{M_\psi}{I_{yy}} r$$

Here τ_r represents shear stress in X-Z plane normal to radius vector r (along rotation about Y axis i.e. direction ψ and I_{yy} is the polar moment of inertia of the base contact surface of the plate about Y-axis passing through center of the pad.

Rotation of the pad (about Y axis passing through center of the pad) $= \psi$
Shear displacement (normal to vector r) at the same point is $= r\psi$
Thickness of pad $= t$

Shear strain at the same point $\gamma_r = \frac{r\psi}{t}$

Shear Modulus G is $G_x = G_z = \frac{Shear\ Stress}{Shear\ Strain} = \frac{\tau_r}{\gamma_r} = \frac{\frac{M_\psi}{I_{yy}} r}{\left(\frac{r\psi}{t}\right)} = \frac{M_\psi t}{I_{yy}\psi}$

Substituting $G = \frac{E}{2(1+\nu)}$ we get Torsional Stiffness k_ψ as

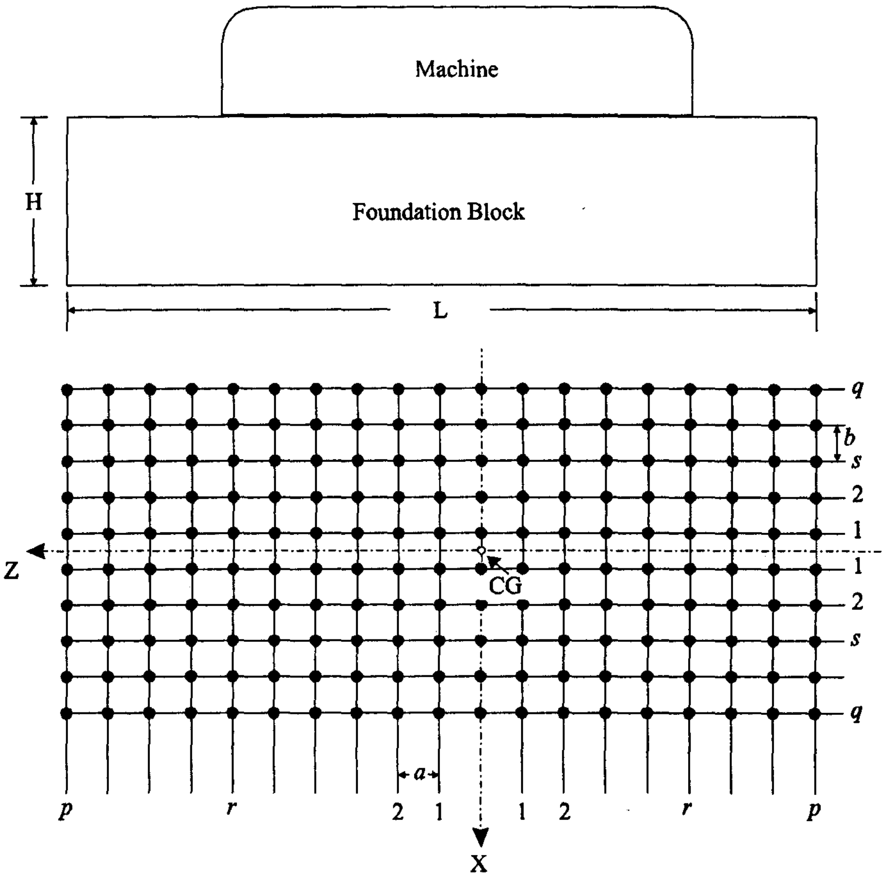
$$k_\psi = \frac{M_\psi}{\psi} = \frac{G I_{yy}}{t} = \frac{E}{2(1+\nu)} \frac{I_{yy}}{t} \qquad (5.5\text{-}18)$$

5.5.3 Foundation Supported over a set of Springs

This derivation may be found useful incase the foundation is supported directly over a set of springs having vertical and translational stiffness. These springs are mathematically represented as Equivalent springs, one along each of the six DOFs i.e. Three Translational Springs and Three Rotational springs.

Consider a foundation block of Length L, width B and Height H supporting a machine of mass m. Consider that the foundation is supported over a set of n springs.

Consider that there are odd numbers of springs $2p+1$ in each row along length and even number of springs $2q$ in each row along width of the foundation. Let the spacing of springs along length of the foundation be a and that along width be b. Consider that each spring has a vertical stiffness of k_v and horizontal stiffness of k_h. The arrangement is as shown in Figure 5.5-5



Spring Locations in Plan

Figure 5.5-5 Foundation Supported Over a set of Springs
Springs Spacing along Length a & Number of Springs $2p + 1$
Springs Spacing along width - b & Number of Springs $2q$

For the purpose of analysis, these springs need to be mathematically represented as equivalent springs one along each of the six DOFs i.e. Three Translational Springs and Three Rotational springs.

Equivalent Stiffness at CG of Base area of Foundation Block

Number of springs along Length	$2p+1$
Number of springs along Width	$2q$
Total number of springs	$n=(2p+1)\times 2q$
Vertical Stiffness of each spring along Y	k_v
Lateral Stiffness of each spring along X / Z	k_h

Equivalent Translational Stiffness

Equivalent Translational Stiffness is the summation of stiffness of all the springs in respective directions.

Equivalent Vertical Stiffness along Y	$k_y = n\times k_v$
Equivalent Lateral Stiffness along X	$k_x = n\times k_h$
Equivalent Vertical Stiffness along Z	$k_z = n\times k_h$

Equivalent Rotational Stiffness

Let us now compute Equivalent Rotational Stiffness k_θ about X, k_ϕ about Z and k_ψ about Y-axis.

Equivalent Rocking Stiffness k_θ about X

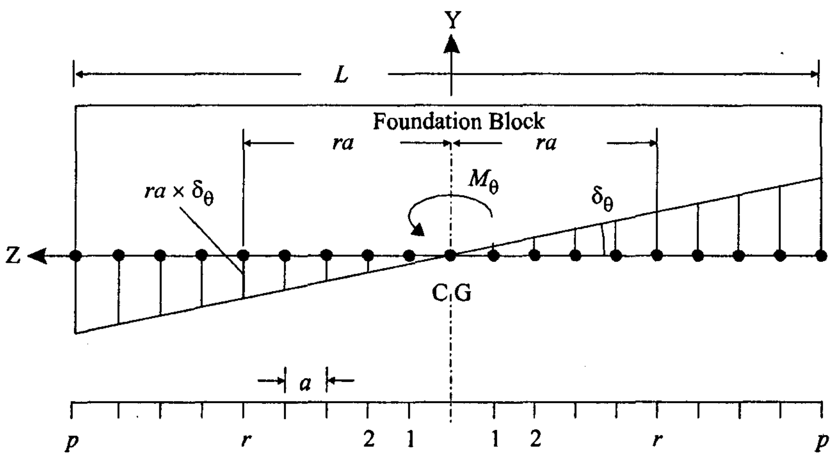


Figure 5.5-6 Rotation about X-axis

Consider a moment M_θ applied at the CG of the base of the foundation about X-axis. Let the resulting rotation of the foundation be δ_θ . This rotation of the foundation block produces vertical deflection in each of the spring on either side of the CG. The rotation and the corresponding deflection is as shown in Figure 5.5-6

Consider spring on the LHS of CG

Distance of r^{th} row spring from CG	ra
Rotation of the foundation Block	δ_θ
The deflection of the spring at the r^{th} row	$\delta_r = ra\delta_\theta$
Force developed in the spring	$\delta_{F_r} = k_v\delta_r = k_v ra\delta_\theta$

Resisting Moment developed

Resisting Moment by the spring at LHS & RHS	$2\times k_v ra\delta_\theta \times ra$
Number of spring in r^{th} row	$2q$
Total Resisting Moment developed by r^{th} row springs	$\delta_{M_r} = 2q\times 2k_v(ra)^2\delta_\theta$

Total Moment Developed by all the springs

$$M_r = \sum_{r=1}^{r=p} 2q\times 2k_v(ra)^2\delta_\theta = 2q\times 2k_v\times a^2\delta_\theta\sum_{r=1}^{r=p}(r)^2$$

$$= 2q\times 2k_v\times a^2\delta_\theta\times\frac{p(p+1)(2p+1)}{6} = 2q\times(2p+1)\times k_v\times a^2\delta_\theta\times\frac{p(p+1)}{3}$$

$$= k_y\times a^2\delta_\theta\times\frac{p(p+1)}{3}$$

Equating resisting moment with applied moment, we get

$$M_\theta = M_r; \qquad M_\theta = k_y\times a^2\delta_\theta\times\frac{p(p+1)}{3}$$

This gives rotational Stiffness k_θ as

$$k_\theta = \frac{M_\theta}{\delta_\theta} = k_y\times a^2\times\frac{p(p+1)}{3} \qquad (5.5\text{-}19)$$

In case number of springs along length is an even number i.e. '$2p$', then the stiffness becomes:

$$k_\theta = \frac{M_\theta}{\delta_\theta} = k_y \times a^2 \times \frac{(2p+1)(2p-1)}{12} \tag{5.5-19a}$$

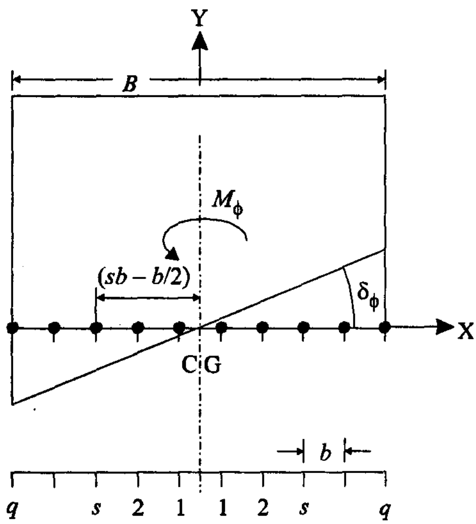


Figure 5.5-7 Rotation about Z-axis

Equivalent Rotational Stiffness k_ϕ about Z

Consider a moment M_ϕ applied at the CG of the base of the foundation about Z-axis (Figure 5.5-7). Let the resulting rotation of the foundation be δ_ϕ. This rotation of the foundation block produces vertical deflection in each of the spring on either side of the CG.

Consider spring on the LHS of CG

Distance of s^{th} row spring from CG	$(sb - b/2)$
Rotation of the foundation Block	δ_ϕ
The deflection of the spring at the s^{th} row	$\delta_s = (sb - b/2) \times \delta_\phi$
Force developed in the spring	$\delta_{F_s} = k_v \delta_s = k_v (sb - b/2)\delta_\phi$
Resisting Moment developed at center	$2 \times \{k_v (sb - b/2)\delta_\phi \times (sb - b/2)\}$
Number of spring in s^{th} row	$(2p+1)$

Total Resisting Moment developed by s^{th} row springs

$$\delta_{M_s} = (2p+1)\times 2k_v (sb - b/2)^2 \delta_\phi$$

Total Moment Developed by all the springs

$$M_s = \sum_{s=1}^{s=q} (2p+1)\times 2k_v (sb-b/2)^2 \delta_\phi = (2p+1)\times 2k_v \times b^2 \delta_\phi \sum_{s=1}^{s=q} (s-1/2)^2$$

$$M_s = (2p+1)\times 2k_v \times b^2 \delta_\phi \sum_{s=1}^{s=q}(s^2 - s + 1/4) = (2p+1)\times 2k_v \times b^2 \delta_\phi \left[\sum_{s=1}^{s=q} s^2 - \sum_{s=1}^{s=q}(s) + \sum_{s=1}^{s=q}(1/4)\right]$$

$$M_s = (2p+1)\times 2k_v \times b^2 \delta_\phi \times \left\{\frac{q(q+1)(2q+1)}{6} - \frac{q(q+1)}{2} + \frac{q}{4}\right\}$$

$$= (2p+1)\times 2k_v \times b^2 \delta_\phi \times \left\{\frac{2q(2q^2+3q+1) - 6q(q+1) + 3q}{12}\right\}$$

$$M_s = (2p+1)\times 2q \times k_v \times b^2 \delta_\phi \times \left\{\frac{(2q+1)(2q-1)}{12}\right\} = k_y b^2 \delta_\phi \left\{\frac{(2q+1)(2q-1)}{12}\right\}$$

Equating resisting moment with applied moment, we get

$$M_\phi = M_s; \qquad M_\phi = k_y \times b^2 \delta_\phi \times \left\{\frac{(2q+1)(2q-1)}{12}\right\}$$

This gives rotational Stiffness k_ϕ as

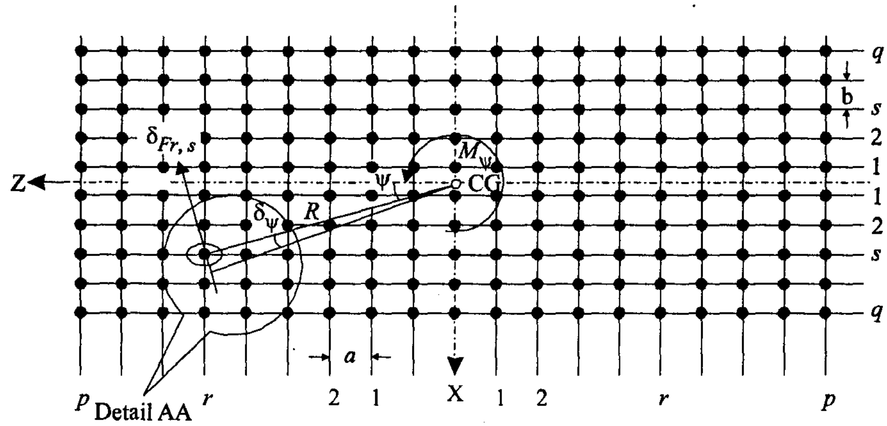
$$k_\phi = \frac{M_\phi}{\delta_\phi} = k_y \times b^2 \times \left\{\frac{(2q+1)(2q-1)}{12}\right\} \qquad (5.5\text{-}20)$$

In case number of springs along width is an odd number i.e. $(2q+1)$, then the stiffness becomes:

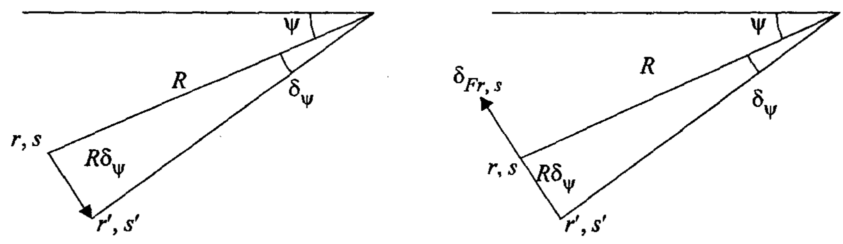
$$k_\phi = \frac{M_\phi}{\delta_\phi} = k_y \times b^2 \times \left\{\frac{q(q+1)}{3}\right\} \qquad (5.5\text{-}20a)$$

Equivalent Torsional Stiffness k_ψ about Y

Consider a moment M_ψ applied at the CG of the base of the foundation about Y-axis. Let the resulting rotation of the foundation be δ_ψ. This rotation of the foundation block produces lateral deflection in each of the spring on either side of the CG along normal to the radius vector joining spring location to the CG of base area.



(a) Deflection due to torsional moment



(b) Detail AA

Figure 5.5-8 Torsion about Y-axis

Consider spring at location r,s on the LHS of CG as shown in Figure 5.5-8. Let R be the distance of this point from CG and consider that this radius vector R makes an angle ψ with respect to Z-axis.

Rotation of the foundation Block about CG δ_ψ

Lateral deflection (normal to radius vector R) of the spring $\delta_{r,s} = R\delta_\psi$

Force developed in the spring $\delta_{F(r,s)} = k_h \times R\delta_\psi$

Resisting Moment developed at center $\delta_{M(r,s)} = k_h \times R^2\delta_\psi$

Total Resisting Moment developed

$$M_{(r,s)} = \sum_{r=-p}^{p} \sum_{s=-q}^{q} \delta_{M(r,s)} \quad \Rightarrow \quad M_{(r,s)} = \sum_{r=-p}^{p} \sum_{s=-q}^{q} k_h \times R^2 \delta_\psi \tag{5.5-21}$$

Distance of r^{th} row spring from CG $\quad ra$

Distance of s^{th} row spring from CG $\quad (sb - b/2)$

This gives $\quad R^2 = (ra)^2 + (sb - b/2)^2$

Substituting in equation (5.5-21), we get

$$M_{(r,s)} = \sum_{r=-p}^{p} \sum_{s=-q}^{q} k_h \times \left((ra)^2 + (sb - b/2)^2\right) \delta_\psi \tag{5.5-22}$$

Solution of this equation yields

$$M_{(r,s)} = \delta_\psi \left[\begin{array}{l} \left\{ k_h a^2 \times 2 \times 2q \times \dfrac{p(p+1)(2p+1)}{6} \right\} \\ + \left\{ k_h b^2 \times 2 \times (2p+1) \times \left(\dfrac{q(q+1)(2q+1)}{6} - \dfrac{q(q+1)}{2} + \dfrac{q}{4} \right) \right\} \end{array} \right]$$

$$M_{(r,s)} = \delta_\psi \left[\begin{array}{l} \left\{ a^2 k_h \times 2q \times (2p+1) \times \dfrac{p(p+1)}{3} \right\} \\ + \left\{ b^2 \times k_h \times 2q \times (2p+1) \times \left(\dfrac{(q+1)(2q+1)}{6} - \dfrac{(q+1)}{2} + \dfrac{1}{4} \right) \right\} \end{array} \right]$$

$$M_{(r,s)} = \delta_\psi \left[\left\{ a^2 k_x \frac{p(p+1)}{3} \right\} + \left\{ b^2 \times k_z \frac{(2q+1)(2q-1)}{12} \right\} \right] \tag{5.5-23}$$

Here $\quad k_x = k_z = k_h \times 2q \times (2p+1)$

We can also get the same solution by considering components of force in X & Z direction and taking moment of these force components about CG and taking summation over all the springs.

Equation (5.5.3-5) gives Torsional Stiffness k_ψ as

$$k_\psi = \frac{M_\psi}{\delta_\psi} = \frac{M_{(r,s)}}{\delta_\psi} = \left[\left\{ a^2 k_x \frac{p(p+1)}{3} \right\} + \left\{ b^2 \times k_z \frac{(2q+1)(2q-1)}{12} \right\} \right] \tag{5.5-24}$$

In case number of springs along width is also odd number i.e. $(2q+1)$, then the stiffness becomes:

$$k_{\psi} = \frac{M_{\psi}}{\delta_{\psi}} = \left[\left\{a^2 k_x \frac{p(p+1)}{3}\right\} + \left\{b^2 \times k_z \frac{q(q+1)}{3}\right\}\right] \quad (5.5\text{-}24a)$$

In case number of springs along length is also even number i.e. $2p$, then the stiffness becomes:

$$k_{\psi} = \frac{M_{\psi}}{\delta_{\psi}} = \left[\left\{a^2 k_x \frac{(2p+1)(2p-1)}{12}\right\} + \left\{b^2 \times k_z \frac{(2q+1)(2q-1)}{12}\right\}\right] \quad (5.5\text{-}24b)$$

5.5.4 Foundation Supported over Piles

For machine foundation application, piles are provided in the following cases:

i. When soil is weak in bearing capacity to withstand pressures due to both static and dynamic loads
ii. When significant loss of soil strength is postulated under dynamic loads on account of critical soil and water table conditions
iii. When it is required to increase natural frequency of the machine foundation system
iv. When dynamic amplitudes are required to be reduced
v. When it is required to stiffen the support system on account of seismic considerations

In each case selection of pile type, pile size, pile depth, number of piles etc is an involved task and is accomplished using standard pile design procedures based on soil data and the load data (both dynamic and static loads). In certain cases, selection of pile type, pile size, pile depth, number of piles etc becomes a tricky issue and for all practical purposes may turn out to be a difficult task. In either case, evaluation of dynamic characteristics of piles is a complex task and suffers with many associated uncertainties.

More often than not, a machine foundation block itself serves as rigid pile cap that connects piles at the top. Evaluation of dynamic characteristics of a single pile, in itself, is a difficult task and evaluation of dynamic characteristics of a group of piles connected by a rigid pile cap becomes complex and calls for many assumptions resulting in added levels of uncertainties.

Even in current era of advanced technology, most of the authors, who have significantly contributed to **Pile Supported Machine Foundations,** do corroborate that:

i. Understanding of Dynamic Behaviour of Group of Piles is still in its Infancy
ii. Evaluation of dynamic characteristics of piles is a complex task and suffers with many associated uncertainties
iii. As the reliability of dynamic characteristics of group of piles is faced with many questions, so shall be the status of computed dynamic response.

Regarding evaluation of stiffness and damping of pile-supported foundations, general observations by various authors, as reported in the literature, are as under:

1. Elastic resistance of pile to vertical loads changes with lapse of time i.e. the elastic resistance offered by a fresh driven pile to vertical loads is different than the resistance offered by it after lapse of some time.
2. Elastic resistance of pile to vertical loads changes with increase in length of pile.
3. Elastic resistance of pile to lateral loads primarily depends upon its cross-section and its fixation length and any increase in length of pile beyond fixation length has no influence on its lateral resistance. The fixation length of a pile is the length of the pile in the soil, where it is assumed fixed when subjected to lateral loads. This is generally of the order of 1 to 1.5 m for all piles -Barkan 1962.
4. Dynamic stiffness of a single pile is generally found to be greater than its static stiffness.
5. Both stiffness and damping of pile have been found to be frequency dependent i.e. these vary with change in frequency. The reliability of response using frequency independent stiffness and damping values, in dynamic domain, would therefore be questionable.
6. Damping increases with increase in pile length.
7. Embedment of pile cap results in increased stiffness and damping of the pile group. However its quantification is not yet established.
8. Damping of group of pile is more frequency dependent than that for a single pile.
9. Dynamic group effect of piles differs considerably from static group effect.
10. Frequency dependence of stiffness and damping of pile group could safely be ignored for translational and rocking modes of vibration.
11. Rocking and Torsional stiffness of individual pile could safely be ignored while evaluating dynamic response of group of piles.
12. Elastic resistance of each pile in a group is a function of pile spacing. Inter-influence of piles is observed to be quite significant. The elastic resistance of each pile increases with the increase in the pile spacing and decreases with the decrease in pile spacing. When the pile spacing becomes sufficiently large, the elastic resistance of each pile in a group approaches the resistance of a single pile.
13. The combined stiffness, for a group of n piles, is not the linear summation of individual stiffness of n piles.
14. The effective stiffness of a single pile (in a group of piles) is its individual stiffness multiplied by an influence factor α_{eff} that depends upon the ratio of pile spacing s to its diameter d.

These common observations lead to the broad conclusions that i) definite gaps exist in understanding the dynamic behaviour of a single pile as well as group of piles and the ii) Dynamic Interaction for group of piles is a very complicated task.

There is huge amount of work available in the literature that appears to be good for R & D purposes and its translation as a design tool is lacking for practical applications in the industry. In view of the limitations and associated uncertainties, practically every author suggests that the method be used with caution till better design methods are available. It goes without saying that judicious engineering judgment needs to be exercised in accepting the results for practical applications.

In light of the above:-

- Author feels challenged in recommending any single approach for dynamic response of pile supported machine foundation systems as the reliability of dynamic properties of group of piles derived from that of single pile is low.
- Notwithstanding the above, it is recommended that Elastic Resistance of a Pile, to both vertical and lateral loads, must necessarily be determined from pile test.
- In author's opinion, most of the approaches suggested in the literature are good enough for R&D purposes and may not be suitable for industry.
- The ground reality is that the industry cannot wait till validated solutions are available.
- It has to continue with the designs with the best available practical approaches/solutions such that the machine performance is acceptable, and
- The designer should be able to complete the task in a specified time schedule with a good level of confidence in his design.

Author, based on his long field experience, however, suggests the following design approach for evaluating equivalent springs for pile-supported foundations:

5.5.4.1 Equivalent Pile Springs

Consider a pile-supported foundation having length L, width B and depth H. It is implied that soil exploration for the site has been done. Based on the load data and soil data for the site, design of the piles for the foundation is done using normal pile design procedures/methods for static loads. This provides data regarding pile type, pile diameter d, pile length l, number of piles n & pile spacing s. Piles are so placed that their spacing s along length and width of the foundation remains same. It is strongly recommended that vertical pile stiffness k_{pv} and lateral pile stiffness k_{ph} of each pile be evaluated from pile test.

Effective pile stiffness: Let us consider that the combined stiffness for a group of n piles is the linear summation of effective pile stiffness of each pile in the pile group, where the effective pile stiffness is considered dependant upon ratio of pile spacing to its diameter and is given as under:

Effective vertical stiffness k_v of each pile $$k_v = \alpha_{eff} \times k_{pv} \quad (5.5\text{-}25)$$

Effective lateral stiffness k_h of each pile $$k_h = \alpha_{eff} \times k_{ph} \quad (5.5\text{-}26)$$

Here α_{eff} is the **influence coefficient** that depends upon ratio of pile spacing to its diameter

Regarding influence coefficient, various relationships given in the literature by many authors have been reviewed. It is noted that there is no consistency and each of the relationship is in variance with the other. This keeps the designers in dilemma. In view of this, an empirical equation is proposed for evaluating influence coefficient α_{eff}. Here influence coefficient is defined as a function of s/d, where s is the pile spacing and d is the pile diameter.

$$\alpha_{eff} = 0.212\left(\frac{s}{d}\right)^{0.65} \tag{5.5-27}$$

This empirical relationship provides a fairly good estimate of influence coefficient of a single pile. The equation gives influence coefficients for $2.0 < \left(\frac{s}{d}\right) < 10.0$ as:

It is noted that these numbers are reasonably in good agreement with those given by Barkan, which are derived based on experimental observations. Comparison is listed in Table 5.5-1.

Table 5.5-1 Influence Coefficients for Piles

$\left\{\frac{\text{Pile Spacing}}{\text{Pile Diameter}}\left(\frac{s}{d}\right)\right\}$	$\left\{\begin{matrix}\alpha_{\text{eff}} \\ \text{Coefficient as Proposed}\end{matrix}\right\}$	$\left\{\begin{matrix}\mu \\ \text{Coefficint After Barkan} \\ \text{Table I-14 pp 48}\end{matrix}\right\}$
0.33		
3	0.43	0.41
4	0.52	
4.5	0.56	0.64
5	0.60	
6	0.68	0.65
10	0.95	

Overall Stiffness of group of piles:

Using the same approach as given in § 5.5.3 for **Foundation Supported on Set of springs**, let us develop the formulations for the pile group.

Linear Stiffness

Equivalent Vertical Stiffness along Y

Taking summation of effective vertical stiffness of each pile over all the piles, we get

$$k_y = n \times k_v \tag{5.5-28}$$

Here, k_v is the Effective Vertical Stiffness of each Pile, n is the Total Number of Piles and k_y is the Total Vertical Stiffness of Pile Group.

Similarly we get,

Equivalent Lateral Stiffness along X $\quad k_x = n \times k_h$ (5.5-29)

Equivalent Lateral Stiffness along Z $\quad k_z = n \times k_h$ (5.5-30)

Here, k_h is the Effective Lateral Stiffness of each Pile, n is the Total Number of Piles and k_x & k_z is the Total Lateral Stiffness of Pile Group in X & Z directions respectively.

Rotational Stiffness

Let us consider that there are odd numbers of piles $2p+1$ along one side of the foundation and even numbers of piles $2q$ along other side of the foundation.

Total number of piles $\quad n = (2p+1)\times(2q)$

Pile spacing (same along length & width of the foundation) $\quad s$

Following the same approach as given in § 5.5.3 for Foundation Supported on Set of springs, we get Equivalent Rotational Stiffness.

Equivalent Rotational Stiffness k_θ about X

$$k_\theta = k_y \times s^2 \times \frac{p(p+1)}{3} \tag{5.5-31}$$

Equivalent Rotational Stiffness k_ϕ about Z

$$k_\phi = k_y \times s^2 \times \left\{ \frac{(2q+1)(2q-1)}{12} \right\} \tag{5.5-32}$$

Here, k_y is the Total Vertical Stiffness of Pile Group, s is Pile Spacing and k_θ & k_ϕ is the total Rotational Stiffness of Pile Group about X & Z axis respectively.

Equivalent Torsional Stiffness k_ψ about Y

$$k_\psi = s^2 k_x \left[\left\{ \frac{p(p+1)}{3} \right\} + \left\{ \frac{(2q+1)(2q-1)}{12} \right\} \right] \qquad (5.5\text{-}33)$$

Here, k_x is the total Lateral Stiffness of Pile Group along X axis and k_ψ is the total Torsional Stiffness of Pile Group about Y- axis. It may be noted that total lateral stiffness of pile group in X and Z direction is the same i.e. $k_x = k_z$. Hence either $k_x or\ k_z$ appears in the equation (5.5-33).

5.5.4.2 Damping

The damping offered by pile-supported foundation does depend upon length of pile and embedment of pile cap. Though the observations confirm increase in value of damping with increase in length of pile as well as embedment of pile cap, its quantification, however, is not yet established. It is to be noted that:

- Damping of group of pile has been found to be more frequency dependent than that for a single pile.
- Damping exhibited at resonance is far different than at non-resonant frequencies.
- Damping of pile-supported system has been found to be less than soil-supported system.

Whether piles are provided for improving load carrying capacity of the weak soil or provided on account of increasing the natural frequency of the machine foundation system, the objective of keeping the natural frequencies away from excitation frequencies is achieved in either case. This, in other words, confirms that the response of the foundation is required at non-resonant frequencies. It is suggested to use a nominal value of about 5% damping for response computation because the quantification of pile damping is not yet fully established. Any value higher than this, in practice, would obviously result in lower amplitudes.

EXAMPLE PROBLEMS (§5.4)

P 5.4-1

For a site, dynamic soil investigation is carried out using Wave Propagation Test. The test data is as under:

Distance between geo-phones	6 m
Shear wave velocity	140 m / s
Mass density of soil	$\rho_s = 2000$ kg/m^3
Poisson's Ratio	$\nu = 0.3$

Compute Design Shear Modulus G_{02} & Design Coefficient of Uniform Compression C_{u02} for a machine foundation with details as under:

Foundation Size	$2\,\text{m} \times 4\,\text{m} \times 4\,\text{m}$ deep
Depth of foundation	3 m
Mass Density of Concrete	2500 kg/m^3
Mass of Machine	20000 kg
Ht. of machine CG above foundation top	0.5 m

Solution:

Mass density of soil	$\rho_s = 2000\ \text{kg/m}^3$
Poisson's Ratio	$\nu = 0.3$

Site Data (Data_01)

Site Static Stress (Overburden Pressure)	$\bar{\sigma}_{01}$
Distance between geo-phones	6 m
Effective considered depth (see § 5.4)	$d_{01} = (6/2) = 3.0\,m$

Overburden Pressure at the considered depth is

$$\sigma_1 = \rho_s \times g \times d_{01} = 2000 \times 9.81 \times 3 \times \frac{1}{1000} = 58.86 \quad \text{kN/m}^2$$

$$\bar{\sigma}_{01} = \sigma_1 = 58.86 \quad \text{kN/m}^2$$

Site Dynamic Shear Modulus

Shear wave velocity V_s 140 m / s

$$G_{01} = \rho_s V_s^2 = 2000 \times 140^2 = 392 \times 10^5\,\text{N/m}^2\ ; \quad 3.92 \times 10^4 \quad \text{kN/m}^2$$

Design Data (Data_02)

Base Contact Area of the Foundation $A_{02} = 2 \times 4 = 8\ \text{m}^2$

Static Stress (Overburden Pressure) $\bar{\sigma}_{02}$:

Width of Foundation = 2 m

Depth of Foundation = $3\ \text{m}$

Effective depth (See §5.4) $d_{02} = 0.5 \times 2 + 3 = 4\ \text{m}$

Mass of foundation block $m_b = 8 \times 4 \times 2500 = 80000\ \text{kg}$

Mass of machine $m_m = 20000\ \text{kg}$

Total Mass $m_{02} = m_b + m_m = 100000\ \text{kg}$

$\sigma_1 = \rho_s \times d_{02} \times g$ = Over burden pressure due to soil at depth d_{02}

$$\sigma_1 = 2000 \times 4 \times 9.81 \times \frac{1}{1000} = 78.480\ \ \text{kN/m}^2$$

σ_2 = Over burden pressure due to test block + machine

$$= \frac{100000 \times 9.81}{8} \times \frac{1}{1000} = 122.625\ \ \text{kN/m}^2$$

$$\bar{\sigma}_{02} = (\sigma_1 + \sigma_2)$$

$$\bar{\sigma}_{02} = (78.480 + 122.625) = 201.105\ \ \text{kN/m}^2$$

Design Shear Modulus G_{02}: Rewriting equations 5.4-1, we get $G_{02} = G_{01} \times \left\{\frac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}\right\}^{0.5}$

Substituting values, we get

$$G_{02} = G_{01} \times \left\{\frac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}\right\}^{0.5} = 3.92 \times 10^4 \times \left\{\frac{201.105}{58.86}\right\}^{0.5} = 7.246 \times 10^4\ \ \text{kN/m}^2$$

Design Coefficient of Uniform Compression C_{u02}

Rewriting equation 5.3-9, we get

$$C_{u02} = \frac{4\, G_{02}\, r_0}{1-\nu} \frac{1}{A_{02}}; \quad \text{Where } r_0 = \sqrt{\frac{A_{02}}{\pi}}$$

$$G_{02} = 7.246 \times 10^4\ \ \text{kN/m}^2;\ \ A_{02} = 8\ \text{m}^2;\ \ r_0 = \sqrt{\frac{8}{\pi}} = 1.5957\ \text{m};\ \ \nu = 0.3$$

Substituting, we get $$C_{u02} = \frac{4 \times 7.246 \times 10^4 \times 1.5957}{(1-0.3) \times 8} = 8.2588 \times 10^4\ \text{kN/m}^3$$

P 5.4-2

For a site, dynamic soil investigation is carried out using Vertical Vibration Resonance Test. The test data is as under:

Size of test block	$1.5\,\text{m} \times 0.75\,\text{m} \times 0.7\,\text{m}$ high
Mass density of concrete test block	$\rho_c = 2500\ \text{kg/m}^3$
Depth of test pit	$4\,\text{m}$
Mass of Oscillator system	$m_0 = 160\,\text{kg}$
Resonant Frequency	$f_z = 30\ \text{Hz}$
Amplitude of Vibration	**200 microns**
Mass density of soil	$\rho_s = 2000\ \text{kg/m}^3$
Poisson's Ratio	$\nu = 0.3$

Compute Design Coefficient of Uniform Compression C_{u02} & Design Shear Modulus for a machine foundation with details as under:

Foundation Size	$2\text{m} \times 4\text{m} \times 4\text{m}$ deep
Depth of foundation	3 m
Mass Density of Concrete	$2500\ \text{kg/m}^3$
Mass of Machine	20000 kg
Ht. of machine CG above foundation top	0.5 m

Solution:

Mass density of concrete test block	$\rho_c = 2500\ \text{kg/m}^3$
Mass density of soil	$\rho_s = 2000\ \text{kg/m}^3$
Poisson's Ratio	$\nu = 0.3$

Site Data (Data_01)

Area of Test Block	$A_{01} = 1.5 \times 0.75 = 1.125 \quad \text{m}^2$
Height of Test Block	$h = 0.7\ \text{m}$
Mass of Test Block (including m/c)	$m_{01} = 2500 \times 1.125 \times 0.7 + 160 = 2128.75 \quad \text{kg}$

Site Coefficient of Uniform Compression

$$C_{u01} = 4\times\pi^2\times f_z^{\,2}\times\frac{m}{A_{01}} = 4\times\pi^2\times 30^2\times\frac{2128.75}{1.125}\times\frac{1}{1000} = 6.723\times10^4 \quad \text{kN/m}^3$$

Site Static Stress

Effective depth $\qquad d_{01} = 4+(0.75/2) = 4.375\ \text{m}$

$$\sigma_1 = \rho_s\times d_{01}\times g = 2000\times4.375\times9.81\times\frac{1}{1000} = 85.8375 \quad \text{kN/m}^2$$

$$\sigma_2 = h\times\rho_c\times g = 0.7\times2500\times9.81\times\frac{1}{1000} = 17.1675 \quad \text{kN/m}^2$$

$$\bar{\sigma}_{01} = \sigma_1+\sigma_2 = 103.005 \quad \text{kN/m}^2$$

Design Data (Data_02)

Area of Foundation Block $\qquad A_{02} = 4\times2 = 8\ \text{m}^2$

Effective depth $\qquad d_{02} = 3+0.5\times2 = 4\ \text{m}$

Mass of foundation block $\qquad m_b = 8\times4\times2500 = 80000 \quad \text{kg}$

Mass of machine $\qquad m_m = 20000 \quad \text{kg}$

Total Mass $\qquad m_{02} = m_b + m_m = 100000 \quad kg$

Static Stress

$$\sigma_1 = \rho_s\times d_{02}\times g\times\frac{1}{1000} = 2000\times4\times9.81\times\frac{1}{1000} = 78.48 \quad \text{kN/m}^2$$

$$\sigma_2 = \frac{m_{02}}{A_{02}}\times g\times\frac{1}{1000} = \frac{100000}{8}\times9.81\times\frac{1}{1000} = 122.625$$

$$\bar{\sigma}_{02} = \sigma_1+\sigma_2 = 201.105 \quad \text{kN/m}^2$$

Design Coefficient of Uniform Compression C_{u02}: Since C_u is directly proportional to G, we can consider variation of C_u with static stress in the same manner as that for G.

$$C_{u02} = C_{u01}\times\sqrt{\frac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}}\times\sqrt{\frac{A_{01}}{A_{02}}} = 6.7231\times10^4\times\sqrt{\frac{201.105}{103.005}}\times\sqrt{\frac{1.125}{8}} = 3.5228\times10^4 \quad \text{kN/m}^3$$

Design Shear Modulus G_{02}

$$G_{02} = C_{u02} \times A_{02} \times \frac{(1-\nu)}{4 \times r_0}; \quad r_0 = \sqrt{\frac{A_{02}}{\pi}} = \sqrt{\frac{8}{\pi}} = 1.5957 \text{ m}$$

Substituting the values, we get

$$G_{02} = 3.5228 \times 10^4 \times 8 \times \frac{(1-0.3)}{4 \times 1.5957} = 3.09 \times 10^4 \quad \text{kN/m}^2$$

P 5.4-3.

For a site, dynamic soil investigation is carried out using Cyclic Plate Load Test. The test data is as under:

Size of Test Plate	$600 \text{ mm} \times 600 \text{ mm}$
Depth of test pit	4 m
Test Results *(Only the last test value is presented here)*	
Pressure	$p = 240 \quad \text{kN/m}^2$
Elastic settlement	$s_e = 1.2$ mm
Mass density of soil	$\rho_s = 2000 \text{ kg/m}^3$
Ht. of machine CG above foundation top	0.5 m
Poisson's Ratio	$\nu = 0.3$

Compute Design Coefficient of Uniform Compression C_{u02} & Design Shear Modulus for a machine foundation with details as under:

Foundation Size	$2\text{m} \times 4\text{m} \times 4\text{m}$ deep
Depth of foundation	3 m
Mass Density of Concrete	2500 kg/m^3
Mass of Machine	20000 kg
Maximum permissible Amplitude	100 microns

Solution:

Mass density of soil	$\rho_s = 2000 \text{ kg/m}^3$
Poisson's Ratio	$\nu = 0.3$
Size of Test Plate	$600 \text{mm} \times 600 \text{mm}$
Contact area of test plate	$A_{01} = 0.6 \times 0.6 = 0.36 \text{ m}^2$
Depth of test pit	4 m
Applied Pressure on the plate	$p = 240 \quad \text{kN/m}^2$
Resulting Elastic settlement	$s_e = 1.2 \text{ mm} = 1.2 \times 10^{-3} \text{ m}$

Site Data (Data_01)

Coefficient of Uniform Compression C_{u01}

Coefficient of Uniform Compression $C_{u01} = \frac{p}{S_e} = \frac{240}{1.2 \times 10^{-3}} = 20 \times 10^4 \quad \text{kN/m}^3$

Static Stress (Overburden Pressure) $\bar{\sigma}_{01}$ for the test

Effective Depth $d_{01} = 4.0 + 0.5 \times 0.6 = 4.3 \quad \text{m}$

$$\sigma_1 = \rho_s \times d_{01} \times g = 2000 \times 4.3 \times 9.81 \times \frac{1}{1000} = 84.366 \quad \text{kN/m}^2$$

σ_2 = Over burden pressure = Applied Pressure 'p' on the test plate = 240 kN/m²

$$\bar{\sigma}_{01} = (84.366 + 240) = 324.366 \quad \text{kN/m}^2$$

Design Data (Data_02)

Area of Foundation Block $A_{02} = 4 \times 2 = 8 \text{ m}^2$

Effective depth $d_{02} = 3 + 0.5 \times 2 = 4 \text{ m}$

Mass of foundation block $m_b = 8 \times 4 \times 2500 = 80000 \quad \text{kg}$

Mass of machine $m_m = 20000 \quad \text{kg}$

Total Mass $m_{02} = m_b + m_m = 100000 \quad \text{kg}$

Static Stress

$$\sigma_1 = \rho_s \times d_{02} \times g \times \frac{1}{1000} = 2000 \times 4 \times 9.81 \times \frac{1}{1000} = 78.48 \quad \text{kN/m}^2$$

$$\sigma_2 = \frac{m_{02}}{A_{02}} \times g \times \frac{1}{1000} = \frac{100000}{8} \times 9.81 \times \frac{1}{1000} = 122.625 \quad \text{kN/m}^2$$

$$\bar{\sigma}_{02} = \sigma_1 + \sigma_2 = 201.105 \quad \text{kN/m}^2$$

Design Coefficient of Uniform Compression C_{u02}

$$C_{u02} = C_{u01} \times \sqrt{\frac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}} \times \sqrt{\frac{A_{01}}{A_{02}}} = 20 \times 10^4 \times \sqrt{\frac{201.105}{324.366}} \times \sqrt{\frac{0.36}{8}} = 3.34 \times 10^4 \quad \text{kN/m}^3$$

Design Shear Modulus

$$G_{02} = C_{u02} \times A_{02} \times \frac{(1-\nu)}{4 \times r_0}; \quad r_0 = \sqrt{\frac{A_{02}}{\pi}} = \sqrt{\frac{8}{\pi}} = 1.5957 \text{ m}$$

$$G_{02} = 3.34 \times 10^4 \times 8 \times \frac{(1-.3)}{4 \times 1.5957} = 2.9303 \times 10^4 \quad \text{kN/m}^2$$

P 5.4-4

For a site, dynamic soil investigation is carried out using Vertical Vibration Resonance Test. The test data is as under:

Size of test block	$1.5 \text{ m} \times 0.75 \text{ m} \times 0.7 \text{ m}$ high
Mass density of concrete test block	$\rho_c = 2500 \text{ kg/m}^3$
Depth of test pit	4 m
Mass of Oscillator system	$m_0 = 160$ kg
Resonant Frequency	$f_z = 30$ Hz
Amplitude of Vibration	200 microns
Mass density of soil	$\rho_s = 2000 \text{ kg/m}^3$
Ht. of machine CG above foundation top	0.5 m
Poisson's Ratio	$\nu = 0.3$

Compute Design Coefficient of Uniform Compression C_{u02} & Design Shear Modulus for a machine foundation with details as under:

Foundation Size	$3 \text{ m} \times 5 \text{ m} \times 5 \text{ m}$ deep
Depth of foundation	4 m
Mass Density of Concrete	2500 kg/m^3
Mass of Machine	20000 kg
Maximum permissible Amplitude	100 microns

Solution:

Mass density of concrete test block	$\rho_c = 2500 \text{ kg/m}^3$
Mass density of soil	$\rho_s = 2000 \text{ kg/m}^3$
Poisson's Ratio of soil	$\nu = 0.3$

Site Data (Data_01)

Area of Test Block	$A_{01} = 1.5 \times 0.75 = 1.125 \quad \text{m}^2$
Height of Test Block	$h = 0.7$ m

Mass of Test Block (including m/c)

$$m_{01} = 2500 \times 1.125 \times 0.7 + 160 = 2128.75 \quad \text{kg}$$

Site Coefficient of Uniform Compression

$$C_{u01} = 4 \times \pi^2 \times f_z^2 \times \frac{m_{01}}{A_{01}} = 4 \times \pi^2 \times 30^2 \times \frac{2128.75}{1.125} \times \frac{1}{1000} = 6.723 \times 10^4 \quad \text{kN/m}^3$$

Site Static Stress

Effective depth $d_{01} = 4 + (0.75/2) = 4.375 \text{ m}$

$$\sigma_1 = \rho_s \times d_{01} \times g = 2000 \times 4.375 \times 9.81 \times \frac{1}{1000} = 85.8375 \quad \text{kN/m}^2$$

$$\sigma_2 = h \times \rho_c \times g = 0.7 \times 2500 \times 9.81 \times \frac{1}{1000} = 17.1675 \quad \text{kN/m}^2$$

$$\overline{\sigma}_{01} = \sigma_1 + \sigma_2 = 103.005 \quad \text{kN/m}^2$$

Design Data (Data_02)

Area of Foundation Block $A_{02} = 5 \times 3 = 15 \text{ m}^2$

Effective depth $d_{02} = 4 + 0.5 \times 3 = 5.5 \text{ m}$

Mass of foundation block $m_b = 15 \times 5 \times 2500 = 187500 \quad \text{kg}$

Mass of machine $m_m = 20000 \quad \text{kg}$

Total Mass $m_{02} = m_b + m_m = 187500 + 20000 = 207500 \quad \text{kg}$

Static Stress $\sigma_1 = \rho_s \times d_{02} \times g \times \frac{1}{1000} = 2000 \times 5.5 \times 9.81 \times \frac{1}{1000} = 107.91 \quad \text{kN/m}^2$

$$\sigma_2 = \frac{m_{02}}{A_{02}} \times g \times \frac{1}{1000} = \frac{207500}{15} \times 9.81 \times \frac{1}{1000} = 135.705$$

$$\overline{\sigma}_{02} = \sigma_1 + \sigma_2 = 243.615 \quad \text{kN/m}^2$$

Design Coefficient of Uniform Compression C_{u02}

Since area of foundation is greater than 10 m^2, consider $A_{02} = 10 \text{ m}^2$ (See § 5.4.2)

$$C_{u02} = C_{u01} \times \sqrt{\frac{\overline{\sigma}_{02}}{\overline{\sigma}_{01}}} \times \sqrt{\frac{A_{01}}{A_{02}}} = 6.7231 \times 10^4 \times \sqrt{\frac{243.615}{103.005}} \times \sqrt{\frac{1.125}{10}} = 3.47 \times 10^4 \quad \text{kN/m}^3$$

Design Shear Modulus G_{02}

Since area of foundation is greater than $10\ m^2$, consider $A_{02} = 10\ \text{m}^2$

$$G_{02} = C_{u02} \times A_{02} \times \frac{(1-\nu)}{4 \times r_0}; \quad r_0 = \sqrt{\frac{A_{02}}{\pi}} = \sqrt{\frac{10}{\pi}} = 1.7841 \quad \text{m}$$

Substituting the values, we get

$$G_{02} = 3.47 \times 10^4 \times 10 \times \frac{(1-0.3)}{4 \times 1.7841} = 3.4016 \times 10^4 \quad \text{kN/m}^2$$

EXAMPLE PROBLEMS (§ 5.5)

P 5.5-1

For the data given in Problem P 5.4-1, compute Equivalent Soil Springs in all six DOFs.

Solution:

Machine Foundation details as given in Problem P 5.4-1:

Foundation Size	$2\ \text{m} \times 4\ \text{m} \times 4\ \text{m}$ deep
Depth of foundation	3 m
Base Contact Area of the Foundation	$A_{02} = 2 \times 4 = 8\ \text{m}^2$
Mass of foundation block	$m_b = 8 \times 4 \times 2500 = 80000 \quad \text{kg}$
Mass of machine	$m_m = 20000 \quad \text{kg}$
Ht. of machine CG above foundation top	$0.5\ m$
Total Mass	$m_{02} = m_b + m_m = 100000 \quad \text{kg}$
Mass density of soil	$\rho_s = 2000\ \text{kg/m}^3$
Poisson's Ratio of soil	$\nu = 0.3$

From the solution to problem P 5.4-1, we get:

Design Shear Modulus	$G_{02} = 7.246 \times 10^4 \quad \text{kN/m}^2$
Design Coefficient of Uniform Compression	$C_{u02} = 8.2588 \times 10^4\ \text{kN/m}^3$

Soil Spring Stiffness

Elastic Half Space method

Length of Foundation	4.0 m
Width of Foundation	2.0 m
L/B ratio of the foundation	$L/B = 2.0$
B/L ratio of the foundation	$B/L = 2/4 = 0.5$

From Figure 5.5-1, for $L/B = 2.0$, we get $\beta_y = 2.2$; $\beta_z = 0.962$

For rocking mode (about Z axis), for $B/L = 0.5$ **we get** $\beta_\phi = 0.45$

For rocking mode (about X axis), for $L/B = 2.0$ we get $\beta_\theta = 0.6$

Vertical (Y-direction)

$$k_y = \beta_y \times \frac{G}{(1-\nu)} \times \sqrt{L \times B} \quad = \quad k_y = 6.44 \times 10^5 \text{ kN/m}$$

Lateral Z –direction

$$k_z = \beta_z \times 4 \times (1+\nu) \times G \times \sqrt{\frac{L \times B}{4}} = k_z = 5.13 \times 10^5 \text{ kN/m}$$

Lateral X –direction

$$k_x = \beta_x \times 4 \times (1+\nu) \times G \times \sqrt{\frac{L \times B}{4}} = k_x = 5.60 \times 10^5 \text{ kN/m}$$

Rocking about Z-axis (ϕ direction)

$$k_\phi = \beta_\phi \times \frac{G}{(1-\nu)} \times B^2 \times L = 7.45 \times 10^5 \text{ kN m/rad}$$

Rocking about X-axis (θ direction)

$$k_\theta = \beta_\theta \times \frac{G}{(1-\nu)} \times B \times L^2 = 19.87 \times 10^5 \text{ kN m/rad}$$

Torsional Mode about Y(ψ direction)

$$k_\psi = \frac{16G}{3} \left\{ \left(\frac{L\,B\left(L^2 + B^2\right)}{6\pi} \right)^{1/4} \right\}^3 = 19.22 \times 10^5 \text{ kN m/rad}$$

Coefficient of Sub grade Reaction Method

Base area of Foundation $A = 4 \times 2 = 8\ \text{m}^2$

Moment of Inertia of Base area:

$$I_{xx} = \frac{1}{12} \times 2 \times 4^3 = 10.667 \quad \text{m}^4; \qquad I_{zz} = \frac{1}{12} \times 4 \times 2^3 = 2.667 \quad \text{m}^4$$

$$I_{yy} = 10.667 + 2.667 = 13.334 \quad \text{m}^4$$

Coefficient of Uniform Compression $C_u = 8.2588 \times 10^4 \quad \text{kN/m}^3$

Coefficient of Uniform Shear $C_\tau = 0.5 \times C_u = 4.1294 \times 10^4 \quad \text{kN/m}^3$

Coefficient of Non-Uniform Compression $C_\phi = 2 \times C_u = 16.57 \times 10^4 \quad \text{kN/m}^3$

Coefficient of Non-Uniform Shear $C_\psi = 0.75 \times C_u = 6.20 \times 10^4 \quad \text{kN/m}^3$

Equivalent Spring Constants

Lateral X –direction $k_x = C_\tau \times A = 4.129 \times 10^4 \times 8 = 3.3 \times 10^5 \quad \text{kN/m}$

Vertical (Y-direction) $k_y = C_u \times A = 8.2588 \times 10^4 \times 8 = 6.61 \times 10^5 \quad \text{kN/m}$

Lateral Z –direction $k_z = C_\tau \times A = 4.129 \times 10^4 \times 8 = 3.3 \times 10^5 \quad \text{kN/m}$

Rocking about X-axis (θ direction)

$$k_\theta = C_\phi \times I_{xx} = 16.57 \times 10^4 \times 10.667 = 17.6 \times 10^5 \quad \text{kN m/rad}$$

Torsional Rotation about Y- axis (ψ direction)

$$k_\psi = C_\phi \times I_{xx} = 6.20 \times 10^4 \times 13.334 = 8.26 \times 10^5 \quad \text{kN m/rad}$$

Rocking about Z-axis (ϕ direction)

$$k_\phi = C_\phi \times I_{zz} = 16.57 \times 10^4 \times 2.667 = 4.41 \times 10^5 \quad \text{kN m/rad}$$

For the sake of academic interest let us compare the stiffness (all six stiffness) as obtained by these two methods. The ratio of stiffness by Elastic Half Space Method to that by Coefficient of Sub-grade Reaction Method for all the six DOFs is:

	k_y	k_z	k_x	k_ϕ	k_θ	k_ψ
Ratio	0.974	1.55	1.70	1.69	1.13	2.33

It is noticed from these numbers that except for vertical stiffness, Elastic Half Space gives sufficiently higher stiffness in all the modes of vibration.

P 5.5-2:

For the data given in Problem P 5.5-1, compute Radiation Damping in all six DOFs for Elastic Half Space method.

Solution:

For evaluating Geometric Damping, we need to compute equivalent radius r_0 and Mass Ratio b for all the six DOFs. Let us first compute equivalent radius in each DOF

Equivalent radius

Vertical (Y-direction) $$r_0 = \sqrt{\frac{L \times B}{\pi}} = \sqrt{\frac{4 \times 2}{\pi}} = 1.5957 \quad \text{m}$$

Equivalent radius in Lateral Z – direction and Lateral X – direction is the same as that for Vertical (Y-direction).

Lateral Z – direction $r_0 = 1.5957$ m

Lateral X – direction $r_0 = 1.5957$ m

Rocking about Z-axis (ϕ direction) $$r_0 = \left(\frac{B^3 \times L}{3\pi}\right)^{1/4} = \left(\frac{2^3 \times 4}{3\pi}\right)^{1/4} = 1.3574 \quad \text{m}$$

Rocking about X-axis (θ direction) $$r_0 = \left(\frac{L^3 \times B}{3\pi}\right)^{1/4} = \left(\frac{4^3 \times 2}{3\pi}\right)^{1/4} = 1.9197 \quad \text{m}$$

Torsional Rotation about Y (ψ direction)

$$r_0 = \left(\frac{L \times B \times \left(L^2 + B^2\right)}{6\pi}\right)^{1/4} = \left(\frac{4 \times 2 \times \left(4^2 + 2^2\right)}{6\pi}\right)^{1/4} = 1.7069 \quad \text{m}$$

Mass Ratio: To compute Mass ratio, we need to compute Mass m and Mass Moment of Inertia M_{mo} (for both mass of the foundation block as well as mass of the machine) along the three translational DOFs and the three rotational DOFs passing through CG of the base area point O.

Foundation Mass	80000 kg
Machine Mass	20000 kg
Total Mass (see solution P 5.5-1)	100000 kg
Thus Mass m along three translational DOFs	$m = 100000$ kg

Mass Moment of Inertia (MMI)

About Z-axis (M_{moz})

$$M_{moz} = \left(\frac{80000}{12}\right) \times \left(2^2 + 4^2\right) + 80000 \times (4/2)^2 + 20000 \times (4 + 0.5)^2 = 858333 \quad \text{kg m}^2$$

About X-axis (M_{mox})

$$M_{mox} = \left(\frac{80000}{12}\right) \times \left(4^2 + 4^2\right) + 80000 \times (4/2)^2 + 20000 \times (4 + 0.5)^2 = 938333 \quad \text{kg m}^2$$

About Y-axis (M_{moy})

MMI of the foundation is directly computed based on its geometrical data and for MMI of machine about Y-Y axis, its Radius of Gyration need to be evaluated. This information is either given by the manufacturer or computed based on its geometrical layout data. In the absence of any given data for the present case, it is assumed that radius of gyration of machine about YY is equal to 75% of equivalent radius of foundation. (This is an assumption made only for this problem).

Radius of Gyration $\qquad r = 0.75 \times r_0 = 0.75 \times 1.5957 \quad = 1.1968 \text{ m}$

$$M_{moy} = \left(\frac{80000}{12}\right) \times \left(4^2 + 2^2\right) + 20000 \times (1.1968)^2 = 161981 \quad \text{kg m}^2$$

Let us now compute Mass Ratios for each DOF.

$$b_x = b_z = b_y = \frac{(1-\nu)}{4} \frac{m}{\rho_s} \frac{1}{r_0^3} = \frac{(1-0.3)}{4} \times \frac{100000}{2000} \times \frac{1}{1.5957^3} = 2.153267$$

$$b_\phi = \frac{3(1-\nu))}{8} \times \frac{M_{moz}}{\rho_s \times r_0^5} = \frac{3(1-0.3))}{8} \frac{858333}{2000 \times 1.3574^5} = 24.44$$

$$b_\theta = \frac{3(1-\nu))}{8} \times \frac{M_{mox}}{\rho_s \times r_0^5} = \frac{3(1-0.3))}{8} \frac{938333}{2000 \times 1.9197^5} = 4.72$$

$$b_\psi = \frac{3(1-\nu))}{8} \times \frac{M_{moy}}{\rho_s \times r_0^5} = \frac{3(1-0.3))}{8} \frac{161981}{2000 \times 1.7069^5} = 1.4673$$

Geometrical Damping Constant

$$\zeta_y = \frac{0.425}{\sqrt{b_y}} = \frac{0.425}{\sqrt{2.153267}} = 0.2896$$

$$\zeta_z = \frac{0.2875}{\sqrt{b_z}} = \frac{0.2875}{\sqrt{2.153267}} = 0.1959$$

$$\zeta_x = \frac{0.28755}{\sqrt{b_x}} = \frac{0.2875}{\sqrt{2.153267}} = 0.1959$$

$$\zeta_\phi = \frac{0.15}{(1+b_\phi)\sqrt{b_\phi}} = \frac{0.15}{(1+24.44)\sqrt{24.44}} = 0.0012$$

$$\zeta_\theta = \frac{0.15}{(1+b_\phi)\sqrt{b_\theta}} = \frac{0.15}{(1+4.72)\sqrt{4.72}} = 0.0120$$

$$\zeta_\psi = \frac{0.5}{(1+2b_\psi)} = \frac{0.5}{(1+21.4673)} = 0.1270$$

P 5.5-3

A machine having Mass of 20000 kg is supported over a concrete block of length 4m, width 2 m and depth 4 m (Machine and Foundation data same as that for Problem P 5.5-1). The concrete block in turn is supported over an elastic cork pad 200 mm thick as shown in Figure 5.5-3. Mass Density of Concrete is $2500\ \text{kg/m}^3$. Elastic Modulus of cork is $E_{cork} = 1.2\times10^5\ \ \text{kN/m}^2$ and that of concrete is $E_{conc} = 3\times10^7\ \ \text{kN/m}^2$. Poisson's ratio of the cork is $\nu_{cork} = 0.04$ and that of concrete is $\nu_{conc} = 0.15$. Compute equivalent springs representing cork stiffness in all six DOFs.

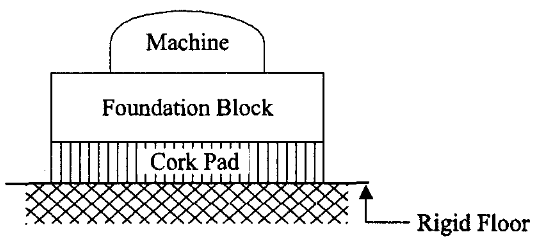


Figure P5.5-3 Foundation Block Supported over cork Pad

Solution:

Area of the cork pad $A = 2\times4 = 8\ \text{m}^2$

Thickness of cork pad $t = 0.2\ \text{m}$

Elastic Modulus of cork $E_{cork} = 1.2\times10^5\ \ \text{kN/m}^2$

Poisson's Ratio of cork $\nu = 0.04$

Moment of Inertia about axes passing through CG of Base Area

About XX $$I_{xx} = \frac{1}{12} \times 2 \times 4^3 = 10.6667 \quad m^4$$

About ZZ $$I_{zz} = \frac{1}{12} \times 4 \times 2^3 = 2.6667 \quad m^4$$

About YY $$I_{yy} = \frac{1}{12} \times (4 \times 2^3 + 2 \times 4^3) = 13.3333 \quad m^4$$

Stiffness of Cork:

Linear Stiffness

Along Vertical Y direction $k_y = \dfrac{E_{cork} \times A}{t} = \dfrac{1.2 \times 10^5 \times 8}{0.2} = 4.8 \times 10^6$ kN/m

Along Lateral Z direction $k_z = \dfrac{E\,A}{2(1+\nu)t} = \dfrac{1.2 \times 10^5 \times 8}{2 \times (1+0.04) \times 0.2} = 2.307 \times 10^6$ kN/m

Along Lateral X direction $k_x = \dfrac{E\,A}{2(1+\nu)t} = \dfrac{1.2 \times 10^5 \times 8}{2 \times (1+0.04) \times 0.2} = 2.307 \times 10^6$ kN/m

Rotational Stiffness

About X- axis $$k_\theta = \frac{E\,I_{xx}}{t} = \frac{1.2 \times 10^5 \times 10.6667}{0.2} = 6.4 \times 10^6 \quad \text{kN m/rad}$$

About Z- axis $$k_\phi = \frac{E\,I_{zz}}{t} = \frac{1.2 \times 10^5 \times 2.6667}{0.2} = 1.6 \times 10^6 \quad \text{kN m/rad}$$

About Y- axis $$k_\psi = \frac{E}{2(1+\nu)} \frac{I_{yy}}{t} = \frac{1.2 \times 10^5}{2 \times (1+0.04)} \times \frac{13.3333}{0.2} = 3.84 \times 10^6 \quad \text{kN m/rad}$$

P 5.5-4

A machine having Mass of 20000 kg is supported over a concrete foundation block of length 5m, width 2 m and depth 2 m. The concrete block in turn is supported over 48 springs (9 springs along length and 5 springs along width) as shown in Figure P 5.5-4. Vertical Stiffness of each spring is $k_v = 1 \times 10^5$ kN/m and Lateral Stiffness is $k_h = 0.6 \times 10^5$ kN/m. Mass Density of Concrete is 2500 kg/m^3. Elastic Modulus of concrete is $E_{conc} = 3 \times 10^7$ kN/m^2. Poisson's ratio of concrete is $\nu_{conc} = 0.15$. Compute equivalent springs at CG of the base of the block in all six DOFs.

Solution:

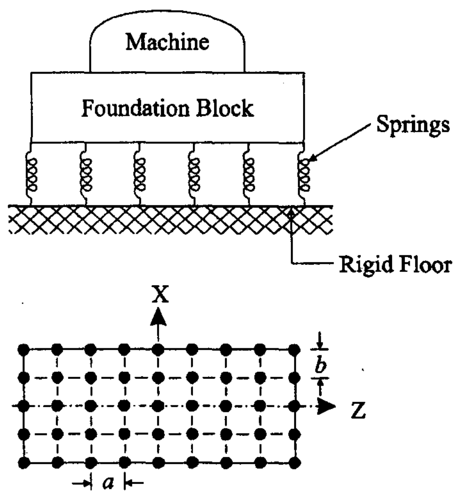
Given Data

Length of the Foundation Block	5 m
Width of the Foundation Block	2 m
Vertical Stiffness of each spring	$k_v = 1\times10^5$ kN/m
Lateral Stiffness of each spring	$k_h = 0.6\times10^5$ kN/m

Springs along length 9 & springs along width 5 & total number of springs = 45

It is seen that there are odd number of springs both along length and width.

Springs along length	9	$2p+1=9;$	$p=4$
Springs along width	5	$2q+1=5;$	$q=2$
Total number of springs		$n=9\times5=45$	
Spacing along Length: There are 9 springs hence 8 spacing		$a=5/8=0.625$ m	
Spacing along Width: There are 5 springs hence 4 spacing		$b=2/4=0.5$ m	



Spring Locations shown in PLAN

Figure P5.5-4 Foundation Supported over 45 Springs

Equivalent Stiffness

Translational Stiffness

Vertical Stiffness $k_y = n \times k_v = 45 \times 1 \times 10^5 = 4.5 \times 10^6$ kN/m

Lateral Stiffness $k_x = k_z = n \times k_h = 45 \times 0.6 \times 10^5 \times = 2.7 \times 10^6$ kN/m

Rotational Stiffness

Rocking Stiffness about X-X (Equation 5.5.3-1)

$$k_\theta = \frac{M_\theta}{\delta_\theta} = k_y \times a^2 \times \frac{p(p+1)}{3}$$

$$k_\theta = 4.5 \times 10^6 \times (0.625)^2 \times \frac{4 \times (4+1)}{3} = 11.72 \times 10^6 \quad \text{kN m/rad}$$

Rocking Stiffness about Z-Z (Equation 5.5.3-2a)

$$k_\phi = \frac{M_\phi}{\delta_\phi} = k_y \times b^2 \times \left\{\frac{q(q+1)}{3}\right\}$$

$$k_\phi = 4.5 \times 10^6 \times (0.5)^2 \times \frac{2 \times (2+1)}{3} = 2.25 \times 10^6 \quad \text{kN m/rad}$$

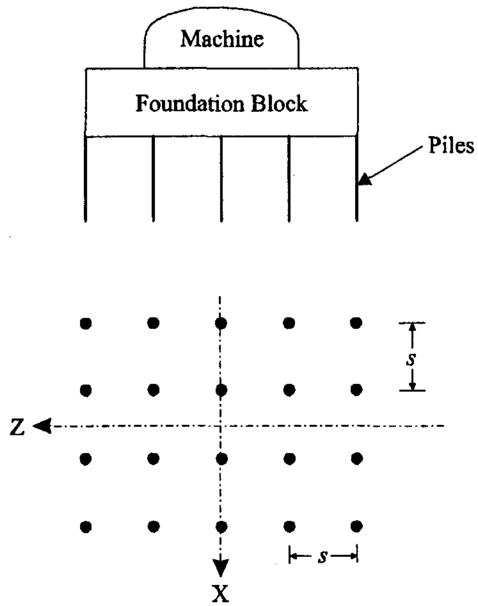
Torsional Stiffness about Y-Y (Equation 5.5.3-6a)

$$k_\psi = \frac{M}{\delta_\psi} = \left[\left\{a^2 k_x \frac{p(p+1)}{3}\right\} + \left\{b^2 \times k_z \frac{q(q+1)}{3}\right\}\right]$$

$$k_\psi = \left\{2.7 \times 10^6 \times \left((0.625)^2 \times \frac{4 \times 5}{3}\right) + 2.7 \times 10^6 \times \left((0.5)^2 \times \frac{2 \times 3}{3}\right)\right\} = 8.38 \times 10^6 \quad \text{kN m/rad}$$

P 5.5.5

A machine having Mass of 20000 kg is supported over a concrete foundation block of length 5m, width 4 m and depth 2 m. The concrete block in turn is supported over 20 Piles (5 piles along length and 4 Piles along width) as shown in Figure P 5.5-5. Each pile is 400 mm dia and 20 m long. Pile spacing is 1.0 m, both along length and width. Vertical Stiffness of each Pile is $k_{pv} = 6.4 \times 10^5$ kN/m and Lateral Stiffness is $k_{ph} = 3.84 \times 10^5$ kN/m. Compute equivalent springs at CG of the base of the block in all six DOFs.



Pile Location shown in PLAN

Figure P5.5-5 Foundation Supported over 20 Piles

Solution:

Total number of piles $n = 20$

Piles along Length = 5; $2p + 1 = 5; \quad p = 2$

Piles along Width = 4; $2q = 4; \quad q = 2$

Pile Spacing $s = 1$; Pile diameter $d = 0.40$; $s/d = 2.5$

Vertical stiffness of each pile $k_{pv} = 6.4 \times 10^5$ kN/m

Lateral stiffness of each pile $k_{ph} = 3.84 \times 10^5$ kN/m

Influence Coefficient for $s/d = 2.5$

$$\alpha_{eff} = 0.212 \times \left(\frac{s}{d}\right)^{0.65} = 0.212 \times (2.5)^{0.65} = 0.3845$$

Effective Vertical Stiffness k_v of each pile $k_v = \alpha_{eff} \times k_{pv}$

$k_v = 0.3845 \times 6.4 \times 10^5 = 2.46 \times 10^5$ kN/m

Effective Lateral Stiffness k_h of each pile $k_h = \alpha_{eff} \times k_{ph}$

$$k_h = 0.3845 \times 3.84 \times 10^5 = 1.476 \times 10^5 \quad \text{kN/m}$$

Equivalent Springs

Linear springs along Vertical Y Direction

$$k_y = n \times k_v = 20 \times 2.46 \times 10^5 = 4.92 \times 10^6 \quad \text{kN/m}$$

Along Lateral X Direction $k_x = n \times k_h = 20 \times 1.476 \times 10^5 = 2.952 \times 10^6$ kN/m

Along Lateral Z Direction $k_z = n \times k_h = 20 \times 1.476 \times 10^5 = 2.952 \times 10^6$ kN/m

Rotational Springs

Rotational Spring Constant about X-axis

$$k_\theta = k_y \times s^2 \times \frac{p(p+1)}{3}$$

$$= 4.92 \times 10^6 \times 1^2 \times \frac{2(2+1)}{3} = 9.84 \times 10^6 \quad \text{kNm/rad}$$

Rotational Spring Constant about Z-axis

$$k_\phi = k_y \times s^2 \times \left\{ \frac{(2q+1)(2q-1)}{12} \right\}$$

$$= 4.92 \times 10^6 \times 1^2 \times \left\{ \frac{(2 \times 2 + 1)(2 \times 2 - 1)}{12} \right\} = 6.15 \times 10^6 \quad \text{kNm/rad}$$

Torsional Spring Constant about Y-axis

$$k_\psi = s^2 k_x \left\{ \frac{p(p+1)}{3} + \frac{(2q+1)(2q-1)}{12} \right\}$$

$$= 1^2 \times 2.952 \times 10^6 \left\{ \frac{2(2+1)}{3} + \frac{(2 \times 2 + 1)(2 \times 2 - 1)}{12} \right\} = 6.344 \times 10^6 \quad \text{kNm/rad}$$

6

DESIGN MACHINE PARAMETERS

- Rotary Machines
- Reciprocating Machines
- Impact Machines
- Impulsive Load Machines
- Amplitudes of Vibration
- Rotor Eccentricity
- Unbalance Force
- Transient Resonance
- Critical Speeds
- Emergency Loads
- Coupling of Machines

Example Problems

Design Machine Parameters

In the context of machine foundation design, a machine would necessarily include:

- A Drive Machine
- A Driven Machine
- A Coupling Device

Machine data is required both for **drive machine and driven machine** along with **coupling** details. The complete knowledge of excitation forces, associated frequencies and load transfer mechanism from the machine to the foundation is a must for correct evaluation of dynamic response. A close interaction between the foundation designer and machine supplier as well as appreciation of each other's limitations, therefore, is essential.

Before we go to the details of machine data for ***Drive Machine, Driven Machine & Coupling Device****, it may be desirable to note the following:*

- *Though the supplier for all the three machines may be a single agency, invariably manufacturer would be different for each machine*
- *Each machine, according to its footprint, has its own base frame and bolting arrangement with the foundation*
- *Each machine is balanced independently as a separate unit*
- *Each machine rotor has its own critical speed*
- *When these machines are coupled together and supplied as a set, the data for individual machine may not fully hold good for the coupled machines. It may need appropriate correction/modification. At times, these machines may be mounted on a common base frame instead of their individual base frame and that will reflect as a change in the mass CG location.*

The equipment drawings and data-sheets supplied by the machine manufacturer do provide a host of information about the machine and out of this, only the information required for the foundation design needs to be selected. In certain cases, some data may have to be processed for design purposes. Thus, for a properly designed foundation, careful determination of those design machine parameters that influence response of machine foundation becomes essential.

The available Machine data, therefore, needs to be suitably converted and translated in to Design Machine Parameters for use in Machine Foundation design.

A typical data set required for each machine (**drive machine, driven machine & coupling**) is listed as under:

For Dynamic Response Analysis

1. Total mass of Machine (including rotating parts), Radius of Gyration and its Overall Centroid location.
2. Mass of rotating parts of the machine, operating speed, height of centerline of rotor from machine base frame, etc.
3. Footprint of machine, base frame details and holding down bolts.
4. Dynamic forces generated by the machine under operating conditions.

Additional information regarding number of blades in case of fans & turbines, number of poles in case of motors etc. may turn out to be helpful in specific cases (**see $ 6.1**),

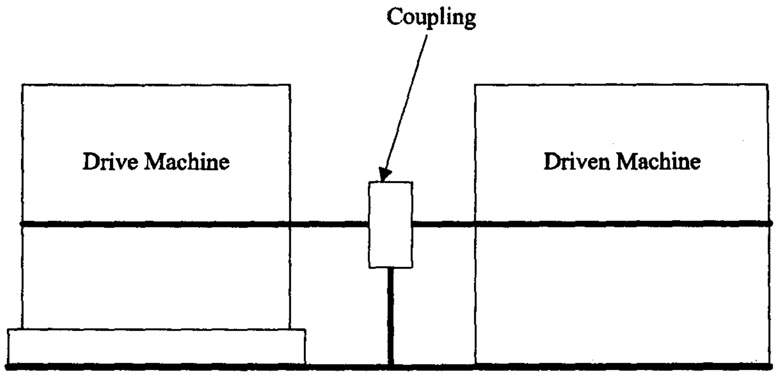


Figure 6.1-1 A Typical Rotary Machine

For Strength Design

1. Static loads from the machine.
2. Equivalent Static forces i.e. dynamic forces converted as Equivalent Static Forces
3. Forces generated under Emergency and Faulted conditions e.g. Bearing Failure Forces, Short Circuit Forces and Forces due to Loss of Blade etc.
4. Forces during Erection, Maintenance & Test Conditions of the machine

In this chapter we refer to machine parameters for

a) Rotary Machines
b) Reciprocating Machines
c) Impact Machines

6.1 PARAMETERS FOR ROTARY MACHINES

Conceptually, a rotary machine comprises of a rotary drive machine, a rotary driven machine connected through a coupling device. A typical schematic arrangement is shown in Figure 6.1-1.

6.1.1 Dynamic Forces

Every rotating machine possesses some amount of residual unbalance even after balancing. This **residual unbalance** is termed as **rotor eccentricity** and the rotor, due to this eccentricity, produces unbalance dynamic force. The unbalance dynamic force therefore is a function of rotor mass, rotor eccentricity and rotor speed.

Rotor generates dynamic forces at all speeds. During start-up it generates dynamic forces right from zero speed to its full operating speed, whereas during shutdown it generates dynamic forces right from full operating speed to halt position (zero speed). The dynamic force shall be at its maximum speed of operation. Dynamic Forces are normally supplied by the manufacturer/supplier. The generated unbalance forces are function of rotor mass, rotor eccentricity and rotor speed. These forces are computed at bearing levels for all possible combinations.

6.1.1.1 Rotor Eccentricity

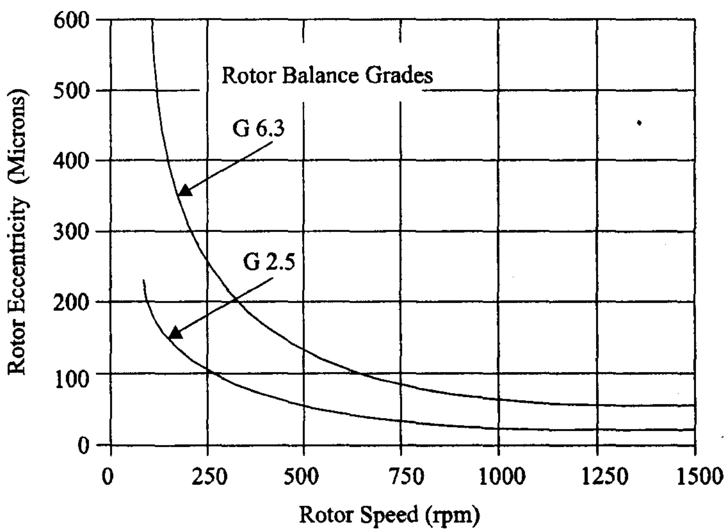


Figure 6.1.1-1 Rotor Eccentricity Vs. Rotor Speed for Rigid Rotors

The rotor (of every rotating machine) is balanced to a required balance quality grade. Balance quality grade for a rotor gets decided based on operating speed and the intended use of the machine. International Standard Organization code 'ISO 1940/1' gives the recommended balance quality grades for rotors (**only Rigid Rotors**) of all types of rotating machines.

The residual balance present in the rotor gives rise to unbalance dynamic forces. Thus the generated unbalance forces are directly proportional to the balance quality grade. Balance quality grade is represented as Gr (e.g. G0.4, G1, G2.5, G6.3, G16, G40 etc.) where the letter G is used as a notation for Grade and r is the number (in mm/sec) that represents product of eccentricity (in mm) and rotation speed ω (in rad/sec). In other words r could also be expressed as e *per* ω Thus ratio r/ω gives eccentricity in mm.

Thus for a rotor balanced to balance quality grade Gr and operating at speed ω rad/sec, the eccentricity of rotor e (in meters) is given as:

$$e = \frac{r}{\omega} \times 10^{-3} \quad \text{m} \qquad (6.1.1\text{-}1)$$

Thus a balance grade $G6.3$ for a rotor operating at 900 rpm ($\omega = 94.25$ rad/s) would mean that the rotor eccentricity is:

$$e = \frac{6.3}{94.25} \times 10^{-3} = 66.8 \times 10^{-6} \text{ m} \quad = 66.8 \text{ microns}$$

Figure 6.1.1-1 gives Rotor Eccentricity in microns vs. Rotor speed for Balance grades $G2.5$ & $G6.3$.

Normally machine manufacturer provides either rotor eccentricity or rotor balance quality grade for the rotor. In certain cases unbalanced forces generated by the rotor are furnished. In the absence of any such information, it is recommended to use the following relationship:

a) For rigid rotors use balance grade as per ISO 1940/1, and

b) For Flexible Rotors $e = \dfrac{500}{N^2}$ m (after Barkan (1962)) (6.1.1-2)

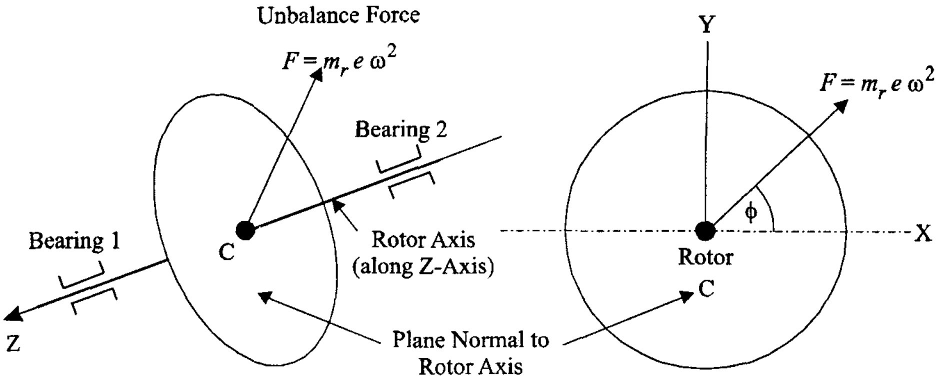
Here e represents eccentricity in meters & N represents operating speed of the rotor in rpm.

6.1.1.2 Unbalance Forces

The eccentricity e represents the residual unbalance left in the rotor after balancing. The rotor generates dynamic unbalance force, which is nothing but the centrifugal force generated by the rotor of mass m_r (in kg) having eccentricity e (in meters), rotating at frequency ω rad/s.

Thus the dynamic unbalance force F in (Newton) is given as

$$F(t) = m_r \times e \times \omega^2 \sin \omega t \qquad \text{N} \tag{6.1.1-3}$$



(a) Rotor Support on Bearings

(b) Unbalance Force acting in a plane normal to Rotor axis

Figure 6.1.1-2 Rotor Unbalance Force

This is also expressed as $F(t) = F_0 \sin \omega t$ where F_0 represents magnitude of the unbalance force in Newton and ω represents the excitation frequency in rad/s.

The magnitude of the dynamic unbalance force is $F_0 = m_r \times e \times \omega^2$ and excitation frequency equals to the running speed of the rotor i.e. ω rad/s or N rpm. This force acts in a plane normal to rotor axis and is directed radially outward from the center of the rotor.

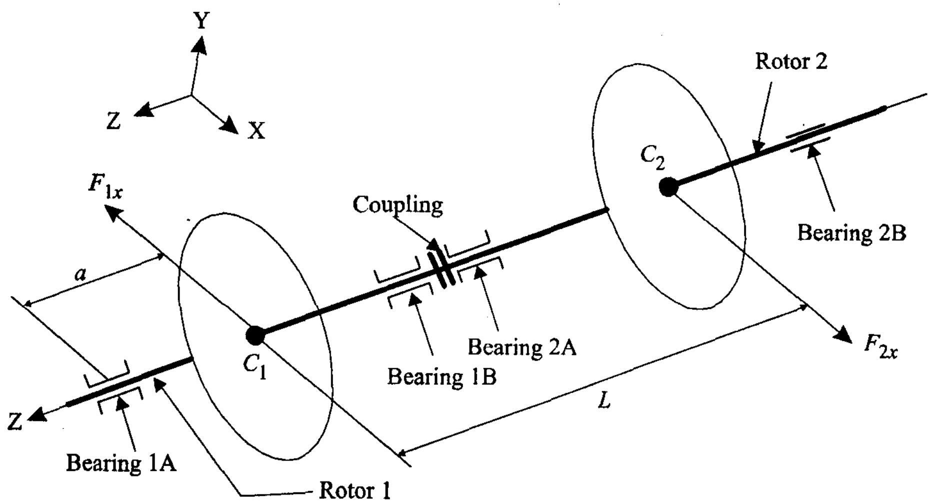
Let us consider a rotor of mass m_r (in kg) having eccentricity e (in meters), rotating at frequency ω rad/sec having rotor axis as Z – axis as shown in Figure 6.1.1-2. Consider at any instant of time t, that the force $F(t) = m_r \times e \times \omega^2 \sin \omega t$ is directed at an angle ϕ from the horizontal axis (X-X) passing through center of the rotor point C. The two components of this force in X-X & Y-Y direction are:

In X-X direction $\quad F_{xx} = m_r\, e\,\omega^2 \cos\phi \quad$ (6.1.1-4)

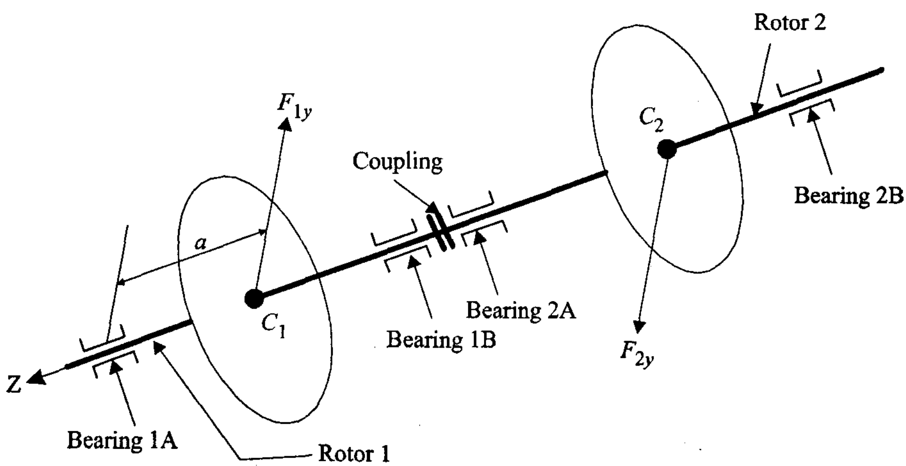
In Y-Y direction $\quad F_{yy} = m_r\, e\,\omega^2 \sin\phi \quad$ (6.1.1-5)

Maximum Horizontal component $(\cos\phi = 1)$ $\quad F_{zz} = m_r\, e\,\omega^2 \quad$ (6.1.1-6)

Maximum Vertical component $(\sin\phi = 1)$ $\quad F_{yy} = m_r\, e\,\omega^2 \quad$ (6.1.1-7)



(a) X Component of Unbalance Force 180° Out of Phase



(b) Y Component of Unbalance Force 180° out of Phase

Figure 6.1.1-3 Machine Having two Rotors - Rotor 1 & Rotor 2 - Unbalance Forces out of Phase in Each rotor

This force is transferred from the rotor to its two bearings in the same ratio as that of the rotor static reactions. If the reactions on the two bearings due to rotor weight are in the ratio of say $a:b$, then the dynamic force transferred from the rotor to the bearings shall also be in the ratio $a:b$. Considering that the bearing pedestals are rigid not to cause any amplification, the transfer of the force from the bearing to the foundation takes place as per law of Statics.

For machines having more than one rotor, the unbalance force generated in each rotor may or may not have same phase angle. Let us examine this also in detail. Let $F_1, F_2, F_3, ...$ represent dynamic forces generated by rotors (rotor 1, rotor 2, rotor 3 etc).

Consider a machine comprising of two rotors, say rotor 1 & rotor 2 having same mass m_{r1} & m_{r2}, eccentricity e_1 & e_2 and speed ω_1 & ω_2 respectively as shown in Figure 6.1.1-2. Let Rotor 1 be supported by Bearing 1A & 1B & Rotor 2 be supported by Bearing 2A & 2B respectively. Let the unbalance forces generated by Rotor 1 & 2 be F_1 & F_2 respectively. Let C_1 and C_2 represent the points of action of forces F_1 & F_2 respectively. Let the distance between C_1 & C_2 be L and that between C_1 and bearing 1A be a.

We can write the Unbalance forces as

$$F_1 = m_{r1}\, e_1\, \omega_1^{\,2}; \qquad F_2 = m_{r2}\, e_2\, \omega_2^{\,2} \qquad (6.1.1\text{-}8)$$

Consider that at any instance of time *t*, the forces F_1 & F_2 are at an angle ϕ_1 & ϕ_2 with respect to X-axis respectively.

Components of the unbalance forces are:

X-Components are $F_{1x} = F_1 \cos\phi_1$; $F_{2x} = F_2 \cos\phi_2$

Y-Components are $F_{1y} = F_1 \sin\phi_1$; $F_{2y} = F_2 \sin\phi_2$

Case 1: Both the forces F_1 & F_2 have same phase angle i.e. $\phi_1 = \phi_2 = \phi$

The total Maximum Reaction along Y-axis shall be $(\sin\phi = 1)$

$$F_{1y} + F_{2y} = F_1 + F_2 \qquad (6.1.1\text{-}9)$$

The total Maximum Reaction along X-axis shall also be $(\cos\phi = 1)$

$$F_{1x} + F_{2x} = F_1 + F_2 \tag{6.1.1-10}$$

Case 2: Both the forces F_1 & F_2 are 180^0 out of phase i.e. $\phi_1 = \phi$ & $\phi_2 = 180 - \phi$

The total Maximum Reaction along Y-axis shall be $(\sin\phi = 1)$

$$F_{1y} - F_{2y} = F_1 - F_2 \tag{6.1.1-11}$$

The total Maximum Reaction along X-axis shall also be $(\cos\phi = 1)$

$$F_{1x} - F_{2x} = F_1 - F_2 \tag{6.1.1-12}$$

In addition, the unbalance forces shall give rise to two couples:

Moment at any point say at bearing 1 (@ distance a (along Z) from center of Rotor 1)

Maximum Moment about Y- axis $\quad M_\psi = F_{2x} \times (L + a) - F_{1x} \times a \quad$ (6.1.1-13)

Maximum Moment about X-axis $\quad M_\theta = F_{2y} \times (L + a) - F_{1y} \times a \quad$ (6.1.1-14)

Thus it is clear that though the generated unbalance forces have components only in X & Y direction, these will also generate moments about Y & X axes. Hence it becomes obvious that it is not enough to compute amplitudes for vibration modes in Y and X translation, but amplitudes must also be computed for rocking (about X-axis) as well as Torsional Mode (about Y axis) for the moments thus generated as above.

> **Caution:** At times it has been noticed that dynamic forces given by some machine suppliers contain an arbitrary **Multiplication Factor**. This is undesirable and must be corrected at this stage itself otherwise it may lead to unrealistic dynamic design of the foundation. Forces generated must be in accordance with rotor eccentricity, rotor speed and rotor mass (as above).

6.1.2 Transient Resonance

6.1.2.1 Unbalance Forces during Start-up and Shutdown

During start-up the rotor generates dynamic forces at all speeds, right from start (zero speed) till full operating speed **(see $ 6.1).** Similarly during shutdown, it generates dynamic forces at all speeds i.e. from full speed right up to halt position.

The dynamic forces at operating speed contribute to **Steady-State Response** of the system, whereas the dynamic forces during startup and shutdown contribute to **Transient Response**. Its significance becomes more for under-tuned foundations where foundation natural frequencies are below operating speed of the machine. With every start-up & shutdown, machine speed comes into resonance with foundation natural frequencies called **Transient Resonance**. Though unbalance force is relatively low at transient resonance, resulting amplitudes become higher on account of resonance.

Maximum magnitude of dynamic force generated by rotor (**see equation 6.1.1-3**) is:

$$F_0 = m_r \times e \times \omega^2 \tag{6.1.2-1}$$

In non-dimensional form, we can write

$$\frac{F_0}{m_r g} = \frac{m_r e \omega^2}{m_r g} = \frac{e\omega^2}{g} \tag{6.1.2-2}$$

LHS term is the ratio of Unbalance Force to the Rotor Weight. Plot of equation 6.1.2-2 for a machine with rotor eccentricity of 50 microns and maximum operating speed of 50 Hz. is shown in Figure 6.1.2-1.

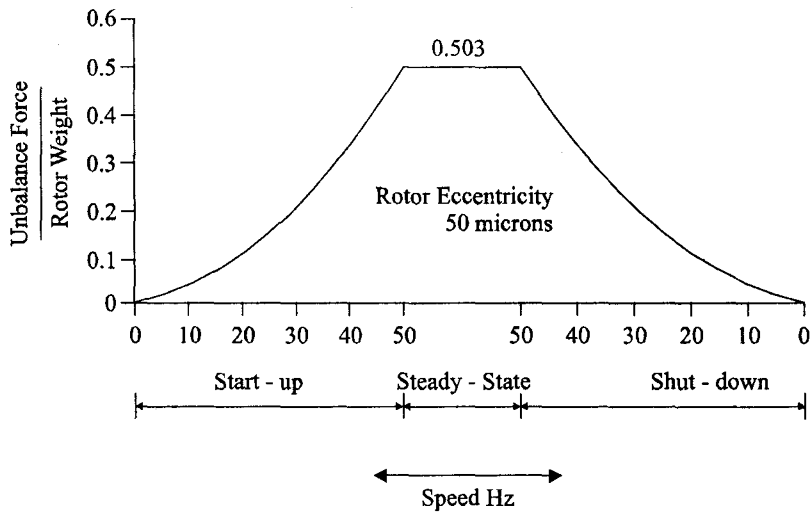


Figure 6.1.2-1 Unbalance Force During Start-up, Steady-State & Shut-Down for Rotor Eccentricity 50 Microns - Max Machine Speed 50 Hz.

It is seen that during start-up, the force rises and reaches its maximum at full operating speed and during shutdown it drops down to zero value at halt position. It is also seen that the maximum value of the force is 0.503 times the weight of the rotor.

For transient resonance, though the unbalance force is lesser but the magnification shall be high because of resonance giving high amplitudes.

6.1.2.2 Nozzle Passing Frequencies

Nozzle passing frequencies are expressed as rotor speed multiplied by number of blades. In case of machines like pumps, fans, turbines, compressors, etc. high vibrations have been encountered when foundation/support system frequencies are in resonance with nozzle passing frequencies.

As these frequencies are multiple of operating speed, possibility of occurrence of transient resonance exists only in over-tuned machines. It is desirable to avoid these frequencies while computing natural frequencies of foundations.

6.1.3 Critical Speeds of Rotors

Critical Speeds of Rotors correspond to flexural frequencies of the rotors. Normally machine supplier furnishes these. If not, one should ask for these.

In some cases, high vibrations have been reported on account of resonance with critical speeds. It is only desirable to avoid these frequencies while computing natural frequencies of foundations.

6.1.4 Rotor Bearing Supports

Generally one comes across the machines having either **Pedestal Bearings** or **End-shield Bearings.** In case of Pedestal Bearings, the pedestal is independently supported on the foundation and the dynamic force from the rotor are transmitted to the foundation at the pedestal support location.

In case of End-shield Bearings, the bearing is housed in the machine casing itself and the dynamic force from the rotor are transmitted to the foundation through the machine stator support points.

This aspect is important from the point of view of **Amplitude Computations** and must be taken care of while mathematical modeling of machine foundation system.

6.1.5 Forces Due To Emergency And Faulted Conditions

Invariably every machine, during its life cycle, sustains very high forces, which occur due to malfunction of one or other features. Such conditions are termed as **Emergency and Faulted Conditions**. Machine develops very high forces during these conditions. Adequacy of foundation must be ensured to withstand these forces. Hence, these forces must be considered for strength design of the foundation only.

Without going into further details, we list some of the known faulted conditions related to machines. These are:

6.1.5.1 Bearing Failure Forces

Cases of Bearing Seizure, on one account or the other, have been reported in the past. In certain cases, inadequate supply or no supply of lube oil to bearings (for whatsoever reason), have resulted in **Seizure of Bearings**. Due to this machine running at full speed comes to grinding halt in very-very short time (may be in seconds or in a couple of minutes). This phenomenon results in very high dynamic forces developed by the rotor. These are termed as **Bearing Failure Forces.**

The damage to the machine obviously cannot be prevented but the objective is that the foundation should structurally be strong enough to withstand such high forces.

It is difficult to quantify these forces specifically. Based on the experience, it is however suggested that a **Force equal to 3 to 5 times the Rotor Weight** should be considered in vertical as well as transverse direction transmitted through rotor bearings to the foundation. It is to be ensured that Foundation should be capable to withstand this force. An increase in the allowable stress to the tune of 50 % could be considered for strength check. Similar increase could also be considered for bearing pressure check.

6.1.5.2 Short Circuit Forces

These forces occur due to short circuit in motors. These are generally furnished by the supplier and should be considered for strength design of the foundation. An increase in the allowable stress to the tune of 25 % could be considered for strength check. Similar increase could also be considered for bearing pressure check.

6.1.5.3 Forces due to Loss of moving part like Blade, Hammer and Fins etc.

For machines such as Fans, Pumps, Compressors, Crushers, etc. often one encounters condition of loss of blade or loss of hammer as the case may be. These give rise to very high forces. An increase in the allowable stress to the tune of 25 % could be considered for strength check. Similar increase could also be considered for bearing pressure check. These forces are furnished by the supplier/manufacturer and should be considered for strength design of the foundation.

6.1.6 Coupling of Machines

Different types of coupling arrangements are seen for different machines. Drive machine could be coupled directly to the driven machine or through a gearbox. When the machines are directly coupled, the operating speed of drive and driven machine remain the same and the unbalance forces for both drive and driven machine develop at the same speed.

On the other hand, when coupling is through gearbox, the operating speeds of drive and driven machine become different (in proportion to gear ratio) and the unbalance forces developed by both drive and driven machine are at different speeds. In this case transient resonance amplitudes are computed with respect to speeds of both the machines.

6.2 PARAMETERS FOR RECIPROCATING MACHINES

Conceptually, a reciprocating machine comprises of i) crank rod, ii) connecting rod and iii) piston (including piston rod). In this type of machine, linear motion of the piston, through a connecting rod, results in rotary motion of the crankshaft.

Consider a single cylinder-reciprocating machine placed in Y-Z plane. Movement of piston is along Z-axis and rotation of the crank about X-axis (X-axis is normal to plane of paper). A typical schematic arrangement showing system under motion is as given in Figure 6.2-1.

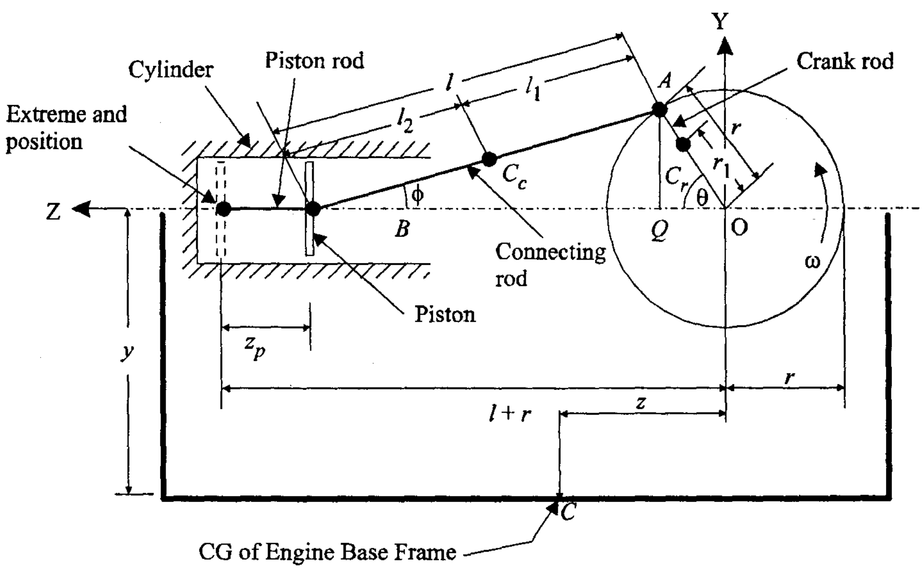


Figure 6.2-1 A Typical arrangement of a Single Cylinder Reciprocating Machine System Under Motion - Position at any time t

6.2.1 Dynamic Forces

The motion of the connecting rod is a complex motion i.e. one end of the connecting rod performs linear motion and the other end performs rotary motion. For evaluation of dynamic forces, let us define the machine parameters as under:

6.2.1.1 Single Cylinder Machine

Consider machine centerline along Z-axis. Y-axis represents vertical direction.

Crank rod:

(Center of rotation point O connected to connecting rod at point A)

Mass of crank rod	m_r
Length of crank rod	r
Distance of its CG (C_r) from center of rotation point 'O'	r_1
Speed of rotation (rad/sec)	ω

Connecting rod:

(End A connected to crank rod & end B connected to piston):

Mass of connecting rod	m_c
Length of connecting rod	l
Distance of CG of connecting rod (C_c) from point A	l_1
Distance of CG of connecting rod (C_c) from point B	l_2

Piston:

Mass (Piston assembly including piston rod, cross head etc)	m_p

Consider the position of piston at any time t is as shown in Figure 6.2-1. Let the piston displacement from its extreme end position be z_p. At this position, connecting rod makes an angle ϕ and crank rod makes an angle θ with machine axis OZ as shown. Rotary motion of the crank rod generates linear motion of the piston.

For computation of dynamic forces, consider that the distributed mass of connecting rod is lumped at points $A \& B$ and that of crank rod is lumped at point $A \& O$ using principle of statics.

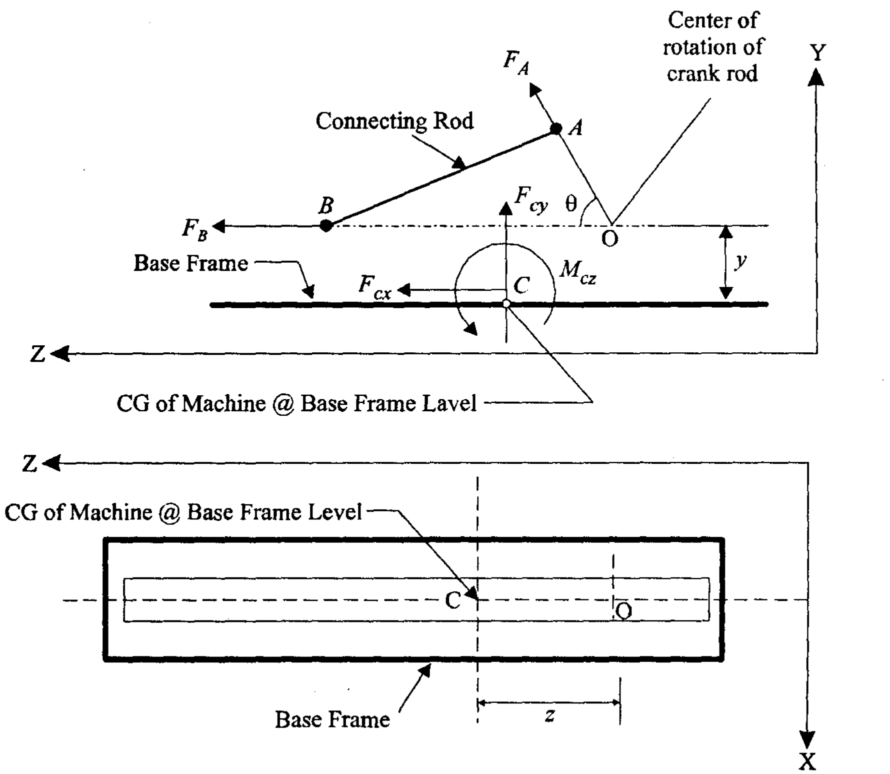


Figure 6.2-2 Single Machine - Position at any Instant *t* - Dynamic Forces Transferred at CG of Machine @ Base Frame Level

Mass at A $$m_A = m_r(r_1/r) + m_c(l_2/l)$$

Mass at B $$m_B = m_c(l_1/l) + m_p$$

Dynamic force

Dynamic force generated at point A $$F_A = m_A \times r \times \omega^2 \qquad (6.2\text{-}1)$$
(This force acts radially outwards along OA)

Dynamic force generated at point B $$F_B = m_B \times \ddot{z}_p \qquad (6.2\text{-}2)$$
(This force acts linearly along OZ)

Let us now evaluate z_p in terms of motion parameters of the machine.

$$z_p = l + r - (r\cos\theta + l\cos\phi) = r(1-\cos\theta) + l(1-\cos\phi) \qquad (6.2\text{-}3)$$

From Figure 6.2-1, we get

Also $AQ = r\sin\theta = l\sin\phi$; this gives $\cos\phi = \left(1 - \frac{r^2}{l^2}\sin^2\theta\right)^{1/2}$

Using series expansion and ignoring higher order terms, it gives

$$\cos\phi = 1 - \frac{1}{2}\frac{r^2}{l^2}\sin^2\theta \qquad \text{Or}$$

$$(1-\cos\phi) = \frac{1}{2}\frac{r^2}{l^2}\sin^2\theta = \frac{1}{2}\frac{r^2}{l^2}\frac{1-\cos 2\theta}{2} = \frac{1}{4}\frac{r^2}{l^2}(1-\cos 2\theta) \qquad (6.2\text{-}4)$$

Substituting 6.2-4 in equation 6.2-3, we get

$$z_p = r(1-\cos\theta) + \frac{1}{4}\frac{r^2}{l}(1-\cos 2\theta) = r\left(1+\frac{r}{4l}\right) - r\left(\cos\theta + \frac{r}{4l}\cos 2\theta\right)$$

Since $\theta = \omega t$; substituting we get

$$z_p = r\left(1+\frac{r}{4l}\right) - r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \qquad (6.2\text{-}5)$$

Differentiating it gives

$$\dot{z}_p = r\omega\left(\sin\omega t + \frac{r}{2l}\sin 2\omega t\right)$$

$$\ddot{z}_p = r\omega^2\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) \qquad (6.2\text{-}6)$$

Substituting in equation 6.2-2, we get

$$F_B = m_B \times \ddot{z}_p = m_B \times \left\{r\omega^2\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right)\right\}\ddot{z}_p$$
$$= \underbrace{m_B r\omega^2(\cos\omega t)}_{primary\,component} + \underbrace{m_B r\omega^2\left(\frac{r}{l}\cos 2\omega t\right)}_{secondary\,component} \qquad (6.2\text{-}7)$$

It is seen that the **Dynamic force generated at point A** that acts radially outwards along OA has only one component at machine speed whereas **Dynamic force generated at point B** that acts along OZ, has two components, one at machine speed and the other at twice the machine speed.

These dynamic forces in turn are transferred to the foundation through the base frame of the machine. Resolving these forces and transferring to CG of Base Frame point C at instant of time t, we get:

$$F_{cz} = F_B + F_A \cos\theta; \quad F_{cy} = F_A \sin\theta; \quad M_{cx} = F_{cz} \times y + F_{cy} \times z \tag{6.2-8}$$

Here $F_A \& F_B$ are as given by equations 6.2-1 and 6.2-7 respectively. M_{cx} is the moment at C about X –axis and $y \& z$ are distances of point O from point C in Y & Z direction respectively. These forces are shown in Figure 6.2-2. From this equation, we get forces:

At $\theta = 0°$, forces and moments acing at point C are:

$$F_{cy} = 0; \quad F_{cz} = (F_A + F_B); \quad M_{cx} = (F_A + F_B) \times y \tag{6.2-9}$$

At $\theta = 90°$, forces and moments acing at point C are:

$$F_{cy} = F_A; \quad F_{cz} = F_B; \quad M_{cx} = F_A \times z + F_B \times y \tag{6.2-10}$$

6.2.1.2 Multi-Cylinder Machine

Consider a multi-cylinder engine having n **cylinders.** The placement of these n cylinders in Plan (X-Z Plane) is as shown in Figure 6.2-3. Let us consider that at any instant of time t, the crank rod in i^{th} cylinder makes an angle θ_i with the O-Z.

Dynamic forces are computed in line with the procedure for a single cylinder machine. The net force developed by the machine in a specific direction is the algebraic sum of the force developed in each cylinder in that direction.

Forces developed in the i^{th} cylinder machine

Force along Z-direction at CG of the base frame level point C:

$$F_{cz\,i} = (F_{Bi} + F_{Ai} \cos\theta_i) \tag{6.2-11}$$

Force along Y-direction at CG of the base frame level point C:

$$F_{cy\,i} = F_{Ai} \sin\theta_i \tag{6.2-12}$$

Moment about X-axis at CG of the base frame level point C:

$$M_{cx\,i} = F_{cz\,i} \times y_i + F_{cy\,i} \times z_i \tag{6.2-13}$$

In addition there shall be one more moment M_{cz} developed at point C about Z-axis.

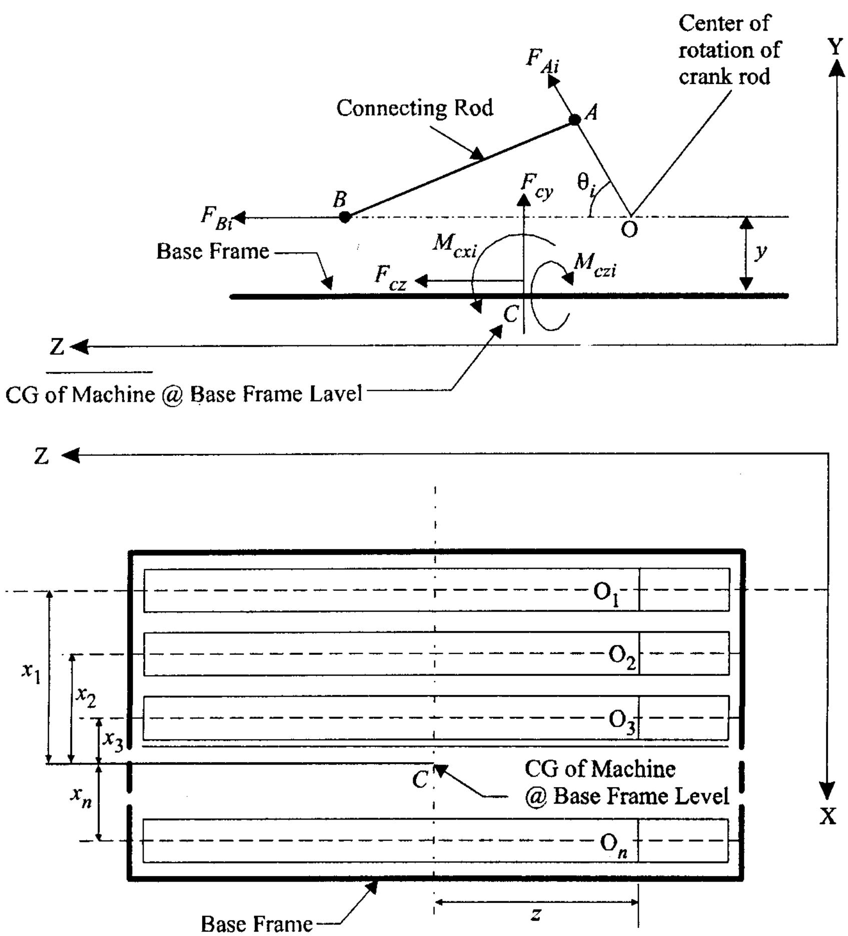


Figure 6.2-3 Multi Cylinder Machine - (*n* Cylinders) Position at any instant *t* - Dynamic Forces Transferred at CG of Machine C @ Base Frame Level

Moment at point C about Z- axis by i^{th} cylinder Machine

$$M_{cz\,i} = (F_{cz\,i} \times x_i) + (F_{cy\,i} \times x_i) \tag{6.2-14}$$

Here x_i, y_i & z_i are the distances of the i^{th} machine from CG of the base frame level point C in X,Y and Z direction respectively.

Total Force at point C in Z direction $$F_{cz} = \sum_{i=1}^{n} F_{cz\,i} \tag{6.2-15}$$

Total Force at point C in Y direction $$F_{cy} = \sum_{i=1}^{n} F_{cy\,i} \tag{6.2-16}$$

Total Moment at point C about X- axis $$M_{cx} = \sum_{i=1}^{n} M_{cx\,i} \tag{6.2-17}$$

Total Moment at point C about Z- axis $$M_{cz} = \sum_{i=1}^{n} M_{cz\,i} \tag{6.2-18}$$

6.2.2 Transient Resonance

In case the rotary forces of the crankshaft are not counterbalanced, transient resonance conditions will be setup during each start-up and shutdown. Thus for under-tuned foundations, Transient

Resonance Amplitudes are to be computed as per provisions given in **$ 6.1.2**.

6.2.3 Forces Due to Emergency and Faulted Conditions

Machine develops very high forces during these conditions. Adequacy of foundation must be ensured to withstand these forces. Thus provisions as given in **$ 6.1.5** are to be considered, if applicable, for strength design of the foundation only.

6.2.4 Coupling of Machines

Provisions as given in **$ 6.1.6** are to be considered, if applicable.

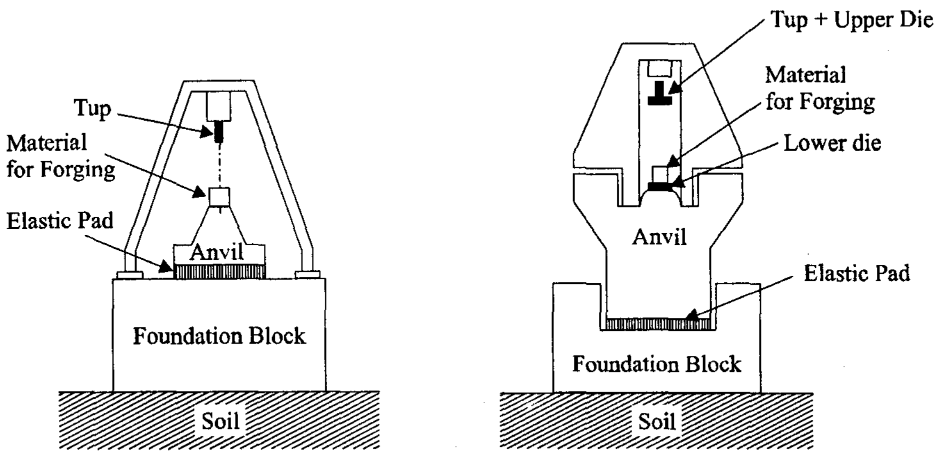
6.3 PARAMETERS FOR IMPACT MACHINES

Various types of **Impact Machines** are in use by the industry. Impact produced by these machines fall under one of the following categories:

i) Machines Producing Repeated Impacts e.g. Forge Hammers & Drop Hammers
ii) Machines Producing Impulse/Pulse Loading e.g. Drop Crushers, Pig Breakers, Jolters, Forging and Stamping Press, etc.

6.3.1 Machines producing repeated Impacts- Forge Hammers

Typical examples of Impact Machine are Forge Hammers. A Forge Hammer comprises of a Tup, an Anvil and a Frame. The complete assembly is mounted over a rigid RCC Foundation, which in turn rests on the soil.



(a) Forge Hammer

Guide Frame attached to Foundation
Elastic Pad below Anvil

(b) Drop Hammer

Guide Frame attached to Anvil
Elastic Pad below Anvil

Figure 6.3-1 A Typical Arrangement of a Hammer Foundation

Drop Hammer is also a sub-set of Forge Hammer. Drop Hammers are used for Die Stampings, whereas Forge Hammers are used for Forging Operations. In case of Drop Hammers, Guide-frame is mounted over Anvil whereas in case of Forge Hammers, Guide-frame and Anvil are supported independently over the foundation block.

The Tup, which is a block of heavy mass, falls from a height and strikes the material, to be forged, placed on the Anvil. The Anvil is invariably placed over an elastic pad and the pad rests on the Foundation Block supported over soil. The elastic pad helps in preventing the bouncing of the Anvil over the Foundation.

The force produced during the strike is termed as the Impact Force. The energy imparted by the impact force results in motion of the Anvil. The energy from the Anvil is then transmitted to the soil through the foundation. Thus for both these machines i.e. Forge and Drop Hammers, Machine parameters, therefore are those parameters that are necessary to compute initial velocity generated by the Anvil as well as stiffness properties of the elastic pad below the Anvil.

Typical parameters required are:

i) Total Mass of the Hammer i.e. Mass of the Tup, Anvil, Die & Frame
ii) Mass of falling part i.e. Tup (also Mass of Upper Die in case of Drop Hammers)
iii) Height of fall for the Tup/ Energy of Impact

iv) Area of the Piston
v) Pressure in the Cylinder
vi) Frequency of Impact
vii) Mass of Anvil (also Mass of Guide Frame if attached to Anvil)
viii) Frequency i.e. Number of Blows/min
ix) Base area of the Anvil
x) Details of Anchor Bolts connecting frame base to the foundation
xi) Thickness of Elastic Pad placed below Anvil and its Elastic Modulus
xii) Coefficient of Restitution/Impact

Depending upon machine type, additional parameters may be needed for evaluating dynamic forces.

6.3.1.1 Dynamic Forces

Vibration of the foundation subjected to impact by the hammer is basically an **Initial Velocity Problem**. We can represent the complete **Hammer-Foundation System** in two parts:

i. A falling Mass m_0 from a height h producing Impact
ii. Remaining System that withstands this impact

Let us first evaluate Initial Velocity of the Falling Mass

Consider that the mass is falling freely. From basic law of dynamics, we write the Initial Velocity of the **freely falling mass** m_0 from a height h as:

$$v_0' = \sqrt{2gh} \qquad (6.3\text{-}1)$$

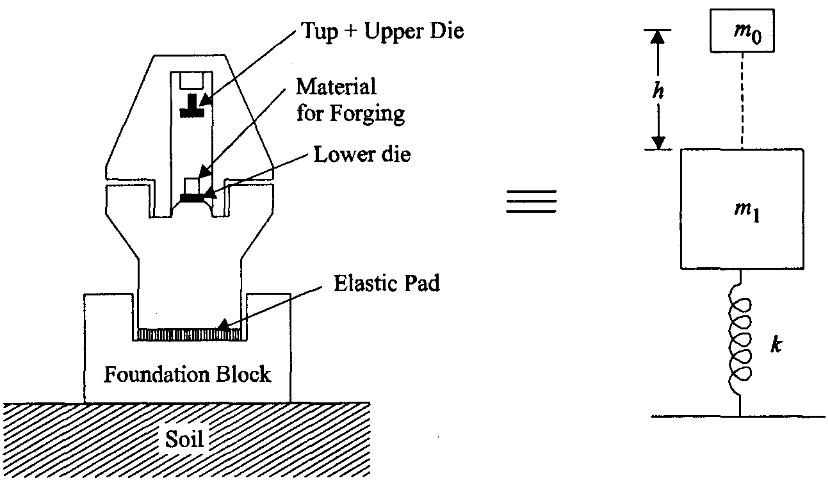
(a) Single Acting Drop Hammers

For a single acting Drop Hammer, initial velocity of the falling mass **(mass of Tup and mass of Top Die)** from a height h is written as:

$$v_0' = \eta\sqrt{2gh} \qquad (6.3\text{-}2)$$

h Represents **total height of fall** of the falling mass
g Represents **acceleration due to gravity**
η Represents **Efficiency of Drop**

Factor η depends upon energy lost in overcoming the friction to the Tup's movement and the resistance of the steam/air counter pressure. From practical considerations, the recommended value of **Efficiency of Drop** η is 0.65.



(a) Drop Hammer
Guide Frame attached to Anvil
Elastic Pad below Anvil

(b) Equivalent Spring Mass System

Figure 6.3-2 Atypical Arrangement of a Hammer Foundation - Equivalent Spring Mass System

(b) Double Acting Hammers

In this case hammer is operated by pneumatic/steam pressure and initial velocity is given as:

$$v_0' = \eta \sqrt{2\, g\, h \times \left(\frac{m_0 \times g + p_s \times A_p}{m_0 \times g} \right)} \tag{6.3-3}$$

The quantity in the bracket represents influence of force on the piston to the initial velocity.

Here:

A_p	Represents area of the piston
p_s	Represents pressure (steam/air) acting on the piston
m_0	Represents total mass of the falling parts
g	Represents acceleration due to gravity

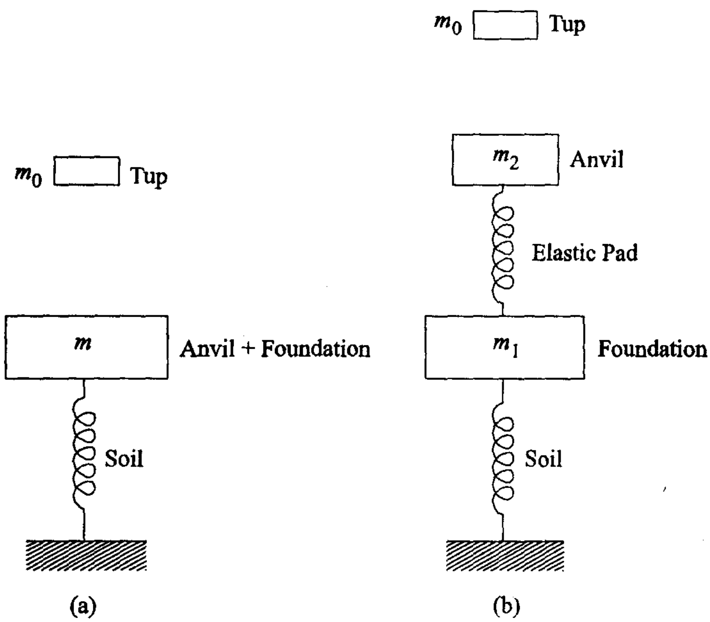


Figure 6.3-3 (a) Foundation Represented as Single Spring Mass System
(b) Foundation Represented as two Spring Mass System

The impact of the Tup is resisted by the Anvil and transferred to soil through the foundation. Let us consider two cases that could be considered representative of Anvil-Foundation System. These are:

i. If there is **no elastic pad** below Anvil, then the Anvil becomes part of the foundation and the impact is resisted by the anvil and foundation together. In this case, Foundation System is represented as a Single Spring Mass System subjected to initial velocity. System is as shown in Figure 6.3-3 (a)
ii. If there is **an elastic pad** below Anvil, then the Anvil first resists the impact. Due to impact, the Anvil develops an initial velocity. In this case, Foundation System is represented as a Two Spring Mass System subjected to initial velocity. System is as shown in Figure 6.3-3 (b)

For both these cases, we need to compute only initial velocity of the Anvil/Foundation as the case may be due to impact of the Tup.

For Single Spring Mass System shown in Figure 6.3-3 (a), initial velocity imparted to the anvil (Anvil + Foundation) by the impact of the Tup is given by equation (2.2.5-3) and the same is reproduced below:

$$v_1 = v_0' \times \frac{(1+e)}{(1+\lambda_1)} \tag{6.3-4}$$

Here $\lambda = m/m_0$ represents ratio of mass of Anvil + Foundation to mass of Tup.

For Two Spring Mass System shown in Figure 6.3-3 (b), initial velocity of the anvil is given by equation (3.2.1-20) and the same is reproduced below:

$$v_2 = v_0' \times \frac{(1+e)}{(1+\lambda_2)} \tag{6.3-5}$$

Here $\lambda_2 = m_2/m_0$ represents ratio of mass of Anvil to mass of Tup.

Here e represents **Coefficient of Restitution** that depends upon properties of the material of the masses m_0 & m or m_0 & m_2 as the case may be. For perfectly plastic central impact, the value of e is zero and for perfectly elastic central impact e is equal to unity. For real bodies in practice, the value lies in the range $0 < e < 1$ and for all practical purposes it's reasonably good to use $e = 0.5$.

6.3.2 Machines Producing Impulse/Pulse Loading

i) Forging and Stamping Press

These presses may be of Hydraulic type, Friction Type or Eccentric. Generally these are very large capacity presses having pressing capacity in the range of a few thousand tons say 10,000 t. Though dynamic forces transmitted to the foundation are small because of low speed of operation, stresses due to impact/shock is of significant order and may cause overloading to the order of 50 to 100% of the material to be forged.

ii) Drop Weight Crushers

These crushers, through dropping of ram, impart very high kinetic energy to the foundation resulting in transmission of high vibrations in the soil. Adjoining structures therefore need to be isolated from these crusher foundations. Impact forces are evaluated like those for hammers.

iii) Crushing, Rolling and Grinding Mills

These mills, due to presence of unbalance masses, produce high dynamic forces that in turn are transmitted to soil through foundation. Due to presence of unbalance masses, dynamic forces are evaluated in line with those for rotating machines.

Typical machine parameters/data required are:

i) Mass Parameters

a. Total Mass of machine
b. Mass of cross head
c. Mass of material to be forged

ii) Dynamic Force Parameters

d. Stroke of the press/ height of fall of ram
e. Pressure exerted by the press
f. Load time history of the pulse
g. Frequency of Impact i.e. Number of Blows/min
h. Unbalance force (in case of mills)

iii) Height & Cross section area of steel columns (in case of Press)
iv) Details of Anchor Bolts and other embedded parts

6.3.2.1 Dynamic Forces

The force produced during operation is termed as the Impulsive Force. Two types of pulse loading are considered:

i) Short duration Impulse Loading
ii) Long duration Pulse Loading

Short Duration Impulse Loading: For machines producing Short Duration Impulse Loading, dynamic response depends upon dynamic excitation force and frequency of excitation. It is therefore desirable to consider Frequency of Impact i.e. Number of Blows/min for amplitude computation. Here dynamic magnification factor depends upon ratio of Frequency of Impact to natural frequency. Dynamic force is nothing but Impulse loading I as shown in Figure 6.3-4.

Impulse $$I = F \times \tau \qquad (6.3\text{-}6)$$

In case Force Time History/Impulse Momentum is not defined by the manufacturer, its value may be considered as:

Impulse $$I = m_0 \times v_0 \qquad (6.3\text{-}7)$$

Here m_0 represents total mass of the falling part and $v_0 = \sqrt{2gh}$ is the terminal velocity, h being height of fall.

Natural frequency $p = \sqrt{\frac{k}{m}}$ rad/s

Number of strike per minute N

Frequency of Impact $\omega = \frac{2\pi}{60} N$ rad/s

Frequency ratio $\beta = \frac{\omega}{p}$

Dynamic Magnification $\mu_y = \frac{1}{\left|\left(1 - \beta^2\right)\right|}$ (6.3-8)

Amplitude $y = \frac{I}{mp} \mu_y \times 10^3$ mm (6.3-9)

Long Duration Pulse Loading: For machines producing Long Duration Pulse Loading, pulse shape showing load time history must be considered for computing dynamic magnification for amplitude computation. Dynamic magnification factor depends upon ratio of pulse duration to natural time period of foundation. For such loadings, maximum response reaches in a very short time before system damping gets effective. Depending upon ratio of duration of the pulse to natural time period, maximum response may occur either during the pulse or after the pulse.

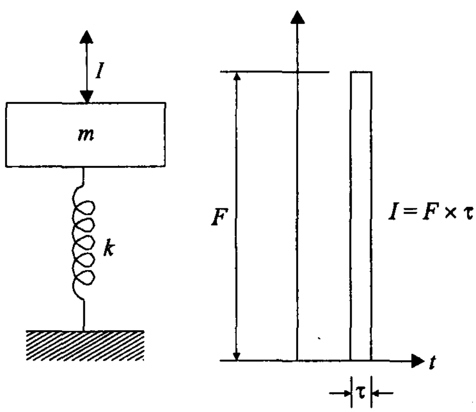


Figure 6.3-4 System Subjected to Impulsive Load

System Response: Response due to pulse/impulse loading is discussed in detail in Chapter 2 (see $ 2.2.6). Response of the system during the pulse duration (Phase I) is the **Forced Vibration Response** and the response after the pulse (Phase II) is the **Free Vibration Response.**

Irrespective of shape of the pulse, for frequency ratio $\beta_y < 1$ i.e. for $\frac{\tau}{T} > \frac{1}{2}$ maximum response occurs during forced vibration phase i.e. Phase I and for $\beta_y > 1$ i.e. for $\frac{\tau}{T} < \frac{1}{2}$, maximum response occurs during Free Vibration Phase i.e. Phase II.

We know that for a spring mass system having stiffness k and mass m

Natural frequency is $p = \sqrt{\frac{k}{m}}$ rad/s. This gives Time Period $T = \frac{2\pi}{p}$.

For Peak Dynamic Force induced by machine as F_y

Amplitude thus becomes $$y = \frac{F_y}{k} \times \xi \times \mu_y \times 10^3 \text{ mm} \quad (6.3\text{-}10)$$

Here ξ is the Fatigue Factor. In case fatigue factor is not defined, it may be taken equal to 2. Magnification Factor μ_y vs. τ / T is shown in Figure 2.2.6-3.

6.4 AMPLITUDES OF VIBRATION

The acceptable norms of vibration tolerance and accordingly the permissible amplitudes of vibration for different types of machines depend upon machine type, its class, its placement in the plant cycle and applicable codes of practices. The acceptance norms may differ from industry to industry and from country to country. Hence it is suggested that one should refer to applicable codes of practices in his respective country or follow the provisions laid down in **International Codes/Standards.** In addition, guidelines/ restrictions given by machine manufacturer must also be given due consideration.

Before setting the limit to the permissible values of the amplitudes, the following considerations must be looked into:

i. Computed amplitudes are always **half amplitudes,** whereas limiting amplitudes, as given by machine manufacturer, are invariably **double amplitudes.**

ii. Limiting amplitudes, given by machine manufacturer, are at machine **bearing locations** whereas computed amplitudes are normally at the **foundation level.**

iii. Machines also can withstand much **higher amplitudes (3 to 5 times higher)** than permissible without getting damaged.

iv. Even when the amplitudes for a given machine are within permissible limits, it could be destructive /unacceptable to adjoining machines/structures.

v. Similar machines would have different vibration limits when used in **different environment** viz. a pump required to supply lube-oil to machine bearing would certainly have stringent vibration limits compared to similar pump for normal pumping applications as failure of pump in earlier case would have serious consequences.

In view of the above, no common guidelines could be defined for setting permissible limits of vibration for all types of machines and for all applications. The governing criteria therefore are the permissible amounts of vibrations that the machine, its surroundings or the persons in the vicinity of the machines can tolerate.

Vibration measurements on the machine are done a) during test at the shop floor at the test bed, b) during commissioning at the site and during normal running of the machine throughout its life cycle at specified time interval. For some machine, vibration monitoring is done continuously 24 hours during operation of machine. Invariably, vibration measurements are taken at all salient points on the machine. These are:

EXAMPLE PROBLEMS – ROTARY MACHINES

P 6.1-1

Compute eccentricity for a rotor having mass $m_r = 1000$ kg, operating speed of 3000 rpm and balance grade $G6.3$

Solution:

Operating speed $N = 3000$ rpm ; or $f = \frac{3000}{60} = 50$ Hz. (cps)

$\omega = 50 \times 2 \times \pi = 314.16$ rad/s

Balance quality grade $G6.3$ (This means $e\omega = 6.3$ mm/sec)

Eccentricity (in - m) $e = \frac{6.3}{314.16} \times 10^{-3} = 2.00 \times 10^{-5}$ m

Thus we can say that eccentricity is $e = 20$ microns

Note: For computation purposes, Eccentricity value must be expressed in meters.

P 6.1-2

Compute unbalance force for the rotor, having mass $m_r = 1000$ kg, balance grade $G6.3$ and rotating at speed of 3000 rpm.

Solution:

Operating speed $N = 3000$ rpm ; or $\omega = \frac{3000}{60} \times 2 \times \pi = 314.16$ rad/s

Balance quality grade (This means $e\omega = 6.3$ mm/s)

Eccentricity corresponding to $G6.3$ **(see P 6.1-1)** $e = 2.00 \times 10^{-5}$ m

Unbalance force $F_t = F_0 \sin \omega t$; $F_0 = m_r \, e \, \omega^2$

$F_0 = 1000 \times 2.0 \times 10^{-5} \times (314.16)^2 = 1974$ N

The force of 1974 N acts at excitation frequency $\omega = 314.16$ rad/s or 3000 rpm

P 6.1-3

A machine has its rotor having mass $m_r = 1000$ kg, operating speed of 3000 rpm and balance grade $G6.3$. Consider that the rotor is along Z-Z axis, supported on two pedestal bearings A & B and the height of centerline of the rotor above the bottom of the bearing pedestal is 800 mm as shown in Figure P 6.1-3. Static reactions R_A & R_B due to rotor at the bearing A and bearing B respectively are in the ratio of 1:1. Compute the unbalance dynamic force a) at the bearing top points and b) at bearing bottom points.

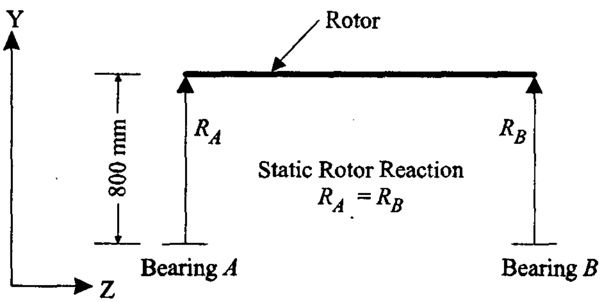


Figure P 6.1-3 Rotor Supported on Bearing A & B

Solution:

Rotor data is same as that in P 6.1-2.

Unbalance force (see P 6.1-2) $= 1974$ N

Since $R_A = R_B$, the reactions due to dynamic force on bearings A & B shall be equal. Thus reaction due to dynamic force at each bearing = $F_A = F_B = 0.5 \times 1974 = 987$ N

a) Unbalance dynamic force at rotor support point of bearing

Max. Vertical Unbalance Force $(\sin\phi = 1)$ along Y-Y at each bearing $F_y = 987$ N

Max. Horizontal Unbalance Force $(\cos\phi = 1)$ along X-X at each bearing $F_x = 987$ N

b) Unbalance dynamic force at bottom of bearing

The vertical force gets transmitted to bearing bottom as it is whereas horizontal force also generates moment at the bearing bottom.

Max. Vertical Unbalance Force (along Y-Y) at each bearing (Same as F_y) = 987 N

Max. Horizontal Unbalance Force (along X-X) at each bearing (Same as F_x) = 987 N

Max. Moment at bearing pedestal bottom (About Z-axis) $M_\phi = 987 \times 0.8 = 789.6$ Nm

P 6.1-4

For the data given in Problem 6.1-3, consider static reactions due to rotor at the two bearing ends are such that $R_A : R_B :: 7:3$. Compute the unbalance dynamic force a) at the bearing top points and b) at bearing bottom points.

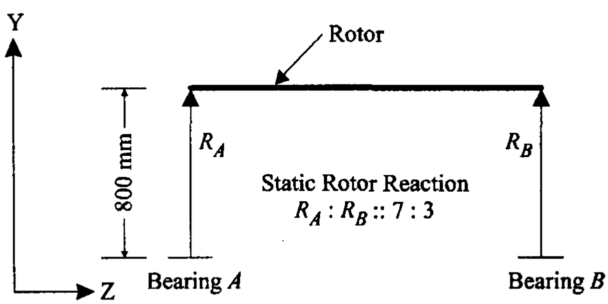


Figure P 6.1-4 Rotor Supported on Bearing *A* & *B*

Solution:

Unbalance force (see Problem 6.1-2) $=1974$ N

a) Unbalance dynamic force at rotor support point of bearing

Max. Vertical Unbalance Force (along Y-Y)

At Bearing A $F_{y@A} = 0.7 \times 1974 = 1381.8$ N

At Bearing B $F_{y@B} = 0.3 \times 1974 = 592.2$ N

Max. Horizontal Unbalance Force (along X-X)

At Bearing A $F_{x@A} = 0.7 \times 1974 = 1381.8$ N

At Bearing B $F_{x@B} = 0.3 \times 1974 = 592.2$ N

b) Unbalance dynamic force at bottom of bearing

Max. Vertical Unbalance Force (along Y-Y) at Bearing A $=1381.8$ N

Max. Vertical Unbalance Force (along Y-Y) at Bearing B $=592.2$ N

Max. Horizontal Unbalance Force (along X-X) at Bearing A $=1381.8$ N

Max. Horizontal Unbalance Force (along X-X) at Bearing B $=592.2$ N

Max. Moment M_ϕ at base of Bearing Pedestal A $M_{\phi@A} = 1318.8 \times 0.8 = 1055.04$ Nm

Maximum moment M_ϕ at base of Bearing Pedestal B $M_{\phi@B} = 592.2 \times 0.8 = 473.76$ Nm

EXAMPLE PROBLEMS – RECIPROCATING MACHINES

P 6.2-1

A horizontal single cylinder reciprocating engine is mounted on a base frame supported by a foundation block as shown in Figure P 6.2-1. Compute unbalanced forces and moments @ CG of Base Frame point C.

The data for the engine is as under:

Mass of the piston & piston rod	19.5 kg
Mass of the crank	15.8 kg
Distance of crank Centroid C_r from axis of rotation point O	150 mm

Mass of connecting rod	9.7 kg
Length of connecting rod	650 mm
Distance of connecting rod Centroid C_c from point B	400 mm
Crank Radius	270 mm
Operating Speed of the engine	300 rpm
Distance of point O from Base frame Centroid (along Z)	1200 mm
Distance of point O from Base frame Centroid (along Y)	500 mm

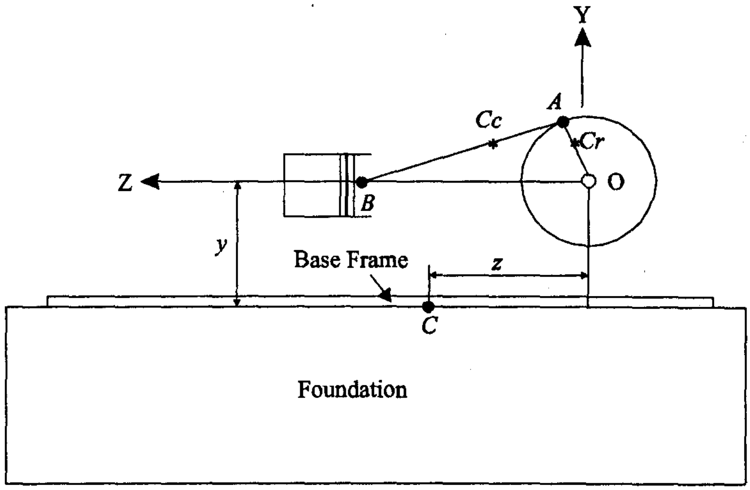


Figure P6.2-1 Single Cylinder Horizontal Reciprocating Engine

Solution

Rewriting basic data in to working units:-

Mass of the piston & piston rod	m_p	19.5 kg
Mass of the crank	m_r	15.8 kg
Distance of crank Centroid Cr from axis of rotation point O	r_1	0.15 m
Mass of connecting rod	m_c	9.70 kg
Length of connecting rod	l	0.65 m
Distance of connecting rod Centroid Cc from point B	l_2	0.40 m
Crank Radius	r	0.27 m
Distance of point O from Base Frame Centroid (along Z)	z	1.20 m

Distance of point O from Foundation Centroid (along Y) y 0.50 m

Operating Speed $\omega = \dfrac{300}{60} \times 2\pi = 31.42$ rad/s

Computation of Masses at points A & B

Mass at A: $m_A = m_r \dfrac{r_1}{r} + m_c \dfrac{l_2}{l} = 15.8 \times \dfrac{0.15}{0.27} + 9.7 \times \dfrac{0.4}{0.65} = 14.75$ kg

Mass at B: $m_B = m_c \dfrac{(l - l_2)}{l} + m_p = 9.7 \times \dfrac{(0.65 - 0.4)}{0.65} + 19.5 = 23.23$ kg

Unbalance Forces:

a) When crank shaft is parallel to Z axis - Crank makes an angle $\theta = 0°$ with Z axis

i) Force generated by mass m_A at A

Along Y axis $F_{Ay} = 14.75 \times 0.27 \times 31.42^2 \sin(0) = 0.0$ N @ $\sin\omega$ t

Along Z axis $F_{Az} = 14.75 \times 0.27 \times 31.42^2 \cos(0) = 3929.8$ N @ $\cos\omega$ t

ii) Force generated by mass m_B at B

$$F_{Bz1} = 23.23 \times 0.27 \times 31.42^2 = 6190.5 \text{ N @ } \cos\omega \text{ t}$$

$$F_{Bz2} = 6190.5 \times \left(\frac{0.27}{0.65}\right) = 2571.5 \text{ N @ } \cos 2\omega \text{ t}$$

Forces Transferred at CG of Base Frame (C)

$F_{cy} = F_{Ay} = 0.0$

$F_{cz1} = F_{Az} + F_{Bz1} = 3929.8 + 6190.5 = 10120.3$ N @ $\cos\omega$ t

$F_{cz2} = F_{Bz2} = 2571.5$ N @ $\cos 2\omega$ t

$M_{cx1} = F_{cz1} \times y = 10120.3 \times 0.5 = 5060.15$ Nm @ $\cos\omega$ t

$M_{cx2} = F_{cz2} \times y = 2571.5 \times 0.5 = 1285.75$ Nm @ $\cos 2\omega$ t

b) When crank shaft is parallel to Y axis - Crank makes an angle $\theta = 90^\circ$ with Z axis

i) Force generated by mass m_A at A

Along Y axis $F_{Ay} = 14.75 \times 0.27 \times 31.42^2 \sin(\pi/2) = 3929.8$ N @ $\sin\omega t$

Along Z axis $F_{Az} = 14.75 \times 0.27 \times 31.42^2 \cos(\pi/2) = 0.0$ N @ $\cos\omega t$

ii) Force generated by mass m_B at B

$$F_{Bz1} = 6190.5 \text{ N} \ @ \ \cos\omega t$$

$$F_{Bz2} = 2571.5 \text{ N} \ @ \ \cos 2\omega t$$

Forces Transferred at CG of Base Frame (C)

$$F_{cy} = F_{Ay} = 3929.8 \text{ N} @ \sin\omega t$$

$$F_{cz1} = F_{Bz1} = 6190.5 \text{ N} @ \cos\omega t$$

$$F_{cz2} = F_{Bz2} = 2571.5 \text{ N} @ \cos 2\omega t$$

$$M_{cx1} = F_{Ay} \times z + F_{Bz1} \times y = 3929.8 \times 1.2 + 6190.5 \times 0.5 = 7811 \text{ Nm} @ \cos\omega t$$

$$M_{cx2} = F_{Bz2} \times y = 2571.5 \times 0.5 = 1285.75 \text{ Nm} @ \cos 2\omega t$$

For the foundation design, these forces are finally transferred at DOF location of the foundation. Amplitudes of vibration are evaluated for both these conditions as above.

P 6.2-2

A horizontal twin cylinder reciprocating engine is mounted on a base frame supported by a foundation block as shown in Figure P 6.2-2. The data for each cylinder of engine is same as that of example P 6.2-1. Consider the crank angles as $\alpha = 0 \,\&\, \alpha = 180^\circ$ for 1st and 2nd cylinder respectively. Compute unbalanced forces and moments @ CG of Base Frame point C. Both the cylinders are placed equidistant from Z axis along X as shown in figure.

Distance of point O from Base frame Centroid (along Z)	**1200 mm**
Distance of point O from Base frame Centroid (along Y)	**500 mm**
Distance of point O of each cylinder from Base frame Centroid (along X)	**800 mm**

Solution

Distance of point O from Base Frame Centroid (along Z) $z = 1.20$ m

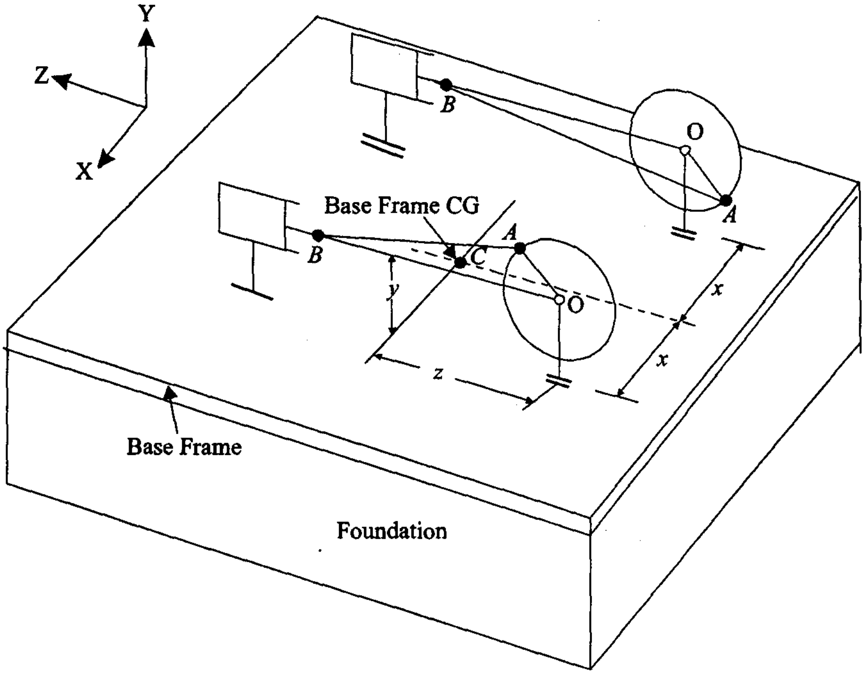


Figure P 6.2-2 Twin Cylinder - Horizontal Reciprocating Engine with Crank angle 0° & 180°

Distance of point O from Foundation Centroid (along Y) $y = 0.50$ m

Distance of point O of from Base frame Centroid (along X) $x = 0.80$ m

Operating Speed $\omega = \frac{300}{60} \times 2\pi = 31.42$ rad/s

Masses at points A & B for each cylinder

Mass at A : $m_A = 14.75$ kg Mass at B : $m_B = 23.23$ kg

Unbalance Forces:

a) When crank shaft is parallel to Z axis – consider that crank of cylinder 1 makes an angle $\theta = 0^\circ$ with Z axis

Cylinder 1: $\theta = 0^\circ$

i) Force at A $F_{Ay} = 0.0$; $F_{Az} = 3929.8$ N @ $\cos \omega t$

ii) Force at B $F_{Bz1} = 6190.5 \text{ N} @ \cos\omega t$

$F_{Bz2} = 2571.5 \text{ N} @ \cos 2\omega t$

Cylinder 2: When crank of cylinder 1 makes an angle $\theta = 0^\circ$ with Z axis, crank of cylinder 2 makes angle $\theta = 180^\circ$

iii) Force at A $F_{Ay} = 0.0$; $F_{Az} = -3929.8 \text{ N} @ \cos\omega t$

iv) Force at B $F_{Bz1} = -6190.5 \text{ N} @ \cos\omega t$

$F_{Bz2} = -2571.5 \text{ N} @ \cos 2\omega t$

Forces Transferred at CG of Base Frame (C)

$$F_{cy} = \overbrace{F_{Ay}}^{\text{Cyl 1}} + \overbrace{F_{Ay}}^{\text{cyl 2}} = 0.0 @ \cos\omega t$$

$$F_{cz1} = \overbrace{F_{Az} + F_{Bz1}}^{\text{cyl 1}} + \overbrace{F_{Az} + F_{Bz1}}^{\text{cyl 2}}$$
$$= (3929.8 + 6190.5) + (-3929.8 - 6190.5) = 0 \text{ N} @ \cos\omega t$$

$$F_{cz2} = \overbrace{F_{Bz2}}^{\text{cyl 1}} + \overbrace{F_{Bz2}}^{\text{cyl 2}} = (2571.5) + (-2571.5) = 0.0 \text{ N} @ \cos 2\omega t$$

$$M_{cx1} = \overbrace{(F_{Az} + F_{Bz1}) \times y}^{\text{cyl 1}} + \overbrace{(F_{Az} + F_{Bz1}) \times y}^{\text{cyl 2}}$$
$$= \overbrace{(3929.8 + 6190.5) \times 0.5}^{\text{cyl 1}} + \overbrace{(-3929.8 - 6190.5) \times 0.5}^{\text{cyl 2}} = 0.0 \text{ Nm} @ \cos\omega t$$

$$M_{cx2} = \overbrace{F_{Bz2} \times y}^{\text{cyl 1}} + \overbrace{F_{Bz2} \times y}^{\text{cyl 2}} = \overbrace{2571.5 \times 0.5}^{\text{cyl 1}} + \overbrace{-2571.5 \times 0.5}^{\text{cyl 2}} = 0.0 \text{ Nm} @ \cos 2\omega t$$

Equal and opposite forces of cylinder 1 & 2 along Z produce Torsional Moment at C about Y axis

At engine order frequency, we get

$$M_{cy1} = \overbrace{(F_{Az} + F_{Bz1}) \times x}^{\text{cyl 1}} + \overbrace{(F_{Az} + F_{Bz1}) \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{(3929.8 + 6190.5) \times 0.8}^{\text{cyl 1}} + \overbrace{(-3929.8 - 6190.5) \times (-0.8)}^{\text{cyl 2}} = 16192.5 \text{ Nm @ } \cos\omega t$$

At 1st harmonic, we get

$$M_{cy2} = \overbrace{F_{Bz2} \times x}^{\text{cyl 1}} + \overbrace{F_{Bz2} \times x}^{\text{cyl 2}}$$

$$= \overbrace{(2571.5) \times 0.8}^{\text{cyl 1}} + \overbrace{(-2571.5) \times (-0.8)}^{\text{cyl 2}} = 4114.4 \text{ Nm @ } \cos 2\omega t$$

b) When crank shaft is parallel to Y axis – Consider that Crank of cylinder 1 makes an angle $\theta = 90^\circ$ with Z axis

Cylinder 1:

v) Force at A $F_{Ay} = 3929.8\text{ N @ } \sin\omega t$; $F_{Az} = 0.0$

vi) Force at B $F_{Bz1} = 6190.5 \text{ N @ } \cos\omega t$; $F_{Bz2} = 2571.5 \text{ N @ } \cos 2\omega t$

Cylinder 2:

vii) Force at A $F_{Ay} = -3929.8\text{ N @ } \sin\omega t$; $F_{Az} = 0.0$

viii) Force at B $F_{Bz1} = -6190.5 \text{ N @ } \cos\omega t$; $F_{Bz2} = -2571.5 \text{ N@ } \cos 2\omega t$

Forces Transferred at CG of Base Frame (C)

$$F_{cy} = \overbrace{F_{Ay}}^{\text{cyl 1}} + \overbrace{F_{Ay}}^{\text{cyl 2}} = 3929.8 - 3929.8 = 0.0 \text{ N@} \sin\omega t$$

$$F_{cz1} = \overbrace{F_{Bz1}}^{\text{cyl 1}} + \overbrace{F_{Bz1}}^{\text{cyl 2}} = 6190.5 - 6190.5 = 0.0 \text{ N@} \cos\omega t$$

$$F_{cz2} = \overbrace{F_{Bz2}}^{\text{cyl 1}} + \overbrace{F_{Bz2}}^{\text{cyl 2}} = (2571.5) + (-2571.5) = 0.0 \text{ N@} \cos 2\omega t$$

$$M_{cx1} = \overbrace{(F_{Bz1}) \times y + F_{Ay} \times z}^{\text{cyl 1}} + \overbrace{(F_{Bz1}) \times y + F_{Ay} \times z}^{\text{cyl 2}}$$

$$= \overbrace{(6190.5) \times 0.5 + (3929.8) \times 1.2}^{\text{cyl 1}} + \overbrace{(-6190.5) \times 0.5 + (-3929.8) \times 1.2}^{\text{cyl 2}} = 0.0 \text{ Nm @ } \cos\omega \text{ t}$$

$$M_{cx2} = \overbrace{(F_{Bz1}) \times y}^{\text{cyl 1}} + \overbrace{(F_{Bz1}) \times y}^{\text{cyl 2}}$$

$$= \overbrace{(2571.5) \times 0.5}^{\text{cyl 1}} + \overbrace{(-2571.5) \times 0.5}^{\text{cyl 2}} = 0.0 \text{ Nm @ } \cos 2\omega \text{ t}$$

Equal and opposite forces of cylinder 1 & 2 at point B along Z produce Torsional Moment at C about Y axis

At engine order frequency, we get

$$M_{cy1} = \overbrace{F_{Bz1} \times x}^{\text{cyl 1}} + \overbrace{F_{Bz1} \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{6190.5 \times 0.8}^{\text{cyl 1}} + \overbrace{(-6190.5) \times (-0.8)}^{\text{cyl 2}} = 9904.8 \text{ Nm @ } \cos\omega \text{ t}$$

At 1[st] harmonic, we get

$$M_{cy2} = \overbrace{F_{Bz2} \times x}^{\text{cyl 1}} + \overbrace{F_{Bz2} \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{2571.5 \times 0.8}^{\text{cyl 1}} + \overbrace{(-2571.5) \times (-0.8)}^{\text{cyl 2}} = 4114.4 \text{ Nm @ } \cos 2\omega \text{ t}$$

Equal and opposite forces of cylinder 1 & 2 at point A along Y produce Rocking Moment at C about Z axis

$$M_{cz} = \overbrace{F_{Ay} \times x}^{\text{cyl 1}} + \overbrace{F_{Ay} \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{(3929.8) \times 0.8}^{\text{cyl 1}} + \overbrace{(-3929.8) \times (-0.8)}^{\text{cyl 2}} = 6287.7 \text{ Nm @ } \sin\omega \text{ t}$$

For the foundation design, these forces are finally transferred at DOF location of the foundation. Amplitudes of vibration are evaluated for both these conditions as above.

P 6.2-3

A vertical single cylinder reciprocating engine is mounted on a base frame supported by a foundation block as shown in Figure P 6.2-3. The data for the engine is same as that for P 6.2-1. The motion of the piston is along Y axis. Compute unbalanced forces and moments @ CG of Base Frame point C.

Distance of point O from Base frame Centroid (along Y) 500 mm

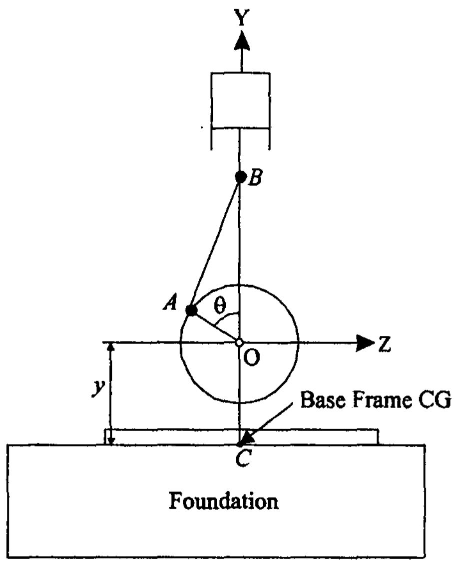


Figure P 6.2-3 Single Cylinder - Vertical Reciprocating Engine

Solution:

Computation of Masses at points A & B

Mass at A : $m_A = 14.75$ kg Mass at B : $m_B = 23.23$ kg

Unbalance Forces:

a) When crank shaft is parallel to Y axis - Crank makes an angle $\theta = 0°$ with Y axis

i) Force generated by mass m_A at A

Along Y axis $F_{Ay} = 3929.8$ N $@\cos\omega t$

Along Z axis $F_{Az} = 0.0$ N $@\sin\omega t$

ii) Force generated by mass m_B at B

Along Z axis $F_{By1} = 6190.5$ N @ $\cos\omega t$ at engine order frequency

Along Z axis $F_{By2} = 2571.5$ N @ $\cos 2\omega t$ at First Harmonic

Forces Transferred at CG of Base Frame (C)

$$F_{cy1} = F_{Ay} + F_{By1} = 10120.3 \text{ N } @\cos\omega t$$

$$F_{cy2} = F_{By2} = 2571.5 \text{ N } @\cos 2\omega t$$

$$F_{cz} = F_{Az} = 0.0 \text{ N} @ \sin\omega t$$

$$M_{cx} = F_{Az} \times y = 0.0 \text{ Nm } @ \sin\omega t$$

b) When crank shaft is parallel to Z axis - Crank makes an angle $\theta = 90°$ with Y axis

iii) Force generated by mass m_A at A

Along Y axis $F_{Ay} = 0.0$ N @ $\cos\omega t$

Along Z axis $F_{Az} = 3929.8$ N @ $\sin\omega t$

iv) Force generated by mass m_B at B

Along Z axis $F_{By1} = 6190.5$ N @ $\cos\omega t$

Along Z axis $F_{By2} = 2571.5$ N @ $\cos 2\omega t$

Forces Transferred at CG of Base Frame (C)

$$F_{cy1} = F_{By1} = 6190.5 \text{ N } @\cos\omega t$$

$$F_{cy2} = F_{By2} = 2571.5 \text{ N } @\cos 2\omega t$$

$$F_{cz} = F_{Az} = 3929.8 \text{ N } @\sin\omega t$$

$$M_{cx} = F_{Az} \times y = 3929.8 \times 0.5 = 1964.9 \text{ Nm } @\sin\omega t$$

For the foundation design, these forces are finally transferred at DOF location of the foundation. Amplitudes of vibration are evaluated for both these conditions as above.

P 6.2-4

A vertical twin cylinder reciprocating engine is mounted on a base frame supported by a foundation block as shown in Figure P 6.2-4. The data for each cylinder of engine is same as that of example P6.2-1. Motion of the piston is along Y and crank moves in Y-Z plane. Consider the crank angles as $\alpha = 0 \& \alpha = 180°$ for 1st and 2nd cylinder respectively. Compute unbalanced forces and moments @ CG of Base Frame point C. Both the cylinders are placed such that the eccentricity along Z with respect to CG of Base frame is zero and the cylinders are equidistant from point C along X as shown in Figure.

Distance of point O from Base frame Centroid (along Y)	500 mm
Distance of point O of each cylinder from Base frame Centroid (along X)	800 mm

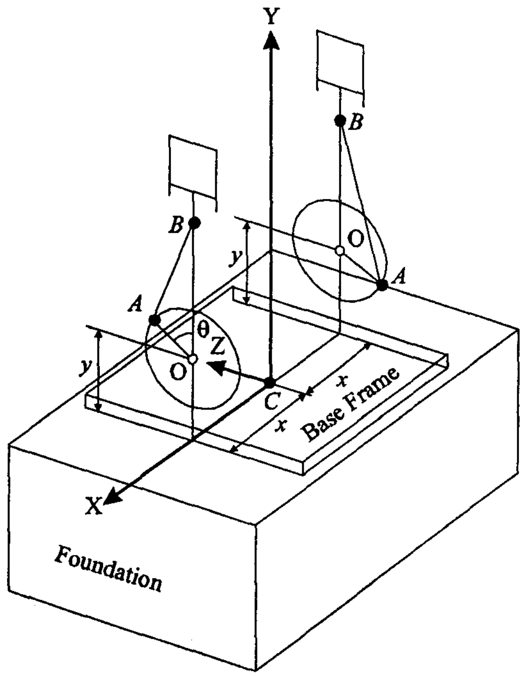


Figure P 6.2-4 Twin Cylinder Vertical Reciprocating Engine with Crank angle 0° & 180°

Solution

Distance of point O from C along Y (for each cylinder)	y	0.50 m
Distance of point O from C along X (for each cylinder)	x	0.80 m

Operating Speed $\omega = \frac{300}{60} \times 2\pi = 31.42$ rad/s

Masses at points A & B for each cylinder

$m_A = 14.75$ kg $\qquad m_B = 23.23$ kg

Unbalance Forces:

a) When crank shaft is parallel to Y axis – consider that crank shaft of cylinder 1 makes an angle $\theta = 0°$ with Y axis

Cylinder 1: $\theta = 0°$

i) Force at A $F_{Ay} = 3929.8$ N @ $\cos\omega t$; $F_{Az} = 0.0$ N @ $\sin\omega t$

ii) Force at B $F_{By1} = 6190.5$ N @ $\cos\omega t$; $F_{By2} = 2571.5$ N @ $\cos 2\omega t$

Cylinder 2: When crank of cylinder 1 makes an angle $\theta = 0°$ with Y axis, crank of cylinder 2 makes angle $\theta = 180°$

iii) Force at A $F_{Ay} = -3929.8$ N @ $\cos\omega t$; $F_{Az} = 0.0$ N @ $\sin\omega t$

iv) Force at B $F_{By1} = -6190.5$ N @ $\cos\omega t$; $F_{By2} = -2571.5$ N @ $\cos 2\omega t$

Forces Transferred at CG of Base Frame (C)

$$F_{cy1} = \overbrace{F_{Ay} + F_{By1}}^{\text{cyl 1}} + \overbrace{F_{Ay} + F_{By1}}^{\text{cyl 2}}$$
$$= (3929.8 + 6190.5) + (-3929.8 - 6190.5) = 0.0 \text{ N @}\cos\omega t$$

$$F_{cy2} = \overbrace{F_{By2}}^{\text{cyl 1}} + \overbrace{F_{By2}}^{\text{cyl 2}} = (2571.5) + (-2571.5) = 0.0 \text{ N @ } \cos 2\omega t$$

$$F_{cz} = \overbrace{F_{Az}}^{\text{Cyl 1}} + \overbrace{F_{Az}}^{\text{cyl 2}} = 0.0 + 0.0 = 0.0 \text{ N @ } \sin\omega t$$

$$M_{cx} = \overbrace{F_{Az} \times y}^{\text{cyl 1}} + \overbrace{F_{Az} \times y}^{\text{cyl 2}} = 0.0 \text{ Nm @ } \sin\omega t$$

Equal and opposite forces of cylinder 1 & 2 at point A along Z produce Torsional Moment at C about Y axis

$$M_{cy} = \overbrace{F_{Az} \times x}^{\text{cyl 1}} + \overbrace{F_{Az} \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{0.0 \times 0.8}^{\text{cyl 1}} + \overbrace{0.0 \times (-0.8)}^{\text{cyl 2}} = 0.0 \text{ Nm } @ \sin\omega t$$

Equal and opposite forces of cylinder 1 & 2 at point A & point B along Y produce Rocking Moment at C about Z axis at engine order frequency as well as it's 1st harmonic.

$$M_{cz1} = \overbrace{(F_{Ay} + F_{By1}) \times x}^{\text{cyl 1}} + \overbrace{(F_{Ay} + F_{By1}) \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{(3929.8 + 6190.5) \times 0.8}^{\text{cyl 1}} + \overbrace{(-3929.8 - 6190.5) \times (-0.8)}^{\text{cyl 2}} = 16192.5 \text{ Nm } @ \cos\omega t$$

$$M_{cz2} = \overbrace{(F_{By2}) \times x}^{\text{cyl 1}} + \overbrace{(F_{By2}) \times (-x)}^{\text{cyl 2}}$$

$$= \overbrace{(2571.5) \times 0.8}^{\text{cyl 1}} + \overbrace{(-2571.5) \times (-0.8)}^{\text{cyl 2}} = 4114.4 \text{ Nm } @ \cos 2\omega t$$

b) When crank shaft is parallel to Z axis i.e. Crank of cylinder 1 makes an angle $\theta = 90°$ with Y axis

Cylinder 1:

v) Force at A $F_{Ay} = 0.0 \text{ N} @ \cos\omega t$; $F_{Az} = 3929.8 \text{ N} @ \sin\omega t$

vi) Force at B $F_{By1} = 6190.5 \text{ N} @ \cos\omega t$; $F_{By2} = 2571.5 \text{ N} @ \cos 2\omega t$

Cylinder 2:

vii) Force at A $F_{Ay} = 0.0 \text{ N} @ \cos\omega t$; $F_{Az} = -3929.8 \text{ N} @ \sin\omega t$

viii) Force at B $F_{By1} = -6190.5 \text{ N} @ \cos\omega t$; $F_{By2} = -2571.5 \text{ N} @ \cos 2\omega t$

Forces Transferred at CG of Base Frame (C)

$$F_{cy1} = \overbrace{F_{By1}}^{\text{cyl 1}} + \overbrace{F_{By1}}^{\text{cyl 2}} = 6190.5 - 6190.5 = 0.0 \text{ N } @ \cos\omega t$$

$$F_{cy2} = \overbrace{F_{By2}}^{\text{cyl 1}} + \overbrace{F_{By2}}^{\text{cyl 2}} = (2571.5) + (-2571.5) = 0.0 \text{ N } @ \cos 2\omega t$$

$$F_{cz} = \overbrace{F_{Az}}^{\text{cyl 1}} + \overbrace{F_{Az}}^{\text{cyl 2}} = 3929.8 - 3929.8 = 0.0 \text{ N } @ \cos\omega t$$

$$M_{cx} = \overbrace{F_{Az} \times y}^{\text{cyl 1}} + \overbrace{F_{Az} \times y}^{\text{cyl 2}} = \overbrace{3929.8 \times 0.5}^{\text{cyl 1}} + \overbrace{-3929.8 \times 0.5}^{\text{cyl 2}} = 0.0 \ \text{Nm} \ @ \sin\omega t$$

Equal and opposite forces of cylinder 1 & 2 at point A along Z produce Torsional Moment at C about Y axis

$$M_{cy} = \overbrace{F_{Az} \times x}^{\text{cyl 1}} + \overbrace{F_{Az} \times (-x)}^{\text{cyl 2}} = \overbrace{3929.8 \times 0.8}^{\text{cyl 1}} + \overbrace{(-3929.8) \times (-0.8)}^{\text{cyl 2}}$$
$$= 6287.7 \ \text{Nm} \ @ \sin\omega t$$

Equal and opposite forces of cylinder 1 & 2 at point B along Y produce Rocking Moment at C about Z axis

$$M_{cz1} = \overbrace{F_{By1} \times x}^{\text{cyl 1}} + \overbrace{F_{By1} \times (-x)}^{\text{cyl 2}}$$
$$= \overbrace{(6190.5) \times 0.8}^{\text{cyl 1}} + \overbrace{(-6190.5) \times (-0.8)}^{\text{cyl 2}} = 9904.8 \ \text{Nm} \ @ \cos\omega t$$
$$M_{cz2} = \overbrace{F_{By2} \times x}^{\text{cyl 1}} + \overbrace{F_{By2} \times (-x)}^{\text{cyl 2}}$$
$$= \overbrace{(2571.5) \times 0.8}^{\text{cyl 1}} + \overbrace{(-2571.5) \times (-0.8)}^{\text{cyl 2}} = 4114.4 \ \text{Nm} \ @ \cos 2\omega t$$

For the foundation design, these forces are finally transferred at DOF location of the foundation and amplitudes of vibration are evaluated for both these conditions as above.

EXAMPLE PROBLEMS - IMPACT MACHINES

P 6.3-1

For the system shown in Figure P 6.3-1, compute velocity developed by the Anvil **after impact. Assume Efficiency of Drop** $\eta = 0.65$ **and coefficient of Restitution** $e = 0.5$ **.**

Solution:

The data for a hammer is as under:

Mass of Tup	$m_0 = 3500 \quad \text{kg}$
Mass of Anvil	$m_2 = 80000 \quad \text{kg}$
Height of fall	$h = 2 \text{ m}$

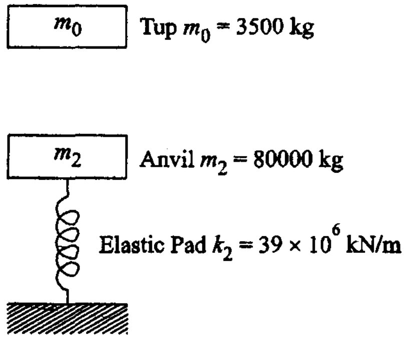


Figure P6.3-1 A Spring Mass System Subjected to Impact

Stiffness of Elastic Pad below Anvil $k_2 = 3.9 \times 10^6$ kN/m

Efficiency of Drop $\eta = 0.65$

Coefficient of Restitution for the impact $e = 0.5$

$m_2 = 80000$ kg; $m_0 = 3500$ kg; $\eta = 0.65$

$k_2 = 3.9 \times 10^6$ kN/m; $h = 2$ m; $e = 0.5$

Initial Velocity of mass m_2

Velocity of mass m_0 before impact

$$v_0' = 0.65\sqrt{2gh} = 0.65\sqrt{2 \times 9.81 \times 2} = 4.07 \quad \text{m/s}$$

$$\lambda_2 = \frac{m_2}{m_0} = \frac{80000}{3500} = 22.857$$

Velocity of mass m_2 after impact

$$v_2 = v_0' \times \frac{(1+e)}{(1+\lambda_2)} = 4.07 \times \frac{(1+0.5))}{(1+22.86)} = 0.256 \quad \text{m/sec}$$

P 6.3-2

Compute the velocity developed by the Anvil for a double acting hammer having data as under:

Mass of Tup	2000 kg
Mass of upper die	500 kg
Height of Tup Stroke	$h = 1.0$ m
Area of Piston	$A_p = 0.2 \text{ m}^2$
Steam Pressure	$p_s = 1000 \text{ kN/m}^2$
Mass of Anvil	40000 kg
Mass of Frame	15000 kg
Anvil Base Area	6 m^2
Elastic Pad below Anvil:	
Elastic Modulus	$E_p = 3.0 \times 10^5 \text{ kN/m}^2$
Thickness of Pad	$t = 0.4$ m
Efficiency of Drop	$\eta = 0.65$
Coefficient of Restitution for the impact	$e = 0.5$

Solution:

Initial Velocity developed by Anvil

Let us designate falling mass to be m_0

$$m_0 = 2000 + 500 = 2500 \text{ kg}$$

Velocity of mass m_0 before impact v_0' (see equation 6.3-3)

$$v_0' = \eta\sqrt{2\,g\,h \times \left(\frac{m_0 \times g + p_s \times A_p}{m_0 \times g}\right)}$$

$$v_0' = 0.65\sqrt{2 \times 9.81 \times 1 \times \left(\frac{2500 \times 9.81 + 1000 \times 10^3 \times 0.2}{2500 \times 9.81}\right)} = 8.71 \text{ m/s}$$

Velocity of mass m_2 after impact v_2

Let us designate mass of Anvil along with mass of frame as m_2

$$m_2 = 40000 + 15000 = 55000 \quad \text{kg}$$

$$\lambda_2 = \frac{m_2}{m_0} = \frac{55000}{2500} = 22$$

$$v_2 = v_0' \times \frac{(1+e)}{(1+\lambda_2)} = 8.71 \times \frac{(1+0.5))}{(1+22)} = 0.568 \quad \text{m/sec}$$

7

DESIGN FOUNDATION PARAMETERS

- Foundation Type
- Foundation Material
- Foundation Eccentricity
- Foundation Tuning
- Isolation from adjoining Structures
- Vibration Limits
- Block Foundation
- Frame Foundation

DESIGN FOUNDATION PARAMETERS

There are many foundation-associated parameters viz. Foundation Type, Foundation Material, Foundation Eccentricity, Foundation Tuning, Vibration Limits, Foundation Sizing, & Stiffness, Strength adequacy etc that are briefly introduced in Chapter 1. Discussion is oriented to highlight their influence on machine foundation response.

7.1 FOUNDATION TYPE

Machine type and its characteristics play a significant role while selecting the type of foundation. Other parameters like high dynamic response; weak soils, etc may also influence **foundation type and size.** Most commonly used foundations in the industry, are **Block Foundations and Frame Foundations** and only these are addressed here in this chapter.

Block Foundation: Block Foundations have commonly been used for supporting all types of machines viz. rotary, reciprocating & impact machines, irrespective of their speed of operation. In this case, machine is mounted over a solid block, generally made of concrete. The foundation block in turn rests on the soil. A typical **Block Foundation** is shown in Figure 7.1-1.

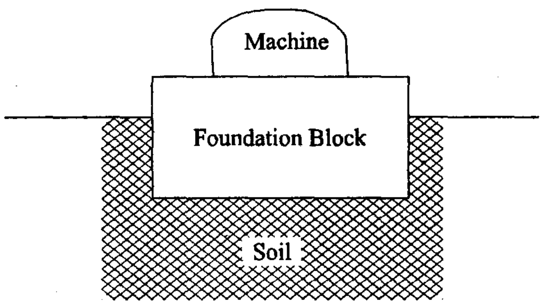


Figure 7.1-1 A Typical Block Foundation

Frame Foundation: Frame foundations are used for turbo-generators, turbo-compressors and various other machines whose mechanical system requires frame type of supporting system.

The foundation is a 3-D frame structure, having base raft, a set of columns and beams and top deck to support the machine. These frame foundations have a number of cross frames in transverse direction tied up by longitudinal beams and/or thick slab components.

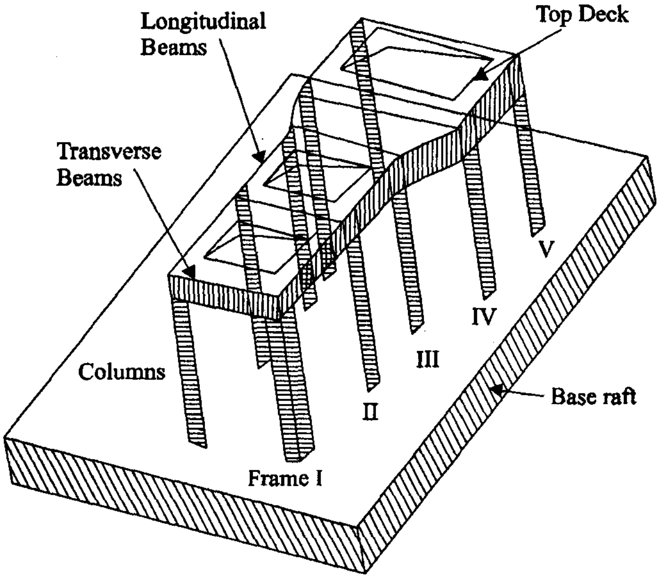


Figure 7.1-2 A Typical Frame Foundation

Machine is supported over a RCC deck called top deck that in turn is supported by a set of columns. These columns are attached to base raft that rests either directly over soil or through a set of piles. Suitable openings are provided in the top deck for taking out piping, locating other equipment directly below the machine and so on. A typical frame foundation supporting the machine is shown in Figure 7.1-2.

7.2 FOUNDATION MATERIAL

Most common material used for machine foundation is Reinforced Cement Concrete (RCC). In specific cases, **Structural Steel** has also been used for **Frame Foundations**. As the percentage of Structural Steel Frame Foundations for real application is much less compared to RCC Frame foundations, the **discussion is restricted to RCC foundations only**.

7.2.1 Concrete

Properties required are: Mass Density, Dynamic Elastic Modulus (for dynamic analysis), Static Elastic Modulus (for strength analysis) and Poisson's Ratio. In addition allowable stresses (bending compression, bending tension, direct compression, direct tension and shear) are needed for strength analysis.

I. Grade of concrete:

The grade of concrete depicts its Characteristic Compressive Strength. For example, concrete grade $M\,20$ corresponds to the Characteristic Compressive Strength f_{ck} of 20 N/mm^2 for 150 mm Cube at 28 days. Letter M followed by two numeral digits designates grade of concrete e.g. $M20$, $M\,25$, $M\,30$. The two digits just after letter $\boldsymbol{M}$ represent the Characteristic Strength of that grade of concrete in MPa or N/mm^2.

Characteristic Strength f_{ck} for any concrete grade represents its Compressive Strength in N/mm^2 of 150 mm cube at 28 days. This nomenclature is as per **Indian Standard Code of Practice - IS 456.**

Conc. Grade	$M\,20$	$M\,25$	$M\,30$
Characteristic Strength* f_{ck} N/mm^2	20	25	30
Mass Density of Concrete kg/m^3 (all grades)	2500		
Poisson's Ratio (all grades)	0.15		

The Grades of Concrete recommended for Machine Foundation are $M\,20$, $M\,25$, $M\,30$. It is generally considered good enough to use $M\,20$ grade concrete for Block Foundations and $M\,25$ *or* $M\,30$ grade for Frame foundations. Higher grades, if desired, may also be used.

II. Elastic Modulus

The elastic modulus of concrete varies with grade of concrete. The values of Elastic Modulus (kN/m^2) both static and dynamic, as given by IS 2974, reproduced hereunder lie in $\pm 20\,\%$ range (approximately).

	E_{static}		E^* *dynamic*
$M20$	20×10^6	24×10^6	25×10^6 *to* 30×10^6
$M25$	22×10^6	26×10^6	28×10^6 *to* 34×10^6
$M30$	25×10^6	29×10^6	31×10^6 *to* 37×10^6

There are different schools of thought regarding **Dynamic Modulus of Elasticity (**$E^*_{dynamic}$**)**. For **dynamic response** computation, recommendations given by various codes, handbooks and textbooks, have been found to be in variance. Some codes recommend use of **dynamic modulus of elasticity** for **dynamic analysis** whereas some are silent about it.

As the design value of Elastic modulus of concrete is strain dependent, reported test results do not indicate any appreciable variation between Dynamic Elastic Modulus and Static Elastic Modulus of concrete at low strain levels. Author has also performed tests on certain foundations and observed that strain levels developed in the machine foundations under dynamic operating conditions are of low order. Similar observations are reported by some other authors too.

In view of the above, **author recommends use of Static Elastic Modulus** for all machine foundation design computations except those cases where associated strain levels during dynamic response are likely to be of higher order.

III. Permissible Stresses (All Stresses in MPa or N/mm^2)

Besides Elastic modulus, stress Properties of Concrete with respect to its grade may differ from country to country depending upon their practices and applicable standards. The Properties of **Concrete** given below are as listed in Indian Standard Code of Practice **IS 456**.

Conc. Grade	$M20$	$M25$	$M30$
Tensile Stress	2.8	3.2	3.6
Bending Compression Stress	7.0	8.5	10.0
Direct Compression	5.0	6.0	8.0
Bond Stress (Plain Bars)	0.8	0.9	1.0
Bond Stress (Deformed Bars)	1.28	1.44	1.6

7.2.2 Reinforcement

Both Mild Steel Bars, as well as High Yield strength Deformed Bars are recommended. Here again, stress Properties of Reinforcing bars may differ from country to country depending upon their practices and applicable standards. The Properties of **Reinforcing bars** given below are as listed in Indian Standard Code of Practice **IS 456**

As per Indian Standard Code of practice, the Mild Steel Bars conforming to IS 432 (Part 1 & Part 2) and High Yield strength Deformed Bars conforming to IS 1786 meet the requirements. The Properties of **Reinforcement Steel** (conforming to IS 432 and IS 1786) given below are as listed in Indian Standard Code of Practice **IS 456.**

Allowable Stresses (MPa or N/mm^2)

Steel Reinforcing Bars	**Mild Steel (IS 432)**	**High Yield (Deformed) Bars (IS 1786)**
Tensile Stress		
Up to 20 mm	140	230
Over 20 mm	130	230

Compressive Stress (MPa or N/mm^2)

Up to 20 mm	140	190
Over 20 mm	130	190

Designer's Note: Designers are advised to use equivalent grades of concrete and steel as applicable in their respective countries.

7.3 FOUNDATION ECCENTRICITY

It is one of the controlling parameter for sizing the foundation. It is defined as the distance between center of mass and center of stiffness along either length or width of the foundation. Thus the eccentricity along length may be different than that along width of the foundation.

Eccentricity it is defined as the distance between **center of mass** of overall system (machine + foundation) and **center of stiffness** (i.e. CG of the base contact area of the foundation with the soil). All components of machine foundation system that contribute to inertia must be included in computing combined CG. In other words machine mass, mass of base frame, mass of grout (if considered significant), and soil mass (if any) over the extended potion of the foundation base and foundation mass (duly accounting for all large openings) must be considered while computing eccentricity.

Absence of eccentricity not only ensures uniform pressure on the soil but also makes vertical mode decoupled from translational and rocking modes (see § **3.3.2.1).** The presence of eccentricity results in generating rocking modes of vibration. For example, for a foundation in X-Z plane, the presence of eccentricity along Z direction induces moments about X-axis and eccentricity along X direction induces moments about Z-axis. These moments contribute to amplitudes in rocking modes of vibration. For higher eccentricity values, these modes may turn out to be significant modes of vibration and if not properly accounted could turn out to be a cause for high vibration. **That's why it is desirable to control foundation eccentricity while sizing the foundation (see § 7.3).**

Eccentricity should be kept to a bare minimum but in no case should exceed 5 % of the corresponding dimension of foundation i.e. eccentricity along X direction should be within 5 % of foundation dimension along X and so is the case along Z. For very long foundation dimensions, it is recommended to keep the eccentricity within 2 %.

7.4 FOUNDATION TUNING

The three parameters that influence natural frequency of foundation are Machine, Soil and Foundation and any variation in the design data of these main components affects natural frequency of the machine foundation system.

It is also true that the possibility of variation in the design data vis-à-vis actual data cannot be ruled out. For example **variation in soil data** may be attributed to soil test methods, test agency, type and quality of instrumentation used, effect of embedment, influence of ground water table etc. Similarly **variation in machine data** may be attributed to the change in design data vis-à-vis actual data supplied with the machine, last minute change of vendor for the supply etc.

Such variations put a question mark on the confidence level of the computed values of natural frequencies, amplitudes and the transmitted force. It is for these reasons that computed natural frequencies are kept **sufficiently away** from operating speeds to avoid resonance. In other words computed natural frequencies should have sufficient **frequency margin** with respect to operating speed so as not to encounter direct or indirect resonance.

Based on practical experience, it is recommended to keep **the frequency margin as** $\pm 20\,\%$ i.e. the combined natural frequencies of the foundation should be at least **20 %** away from operating speeds and preferably from their harmonics too.

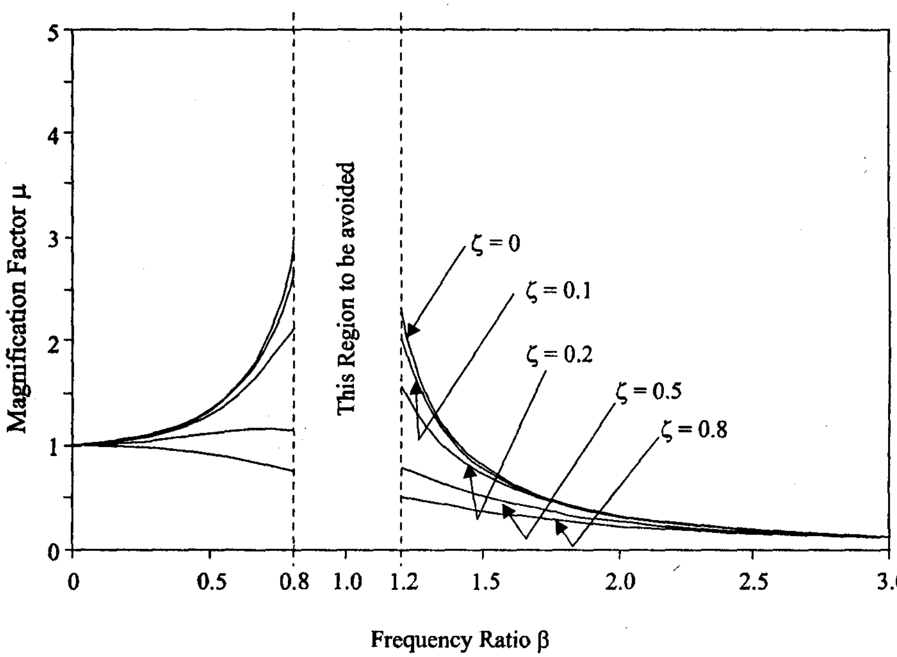


Figure 7.4-1 Magnification Factor μ vs. Frequency Ratio β
$\pm$ 20% Frequency Margin Region

Just for the sake of understanding, Figure 2.2.2-4 giving variation of Magnification Factor 'μ' with Frequency Ratio β has been reproduced as Figure 7.4-1 by blocking the zone for $\pm 20\ \%\ \beta$ i.e. $0.8 < \beta > 1.2$. It is seen from the Figure that the frequency separation of $\pm 20\,\%$ ensures a significant reduction in magnification factor even for zero damping thus ensuring reduction in amplitudes of vibration.

7.4.1 Under-Tuned Foundation

For **high-speed machines** it is desirable to design the foundation as **under-tuned foundation** by keeping its vertical natural frequency below the operating speed of the machine. Such under-tuned foundations, however, would always face resonance during every startup and shutdown of the machine on account of its natural frequency being less than operating speed. Such a condition is termed as **transient resonance** and computation **of amplitudes during transient resonance is therefore a must.**

7.4.2 Over-Tuned Foundation

For **low speed machines** it is desirable to have **over-tuned foundation i.e.** keeping its vertical natural frequency above the operating speed of the machine. For such foundations it is desirable to check for likelihood of **resonance with higher harmonics of the machine**. It is however desirable, though not essential, to avoid resonance with these higher harmonics. If not, these could turn out to be the cause of high vibration on account of resonance with 2-X & 3-X frequency components i.e. frequencies corresponding to twice the operating speed and thrice the operating speed of the machine respectively.

At times difficulty is encountered with medium-speed machines. Here, one or more natural frequencies fall in the operating speed range and it becomes difficult to maintain $\pm 20\,\%$ margin with the operating speed. In such a case the dynamic amplitudes are computed with $\pm 20\,\%$ values of the computed natural frequencies so as to account for possibility of any direct/indirect resonance. If amplitudes show higher values, it is desirable to re-size the foundation (within layout constraints) and reanalyze till satisfactory results are obtained.

7.5 ISOLATION FROM ADJOINING STRUCTURES

Foundation must be isolated from adjoining structures, their foundations as well as from the operating floors. A clear air gap of 25 mm to 100 mm all around the foundation must be maintained. If this condition is not met, the reliability of results of the analysis i.e. frequencies and amplitudes becomes questionable.

For under-tuned foundations, flexible bellows must be provided for all inward and outward piping connected to machine otherwise the piping stiffness would definitely influence the natural frequency as well as amplitudes of vibration and it may not be feasible to account for this effect in mathematical modeling of machine foundation system.

7.6 OTHER MISCELLANEOUS EFFECTS

Pockets, Notches, Projections: All such effects like pockets, notches, projections, primary and secondary grout, etc. are to be accounted for mass effect only. Their influence on stiffness, since not significant, is normally ignored.

High vibrations have been witnessed by the author on account of carbonization of the grout. All efforts to control the vibration failed till grout was replaced under HP turbine Seating Plate. It is recommended to use only non-shrink grout and carry out periodic monitoring for grout health.

7.7 VIBRATION LIMITS IN MACHINE FOUNDATION DESIGN

No common guidelines could be defined for setting permissible limits of vibration for all types of machines and for all applications. The governing criteria therefore are the permissible amounts of vibrations that the machine, its surroundings or the persons in the vicinity of the machines can tolerate.

Based on the information available in the literature, and also based on the measured vibration records on various types of machines, general recommendations of permissible/allowable amplitudes for machine foundation design for different machines are given as under:

Table 7.7-1 Permissible Amplitudes

Machine Type	**Permissible Amplitudes Microns**
Foundations for Rotary Type Machines	
Low Speed Machines (100 to 1500 rpm)	
Operating Speed 100 to 500 rpm	200 to 80
Operating Speed 500 to 1500 rpm	80 to 40
Medium Speed Machines	
Operating Speed 1500 to 3000 rpm	40 to 20
High Speed Machines	
Operating Speed 3000 to 10000 rpm and above	20 to 5

Foundations for Reciprocating Type Machines	
Machines (300 to 1500 rpm)	1000 to 200
Machines (100 to 300 rpm)	1000
Foundations for Impact Type Machines	
Hammer Foundations	1000 to 4000
Foundations for Hammer Crushers	
Operating Frequency up to 300 rpm	300
Operating Frequency above 300 rpm	100

7.8 BLOCK FOUNDATION

The foundation should have adequate strength to withstand forces imparted by the machine and should be able to withstand other environmental effects like wind & earthquake.

7.8.1 Foundation sizing

Foundation should be so dimensioned such that the derived **eccentricity,** in both the lateral directions, is bare minimum (**see** § 7.3). As far as possible, the **eccentricity** should be close to zero and in no case it should exceed 5 % of the base dimension in the respective direction. For **very long foundations** having L/B >3, it is further desirable (but not necessary) that the eccentricity be contained within 2 % **along the length,** whereas the eccentricity along width may still remain within 5 %.

Foundation should extend by a reasonable margin (say minimum 100 mm or more depending upon layout constraints) on all sides of machine base frame and in no case; machine base frame should protrude outside foundation boundaries.

The pressure developed in the soil due to static loads should preferably be below 80% of the allowable safe bearing capacity keeping balance 20 % as the margin for pressure produce by dynamic forces. However for machines producing high dynamic force, the bearing pressure and stability is checked both for static and dynamic loads **(see § 5.2.5)**.

Though, from strength point of view it may appear adequate to have foundation mass equal to that of the machine, a higher mass ratio is desirable as it indirectly helps in keeping the foundation eccentricity low even if there are minor variations in actual machine data vis-à-vis design data.

Foundation Mass Ratio: It is ratio of the mass of foundation to that of machine. Recommended guidelines for different machine types are as under:

- **Rotary Machines**: Foundation mass ratio of 2.5 to 4 is generally considered adequate for rotary machines
- **Reciprocating Machines**: The foundation mass ratio required for reciprocating machines is much larger than that for rotary machines. This ratio could be as high as 8 and for specific cases, based on practical considerations, this ratio could as well be as low as 1.5
- **Impact Machines:** The foundations for impact machines require provision of adequate depth of concrete below the Anvil and it is linked with the mass of the falling part. It is recommended to have foundation thickness of at least 1.0 m, below the Anvil, for falling mass of 1000 kg. Foundation thickness of about 2.5 m to 3 m is recommended for higher falling mass, say, 6000 kg and above. Mass of Anvil is generally in the range of 20 to 25 times that of the falling mass and the mass of the foundation should be 2.5 to 3 times that of the anvil. For unfavourable soil conditions, this ratio could be 4 to 5 times or even higher

7.8.2 Foundation Stiffness

Foundation parameters that govern the dynamic response are its mass and its area in contact with the soil. In specific cases, projected parts of the foundation having finite stiffness also influence dynamic response. There are no other foundation related parameters that influence the response.

The rigidity the foundation is much higher compared to that of the soil supporting it. The elastic deformation of the block, under the influence of static and dynamic forces, is of negligible order compared to that of the soil. The foundation, therefore, is considered as a rigid body consisting of mass only. In other words, foundation is considered as non-elastic (rigid) inertia body. In case the foundation has **some structural members whose stiffness is comparable** with that of the soil, **such members** should be modeled for their **stiffness as well as mass effect.**

One can represent the system as an assemblage of spring mass system (SDOF or MDOF) or one can use FE modeling technique, as the case may be and analyse the system for its dynamic response.

7.8.3 Strength Design

Since the block foundation behaves like a rigid body supported on soft media like soil, invariably the block foundations would turn out to be having adequate strength vis-à-vis forces imparted by the machine.
For strength and stability analysis, material parameters required are the same as those required for analysis & design of any other structure. Permissible Stresses for design, both for concrete and steel, are to be taken as per applicable design codes.

Strength design is done considering forces and moments on the foundation due to **Static Loads, Dynamic Loads, Emergency loads and applicable Earthquake/Wind loads.** Necessary reinforcement is provided to withstand these forces and moments **as per applicable codes of practice.** When Emergency/Earthquake/Wind loads are considered, the permissible stresses shall be enhanced by 25 %.

In case the foundations has any extensions/projections, these must be designed for forces and moments at critical sections due to **Static, Dynamic and Emergency** loads and necessary reinforcement is provided as such projections behave elastically and not like a rigid body.

Anchor Bolts: All anchor bolts should be checked for pullout force caused due to Static, Dynamic and Emergency loads.

Though not necessary, but recommended to perform **overall stability check** for the foundation after the design process is complete.

7.8.4 Minimum Reinforcement

Block Foundation: Provisions for minimum reinforcement have been found to vary from 25 to 50 kg/m^3 of volume of concrete for foundations of different machines. From his experience, the author recommends as under:

- Diameter of the reinforcing bar shall not be less than 12 mm and spacing of bars shall not exceed 200 mm. Where thickness of concrete exceeds $1\,m$, additional layer of reinforcement (both ways) to be provided as shrinkage reinforcement
- Overall steel quantity should not be less than 25 kg/m^3 of concrete.
- Further, it must be ensured that all faces are covered with two-way reinforcement.
- All faces of the openings, pockets, cot-outs etc must be reinforced appropriately with the same provisions as above.

More often than not, one may find that minimum reinforcement, as recommended above, is adequate to withstand the applied/generated forces. Those **extensions/projections** that make the concrete section to behave elastically must be adequately reinforced in accordance with the strength design requirements (**see § 7.3.8**).

7.9 FRAME FOUNDATION

Dynamic Behaviour of Frame Foundation is relatively complex compared to that of Block foundation. There are many foundation related parameters that significantly influence the response viz. stiffness of frame structure, individual vibration characteristics of frame columns, frame beams, cantilever projections etc. Due attention, therefore, must be paid to these aspects while sizing the foundation.

7.9.1 Foundation Sizing

Foundation Plan Layout containing foundation dimensions, details of cut-outs, trenches, notches, depressions, pedestals, machine static loads, dynamic loads etc are provided by customer/machine supplier. In addition, machine loads at base raft, intermediate deck (if any) and other locations are also provided by customer/machine supplier.

For **Frame foundations**, there are **two connotations** to the term **eccentricity** as given below:

i) **Overall Eccentricity:** It is defined as the distance between **center of mass** of overall system (machine + foundation) and **center of stiffness** (i.e. CG of the base contact area of the foundation with the soil) as given in § **7.3.** This should be restricted to 5 % of the respective base dimension of the foundation.

ii) Top Deck Eccentricity: It is defined as the distance between **center of mass** C_m (i.e. combined CG of machine mass, top deck mass and 23% of column mass) and **center of stiffness** of frames C_k in transverse (perpendicular to machine axis) as well as longitudinal direction (along machine axis). It is desirable that this eccentricity should be restricted to 1 % of the respective dimension of the top deck.

General recommendations for sizing of various elements of foundation are as under:

i) **Top Deck:** Top Deck comprises of transverse beams, longitudinal beams, slab connecting these beams, projections on all sides of beams, depressions, cut-outs, notches, openings etc.

  a) Weight of the top deck should in no case be less than weight of the machine
  b) Span to depth ratio of the beams should be 3 to 4.
  c) Depth to width ratio for the beams should be 1 to 1.5
  d) Extent of cantilever projections (in plan) should not be more than half the width of corresponding beam
  e) Depth of slab should invariably be same as that of the encompassing beams except the areas where recess or depressions are provided to accommodate machine

Though from strength consideration, it is possible to design slender sections of the members, yet it is recommended to follow the above guide line as it helps in controlling overall eccentricity.

ii) **Columns:** Total weight of all the columns should be close to weight of the machinery. This condition is desirable but not essential. Column sizes (as marked on the layout drawing) are generally provided by the customer/supplier. More often than not, it has become the practice by the designers to stick to these dimensions.

Such a practice is undesirable and must be discouraged. These should be taken as indicative only and the designer must assess the validity of these sizes keeping in view the followings:

a) It is desirable that **Center of Stiffness** of all the Frames should coincide with **Center of Mass** of machine and top deck.

b) Lateral natural frequencies of each of the column (along transverse as well as longitudinal directions), considering fixed-fixed at top and bottom, should not coincide with Machine frequency or its harmonics.

iii) **Base Raft:**

a) Raft plan dimensions are selected such that Bearing Pressure is well within the allowable safe bearing capacity of the soil keeping a reasonable margin of about 30% to 40 % for pressure induced by dynamic loads **(see § 5.2.5).**

b) Overall eccentricity between center of mass and center of stiffness should lie within 5% of the respective base dimension in plan. For very large dimensioned foundations, eccentricity should, as far as possible, be restricted to 2 % only

c) While finalizing base raft thickness, it is to be borne in mind that differential elastic deformation of the base raft (at the frame support locations) is highly undesirable. It directly influences machine alignment of various machine segments, i.e. it contributes to misalignment resulting in higher dynamic forces and thereby higher amplitudes of vibration. Hence, base raft must have adequate thickness so as not to generate any differential settlement/deformation. A general guide line is that weight of the base raft should be about twice that of the machine weight.

7.9.2 Stiffness Parameters for Frame Foundation

Unlike block foundation, elastic deformation of the structural members of frame foundation is of finite order. Thus, the foundation is considered as an elastic body consisting of both stiffness and mass. Whereas mass and stiffness of foundation are duly taken care of using FE modeling, these are to be evaluated for each of the structural element of the foundation while considering it as lumped-mass model using SDOF or MDOF system. It is therefore desirable to explicitly quantify such influences for lumped-mass modeling of frame foundations. Generalised mass for various structural elements like beams, columns, portal frames etc, using kinetic energy equivalence, is covered in Chapter 2 and 3.

Mass contribution of beam and column of a typical frame is different for vertical vibration than for lateral vibration **(see chapter 2 & 3)**. Further, the mass and stiffness contribution do depend upon whether the model is SDOF or MDOF System.

For cases, where, machine is supported over a column or a beam, mass content of support system is described in **Chapter 2.0**. For portal frames **generalized stiffness** and **generalized mass** contributions are described as under:

For a structural portal frame supporting mass m at frame beam center and constrained to move only in X-Y Plane, evaluation of generalised mass and stiffness considering frame as SDOF system is covered in Chapter 2 and that for a Two DOF System is covered in Chapter 3. There are a couple of factors that make Frame Foundations different from normal structural frame such as shear deformation on account of lower span to depth ratio, provision of haunches at beam column junction etc. Hence the formulations for normal structural frame need to be suitably modified for Frame Foundations. A typical portal frame with haunches is shown in Figure 7.9-1.

> **Some of the scientists / authors have suggested corrections to frame span and height on account of larger column widths and beam depths. It may not be out of place to mention that variation in frequency due to correction applied to frame center line dimensions H & L is only of the order of 2 to 3 % and hence can be ignored as well.**

7.9.2.1 Haunches

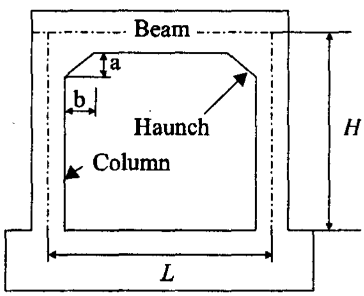
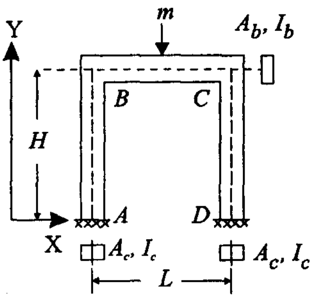


Figure 7.9-1 A Typical Frame with Haunches

Provision of haunches at beam column junction at top deck soffit is a common phenomenon. Though its contribution to mass effect is insignificant, its influence on stiffness has been reported to be of significant order and it depends upon size of the haunch.

Based on Finite element analysis of a number of portal frames with different haunch sizes (haunch width b varying from 5 to 10 % of span and haunch depth a varying from 5 to 10 % of frame height), following observations are made:

i) The presence of haunches tends to increase Lateral Stiffness of the frame. The increase in lateral frequency is in the range of 3% to 6% depending upon haunch sizes.
ii) The presence of haunches does not show any influence on vertical stiffness. The influence on vertical frequency is very insignificant (of the order of 0.2 to 0.3 %).
iii) There is significant influence on higher order column frequencies



(a) Portal Frame

Figure 7.9-2 A typical Portal Frame Supporting Machine Mass at Beam Center

7.9.2.2 Shear Deformation

Since span to depth ratio of beams is relatively low for frame foundation the frame beam behaves like a deep beam making the shear deformation influence significant. This aspect therefore must be included while evaluating stiffness of frame beam in vertical direction.

For a typical Frame as shown in Figure 7.9-2, the shear deformation of the beam is given as

$$\overbrace{y_2}^{\text{Shear}} = \frac{3L}{8GA_b} \tag{7.9-1}$$

This is added to flexural deformation of the beam for computing beam stiffness. Let us compute stiffness k_y of the frame.

Beam to Column Stiffness ratio **(see § 2.1.1-4-5)** $k = \dfrac{k_b}{k_c} = \dfrac{I_b/L}{I_c/H}$

Vertical deformation under unit load at beam center (see equation see equation 2.1.1-34)

i) Flexural Deformation of frame beam $\overbrace{y_2}^{\text{Flexural}} = \dfrac{L^3}{96EI_b} \times \dfrac{2k+1}{k+2}$

ii) Shear deformation of the beam $\overbrace{y_2}^{\text{Shear}} = \dfrac{3L}{8GA_b}$

Total deformation at Frame Beam Center under Unit Load

$$y_2 = \overbrace{y_2}^{\text{Flexural}} + \overbrace{y_2}^{\text{Shear}} = \left(\frac{L^3}{96\,E\,I_b} \times \frac{2k+1}{k+2} \right) + \frac{3L}{8GA_b}$$

Stiffness $$k_2 = \frac{1}{y_2} \qquad (7.9\text{-}2)$$

7.9.3 Strength Design

For **frame foundations**, reinforcement is provided as dictated by strength design of structural members i.e. columns, beams, slabs etc. It is generally recommended to check the strength adequacy considering forces and moments on the foundation due to **Static Loads, Dynamic Loads, Emergency loads and applicable Earthquake/Wind loads.** Foundation must also be checked for thermal stresses wherever applicable.

Necessary reinforcement is provided to withstand these forces & moments **as per applicable codes of practice.** When emergency/earthquake/wind loads are considered, the permissible stresses shall be enhanced by 25 %.

7.9.4 Minimum Reinforcement

Generally speaking, reinforcement in the range of $100 \text{ to } 120$ kg/m^3 has been found to meet the structural safety requirements for frame foundations. This is however a guideline and actual steel requirement should be based on design.

- Reinforcement for **top deck and columns** to be in the range of 100 to 120 kg/m^3
- Reinforcement for **base raft** to be in the range of 70 to 80 kg/m^3

These figures are considered adequate even for estimation and found handy for cross checking.

PART - III

DESIGN OF FOUNDATIONS
FOR
REAL LIFE MACHINES

8. Modeling & Analysis
9. Foundation for Rotary Machines
10. Foundation for Reciprocating Machines
11. Foundations for Impact Type Machines

8

MODELING AND ANALYSIS

- Manual Computational Method
- Finite Element method
- Dynamic Analysis
- Strength Analysis and Design

- **Example problems**

For better clarity, all Figures related to FE analysis, including animations of frequencies and mode shapes, in color, are given in the CD attached at the end of the handbook

MODELING AND ANALYSIS

Every foundation designer should remember that he is dealing with **machines weighing several tons** and is required to design the **foundations having dimensions of several meters** but **amplitudes restricted to only a few microns.** The designer therefore must clearly understand the **assumptions, approximations and simplifications made during modeling and recognize their influence on the response.**

It is this aspect that makes **modeling and analysis very important part of the design.**

A physical system is represented by mathematical model with the basic objective that mathematical model should be compatible with prototype. For each mathematical representation, a host of assumptions, approximations are made. The extent of complexity introduced in mathematical model directly influences the reliability of results. In broad sense, mathematical representation not only depends upon Machine and Foundation parameters but also depends upon Analysis Tools.

Let us consider some of the combinations of Machine Type, Foundation Type and Analysis Tools and consider level of modeling for each case:

i) Low RPM Machines (300 to 1500 RPM)

Low RPM machines generally develop very high unbalance forces and permissible amplitude limits are also high. In case of resonance, these give rise to very high amplitudes. For such machines, one should not attempt isolation both for block and frame foundation.

a) Low RPM Machines on Block Foundations: Foundation lowest natural frequency normally turns out to be much higher than machine frequency thus ruling out any possibility of response magnification. Hence one can resort to very simple model. Care should be taken to avoid direct resonance with engine order frequencies of 1st, 2nd & 3rd order i.e. if engine frequency is ω, one should avoid having foundation natural frequencies as $1\omega, 2\omega \& 3\omega$.

b) Low RPM Machines on Frame Foundations: From operational constraints dictated by plant lay out, at times these machines are mounted on frame foundations. In view of very high unbalance forces, foundation must have large size columns and beams. In view of heavy machine mass, lateral natural frequencies of frame foundations are low with likely possibility of resonance with machine frequency. Hence, one must include all those structural elements which are likely to contribute to low frequency. Any approximation to ignore such elements while modeling may lead to overall high vibrations. In such cases, effect of soil must be included in the model as presence of soil tends to lower down the 1st frequency of the system. Here again, care should be taken to avoid direct resonance with engine order frequencies of 1st, 2nd & 3rd order.

ii) Medium RPM Machines (1500 to 3000 RPM)

Medium RPM machines have stringent balancing requirements resulting in lower rotor eccentricity and thereby lower unbalance forces. Permissible amplitude limits are also low for these machines. In view of low permissible amplitudes, foundation sizing should be so done so as to have foundation eccentricity to bare minimum.

These machines are very sensitive to rotor alignment. This necessitates that beams supporting bearing must not structurally deflect to cause misalignment. Further, base raft should also be sufficiently thick so as to permit overall settlement but certainly no differential settlement.

In view of lower foundation frequencies, there is a possibility of recording higher transient amplitudes. Further there is every possibility of resonance with rotor critical speeds. It is desirable, though not essential, to look in to these aspects too.

Provision of Vibration Isolation System (VIS) for such machines turns out to be quite effective.

a) Medium RPM Machines on Block Foundations: Machine must be modeled along with the foundation. As computed natural frequencies are comparatively of lower order than machine frequency, even a simple model would be adequate. Care should be taken to avoid direct resonance with sub harmonics i.e. $\omega/2 \& \omega/3$ that develop due to certain bearing phenomenon.

b) Machines on Frame Foundations: Machine must be modeled along with the foundation. Such foundations have generally strong top deck but relatively slender column sizes. On account of higher column height, lateral natural frequencies turn out to be low. Higher structural natural frequencies are of comparable order to machine frequency having likelihood of being in resonance with machine frequencies. Further, there is every possibility that higher mode foundation frequencies (mainly higher beam and column modes) may come in resonance with engine order frequencies.

A higher order mathematical model is therefore essential. Effect of soil can conveniently be ignored unless warranted by specific soil characteristics.

iii) High RPM Machines (above 3000 RPM)

High RPM machines have highly stringent balancing requirements. Rotor eccentricity is quite low. Unbalance forces are also low and so are permissible amplitudes. These machines are very sensitive to rotor alignment; hence beams supporting bearing, if not properly designed, could become a source for misalignment. Further, base raft should also be sufficiently rigid so as not to permit any differential settlement especially at frame location points. Foundation eccentricity should be bare minimum. During every start-up and shut-down, there is every possibility of high transient amplitudes at engine sub harmonics, foundation frequencies as well as at rotor critical speeds. Though not essential, it is recommended that these aspects should also be looked in to. Higher order structural frequencies in case of frame foundation must be evaluated to avoid any direct resonance.

Provision of Vibration Isolation System (VIS) for such machines turns out to be quite effective.

8.1 MANUAL COMPUTATIONAL METHOD

8.1.1 Block Foundations

For machines on block foundations, it is good enough to use the procedures and formulations described in Chapters 2 & 3. Whereas majority of the machine and foundation aspects are well taken care of by these procedures, yet there are some aspects, as given below, that can not be fully managed by these manual computational methods:

Foundation Eccentricity: If foundation eccentricity is higher than permissible, the vertical mode of vibration will no longer remain un-coupled from lateral and rotational modes. Hence equations of motion given by equation 3.4.2-2 (as reproduced below) **will not remain valid.**

$$\begin{bmatrix} m & 0 & -mh \\ 0 & m & 0 \\ -mh & 0 & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ y \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin\omega t \\ F_y \sin\omega t \\ M_\phi \sin\omega t \end{Bmatrix}$$

In such a case one has to use equation of motion given by equation (3.4.2-10) as reproduced below:

$$\begin{bmatrix} m & 0 & -mh \\ 0 & m & ma \\ -mh & ma & M_{moz} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & (k_\phi - mgh) \end{bmatrix} \begin{Bmatrix} x \\ y \\ \phi \end{Bmatrix} = \begin{Bmatrix} F_x \sin\omega t \\ F_y \sin\omega t \\ (-s F_x)\sin\omega t \end{Bmatrix}$$

Getting closed-form solutions for these equations is not that simple and computations may turn out to be complex. Further getting **Transient Response History** may be a tedious task, though it is possible to evaluate transient response at any of the defined frequency.

It is therefore recommended to use FE analysis, wherever feasible, to include all these aspects. Further it gives improved reliability on account of lesser number of approximations/assumptions. It also permits visualization of animated mode shapes, view response amplitude build-up and viewing of stress concentration locations.

8.1.2 Frame Foundations

Analytical Procedures for machines on frame foundations as given in Chapters 2 & 3 provide clear insight into the free and forced vibration of portal frame subjected to dynamic loading.

It may not be out of place to mention that the formulations cover only standard frames i.e. frame beam is a rectangular in cross-section having machine mass at its center. The premise, that longitudinal beams of a frame foundation are flexible enough to permit transverse frames vibrate independently, does not hold good for real life machines. These are very ideal cases and most of the **real life machine foundations do not fall under this category**. Some of the aspects that can not be suitably accounted for by manual computational methods are:

- **Haunches**
- **Machine mass at beam off center locations**
- **Beams extended as cantilever on one side/both sides of frame beam**
- **Beams inclined in elevation supporting heavy machine mass**
- **No frame beam at column locations**
- **Higher order frame column vibration frequencies**
- **Presence of solid thick deck within the frames**
- **Depression/recess in the top deck**

Based on many design studies carried out by author, it is observed that:

1. Variation in natural frequencies a frame by Manual method compared to FE method is of the order of 10 to 20 %.
2. **FE analysis** confirms presence of **many additional frequencies** (3 to 4 frequencies) between 1st vertical and 2nd vertical mode as computed by **Manual Method of Analysis.** These additional frequencies lie well within operating range of the medium RPM machines and may significantly contribute to response.
3. An example Problem P 8.1-1 at the end of this chapter is included for this purpose. The observations made clearly **highlight limitations** of **Manual Method of Analysis** for design of **Frame Foundations.**
4. In recognition of higher reliability by FE Method, and the fact that manual method gives results that are in variance by 10 to 20 % compared to FE Analysis, it is suggested that no corrections need to be applied on account of either frame center line dimensions or inclusion of haunches etc. All corrections put together shall easily get absorbed by the available margins.

It is therefore recommended to use FEM analysis with appropriate element types for modeling of Frame Foundation. It is also recommended to use analytical approach to evaluate free vibration response for each frame to get a first hand feeling of the frequency range of frames vis-à-vis operating frequency, their sub and super harmonics.

8.2 FINITE ELEMENT METHOD

Finite Element (FE) is the most commonly accepted analysis tool for solution of engineering problems. Effective **Pre & Post-processing** capabilities make modeling and interpretation of results simple. It is relatively easy to incorporate changes if any and re-do the analysis without much loss of time. Viewing of **Animated Mode Shapes** and **Dynamic Response** makes understanding of the dynamic behaviour of the machine foundation system, relatively simpler.

Design of machine foundation involves consideration of Machine, Foundation and Soil together as a system subjected to applied or generated dynamic forces. Development of specific FE based package for design of machine foundation is generally not feasible on account of a) tight project schedules and b) validation of results.

Use of **Commercially Available Packages** is more effective for design offices. There are many issues that need careful examination before finalizing the package e.g. user friendliness, pre-processor capabilities (modeling capabilities), analysis capabilities, post-processor capabilities (processing of results) etc but the most important one is the validation of results. Every package is a **Black Box** for the user and it has its associated **limitations** some of which are explicit and some implicit. Validation, for some sample known cases, therefore, becomes a must before one accepts the results.

Author himself has used many commercially available packages for analysis and design of Machine Foundation during the course of his professional carrier.

8.2.1 Mathematical Modeling

Most of the issues related to modeling have been discussed in 8.1.

A machine foundation involves modeling of Machine, Foundation and Soil. Finite Element Method (FEM) enables modeling of Machine, Foundation and Soil in one go that brings behaviour of the machine foundation system closer to the prototype resulting in improved reliability.

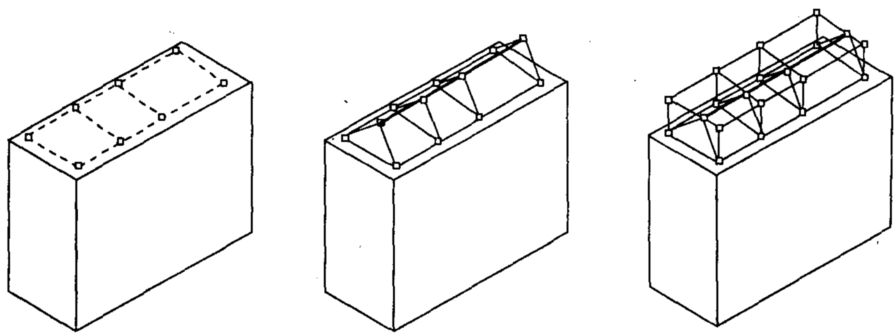
Rigid Beam Elements are used for modeling the machine whereas **Solid Elements** are used for modeling the Foundation. In case soil is represented as continuum, it is also modeled using Solid Elements. In case Soil is represented by equivalent s**prings** it could be modeled using spring elements/boundary elements.

Note: The terminology used here may not comply with the terminology of each package. Readers may modify the terminology in accordance with that of the package under use.

Modeling of each of the constituent is an art in itself and is briefly discussed here under.

8.2.2 Machine

Machine is relatively rigid compared to foundation and soil. It is considered contributing to mass only with its CG lying above Foundation level.



(a) Machine Mass lumped at Foundation top

(b) Machine Masses lumped at Machine CG level

(c) Rotor and Stator Modeled separately - Masses lumped at respective CG level

Figure 8.2-1 Modeling of Machine with Foundation

While modeling the machine, the broad objective is to represent the machine in such a way that its mass is truly reflected and overall mass CG of the model matches with that of the prototype. Thus, modeling of machine with **Rigid Links / Rigid Beam Elements** is considered good enough. Machine mass is considered lumped at appropriate locations so as to simulate the CG location. This should be cross checked with the mass distribution given by the supplier/manufacturer.

Be it a block foundation or a Frame Foundation, lumping of Machine Mass at foundation top level is **not desirable** as this will result in mismatch of the CG of machine mass (in vertical direction) of model with that of the prototype. Figure 8.2-1a shows such lumping for a typical block foundation. Such a representation does affect mass moment of inertia and thereby natural frequencies and response. It is therefore essential that CG of the machine mass in vertical direction must be matched with that of the prototype as given by manufacturer. Machine mass should be lumped at appropriate level above the foundation as shown in Figure 8.2-1b. Similar concept should be used for modeling bearing pedestals.

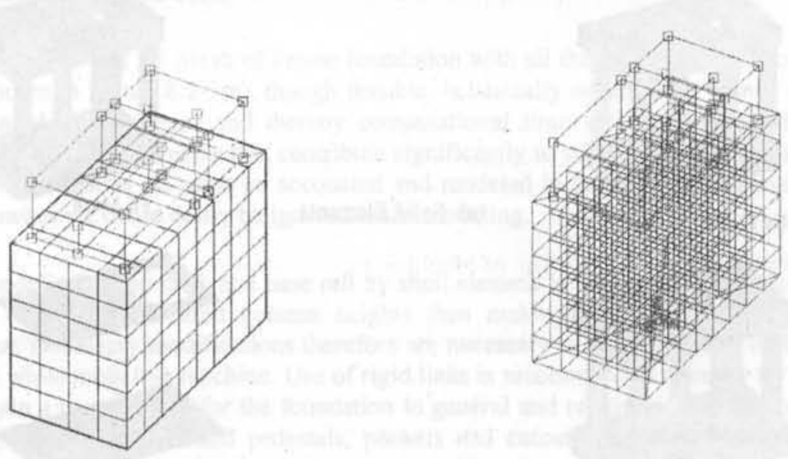
For **Advanced Modeling**, it is desirable to model the **Rotor and Stator** independently. **Rotor** is represented using a set of beam elements with corresponding section and material properties that represent variation of rotor section along machine axis, whereas **Stator** is modeled using Rigid Links with stator mass lumped at appropriate locations such that CG of mass matches with that provided by supplier. Rotor support at the **Bearing locations** should be modeled with corresponding stiffness and damping properties offered by bearings. Model is as shown in Figure 8.2-1c. The **Bearing Pedestals,** however, are modeled as rigid links.

8.2.3 Foundation

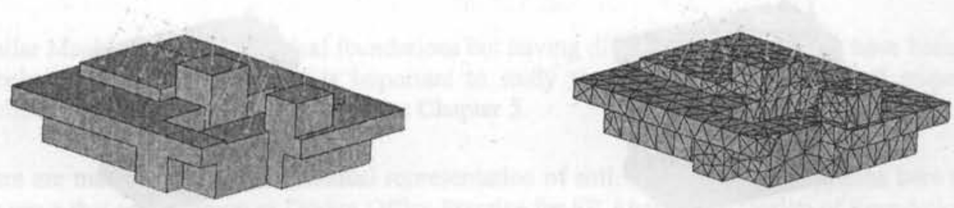
8.2.3.1 Block Foundation

A foundation block is a solid mass made of RCC with required openings, depressions, raised pedestals, cutouts, bolt pockets and extended cantilever projections.

Solid Elements are good enough for modeling foundation block. A coarse mesh for the block and relatively finer mesh in the vicinity of openings, pockets and cutouts is considered reasonably OK. Solid Model & FE Mesh of a typical foundation block is shown in Figure 8.2-2.



Solid Model A Typical Block Foundation FE Mesh



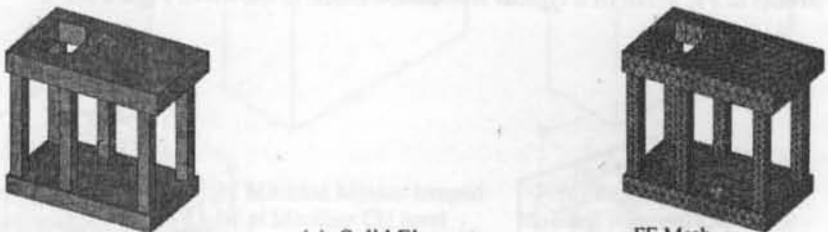
Solid Model A Typical Fan Foundation FE Mesh

(Portion below Ground Level not shown)

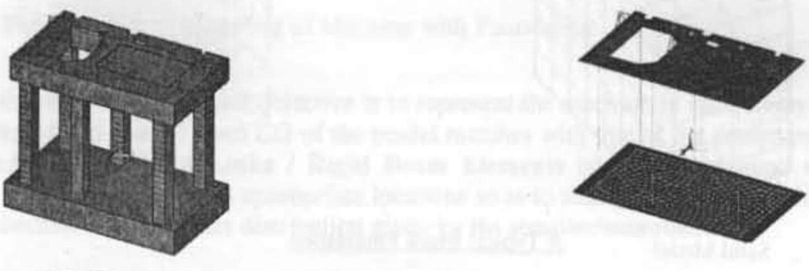
Figure 8.2-2 Foundation Block – Solid Model and FE Mesh

Generally speaking, modeling the foundation block with 8-noded Brick Elements or 10-noded Tetrahedral Elements works reasonably well and is considered good enough. A higher order solid element would increase the size of the model – requiring more computational time & power– while improvement in the result's accuracy may only be marginal. Choice of element size is fairly subjective as it is problem dependent. It is therefore not possible to define firm guidelines regarding choice of right element size that will be applicable to all types of problems. The judgment of optimum mesh density, however, would emerge after experience.

8.2.3.2 Frame Foundation



Solid Model (a) Solid Elements FE Mesh



Geometric Model (b) Shell & Beam Elements FE Mesh

Solid Model FE Mesh

(c) A Typical Top Deck View
Openings, notches, cutouts, pockets etc.

Figure 8.2-3 Frame Foundation – Solid Element Model & Shell Beam Model

A **Frame Foundation** comprises of Base Raft, Set of Columns (Number of Frames), Top Deck consisting of Beams (Longitudinal and Transverse) and Slabs. Top deck is made of RCC with required openings, depressions, raised pedestals, cutouts, bolt pockets and extended cantilever projections. In certain cases, haunches may also be provided between column and top deck.

There are many ways of representing model of a Frame foundation. One can model using Beam Elements, Shell Elements, Solid Elements or a combination of all these. Models with solid elements as well as Beam & Shell elements are shown in Figure 8.2-3 (a) & (b) respectively. Each modeling style, however, shall have associated limitations. For example, while modeling using solid elements, one may not be able to get bending moments and shear forces in the columns, beams and slabs needed for structural design of these members. Whereas it is possible to get bending moments and shear forces in flexural members like beams, columns, slabs etc, it however does not permit inclusion of effect like haunches, depressions, cut-outs, raised blocks, projections etc as shown Figure 8.2-3(c).

It may be noted that FE Mesh of Frame foundation with all the openings, pockets, cutouts, notches etc as shown in Figure 8.2-3(c), though feasible, is basically undesirable. It may unnecessarily add to increased problem size and thereby computational time without any significant gain in the results. Only those elements that contribute significantly to stiffness and mass like large openings, sizeable depressions etc must be accounted and modeled in detail whereas elements like pockets, small notches etc could easily be ignored while modeling.

Since modeling of top deck and base raft by shell element is done at their mid surface locations, it usually results in increased column heights thus making the system more flexible than the prototype. Necessary modifications therefore are necessary to overcome this deficiency. Similar is the case while modeling machine. Use of rigid links is recommended to cover up such deficiencies. Here again a coarse mesh for the foundation in general and relatively finer mesh in the vicinity of openings, depressions, raised pedestals, pockets and cutouts is considered reasonably OK. The judgment of optimum mesh density, however, would emerge after experience

8.2.4 Soil

Similar Machines having identical foundations but having different.soil conditions have been found to behave differently. Hence it is important to study the affect of soil on overall response of machine. For elastic properties of soil, see Chapter 5.

There are many ways of mathematical representation of soil. We limit our discussion here to only two ways that are common as Design Office Practice for FE Analysis & Design of Foundations.

i) Soil represented by a set of equivalent springs
ii) Soil represented as continuum

i) **Soil represented by a set of equivalent springs:** Two types of representations are commonly used in FE modeling the foundation:

a) Soil is represented by a 3 Translational Springs and 3 Rotational Springs attached at CG of the Base. This kind of representation yields results (Frequencies and Amplitudes) that are found to be in close agreement with manual computations. This type of representation is shown in figure 8.2-4 (a).
b) Soil is represented by a set of 3 Translational Springs attached at each node at the Base of the Foundation, in contact with the soil. This kind of representation provides an upper bound to overall rotational stiffness offered by soil about X, Y & Z axes. This type of representation is shown in Figure 8.2-4 (b).

Stiffness Properties of these equivalent springs are discussed in detail in Chapter 5.

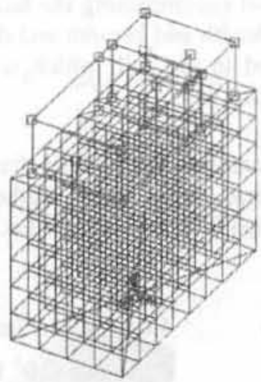
ii) **Soil Represented as Continuum:** Soil domain, in true sense, is an infinite domain and for analysis purposes, it becomes **necessary** to confine it to a **finite domain** when soil is considered as continuum. The broad issues, that need to be addressed are:

- What should be the extent of soil domain to be considered for modeling?
- Whether to consider soil domain only below the foundation base (in which case foundation becomes un-embedded) or to consider the foundation embedded in to the soil domain (in which case foundation becomes embedded in to the soil)?

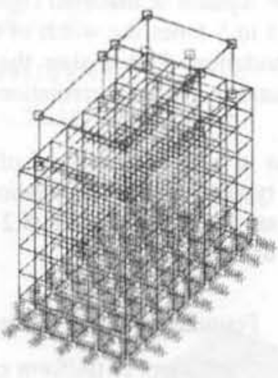
Extent of Soil Domain: For FE Modeling, it is well known that a narrow domain with fixed boundaries is not likely to represent realistic soil behaviour whereas a very large domain would result in increased problem size. It is therefore necessary to find an optimum value that should reflect realistic behaviour of soil without significant loss in accuracy.

Let us consider a stand alone isolated foundation. Different designers adopt their own thumb-rule practices while deciding on the extent of soil domain to be modeled with the foundation. The extent of soil domain has been found to vary from 3 to 8 times the width of the foundation to be provided on all 5 side of the foundation. It is to be noted that such a consideration is good enough for the **academic purposes only.** In a real industrial situation, no foundation could remain isolated from other equipment/structure foundations within this finite soil domain. In other words, many other equipment/structure foundations would exist within the range of 3 to 8 times the dimension of the foundation in each X, Y & Z direction. Thus, in the author's opinion, the computed behaviour of a foundation as a stand-alone foundation is likely to differ with the actual one. It is also true that **modeling** of all the equipment and structure foundations of a project, **in one single go,** is neither feasible nor necessary. Here too, a mesh consisting of solid elements is good enough. As the soil domain is very large compared to foundation, a relatively coarser mesh of the soil is considered to be adequate. Refinement of the mesh size may be adopted if considered necessary for specific cases. The choice of element size remains subjective.

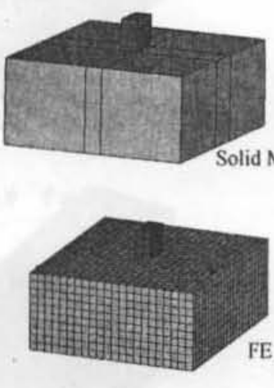
The precise decision on extent of soil domain still remains a question mark. Even from academic side there is no definite answer to this issue. It is also true that a practicing engineer, in view of tight time schedule, can neither afford to do R&D nor can ignore the problem. In **author's considered opinion,** soil domain equal to 3 to 5 times the lateral dimensions in plan on either side of the foundation and 5 times along the depth should work out to be reasonably OK. The finite soil domain is modeled along with the foundation block using FE idealization. Appropriate soil properties in terms of Elastic Modulus/Shear Modulus and Poisson's Ratio are assigned to the soil. If soil profile indicates presence of layered media, appropriate soil properties are assigned to respective soil layers with variation in soil properties along length, width and depth of the soil domain.



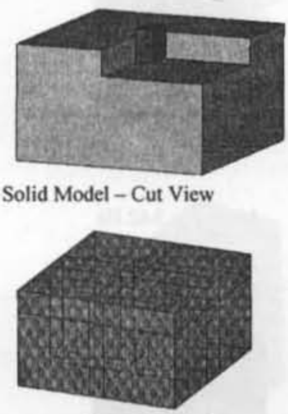
(a) Soil represented by a set of three translational springs, k_x, k_y, k_z and three rotational springs k_θ, k_ψ, k_φ applied at CG of Base of the foundation



(b) Soil represented by a set of three translational springs k_x, k_y, k_z applied at each node in contact with the soil at the foundation base



(c) Soil represented by a continuum below the foundation base extending three times the width of the foundation along length and width and 5 times the depth of the foundation along depth



(d) Soil represented by a continuum starting from the ground level extending three times the width of the foundation along length and width and 5 times the depth of the foundation along depth

Figure 8.2-4 Various Methods of Soil Representation for FE Modeling

Un-Embedded and Embedded Foundation: While modeling soil along with foundation, the two cases arise:

i) Soil domain is modeled below the foundation up to 3 to 5 times the width of the foundation along length and breadth and depth of the foundation. This makes the foundation un-embedded in the soil. This representation is shown in Figure 8.2-4 (c).

ii) Soil domain is modeled right from the ground level encompassing the foundation up to 3 to 5 times the width of the foundation along length and breadth and depth of the foundation. This makes the foundation embedded in the soil, which is a realistic situation. This representation is shown in Figure 8.2-4 (d).

To investigate as to how each method of soil representation compares with others, free vibration analysis of a typical Block foundation has been performed using each method of soil representation, as shown in figure 8.2-4, having same/compatible soil properties. The data considered is as under:

- Foundation Block dimensions (along Z, X, Y) $4\text{m} \times 2\text{m} \times 3.75\text{m}$ deep
- Coefficient of uniform compression $C_u = 4.48 \times 10^4$ kN/m^3

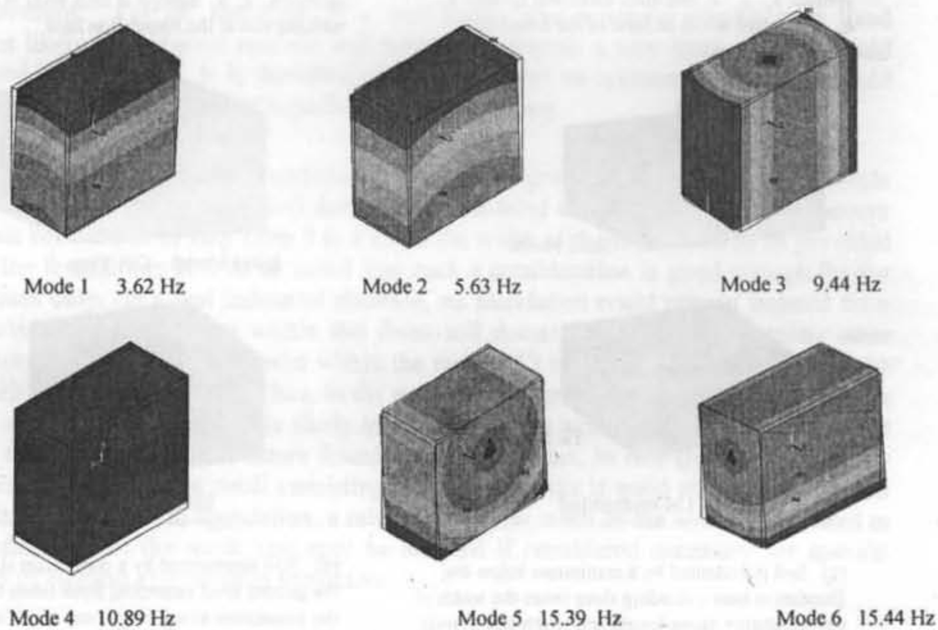


Figure 8.2-5 **Frequencies and Mode shapes – Soil represented by 3 translational and 3 rotational springs at CG of foundation base**

- **Soil spring stiffness**
    - i. Translational $k_y = 35.84 \times 10^4$ kN/m; $k_x = k_z = 17.97 \times 10^4$ kN/m
    - **ii.** Rotational
        - **a.** $k_\theta = 95.5 \times 10^4$ kNm/rad (about X)
        - **b.** $k_\psi = 44.8 \times 10^4$ kNm/rad (about Y)
        - **c.** $k_\phi = 23.9 \times 10^4$ kNm/rad (about Z)
- $\rho_{\text{soil}} = 1.8\ \text{t/m}^3$; $\nu_{\text{soil}} = 0.33$; $E_{\text{soil}} = 89218\ \text{kN/m}^2$
- $\rho_{\text{conc}} = 2.5\ \text{t/m}^3$; $\nu_{\text{conc}} = 0.15$; $E_{\text{conc}} = 2 \times 10^7\ \text{kN/m}^2$

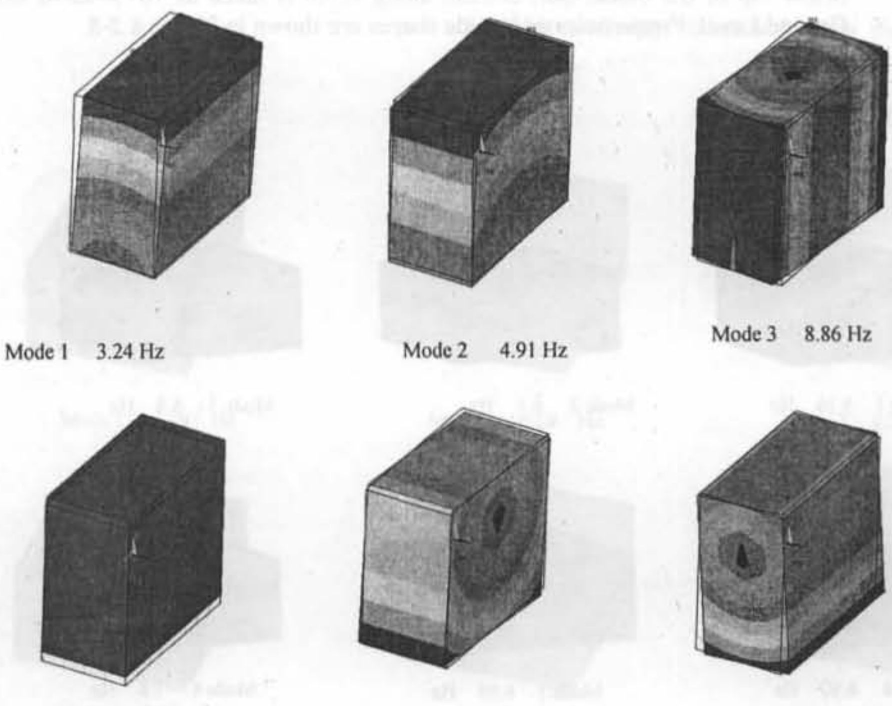


Figure 8.2-6 Frequencies and Mode shapes – Soil represented by 3 Equivalent translational springs at each node at foundation base

Case-1 Soil represented by a set of six springs attached at the CG of Base of Foundation. Frequencies and mode shapes are shown in Figure 8.2-5.

Case-2 Soil represented by a set of three springs attached at each node in contact with soil at Foundation base of Foundation. In total 45 nodes are considered in contact with the soil. Translational stiffness at each node is therefore (1/45) of k_x, k_y, k_z as given above. Frequencies and mode shapes are shown in Figure 8.2-6.

Case-3 Soil represented as continuum below Foundation base level i.e. un-embedded foundation. Soil domain considered is 10 m on all the five sides of the foundation. Frequencies and mode shapes are shown in Figure 8.2-7.

Case-4 Soil represented as continuum right from ground level all around the foundation i.e. embedded foundation. Here again, soil domain considered is 10 m on all the four sides (in plan) of the foundation. Ground Level is considered at 0.75 m below top of the block. Soil domain along depth is taken as 10+3=13 m from Ground Level. Frequencies and mode shapes are shown in Figure 8.2-8.

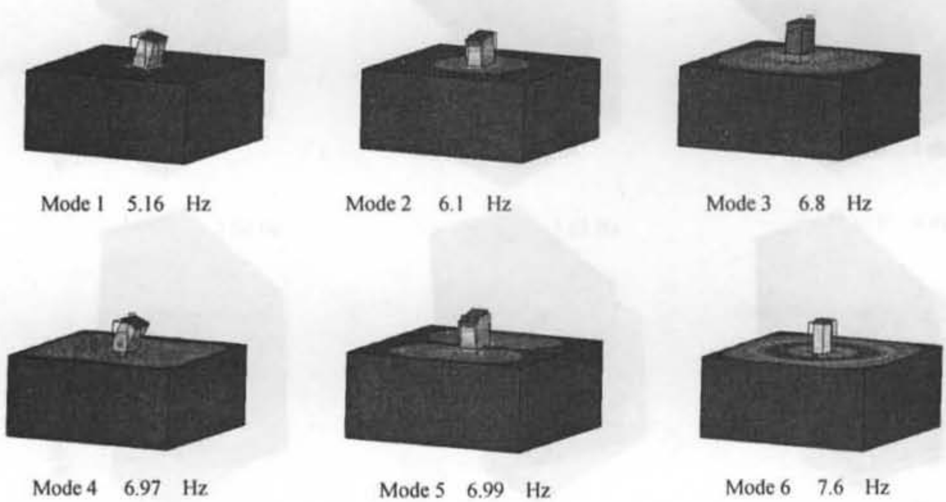


Figure 8.2-7 Frequencies and mode shapes - Soil represented as continuum below foundation base up to 5 times the width all around as well as along depth

Frequencies and mode shapes are listed in **Table 8.2-1**

Table 8.2-1 **Mode shapes & Frequencies - Frequencies in Hz**

	Soil Representation Type	Pridominant Mode Direction						Figure Reference
		x	y	z	θ	ψ	ϕ	
1	Soil represented by 6 spring (3 linear & 3 rotational) @CG of foundation base	15.4	10.89	15.39	5.63	9.44	3.62	Fig 8.2-5
2	Soil represented by 3 equivalent linear spring @each node @ foundation base	15.2	10.99	14.2	4.91	8.86	3.24	Fig 8.2-6
3	Soil Continuum - Foundation considered as unembedded	5.16*	6.8*	6.1*	6.99*	7.6*	6.97*	Fig 8.2-7
4	Soil Continuum - Foundation considered as embedded	6.52	5.96*	6.29	7.03*	7.3	7.23	Fig 8.2-8

* Modes not clearly identifiable from the figure

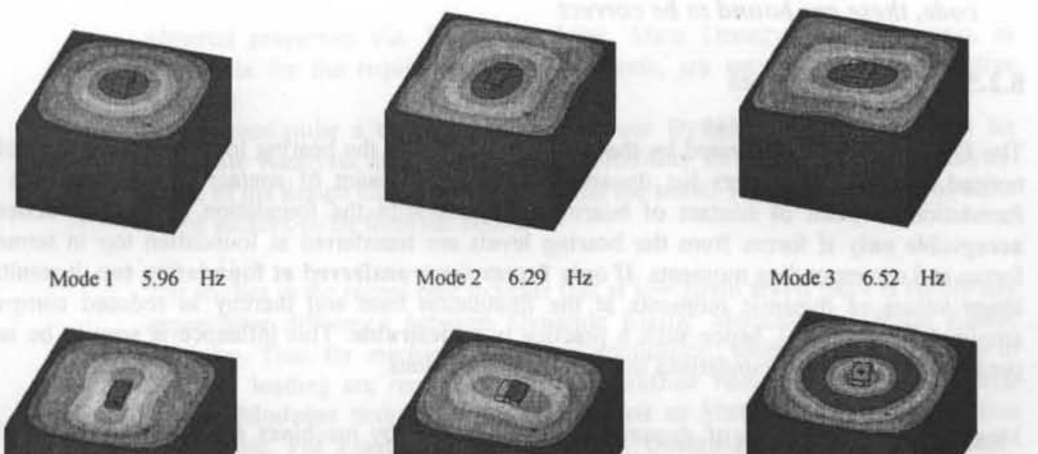


Figure 8.2-8 Frequencies and mode shapes - Soil represented as continuum from Ground Level all around the foundation as well as along depth up to 5 times the width of the foundation

The comparison reveals interesting observations that are as under:

i. Translational mode frequencies for case 3 & 4 i.e. when soil is considered as continuum are much lower than those obtained for case 1 & 2
ii. Variation in rotational frequencies of case 3 &4 is also significant compared to those of case 1 & 2
iii. For case 2, both linear as well as rotational frequencies are marginally lower than those for case 1

Author' observation, based on field measured data, is that measured frequencies are close to those obtained by soil models as in case 1 & 2. For Block foundations, since soil flexibility is a controlling parameter that governs its response, author recommends modeling of soil as in case 1 & 2 only. In view of the above observations, modeling of soil as continuum is NOT RECOMMENDED for Block Foundations., Designers, however, may take their own decisions on need basis.

Whichever modeling criteria is finally chosen by the designer, it is strongly recommended that validation of FE results with manual computation must be done for very simple problem using same modeling criteria, before the modeling criteria is adopted for actual design. Such a caution is essential as often one tends to feel that whatsoever results are obtained by using computer code, these are bound to be correct

8.2.5 Dynamic Forces

The Dynamic Forces generated by the rotor are applied at the bearing location points. It is often noticed that many suppliers list dynamic forces at the point of contact of machine with the foundation or point of contact of bearing pedestals with the foundation. **Such a practice is acceptable only** if forces from the bearing levels are transferred at foundation top in terms of forces and corresponding moments. **If only forces are transferred at foundation top,** it results in lower values of dynamic moments at the foundation base and thereby in reduced computed amplitudes than actual, hence such a practice is **undesirable**. This influence is seen to be more predominant in Block Foundations than Frame Foundations.

Magnitude and directions of dynamic forces generated by machines are discussed in detail in Chapter 6. Though, it is possible to consider more than one set of the dynamic forces simultaneously (if applicable), it would be desirable to consider these forces independently and total response is obtained by finally adding the response of each case.

8.2.6 Boundary Conditions

Having finalized machine and foundation mass data, the first six modes of a block foundation depend primarily on stiffness properties of soil. It is therefore essential to choose and apply right boundary conditions to the FE Model to represent near to realistic situation. One needs to focus at the following:

i) Support conditions at interface between Foundation and Soil
ii) Support conditions at Soil terminating boundary

Interface between Foundation and Soil: In case soil is represented by **equivalent springs** (Case 1 & 2 above) applied at the base of the foundation, one end of each spring is connected to foundation and the other end is restrained. In case soil is modeled as a **continuum** i.e. a finite soil domain, the displacement restraints are recommended as under:

a) It is recommended to apply roller boundary conditions at all the nodes on the terminating boundary of soil domain.

b) At the vertical interface between soil and foundation, it is advisable to apply roller boundary conditions at all the nodes of soil domain. This is due to the consideration that a loss of contact develops at the vertical interface between foundation and soil during course of operation of machine.

c) All nodes at the interface of foundation to the soil at the base should be merged.

8.2.7 Material Data

i) **Foundation:**

a. Material properties viz. Elastic Modulus, Mass Density, Poisson's ratio, as applicable for the required grade of concrete, are assigned to the respective elements.
b. It has been quite a common practice to use **Dynamic Elastic Modulus** for **Dynamic Analysis** and **Static Elastic Modulus** for Static/Strength Analysis. Some of the design Codes also recommend the same. However opinions of some of the authors differ from the above.

The studies in the recent past indicate that at low strain levels there is hardly any appreciable difference between Dynamic Elastic Modulus and Static Elastic Modulus. Thus for machine foundation application, where strain levels due to dynamic loading are reasonably low, the **author recommends use of Static Elastic Modulus** both for **Dynamic as well as Static Analyses** of Machine Foundation. For **Elastic Modulus** refer to **"Design Foundation Parameters"** given in Chapter 7.

ii) **Soil:**

Values of Shear Modulus / Elastic Modulus, Poisson's Ratio and Mass Density, assigned to soil media, should be in accordance with the corresponding soil report. Same values should be used to compute soil springs. General values (only for reference purposes) are given in Chapter 5 **"Design Sub-grade Parameters"**.

8.2.8 Degree of Freedom - Incompatibility:

Such a problem is common in FE modeling of physical systems and arises when two elements having different DOF per node are attached at a particular node. If the corrective action is not taken, it will end up in giving wrong results of n^{th} order.

Foundation Block is modeled using Solid Elements that has **3 DOF** per node whereas Boundary Elements (springs) representing soil stiffness have **6 DOF** per node. Once a Boundary Element is attached at a node to the foundation block modeled by solid elements, it causes **Degree of Freedom Incompatibility** at that particular node.

Similar situation occurs at the interface of i) beam elements and solid elements ii) rigid links & solid elements etc. It is not only desirable but necessary to resolve this issue at the modeling stage itself otherwise the resulting information may be highly misleading. The process of resolving this issue is package dependent and should be undertaken in line with the provisions in the respective CAE package.

8.3 DYNAMIC ANALYSIS

Dynamic analysis includes evaluation of:

- **Free Vibration Response**
- **Forced Vibration Response**

8.3.1 Free Vibration Response

It is recommended to review free vibration results before proceeding for forced response. Review of natural frequencies and associated mode shapes provides understanding of the likely behaviour of the foundation. At times, erroneous results may be encountered, on one count or the other while evaluating dynamic response and the entire sequence of results may become misleading. In view of this, author **strongly recommends** the following:

a) Before attempting dynamic analysis, it is desirable to conduct static analyses subjected to 1 g acceleration load (g stands for acceleration due to gravity) in each X, Y and Z direction. In other words, the machine foundation system is analyzed for self weight alone (self weight of machine & foundation) acting in X, Y or Z direction (one at a time).

b) After ensuring that the displacements due to 1 g static load are of acceptable order (i.e. as expected), one should proceed with the free vibration analysis.

c) In case free vibration results show a pattern that does not appear to be logical, it is an indication to precisely review the mathematical model and make necessary amendments and repeat steps as above.

8.3.1.1 Frequencies & Mode Shapes

It is frequently asked question as to how many frequencies are required to be extracted for a foundation. The general criterion is that highest frequency evaluated should be at least 20% higher than operating speed. However, it primarily depends upon foundation type and range of operating frequency.

Block Foundation: Not withstanding the above, evaluation of frequencies corresponding to first 6-Modes is considered good enough both for **Over-tuned as well as Under-tuned Block Foundations.** First six modes of vibration pertain to rigid body modes of the block and higher modes correspond to flexural deformation of the block.

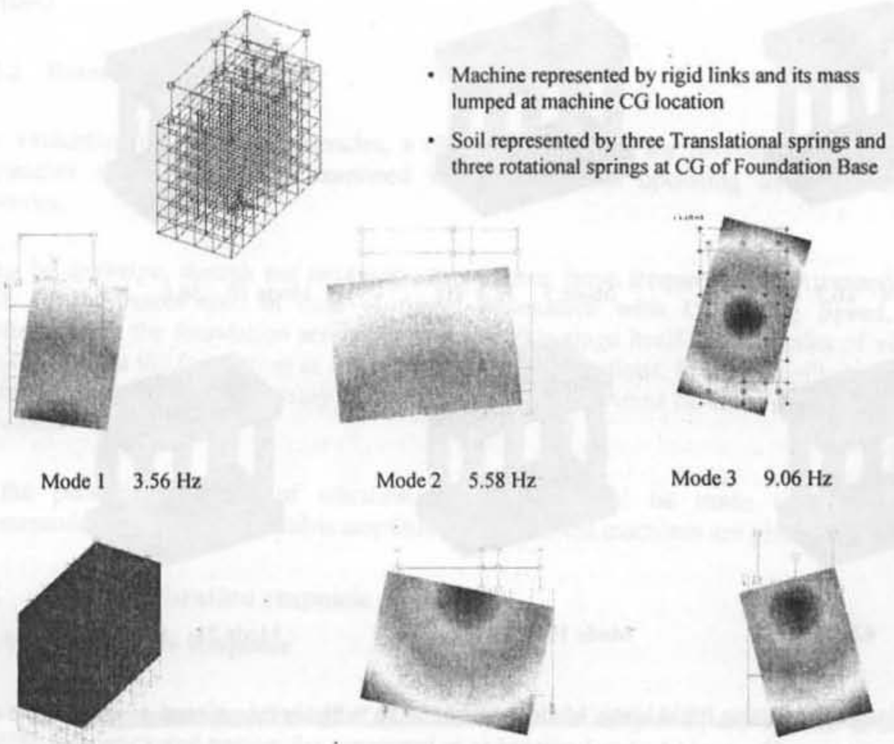
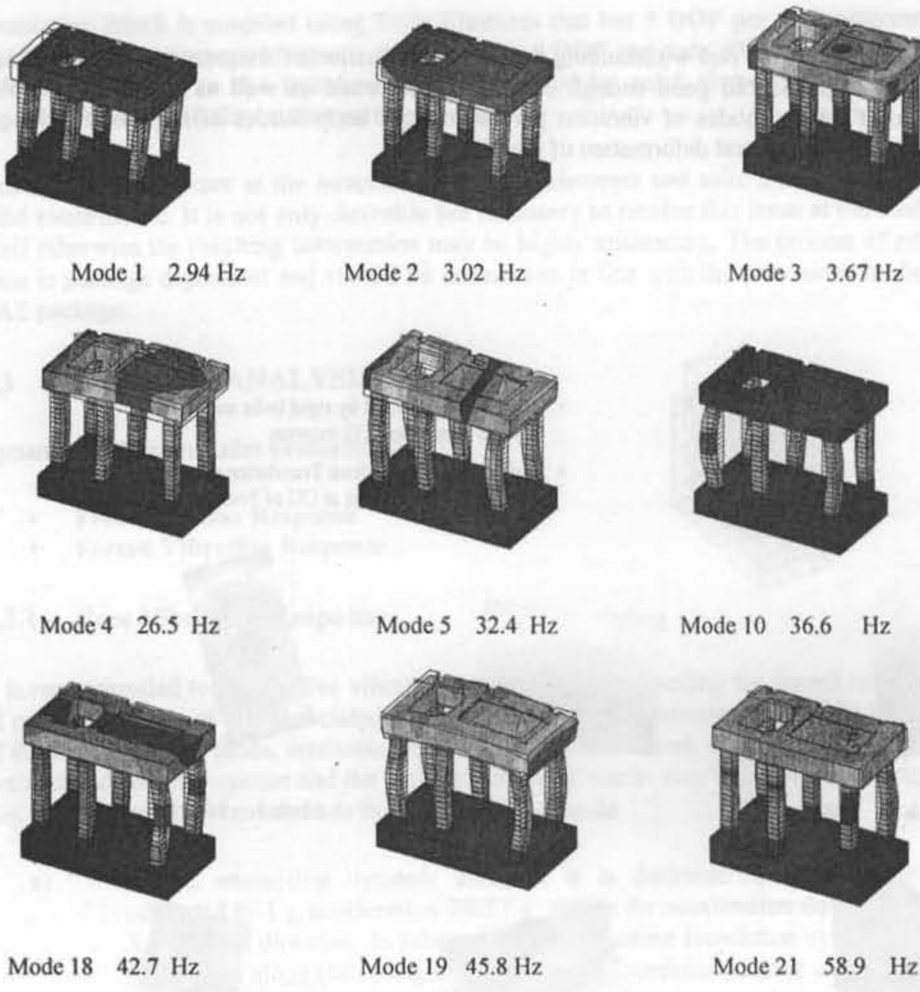


Figure 8.3-1 Frequencies and Mode shapes of a typical Block Foundation

The contribution of higher modes is observed to be practically insignificant except in specific cases such as extended cantilever projections or similar other structural elements where frequency corresponding to elastic deformation mode may lie near to operating frequency. Mode Shapes of a typical foundation block, where soil is represented by a set of equivalent springs is shown in Figure 8.3-1



Machine modeled using Rigid Links. Mode shapes up to 20 % higher than operating speed

Figure 8.3-2 Mode shapes and frequencies of a typical Frame Foundation

Frame Foundation: The number of frequencies to be extracted must follow the criteria as above i.e. the highest extracted frequency should be 20 % more than operating speed. Mode Shapes of a typical frame foundation, considering base raft as fixed at its base are shown in Figure 8.3-2

Observations from mode shapes: Unlike block foundation, study of mode shapes, generally reveals quite interesting information. Let us try to make certain observations from the study of made shapes for the typical foundation, as shown in Fig 4.2-2.

- The uniform colour of top deck in mode 1 (along Z) and mode 2 (along X) are indicative that center of stiffness and center of mass are nearly coincident.
- On the other hand if center of mass and stiffness are not coincident, the colour of top deck deformation would show a mixture of colour depending upon magnitude of displacement. This calls for modification in column stiffness at this stage itself.

There are many more meaningful observations that can be made once all mode shapes are examined

8.3.1.2 Resonance Check

After evaluation of natural frequencies, a check is made for the resonance. Evaluated Natural Frequencies must be critically examined vis-à-vis machine operating frequency and also its harmonics.

It may be desirable, though not necessary, to examine these frequencies with respect to critical speeds of the rotors too. In case of direct **Resonance with Operating Speed**, necessary modifications in the foundation are implemented at this stage itself. Amplitudes of vibration are then computed at the foundation as well as at the bearing locations. In case amplitudes are found to be more than permissible, necessary change in the foundation must be made and the entire analysis is repeated.

For the permissible limits of vibration, reference should be made to applicable codes. Recommended values of permissible amplitudes for different machines are given in Chapter 7.

8.3.2 Forced Vibration response

8.3.2.1 Steady State Response

Dynamic Forces at bearing levels (for dynamic forces see chapter 6) are applied at steady state operating frequency and amplitudes computed at a) bearing levels, b) base of bearing pedestals and c) salient locations at top of the foundation and d) for frame foundation at $1/3^{rd}$ height of the column, mid height of the column and at column top. The forces are applied simultaneously at all bearing levels or these could also be applied one bearing at a time and amplitudes summed up using SRSS (Square Root of Sum of Squares).

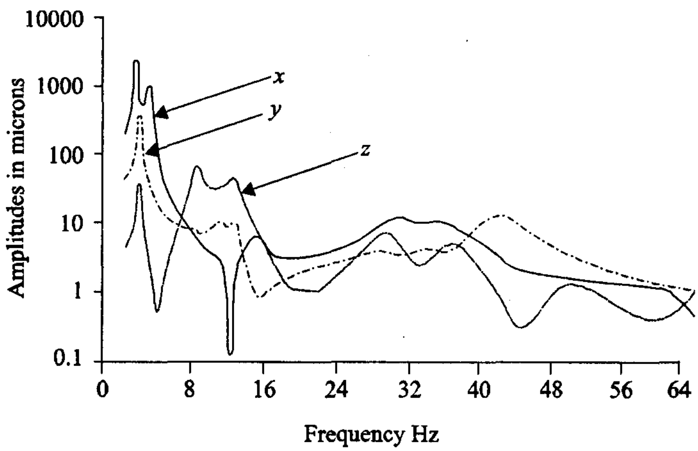


Figure 8.3-3 Transient Response Amplitudes of a Typical Machine

8.3.2.2 Transient Vibration Response

During every start-up and shut down operation of the machine, resonance is noticed at all the system frequencies below operating speed of the machine. This results in sudden rise of amplitude at each frequency of the system and thereafter amplitude diminishes.

It is noteworthy that dynamic force no longer remains constant and changes with change in speed (See chapter 6). It is therefore essential to evaluate the transient resonance amplitudes at bearing levels as well as at other points of interest. Typical plot showing transient amplitudes vs. frequency is shown in Figure 8.3-3

8.4 STRENGTH ANALYSIS & DESIGN

Having ensured that amplitudes are within permissible limits, foundation is checked for strength and stability. Invariably same model, as developed for dynamic analysis, is used for strength analysis. Necessary changes are made for Elastic Modulus properties (i.e. Dynamic Elastic Modulus is changed to Static Elastic Modulus, if considered different).

Stresses in the foundation are computed for all possible load cases including loads due to **Abnormal/Faulted Conditions** (See Chapter 6).

8.4.1 Block Foundations

Block foundations are rigid and have in-built adequate strength to withstand all kinds of forces right from normal operating loads to abnormal loads.

Thus these foundations need not be checked against strength. However anchorage length of holding down bolts either anchored or in pockets, due to earthquake loads, short circuit loads and bearing failure loads must be checked. In addition any cantilever projection supporting equipment mass must also be checked for its strength adequacy.

8.4.2 Frame Foundations

In general, von-mises stresses are reviewed and compared with allowable stresses. However for specific structural components, like cantilever projections, bending & shear stresses may also be reviewed.

Necessary reinforcement is provided in the foundation for the developed Forces and Bending Moments as per applicable codes. For Elastic Modulus properties & allowable stresses, refer Chapter 7. For a typical foundation, stresses developed due to Earthquake Force and Bearing Failure Loads are shown in Figure 8.4-1.

Figure 8.4-1 Frame Foundation – Stresses (mPa) due to Short Circuit Forces, Earthquake force and Bearing failure Loads

EXAMPLE PROBLEMS

P 8.1-1

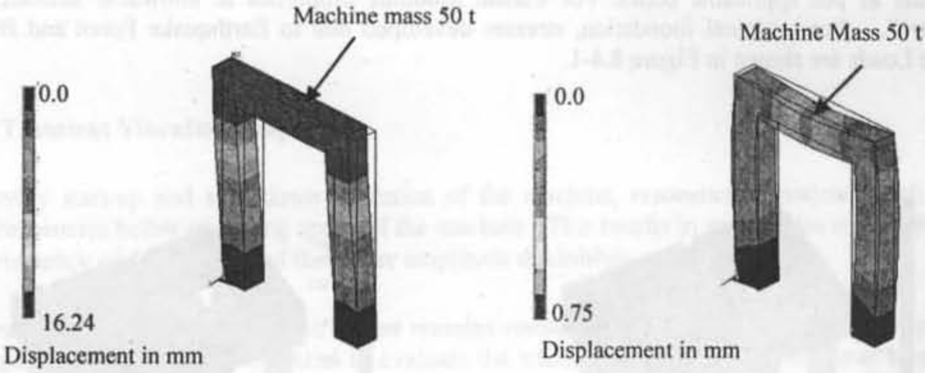
Consider a concrete portal frame of Problem 7.9-1 (as shown in Figure P 7.9-1) using FE Analysis. Frame data is reproduced here.

Portal span	**6.0 m**
Portal height	**10.0 m**
Column	**1.0 m x 1.2 m**
Portal beam	**1.0 m wide x 1.5 m deep**
Machine of mass	**50 t supported at the center of the beam**

Portal is restrained to move only in X-Y plane.

$E_{conc} = 2\times10^7$ kN/m^2 ; $G = 8.7\times10^6$ kN/m^2 **& Mass density** $\rho = 2.5$ t/m^3

Find lateral natural frequencies of the system ignoring Shear Deformations



Static Load 1g along X (Lateral) Static Load 1g along Y (Vertical)

Figure P 8.1-1a Deflection under 1g X force and 1g Y Force

Solution: **FE Model**

- Portal Frame is modeled using Brick Elements
- All nodes at column base are constrained i.e. (fixed) in X, Y & Z.
- All nodes on one face of frame constrained in Z i.e. Frame is allowed to move only in X-Y Plane
- Machine mass located at beam center as lumped mass element

- Shear Deformation not included
- Static Force equal to 1g applied along X & Y (one at a time) – Deflection shown in Figure 8.1-1a
- Solved for Eigenvalue – Mode shapes and Natural Frequencies shown in Figure 8.1-1b

It is recommended to always carry out equivalent static analysis by applying 1g force (here g stands for gravity loading) along X & Y directions (one at a time) to rule out possibility of making slips while modeling/analysis. It is noticed that maximum displacement by 1g load along X is 16.24 mm and for 1 g load along Y it is 0.75 mm. These static deflections correspond to:

$$f_x = \frac{1}{2\pi} \times \sqrt{\frac{g}{\delta_x}} = \frac{1}{2\pi} \times \sqrt{\frac{9810}{16.24}} = 3.91 \text{ Hz}$$

$$f_y = \frac{1}{2\pi} \times \sqrt{\frac{g}{\delta_y}} = \frac{1}{2\pi} \times \sqrt{\frac{9810}{0.75}} = 18.20 \text{ Hz}$$

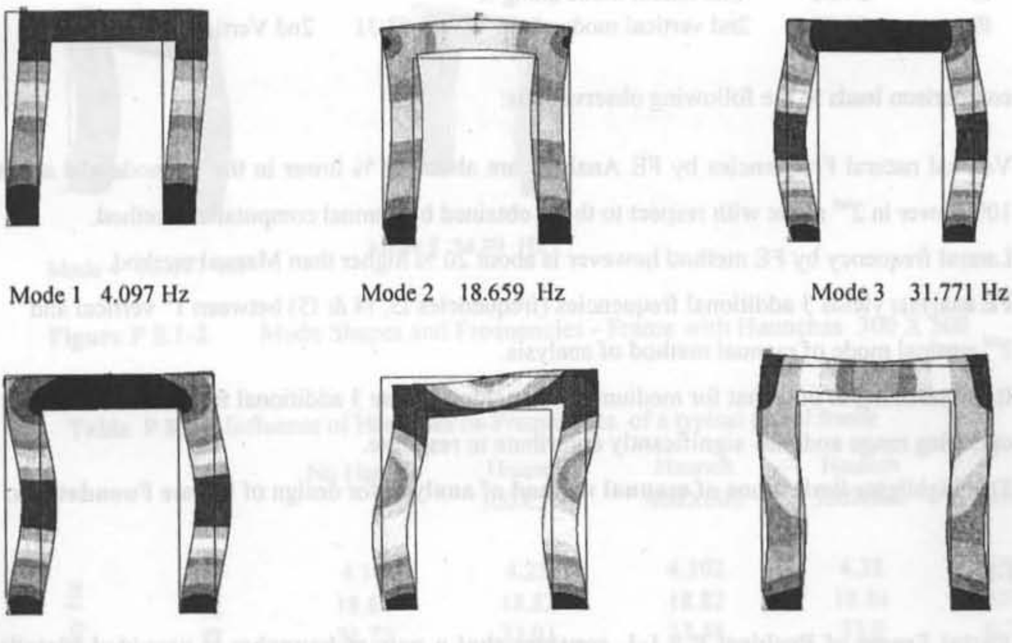


Figure P 8.1-1b Mode Shapes and Frequencies - Portal Frame –Beam Rectangular Cross-section

From Figure P 8.1-1b, we get frequency corresponding to 1st lateral mode (along X) as 4.1 Hz and frequency corresponding to 1st vertical mode is 18.66 Hz. This kind of a check confirms correctness of modeling in broader sense.

For academic interest let us compare the results with those obtained with the manual solution (see Problem P 7.9-1). Comparison of frequencies is shown in **Table P 8.1-1.**

Table P 8.1-1 Comparison of Frequencies - FE Analysis with Manual Computational Method (see Problem P 7.9-1)

Frequency in Hz

	FE Analysis		Manual Method	
Mode #	Frequency	Mode	Frequency	Mode
f1	4.097	1st Lateral mode along X	3.45	Lateral along X
f2	18.66	1st Vertical mode along Y	22.76	1st Vertical along Y
f3	31.77	Column 1st mode along X		
f4	31.99	Column 1st mode along X		
f5	54.86	2nd lateral mode along X		
f6	66.32	2nd vertical mode along Y	73.31	2nd Vertical along Y

The comparison leads to the following observations:

1. Vertical natural Frequencies by FE Analysis are about 20 % lower in the 1st mode and about 10% lower in 2nd mode with respect to those obtained by manual computation method.
2. Lateral frequency by FE method however is about 20 % higher than Manual method.
3. FE analysis yields 3 additional frequencies (frequencies f3, f4 & f5) between 1st vertical and 2nd vertical mode of manual method of analysis.
4. It is interesting to note that for medium rpm machines, these 3 additional frequencies lie within operating range and may significantly contribute to response.
5. This highlights **limitations of manual method of analysis** for design of **Frame Foundations.**

P 8.1-2

For Portal Frame of Problem P 8.1-1, consider that a pair of haunches is provided (details given below). Find lateral natural frequencies of the system.

- **i) Haunch size 300 X 500 mm**
- **ii) Haunch size 400 X 600 mm**
- **iii) Haunch size 500 X 800 mm**

Solution:

Modeling is done on the similar lines as for Problem P 7.9-1. Mode shapes and frequencies for haunch size of 300 X 500 mm are shown in Figure P 8.1-2. Frequencies for other haunch sizes are given in Table P 8.1-2. Just for academic interest, frequencies for plane frame without haunch are also listed in the Table.

Mode 1 4.252 Hz

Mode 2 18.822 Hz

Mode 3 33.014 Hz

Mode 4 33.097 Hz

Mode 5 54.89 Hz

Figure P 8.1-2 Mode Shapes and Frequencies - Frame with Haunches 300 X 500

Table P 8.1-2 Influence of Haunches on Frequencies of a typical portal frame

		No Haunch	Haunch 300X500	Haunch 400X600	Haunch 500X800	% variation
Frequency Hz	f1	4.10	4.25	4.302	4.38	6.91
	f2	18.66	18.82	18.82	18.84	0.97
	f3	31.77	33.01	33.38	33.9	6.70
	f4	31.99	33.1	33.46	33.97	6.19
	f5	54.86	54.89	54.78	54.65	0.38
	f6	66.32	66.23	66.112	65.77	0.83

From the results following salient observations are made:

1. Variation in frequency is of the order of 7 % and 0.4 % in 1st lateral and 2nd lateral mode (mode 1 & mode 5) respectively.
2. A marginal difference of the order of 6 to 7% is however noticed in column lateral frequencies (mode 3 & 4) .

Hence for all manual method of analysis, it may be reasonably OK to ignore haunches and in FE analysis, these will get taken care of automatically.

9

FOUNDATIONS FOR ROTARY MACHINES

- **Design Examples**

  - Block Foundation for a Typical Rotary Machine
  - Frame Foundation for Turbo Generator

For better clarity, all Figures related to FE analysis, including animations of frequencies and mode shapes, in color, are given in the CD attached at the end of the handbook

FOUNDATIONS FOR ROTARY MACHINES

Various types of machines that come under this category have been adequately addressed in Chapter 6. Both Block type foundations as well as Frame Type foundations, normally used to support such machines, have been covered in Chapter 7. Modeling aspects have adequately been covered in Chapter 8. Case studies on various machines and foundations, covering field measurements, failure studies, remedial measures and site feedback are reported in Chapter 14.

A real life machine foundation system is a 3-D system. Machine is supported by the structure/foundation which in turn rests directly on the soil or through piles. The complete system is mathematically modeled and analyzed. Machine generates Dynamic forces in a plane perpendicular to axis of rotation. The system vibrates in all six DOFs and thus requires computation of frequencies and amplitudes corresponding to all six DOF's. **Design Procedures** for a) **Block Foundation** and b) **Frame Foundations** are given hereunder. The application of these design methodic for evaluation of natural frequencies and amplitudes are common for all types of machines irrespective of their speed.

9.1 DESIGN OF BLOCK FOUNDATION

Machine is considered supported by a block foundation resting directly over soil. The complete system is mathematically modeled and analyzed for natural frequencies and amplitudes.

Summary of Design Steps

1. Sizing of Foundation
2. Equivalent Soil Stiffness
3. Dynamic Forces
4. Analysis

I. Dynamic Analysis
   a. Natural Frequencies
   b. Dynamic Amplitudes
      - Steady State Amplitudes
      - Transient Amplitudes

II. Strength and Stability Analysis

a. Equivalent Static Forces (Normal Operating Conditions)
b. Bearing Failure Loads (Abnormal conditions)
c. Handling loads
d. Short Circuit Loads
e. Environmental Loads e.g. Earthquake Loads, Wind Loads etc.
f. Thermal Loads (if any)

Required Input Data

a) Foundation Data
    i) Foundation outline geometry, Levels etc
    ii) Cut-outs, pockets, trenches, notches, projections etc
b) Machine Data
    i) Machine Layout
    ii) Machine Load Distribution at Load Points
    iii) Machine Dynamic Loads
        a. Magnitude of Dynamic Loads
        b. Location of application
        c. Associated excitation Frequencies
    iv) Other Loads like Short Circuit Torque, Bearing Failure Loads etc.
    v) Allowable Amplitudes at Bearing Locations
c) Soil Data
    i) Site Specific Dynamic Soil Data
    ii) Soil type and its basic characteristic properties
    iii) Bearing capacity
    iv) Depth of water table
    v) Liquefaction potential
d) Environmental Data
    i) Site related Seismic data
    ii) Wind Load Data

At this stage it is implied that:

- **Site Soil data** is converted to **Design Sub-grade Parameters** duly accounting for affects of overburden pressure and area in line with provisions given in Chapter 5.
- **Machine data** is converted to **Design Machine Parameters** in line with provisions given in **Chapter 6**
- **Foundation data** is converted to **Design Foundation Parameters** in line with provisions given in **Chapter 7**
- Intricacies of **Modeling and Analysis,** as given in Chapter 8, have been well understood

The mathematical representation of a typical foundation is shown in Figure 9.1-1. Here point O represents CG of Base Area of Foundation in contact with soil. **This point is also termed as Degree of Freedom (DOF) Location.**

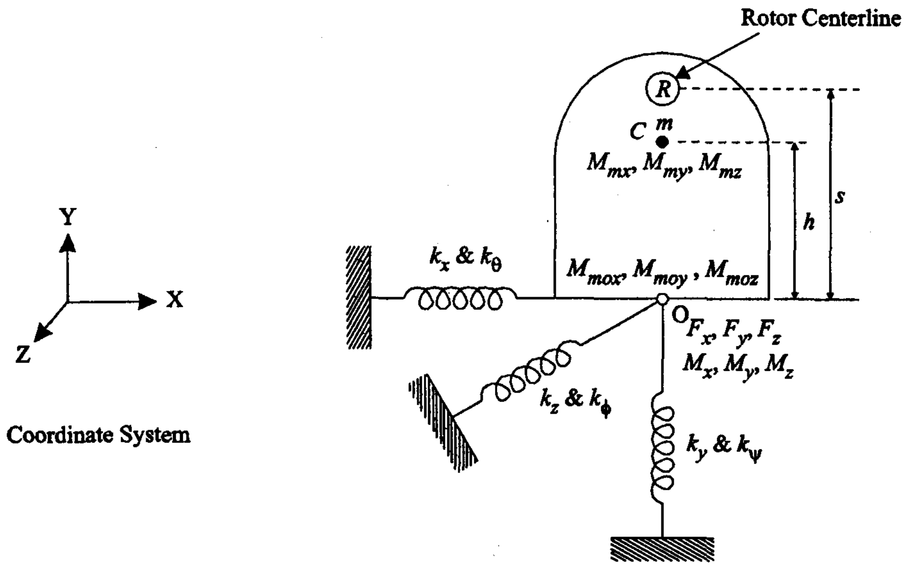


Figure 9.1-1 Mathematical Model of a Typical Block Foundation

Design Data: The design data at this stage is summarized as under:

Mass

Total Mass of Machine and Foundation	m
Height of Overall Centroid C from O	h

Mass Moment of Inertia (Machine+ Foundation) @ Overall Centroid C

Mass Moment of Inertia about X axis	M_{mx}
Mass Moment of Inertia about Y axis	M_{my}
Mass Moment of Inertia about Z axis	M_{mz}

Mass Moment of Inertia (Machine+ Foundation) @ DOF Location O

Mass Moment of Inertia about X axis	M_{mox}
Mass Moment of Inertia about Y axis	M_{moy}

Mass Moment of Inertia about Z axis M_{moz}

Area and Moment of Inertia of Foundation Base in contact with soil

Area of Foundation A

Moment of Inertia about X I_{xx}

Moment of Inertia about Y I_{yy}

Moment of Inertia about Z I_{zz}

Equivalent Soil Stiffness at the foundation base level (at DOF location point O) duly corrected for a) area effect and b) overburden pressure effect

Translational Soil Stiffness along X k_x

Translational Soil Stiffness along Y k_y

Translational Soil Stiffness along Z k_z

Rotational Soil Stiffness about X k_θ

Rotational Soil Stiffness about Y k_ψ

Rotational Soil Stiffness about Z k_ϕ

Operating Frequency/Frequencies of machine ω_1, ω_2 etc

Dynamic Loads

- For FE Analysis, Dynamic Forces need to be specified only at respective bearing locations.
- For manual method of computation, Dynamic Forces acting at bearing locations are transferred at DOF Location point O in terms of Forces and Moments.
- One can have as many sets of forces and moments as number of excitation frequencies

Here we describe forces and moments @ DOF location **point O for manual method of computation.**

Forces @ DOF location **point** O along X, Y & Z direction F_x , F_y & F_z

Moments about X, Y & Z @ DOF location **point** O M_x , M_y & M_z

9.1.1 Dynamic Analysis

The dynamic analysis of a machine foundation system involves computation of natural frequencies and amplitudes of vibration.

From this stage onwards, one can choose either Finite Element Method of Analysis (Chapter 8) or Manual Method of Analysis (Chapters 2 &3).

Natural Frequencies: The machine foundation system undergoes Six Modes of Vibration i.e. three Translational Modes and three Rotational Modes (see chapter 3). Natural frequencies corresponding to these six modes of vibration are reproduced as under:

1. Motion along Y (Vertical direction): This vibration mode is always uncoupled. We get Natural frequency corresponding to Vertical Mode of Vibration (along Y) – (see equation 3.1.2-6):

Vertical Natural frequency
$$p_y = \sqrt{\frac{k_y}{m}} \tag{9.1-1}$$

2. Motion about Y (Torsional): This vibration mode is also uncoupled. We get Natural frequency corresponding to Torsional Mode of Vibration (about Y) as (see equation 3.4.2-8):

Torsional Natural frequency
$$p_\psi = \sqrt{\frac{k_\psi}{M_{moy}}} \tag{9.1-2}$$

3. Motion in X-Y Plane - (Translation along X and Rocking about Z - $x \& \phi$ modes) - This vibration mode is always coupled (see 3.3.2-8c). We get natural frequencies as:

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2} \tag{9.1-3}$$

$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2} \tag{9.1-4}$$

Here $\gamma_z = \dfrac{M_{mz}}{M_{moz}}$; $p_x^2 = \dfrac{k_x}{m}$; $p_\phi^2 = \dfrac{k_\phi}{M_{moz}}$

4. Motion in Y-Z Plane – (Translation along Z and Rocking about X - $z\&\theta$ modes) - This vibration mode is always coupled. On the similar lines, as for the motion in X-Y plane given in (3) above, we get frequency as:

$$p_1^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) - \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x\, p_z^2 p_\theta^2} \qquad (9.1\text{-}5)$$

$$p_2^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) + \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x\, p_z^2 p_\theta^2} \qquad (9.1\text{-}6)$$

Here $\gamma_x = \frac{M_{mx}}{M_{mox}}$; $p_z^2 = \frac{k_z}{m}$; $p_\theta^2 = \frac{k_\theta}{M_{mox}}$

As far as possible, effort is made to ensure that these frequencies are not in direct resonance with operating speed/speeds of the machine. In fact these frequencies should preferably be away by a margin of $\pm 20\%$ from operating speed/speeds. In case resonance is noticed, it may be desirable to suitably alter the foundation dimensions and repeat the computations till the natural frequencies are found to be away from operating speed/speeds of the machine.

9.1.2 Amplitudes of Vibration

Vibration Amplitude is the response of the Machine Foundation System subjected to unbalance force acting on the machine. When the natural frequencies are in resonance with excitation frequency, damping plays a significant role and amplitudes need to be computed considering system with damping. However, when natural frequencies are not in resonance with operating speed, the damping has hardly any influence on the response and it is good enough to compute amplitudes for undamped conditions.

Response Computation using FE Analysis: For response computation, these unbalance forces are applied directly at the bearing level locations. Amplitudes at desired locations viz. Foundation top or bearing levels are obtained directly.

Response Computation using Manual Methods of Analysis: While evaluating response using manual method of analysis, these unbalance forces are transferred at the DOF location (CG of base area of foundation in contact with the soil i.e. **point** O). Thus we get three force components $F_x, F_y \& F_z$ and three moment components $M_\theta, M_\psi \& M_\phi$ @ **point** O. Amplitudes are evaluated at DOF location **point** O. Amplitudes at any other location viz. at foundation top or at bearing locations are computed using geometrical relationships. Amplitudes at DOF **point** O are reproduced as under:

9.1.2.1 Amplitudes (undamped)

Motion along Y (vertical) and motion about Y (Torsional): For Uncoupled Modes i.e. Vertical motion along Y and torsional motion about Y, amplitudes are given by equations 3.4.2-5 & 3.4.2-8 and these are reproduced here as under:

i) Applied dynamic force $F_y \sin \omega t$

Amplitude $$y_o = \delta_y \frac{1}{\left|\left(1-\beta_y^2\right)\right|} \quad (9.1\text{-}7)$$

Here $\delta_y = \frac{F_y}{k_y}$; $\beta_y = \frac{\omega}{p_y}$ & $p_y = \sqrt{k_y/m}$

ii) Applied dynamic moment $M_\psi \sin \omega t$

Amplitude $$\psi_o = \delta_\psi \frac{1}{\left|\left(1-\beta_\psi^2\right)\right|} \quad (9.1\text{-}8)$$

Here $\delta_\psi = \frac{M_\psi}{k_\psi}; \beta_\psi = \frac{\omega}{p_\psi}$ & $p_\psi = \sqrt{k_\psi / M_{moy}}$

Absolute value of magnification factor is considered in above equations so as to get positive value of amplitudes. This is done only for uncoupled modes i.e. motion along Y and motion about Y.

Motion in X-Y plane: For coupled modes i.e. translation along X and rocking about Z, considering one force at a time, we get:

iii) Applied dynamic force $F_x \sin \omega t$

Amplitudes
$$x_o = \delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$
$$\phi_o = -\delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \quad (9.1\text{-}9)$$

iv) Applied dynamic moment $M_\phi \sin \omega t$

Amplitude
$$x_o = -h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$
$$\phi_o = \delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (9.1\text{-}10)$$

Here $\delta_x = \frac{F_x}{k_x}$ & $\delta_\phi = \frac{M_\phi}{k_\phi}$, $\beta_x \,\&\, \beta_\phi$ are frequency ratios corresponding to limiting frequencies $p_x \,\&\, p_\phi$ and $\beta_1 \,\&\, \beta_2$ are frequency ratios corresponding to natural frequencies $p_1 \,\&\, p_2$ (see equations 9.1-3 & 9.1-4 for $p_1 \,\&\, p_2$).

Motion in Y-Z plane: For coupled modes i.e. translation along Z and rocking about X, we get:

v) Applied dynamic force $F_z \sin \omega t$

Amplitude
$$z_o = \delta_z \frac{\left(1-\beta_\theta^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$
$$\theta_o = \delta_z \frac{mh}{M_{mox}} \frac{\beta_\theta^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (9.1\text{-}11)$$

vi) Applied dynamic moment $M_\theta \sin \omega t$

Amplitude
$$z_o = h\delta_\theta \frac{\beta_z^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$
$$\theta_o = \delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)} \qquad (9.1\text{-}12)$$

Here $\delta_z = \frac{F_z}{k_z}$ & $\delta_\theta = \frac{M_\theta}{k_\theta}$; $\beta_z \,\&\, \beta_\theta$ are frequency ratios corresponding to limiting frequencies $p_z \,\&\, p_\theta$ and $\beta_1 \,\&\, \beta_2$ are frequency ratios corresponding to natural frequencies $p_1 \,\&\, p_2$ ($p_1 \,\&\, p_2$ are as given by equations 9.1-5 & 9.1-6).

Note: The amplitudes given by equations 9.1-7 to 9.1-12 are amplitudes @ point O for undamped system (as the mathematical formulation is developed for undamped system).

9.1.2.2 Amplitudes at resonance

Whenever natural frequency corresponding to a specific mode lies within $\pm 20\%$ of normal operating speed of the machine, foundation is considered to be in RESONANCE for that particular mode of vibration. Should such a condition exist, amplitudes are to be computed under damped conditions. Amplitudes under resonance condition @ point O are given in § **3.4.2.1.1** and are reproduced here as under:

Uncoupled Modes:

Motion along Y (vertical) and motion about Y (Torsional):

i) Applied dynamic force $F_y \sin \omega t$ - Resonance in vertical mode

$$y_o = \delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y \zeta\right)^2}} \tag{9.1-13}$$

ii) Applied dynamic moment $M_\psi \sin \omega t$ - Resonance in Torsional mode

$$\psi_o = \delta_\psi \frac{1}{\sqrt{\left(1-\beta_\psi^2\right)^2 + \left(2\beta_\psi \zeta\right)^2}} \tag{9.1-14}$$

ζ is damping constant

Coupled Modes:

Motion in X-Y plane: For natural frequencies p_1 & p_2, see equations 9.1-3 & 9.1-4.

iii) Applied dynamic force $F_x \sin \omega t$

a) Resonance with 1st natural frequency p_1 i.e. $0.8 < \beta_1 < 1.2$

$$x_o = \left[\delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(\sqrt{\left(1-\beta_1^2\right)^2 + \left(2\beta_1\zeta\right)^2}\right) \times \left(1-\beta_2^2\right)}\right]; \; \phi_o = \left[-\delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(\sqrt{\left(1-\beta_1^2\right)^2 + \left(2\beta_1\zeta\right)^2}\right) \times \left(1-\beta_2^2\right)}\right]$$

................................(9.1-15a)

Note 1: If term $\left(1-\beta_1^2\right)$ is negative then term $\left(\sqrt{\left(1-\beta_1^2\right)^2 + \left(2\beta_1\zeta\right)^2}\right)$ should also be negative.

Retaining sign is important from the point of view of overall response evaluation which is vector sum of corresponding response quantities.

b) Resonance with 2[nd] natural frequency p_2 i.e. $0.8 < \beta_2 < 1.2$

$$x_o = \left[\delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}\right)\times\left(1-\beta_1^2\right)}\right]; \quad \phi_o = -\left[\delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}\right)\times\left(1-\beta_1^2\right)}\right]$$

.... (9.1-15b)

Note 2: If term $\left(1-\beta_2^2\right)$ is negative then term $\left(\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}\right)$ shall also be negative (see Note 1).

iv) Applied dynamic Moment $M_\phi \sin \omega t$

a) Resonance with 1[st] natural frequency p_1 i.e. $0.8 < \beta_1 < 1.2$

$$x_o = -\left[h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_2^2\right)\times\left(\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}\right)}\right]; \quad \phi_o = \left[\delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_2^2\right)\times\left(\sqrt{\left(1-\beta_1^2\right)^2+\left(2\beta_1\zeta\right)^2}\right)}\right]$$

....................................(9.1-16a)

Note 3: (see Note 1)

b) Resonance with 2[nd] natural frequency p_2 i.e. $0.8 < \beta_2 < 1.2$

$$x_o = -\left[h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_1^2\right)\times\left(\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}\right)}\right]; \quad \phi_o = \left[\delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_1^2\right)\times\left(\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}\right)}\right]$$

..................................(9.1-16b)

Note 4: (see Note 2)

Motion in Y-Z plane: Here we use natural frequencies as given by equations 9.1-5 & 9.1-6.

v) Applied dynamic force $F_z \sin \omega t$

a) Resonance with 1st natural frequency p_1 i.e. $0.8 < \beta_1 < 1.2$

$$z_o = \left[\delta_z \frac{\left(1-\beta_\theta^2\right)}{\left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right) \times \left(1-\beta_2^2\right)} \right]; \; \theta_o = \left[\delta_z \frac{mh}{M_{max}} \frac{\beta_\theta^2}{\left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right) \times \left(1-\beta_2^2\right)} \right]$$

..........................(9.1-17a)

Note 5: (see Note 1)

b) Resonance with 2nd natural frequency p_2 i.e. $0.8 < \beta_2 < 1.2$

$$z_o = \left[\delta_z \frac{\left(1-\beta_\theta^2\right)}{\left(\sqrt{\left(1-\beta_2^2\right)^2 + (2\beta_2\zeta)^2}\right) \times \left(1-\beta_1^2\right)} \right]; \; \theta_o = \left[\delta_z \frac{mh}{M_{max}} \frac{\beta_\theta^2}{\left(\sqrt{\left(1-\beta_2^2\right)^2 + (2\beta_2\zeta)^2}\right) \times \left(1-\beta_1^2\right)} \right]$$

.... (9.1-17b)

Note 6: (see Note 2)

vi) Applied dynamic moment $M_\theta \sin \omega t$

a) Resonance with 1st natural frequency p_1 i.e. $0.8 < \beta_1 < 1.2$

$$z_o = \left[h\delta_\theta \frac{\beta_z^2}{\left(1-\beta_2^2\right) \times \left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right)} \right]; \; \theta_o = \left[\delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_2^2\right) \times \left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right)} \right]$$

.............................(9.1-18a)

Note 7: (see Note 1)

b) Resonance with 2nd natural frequency p_2 i.e. $0.8 < \beta_2 < 1.2$

$$z_o = \left[h\delta_\theta \frac{\beta_z^2}{\left(1-\beta_1^2\right) \times \left(\sqrt{\left(1-\beta_2^2\right)^2 + (2\beta_2\zeta)^2}\right)} \right]; \; \theta_o = \left[\delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right) \times \left(\sqrt{\left(1-\beta_2^2\right)^2 + (2\beta_2\zeta)^2}\right)} \right]$$

.............................(9.1-18b)

Note 8: (see Note 2)

9.1.2.3 Amplitudes at foundation top

The amplitudes $x_o, y_o, z_o, \theta_o, \psi_o \,\&\, \phi_o$ (as obtained above) are at DOF location point O. Let $\bar{x}_f, \bar{y}_f \,\&\, \bar{z}_f$ represent amplitudes at center of Top of Foundation. Amplitudes at foundation top are obtained using law of statics as shown in Figure 9.1-2.

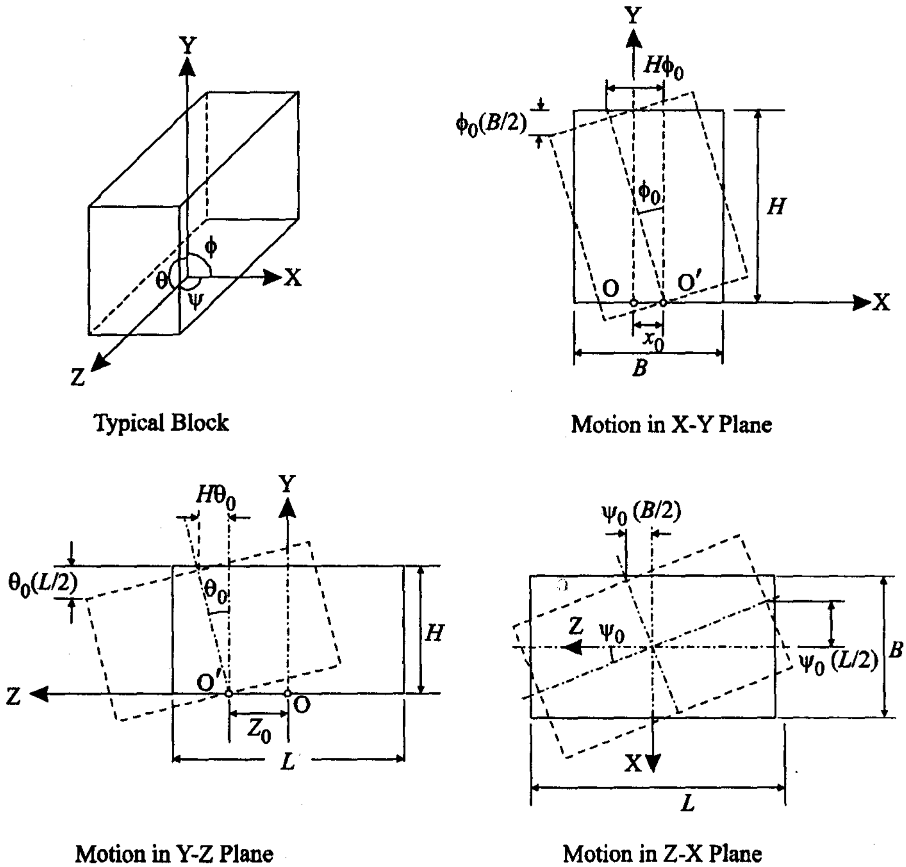


Figure 9.1-2 Amplitude Components at Foundation Top

Amplitudes at foundation top (at center)

Amplitude $\bar{x}_f$ due to $x_0 \,\&\, \phi_0$ $\quad$ $\bar{x}_{f(\max)} = \left|(x_o - H\phi_o)\right|$ (9.1-19)

Amplitude $\bar{y}_f$ due to y_0 $\quad$ $\bar{y}_{f(\max)} = \left|y_o\right|$ (9.1-20)

Amplitude $\bar{z}_f$ due to z_0 & θ_0 $\bar{z}_{f(\max)} = \left|(z_o + H\,\theta_o)\right|$ (9.1-21)

Amplitudes at corners of foundation top

Let x_{fc}, y_{fc} & z_{fc} represent amplitudes at corner of Top of Foundation. Let L & B represent length and width of the foundation along Z and X axes respectively.

Amplitude x_{fc} & z_{fc} due to ψ_0

$$x_{fc(\max)} = \left|(L/2)\psi_o\right|; \qquad z_{fc(\max)} = \left|(B/2)\psi_o\right| \tag{9.1-22}$$

Amplitude y_{fc} due to ϕ_0 & θ_0

$$y_{fc} = \left|(B/2)\phi_o\right|; \qquad y_{fc} = \left|(L/2)\theta_o\right| \tag{9.1-23}$$

$$y_{fc(\max)} = \left|(L/2)\theta_o\right| + \left|(B/2)\phi_o\right| \tag{9.1-24}$$

Maximum amplitude along X

$$x_{f(\max)} = \bar{x}_{f(\max)} + x_{fc(\max)} = \left|(x_o - H\phi_o)\right| + \left|(L/2)\psi_o\right| \tag{9.1-25}$$

Maximum amplitude along Y

$$y_{f(\max)} = \bar{y}_{f(\max)} + \bar{y}_{fc(\max)} = \left|y_o\right| + \left\{\left|(L/2)\theta_o\right| + \left|(B/2)\phi_o\right|\right\} \tag{9.1-26}$$

Maximum amplitude along Z

$$z_{f(\max)} = \bar{z}_{f(\max)} + z_{fc(\max)} = \left|(z_o + H\,\theta_o)\right| + \left|(B/2)\psi_o\right| \tag{9.1-27}$$

Here x_f, y_f & z_f represent amplitudes @ Foundation top. Quantities L, B & H represent Length, Breadth and Height as shown in Figure 9.1-2.

On the similar lines, we can evaluate amplitudes at bearing locations too. It should ultimately be ensured that the amplitudes of vibration are within the allowable values. If analysis shows higher amplitudes, it is essential to redesign the foundation and reanalyze the system till one gets acceptable levels of vibration amplitudes.

DESIGN EXAMPLES

Design Examples are those encountered in real life practice. Comparison with **Finite Element Analysis (FEA)** is also given for specific cases to build up the confidence level. Effort is made to highlight the influence of certain slips commonly committed while computing response of the foundation.

Example D 9.1: Foundation for Low Speed Machine (600 rpm)

Design a Block Foundation for a Rotary Machine set consisting of a Drive machine and a Non-Drive Machine, coupled directly. Foundation outline showing Machine-loading diagram, sectional elevation showing machine CG line, rotor-center line and bearing locations, is given in Figure D 9.1-1. Machine, foundation and soil parameters are as under:

Machine Data:

- Weight of Drive Machine (excluding Rotor) 100 kN
- Weight of the Non-Drive Machine (excluding Rotor) 200 kN
- Weight of Drive Machine Rotor 10 kN
- Weight of Non-Drive Machine Rotor 20 kN
- Bearings: Both the rotors have Pedestal bearings
- Weight of bearing pedestal
  - Drive machine 2 kN (each pedestal)
  - Non-drive machine 4 kN (each pedestal)

Consider CG of bearing pedestals and coupling at rotor Centre Line level.

- Weight of Coupling 6 kN
- Rotor Speed 600 rpm
- Balance Grade for both the rotors $G6.3$
- Height of Rotor Centerline above Ground level 2000 mm
- Height of Machine Centroid below rotor centerline 100 mm

Foundation Data

- Length of Foundation Block 5200 mm
- Width of Foundation 2200 mm
- Height of Foundation block is above ground level
  - Drive end side 1000 mm
  - Non drive end side 200 mm
- Mass Density of concrete $\rho_c = 2500$ kg/m^3

Soil Data

- Mass Density $\rho_s = 1800 \text{ kg/m}^3$
- Poisson's Ratio $\nu = 0.25$
- Damping constant $\zeta = 0.1$
- Site Coefficient of Uniform Compression normalized to 10 m^2 Area

  $C_{u01} = 4.6 \times 10^4 \text{ kN/m}^3$
- Site Static Stress $\bar{\sigma}_{01}$ @ 3.5 m depth 100 kN/m^2
- Net Bearing capacity at 3.5 m depth 250 kN/m^2

Data for Strength Design

- Bearing failure force 5 × rotor weight
- Seismic Coefficient $\alpha_h = 0.20$

Anchor Bolts

- Drive machine 4 # - 20 mm dia bolts - Embedded Length of 300 mm
- Non- drive machine by 4 # - 25 mm dia bolts - Embedded Length of 400 mm

Increase in Allowable stress & soil bearing pressure

- For Earthquake condition 25 %
- For Bearing Failure condition 50 %

SOLUTION:

Machine Data:

Machine layout is shown in Figure D 9.1-2. Drive machine weight is distributed at 4 points @ 25 kN each and Non-drive machine weight is distributed at 4 points @ 50 kN each as shown in the Figure.

Weight of Drive Machine (excluding rotor)	$4 \times 25 = 100 \text{ kN}$
Rotor weight	$2 \times 5 = 10 \text{ kN}$
Weight of bearing pedestals	$2 \times 2 = 4 \text{ kN}$
Total (Rotor + pedestal)	$2 \times 7 = 14 \text{ kN}$

Weight of Non-Drive Machine (excluding rotor)	$4\times 50=200$ kN
Rotor weight	$2\times 10=20$ kN
Weight of bearing pedestals	$2\times 4=8$ kN
Total (Rotor + pedestals)	$2\times 14=28$ kN
Weight of coupling	6 kN
Total Machine weight	$100+14+200+28+6=348$ kN

Foundation sizing:

Foundation outline Plan dimension $L=5.2$ m & $B=2.2$ m

Area of foundation block $A_f=5.2\times 2.2=11.44\ \text{m}^2$

Consider foundation weight equal to 3.0 times the machine weight **(see § 9.1.1)**

Desired Foundation Weight $3\times 348=1044$ kN

Mass Density of concrete $\rho_c=2500\ \text{kg/m}^3=2.5\ \text{t/m}^3$

Wt. of foundation above Ground Level

$$\{2.2\times 2.2\times 1.0+3.0\times 2.2\times 0.2+0.6\times 0.8\times 0.8\}\times 2.5\times 9.81=160.5\ \text{kN}$$

Required Foundation Height below GL $H=\dfrac{(1044-160.5)}{11.44\times 9.81\times 2.5}=3.15$ m

Provide Foundation Depth below GL $H=3.5$ m

Weight of foundation $W_f=11.44\times 3.5\times 2.5\times 9.81+160.5=1142$ kN

Total weight of Machine + Foundation $W=1142+348=1490$ kN

Bearing Pressure

Direct Bearing Pressure $q=\dfrac{1490}{11.44}=130.3\ \ \text{kN/m}^2$

Net Bearing Capacity @ 3.5 m depth 250 kN/m^2

Allowable Bearing Capacity $=250+3.5\times 1.8\times 9.81=312\ \text{kN/m}^2$

Wm 1 = 25 kN Wm 2 = 25 kN Wm 3 = 25 kN Wm 4 = 25 kN
Wm 5 = 7 kN Wm 6 = 7 kN Wm 7 = 6 kN Wm 8 = 50 kN
Wm 9 = 50 kN Wm10 = 50 kN Wm11 = 50 kN Wm12 = 14 kN
Wm13 = 14 kN

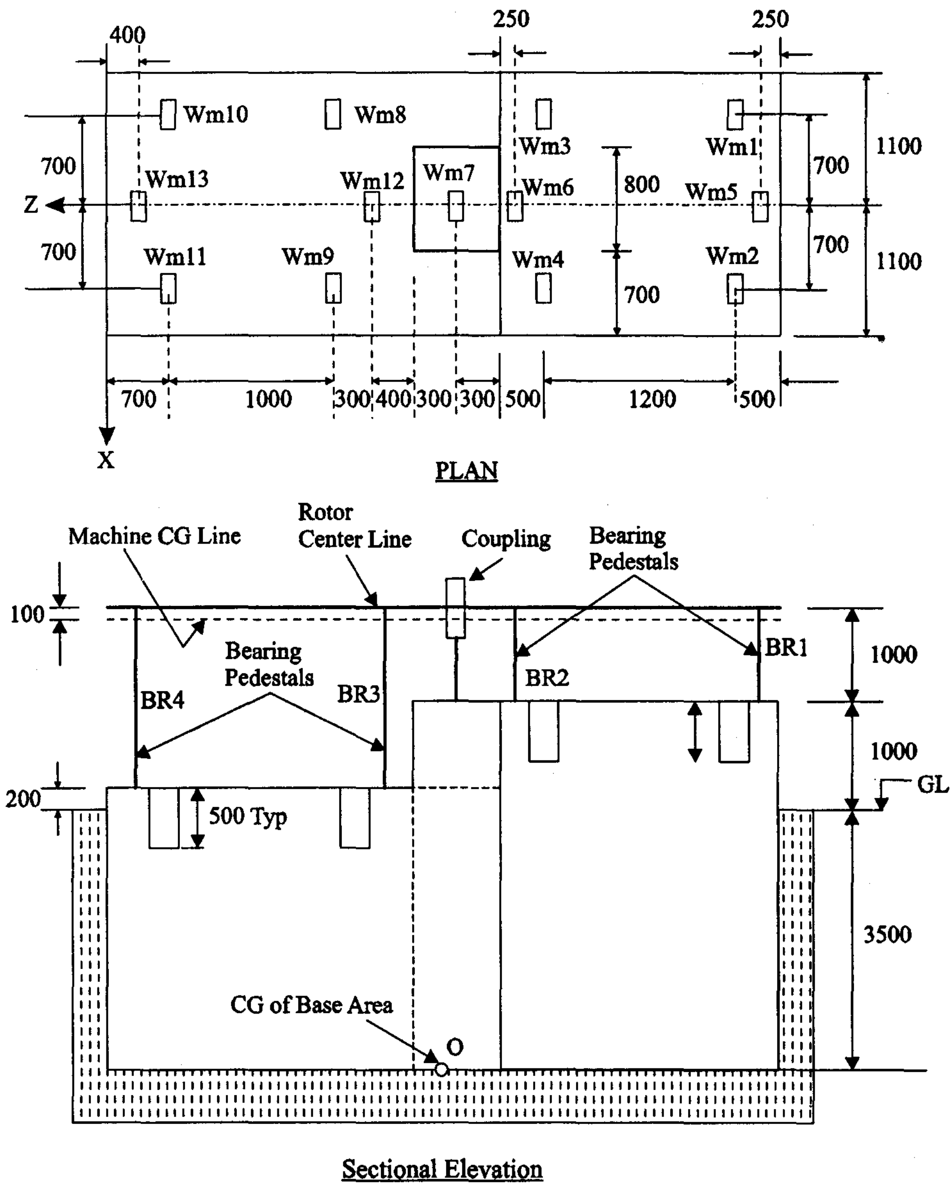


Figure D 9.1-1 Block Foundation Supporting Machines Coupled Directly

Ratio of Bearing Pressure to Bearing Capacity $\frac{130.3}{312} = 0.42$

Margin for other loads (dynamic loads, emergency loads etc.) $= (1-0.42)\times 100 = 58\%$

Margin available $> 30\%$, hence OK

Overall Centroid

Overall Centroid with respect to CG of Base area: Consider CG of Base area point O as shown in Figure D 9.1-2. This is also called DOF location.

a) **Machine**

	Drive M/c				Bearings		Coupling	Non-Drive M/c				Bearings		
W_i	25.0	25.0	25.0	25.0	7.0	7.0	6.0	50.0	50.0	50.0	50.0	14.0	14.0	kN
x_i	0.7	-0.7	0.7	-0.7	0.0	0.0	0.0	0.7	-0.7	0.7	-0.7	0.0	0.0	m
y_i	5.4	5.4	5.4	5.4	5.5	5.5	5.5	5.4	5.4	5.4	5.4	5.5	5.5	m
z_i	-2.1	-2.1	-0.9	-0.9	-2.35	-0.65	-0.1	0.9	0.9	1.9	1.9	0.6	2.2	m

Let $\bar{x}_{mo}, \bar{y}_{mo}, \bar{z}_{mo}$ represent Machine Centroid with respect to CG of Base Area point O. We get

$$\sum W_i = 100+14+6+200+28 = 348\ kN;\quad \sum W_i x_i = 0.0;\quad \sum W_i y_i = 1884;\quad \sum W_i z_i = 147.6$$

$$\bar{x}_{mo} = \frac{\sum W_i x_i}{\sum W_i} = 0;\quad \bar{y}_{mo} = \frac{\sum W_i y_i}{\sum W_i} = 5.414;\quad \bar{z}_{mo} = \frac{\sum W_i z_i}{\sum W_i} = 0.424$$

b) **Foundation**

	Dimension			Distance of CG from Point O		
Block	x	y	z	xi	yi	zi
1	2.2	4.5	2.2	0.0	2.25	-1.5
2	2.2	3.7	3.0	0.0	1.85	1.1
3	0.8	0.8	0.6	0.0	4.10	-0.1

Weight of Foundation

$$W_{f1} = 2.2 \times 2.2 \times 4.5 \times 2.5 \times 9.81 = 534 \quad \text{kN}$$

$$W_{f2} = 3.0 \times 2.2 \times 3.7 \times 2.5 \times 9.81 = 599 \quad \text{kN}$$

$$W_{f3} = 0.6 \times 0.8 \times 0.8 \times 2.5 \times 9.81 = 9.5 \quad \text{kN}$$

$$\text{Total weight} = 1142.5 \text{ kN}$$

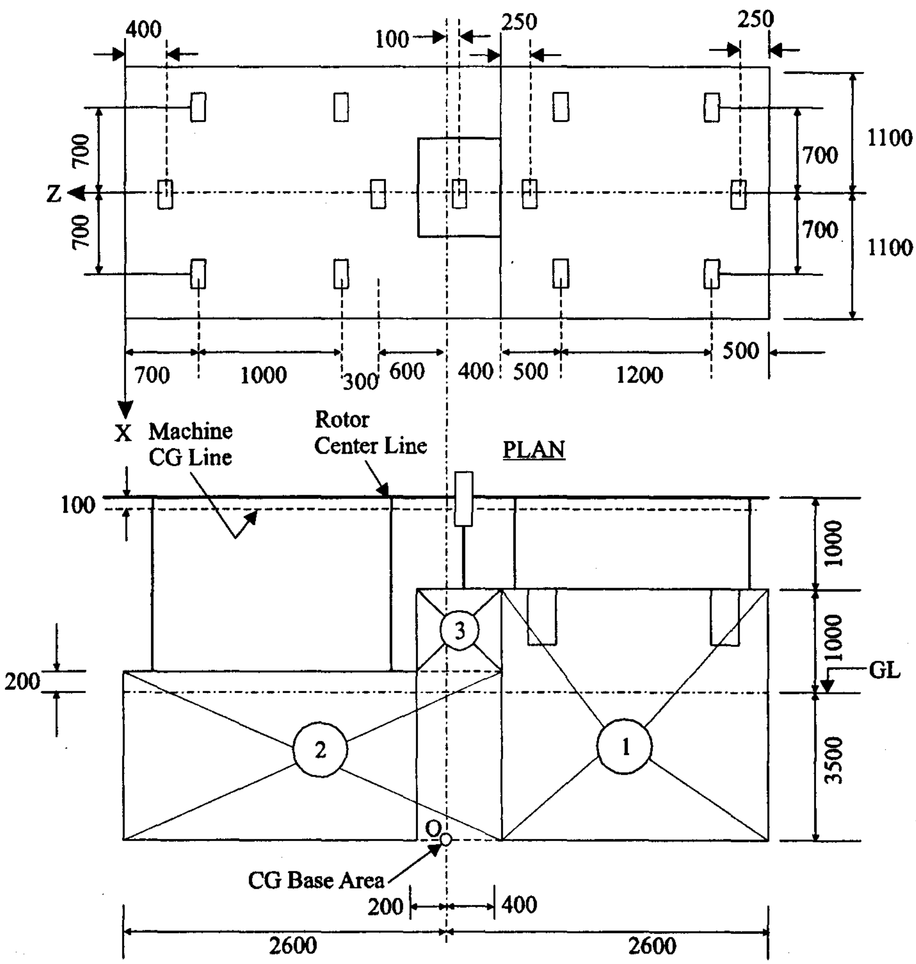


Figure D 9.1-2 Machine Layout with Respect to CG of Base Area O

Let $\bar{x}_{fo}, \bar{y}_{fo}, \bar{z}_{fo}$ represent Foundation Centroid with respect to CG of Base Area point O. We get

$$\sum Wf_i = 534 + 599 + 9.5 = 1142.5 \text{ kN}; \quad \sum W_{fi} x_i = 0.0; \quad \sum W_{fi} y_i = 2360.4; \quad \sum W_{fi} z_i = -143$$

$$\bar{x}_{fo} = \frac{\sum W_i x_i}{\sum W_i} = 0; \quad \bar{y}_{fo} = \frac{\sum W_i y_i}{\sum W_i} = 2.066; \quad \bar{z}_{fo} = \frac{\sum W_i z_i}{\sum W_i} = -0.125$$

Let $\bar{x}_o, \bar{y}_o \,\&\, \bar{z}_o$ represent overall centroid of Machine + Foundation system with respect to CG of Base Area point O

$$\bar{x}_o = \frac{\Sigma\left(W_m \bar{x}_{mo} + W_f \bar{x}_{fo}\right)}{\Sigma\left(W_m + W_f\right)} = \frac{348 \times 0 + 1142.5 \times 0}{348 + 1142.5} = 0$$

$$\bar{y}_o = \frac{\Sigma\left(W_m \bar{y}_{mo} + W_f \bar{y}_{fo}\right)}{\Sigma\left(W_m + W_f\right)} = \frac{348 \times 5.414 + 1142.5 \times 2.066}{1490.5} = 2.8477$$

$$\bar{z}_o = \frac{\Sigma\left(W_m \bar{z}_{mo} + W_f \bar{z}_{fo}\right)}{\Sigma\left(W_m + W_f\right)} = \frac{348 \times 0.424 + 1142.5 \times (-0.125)}{1490.5} = 0.0032$$

Eccentricity

Eccentricity in X-Z plane:

Eccentricity along X-direction $\quad e_x = \left(\bar{x}_o / B\right) \times 100 = \frac{0.0}{2.2} \times 100 = 0.0\% \quad < 5\% \quad \text{OK}$

Eccentricity along Z-direction $\quad e_z = \left(\bar{z}_o / L\right) \times 100 = \frac{0.0032}{5.2} \times 100 = 0.06\% \quad < 5\% \quad \text{OK}$

Both the values of eccentricity are less than 5 %, hence OK

Dynamic Analysis

Site Soil Parameters

Site Coefficient of Uniform Compression (as given) $\quad C_{u01} = 4.6 \times 10^4 \text{ kN/m}^3$

Corresponding base area (given) $\quad A_{01} = 10 \text{ m}^2$

Site Static Stress @ 3.5 m depth (as given) $\quad \bar{\sigma}_{01} = 100 \text{ kN/m}^2$

Design Soil Parameters

Width of Foundation $B = 2.2$ m

Foundation depth Below GL $D = 3.5$ m

Effective depth **(See §5.4)** $d_{02} = 0.5 \times 2.2 + 3.5 = 4.6$ m

Overburden pressure due to soil at depth d_{02} $\sigma_1 = 1.8 \times 4.6 \times 9.81 = 81.23 \text{ kN/m}^2$

Area of Foundation $A_{02} = 2.2 \times 5.2 = 11.44 \text{ m}^2$

Total Wt. of Machine + Foundation 1490 kN

Overburden pressure due to foundation + machine $\sigma_2 = \dfrac{1490}{11.44} = 130.2 \text{ kN/m}^2$

Design Static Stress $\bar{\sigma}_{02} = \sigma_1 + \sigma_2 = (81.23 + 130.2) = 211.43 \text{ kN/m}^2$

Design Coefficient of Uniform Compression $C_{u02} = C_{u01} \times \sqrt{\left(\dfrac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}\right)} \times \sqrt{\left(\dfrac{A_{01}}{A_{02}}\right)}$

Since $A_{02} = 11.44 \text{ m}^2 > 10 \text{ m}^2$; effective $A_{02} = 10 \text{ m}^2$

$$C_{u02} = 4.6 \times 10^4 \times \sqrt{\left(\frac{211.43}{100}\right)} \times \sqrt{\left(\frac{10}{10}\right)} = 6.7 \times 10^4 \text{ kN/m}^3$$

Design Coefficients

Uniform Compression (as given) $C_u = C_{u02} = 6.7 \times 10^4 \text{ kN/m}^3$

Uniform Shear $C_\tau = 0.5 \times C_u = 0.5 \times 6.7 \times 10^4 = 3.35 \times 10^4 \text{ kN/m}^3$

Non-Uniform Compression $C_\theta = C_\phi = 2 \times C_u = 2 \times 6.7 \times 10^4 = 13.4 \times 10^4 \text{ kN/m}^3$

Non-Uniform Shear $C_\psi = 0.75 \times C_u = 0.75 \times 6.7 \times 10^4 = 5.03 \times 10^4 \text{ kN/m}^3$

Soil Stiffness (Equivalent Springs):

Translational Soil Stiffness values along X, Y & Z k_x, k_y, k_z

Rotational Soil Stiffness values about X, Y & Z k_θ, k_ψ, k_ϕ

Foundation Base area $A_f = 11.44 \text{ m}^2$

Moment of Inertia of Base Area about X-axis $I_{xx} = \frac{1}{12} \times 2.2 \times 5.2^3 = 25.78 \quad m^4$

Moment of Inertia of Base Area about Z-axis $I_{zz} = \frac{1}{12} \times 2.2^3 \times 5.2 = 4.614 \quad m^4$

Moment of Inertia of Base Area about Y-axis

$$I_{yy} = I_{xx} + I_{zz} = 25.78 + 4.614 = 30.4 \quad m^4$$

Substituting values, we get:

$$k_x = C_\tau \times A_f = 3.35 \times 10^4 \times 11.44 = 38.32 \times 10^4 \quad \text{kN/m}$$

$$k_y = C_u \times A_f = 6.7 \times 10^4 \times 11.44 = 76.65 \times 10^4 \quad \text{kN/m}$$

$$k_z = C_\tau \times A_f = 38.32 \times 10^4 \quad \text{kN/m}$$

$$k_\theta = C_\theta \times I_{xx} = 13.4 \times 10^4 \times 25.78 = 3.45 \times 10^6 \quad \text{kNm/rad}$$

$$k_\psi = C_\psi \times I_{yy} = 5.03 \times 10^4 \times 30.4 = 1.53 \times 10^6 \quad \text{kNm/rad}$$

$$k_\phi = C_\phi \times I_{zz} = 13.4 \times 10^4 \times 4.614 = 0.62 \times 10^6 \quad \text{kNm/rad}$$

Mass and Mass Moment of Inertia

a) Mass Moment of Inertia about CG of Base Point O

Machine load distribution and locations with respect to point O (see Figure 9.1-2)

i) Machine

	Drive M/c				Bearings		Coupling	Non-Drive M/c				Bearings		
W_i	25.0	25.0	25.0	25.0	7.0	7.0	6.0	50.0	50.0	50.0	50.0	14.0	14.0	kN
x_i	0.7	-0.7	0.7	-0.7	0.0	0.0	0.0	0.7	-0.7	0.7	-0.7	0.0	0.0	m
y_i	5.4	5.4	5.4	5.4	2.7	2.7	2.7	5.4	5.4	5.4	5.4	2.7	2.7	m
z_i	-2.1	-2.1	-0.9	-0.9	-2.35	-0.65	-0.1	0.9	0.9	1.9	1.9	0.6	2.2	m

Total machine Mass $= 348/9.81 = 35.474$ t

Mass Moment of Inertia of Machine

$$M_{mox\_machine} = \sum \left\{ (W_i / g) \times (y_i^2 + z_i^2) \right\} = 1123.1 \quad \text{tm}^2$$

$$M_{moy\_machine} = \sum \{(W_i / g) \times (x_i^2 + z_i^2)\} = 98.3 \quad \text{t m}^2$$

$$M_{moz\_machine} = \sum \{(W_i / g) \times (x_i^2 + y_i^2)\} = 1054.7 \quad \text{t m}^2$$

ii) Foundation

Block	Dimension x	y	z	Distance of CG from Point O xi	yi	zi	Density	Mass
1	2.2	4.5	2.2	0.0	2.25	-1.5	2.5	$2.2 \times 4.5 \times 2.2 \times 2.5 = 54.45$
2	2.2	3.7	3.0	0.0	1.85	1.1	2.5	$2.2 \times 3.7 \times 3.0 \times 2.5 = 61.05$
3	0.8	0.8	0.6	0.0	4.10	-0.1	2.5	$0.8 \times 0.8 \times 0.6 \times 2.5 = 0.96$
								Total Mass = 116.46 t

Mass Moment of Inertia of Foundation

$$M_{mox\_foundation} = \sum \{(m_i / 12)(y^2 + z^2) + m_i (y_i^2 + z_i^2)\} = 926.49 \text{ t m}^2$$

$$M_{moy\_foundation} = \sum \{(m_i / 12)(x^2 + z^2) + m_i (x_i^2 + z_i^2)\} = 310.81 \quad \text{t m}^2$$

$$M_{moz\_foundation} = \sum \{(m_i / 12)(y^2 + x^2) + m_i (y_i^2 + x_i^2)\} = 708.95 \quad \text{t m}^2$$

Total Mass Moment of Inertia about CG of Base Point O

$$M_{mox} = 1123.1 + 926.49 = 2049.6 \quad \text{t m}^2$$

$$M_{moy} = 98.3 + 310.81 = 409.12 \text{ t m}^2$$

$$M_{moz} = 1054.7 + 708.95 = 1763.7 \quad \text{t m}^2$$

b) Mass Moment of Inertia about Overall Centroid

Total Mass (Machine + Foundation) $m = 35.47 + 116.46 = 151.93 \quad \text{t}$

Coordinates of Overall Centroid with respect to CG of Base Area point O

$\bar{x}_o = 0$; $\bar{y}_o = 2.8477$; $\bar{z}_o = 0.0032$

$$M_{mx} = M_{mox} - m\left(\bar{y}_o^{\,2} + \bar{z}_o^{\,2}\right) = \{2049.6 - 151.93 \times (2.8477^2 + 0.0032^2)\} = 817.5 \quad \text{t m}^2$$

$$M_{my} = M_{moy} - m\left(\bar{x}_o^{\,2} + \bar{z}_o^{\,2}\right) = \left\{409.12 - 151.93 \times \left(0 + 0.0032^2\right)\right\} = 409.12 \quad \text{t m}^2$$
$$M_{mz} = M_{moz} - m\left(\bar{y}_o^{\,2} + \bar{x}_o^{\,2}\right) = \left\{1763.7 - 151.93 \times \left(2.8477^2 + 0\right)\right\} = 531.64 \quad \text{t m}^2$$

Ratio of Mass Moment of Inertia at overall centroid to Mass Moment of Inertia at CG of base area point O

$$\gamma_x = \frac{M_{mx}}{M_{mox}} = \frac{817.5}{2049.6} = 0.399; \quad \gamma_y = \frac{M_{my}}{M_{moy}} = \frac{409.12}{409.12} = 1.0; \quad \gamma_z = \frac{M_{mz}}{M_{moz}} = \frac{531.64}{1763.7} = 0.30$$

Natural Frequencies

Limiting Frequencies:

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{38.32 \times 10^4}{151.93}} = 50.22 \text{ rad/s}$$

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{76.65 \times 10^4}{151.93}} = 71.03 \text{ rad/s}$$

$$p_z = \sqrt{\frac{k_z}{m}} = \sqrt{\frac{38.32 \times 10^4}{151.93}} = 50.22 \text{ rad/s}$$

$$p_\theta = \sqrt{\frac{k_\theta}{M_{mox}}} = \sqrt{\frac{3.45 \times 10^6}{2049.6}} = 41.02 \text{ rad/s}$$

$$p_\psi = \sqrt{\frac{k_\psi}{M_{moy}}} = \sqrt{\frac{1.53 \times 10^6}{409.12}} = 61.15 \text{ rad/s}$$

$$p_\phi = \sqrt{\frac{k_\phi}{M_{moz}}} = \sqrt{\frac{0.62 \times 10^6}{1763.7}} = 18.75 \text{ rad/s}$$

Uncoupled Modes: Since vertical and torsional modes (corresponding to $y \,\&\, \psi$ deformation) are uncoupled modes $p_y \,\&\, p_\psi$ also represent the natural frequencies in respective modes.

$$p_y = 71.03 \quad \text{rad/s}; \quad f_y = 11.30 \quad \text{Hz}$$

$$p_\psi = 61.15 \quad \text{rad/s}; \quad f_\psi = 9.73 \quad \text{Hz}$$

Coupled Modes are:

Modes corresponding to $x \& \phi$ deformation (X-Y Plane)

Modes corresponding to $z \& \theta$ deformation (Y-Z Plane)

Natural Frequencies corresponding to $x \& \phi$ deformation see equation

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$

$$p_1^2 = \frac{1}{2\times 0.30}\left(50.22^2 + 18.75^2\right) - \frac{1}{2\times 0.30}\sqrt{\left(50.22^2 + 18.75^2\right)^2 - 4\times 0.30\times 50.22^2\times 18.75^2}$$

$$p_1^2 = 4789.35 - 4469.03 = 320.32$$

$$p_1 = 17.9 \text{ rad/s}; \quad f_1 = 2.84 \text{ Hz}$$

$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$

$$p_2^2 = \frac{1}{2\times 0.30}\left(50.22^2 + 18.75^2\right) + \frac{1}{2\times 0.30}\sqrt{\left(50.22^2 + 18.75^2\right)^2 - 4\times 0.30\times 50.22^2\times 18.75^2}$$

$$p_2^2 = 4789.35 + 4469.03 = 9258.3;$$

$$p_2 = 96.22 \text{ rad/s}; \quad f_2 = 15.3 \text{ Hz}$$

It is noted that since limiting frequency $p_\phi < p_x$, the lower natural frequency $f_1 = 2.84$ Hz shall predominantly correspond to ϕ mode of deformation and $f_2 = 15.3$ Hz shall predominantly correspond to x mode of deformation. On the similar lines, we get natural frequencies corresponding to another coupled mode i.e. mode corresponding to $z \& \theta$ deformation. Substituting $p_z = 50.22,\ p_\theta = 41.02\ \&\ \gamma_x = 0.399$, we get the two natural frequencies as:

$$p_1^2 = \frac{1}{2\times 0.399}\left(50.22^2 + 41.02^2\right) - \frac{1}{2\times 0.399}\sqrt{\left(50.22^2 + 41.02^2\right)^2 - 4\times 0.399\times 50.22^2\times 41.02^2}$$

$$p_1^2 = 5269 - 4139 = 1130;$$

$$p_1 = 33.6 \text{ rad/s}; \quad f_1 = 5.35 \text{ Hz}$$

$$p_2^2 = \frac{1}{2\times 0.399}\left(50.22^2 + 41.02^2\right) + \frac{1}{2\times 0.399}\sqrt{\left(50.22^2 + 41.02^2\right)^2 - 4\times 0.399\times 50.22^2\times 41.02^2}$$

$$p_2^2 = 5269 + 4138.46 = 9407.46;$$

$$p_2 = 96.99 \text{ rad/s}; \quad f_2 = 15.43 \text{ Hz}$$

Here also, since $p_\theta < p_z$, the lower natural frequency $f_1 = 4.57$ Hz shall predominantly correspond to θ mode of deformation and $f_2 = 15.69$ Hz shall predominantly correspond to z mode of deformation.

Machine operating speed 600 rpm = 10 Hz

Rewriting the six natural frequencies (in ascending order) corresponding to six modes of vibration, we get:

			margin with respect to operating speed of 10 Hz
$p_1 = 17.9$ rad/s →	$f_1 = 2.84$ Hz	→ predominantly ϕ mode	71.6 %
$p_2 = 33.6$ rad/s →	$f_2 = 5.35$ Hz	→ predominantly θ mode	46.5 %
$p_3 = 61.15$ rad/s →	$f_3 = 9.73$ Hz	→ uncoupled ψ mode	2.7 %
$p_4 = 71.03$ rad/s →	$f_4 = 11.30$ Hz	→ uncoupled y mode	13.0 %
$p_5 = 96.22$ rad/s →	$f_5 = 15.30$ Hz	→ predominantly z mode	53.0 %
$p_6 = 96.99$ rad/s →	$f_6 = 15.44$ Hz	→ predominantly x mode	54.4 %

It is seen that 3rd & 4th frequencies (f_3 & f_4) are in resonance zone (i.e. frequencies lie within $\pm 20\%$ of operating speed) and rest of the frequencies are sufficiently away from operating speed. Hence amplitudes corresponding to p_3 & p_4 shall be computed with damping whereas for other frequencies, undamped amplitude would be OK.

Unbalance Forces

$$\text{Operating speed of machine} = 600 \quad \text{rpm} = \frac{600 \times 2 \times \pi}{60} = 62.83 \quad \text{rad/s}$$

a) Dynamic force F_1 generated by Drive Machine

Mass of Rotor $= \dfrac{10}{9.81} = 1.02$ t

Excitation frequency $= 62.83$ rad/s

Rotor Balance Grade $= G\,6.3$

Rotor eccentricity $e = (6.3/63.83) = 0.1\ \text{mm} \quad = 0.1\times10^{-3}$ m

Unbalance Force $F_1 = 1.02\times0.1\times10^{-3}\times(63.83)^2 = 0.404$ kN

b) Dynamic force F_2 generated by Non-Drive Machine

Mass of Rotor $= \dfrac{20}{9.81} = 2.04$ t

Rotor Balance Grade $= G6.3$

Rotor eccentricity $e = 0.1\times10^{-3}$ m

Unbalance Force $F_2 = 2.04\times0.1\times10^{-3}\times(62.83)^2 = 0.808$ kN

Dynamic Force – Load cases

Dynamic forces F_1 and F_2 are considered acting in vertical as well as in lateral directions (one at a time). These forces are considered acting (a) in-phase and (b) out of phase. Figure D 9.1-3 shows dynamic forces applied at bearing locations. Bearings 1 & 2 **(Br1 & Br2)** correspond to Drive machine whereas bearings 3 & 4 **(Br3 & Br4)** correspond to Non-drive machine. For each such combination, forces are finally transferred @ point O and amplitudes are computed for each such combination.

Case 1 Forces in-phase acting in (+) X–Direction

Excitation frequency 62.83 rad/s

Force at Bearings Br1 & Br2 each (+X direction) $= (0.404/2) = 0.202$ kN

Force at Bearing Br3 & Br4 each (+X direction) $= (0.808/2) = 0.404$ kN

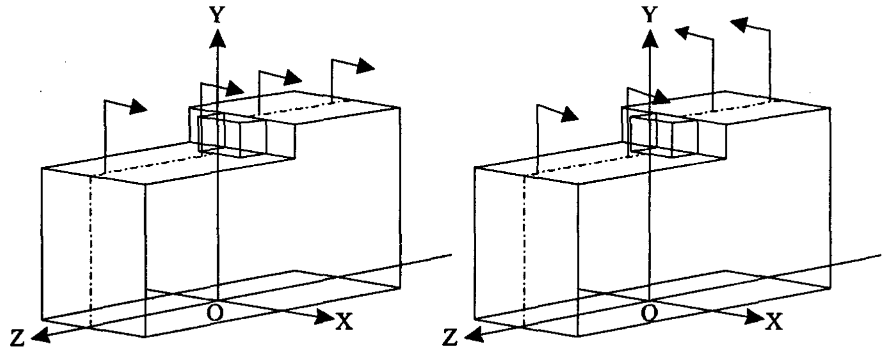
Transferring Forces at CG of Base area point O , we get

(Moment from X to Y, Y to Z and Z to X is positive)

$$F_x = 2\times0.202 + 2\times0.404 = 1.212 \quad \text{kN}$$

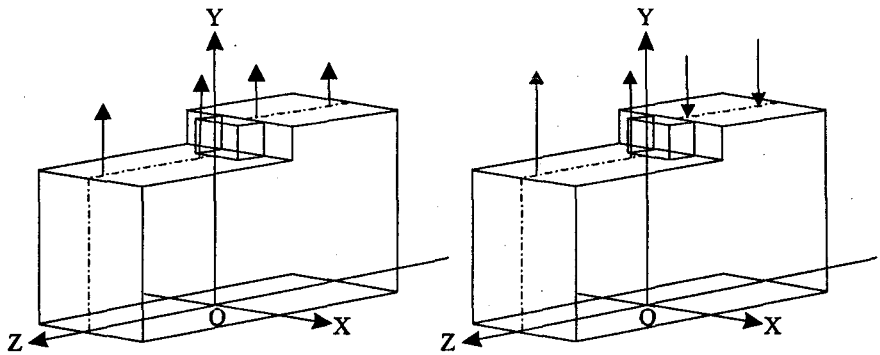
$$M_\phi = -(0.404\times5.5 + 0.404\times5.5 + 0.202\times5.5 + 0.202\times5.5) = -6.66 \quad \text{kNm}$$

$$M_\psi = 0.404\times(2.2+0.6) - 0.202\times(2.35+0.65) = 0.52 \quad \text{kNm}$$



(a) Dynamic Forces - in Phase (b) Dynamic Forces - out Phase

Dynamic Forces applied along X aixs



(c) Dynamic Forces - in Phase (d) Dynamic Forces - out Phase

Dynamic Forces applied along Y aixs

Figure D 9.1-3 Dynamic Forces at Bearing Locations

Case 2 Forces out of Phase acting in X–Direction

Force at Bearings Br1 & Br2 each (-X direction) = –0.202 kN

Force at Bearing Br3 & Br4 each (+X direction) = 0.404 kN

Transferring Forces at CG of Base area point O , we get

$$F_x = 2\times(0.404-0.202) = 0.404 \quad \text{kN}$$

$$M_\phi = -2\times(0.404\times5.5)+2\times(0.202\times5.5) = -2.22 \quad \text{kNm}$$

$M_\psi = 0.404 \times (2.2 + 0.6) + 0.202(0.65 + 2.35) = 1.74 \quad \text{kNm}$

Case 3 Forces F_1 & F_2 acting in-phase acting in Y –Direction

Force at Bearing Br1 & Br2 each (+Y direction) $= 0.202$ kN

Force at Bearing Br3 & Br4 each (+Y direction) $= 0.404$ kN

Transferring Forces at CG of Base area point O, we get

$F_y = 2 \times 0.202 + 2 \times 0.404 = 1.212 \text{ kN}$

$M_\theta = -0.404(2.2 + 0.6) + 0.202(2.35 + 0.65) = -0.525 \quad \text{kNm}$

Case 4 Forces out of Phase acting in Y –Direction

Force at Bearing Br1 & Br2 each (-Y direction) $= 0.202$ kN

Force at Bearing Br3 & Br4 each (+Y direction) $= 0.404$ kN

Transferring Forces at CG of Base area point O, we get

$F_y = -2 \times 0.202 + 2 \times 0.404 = 0.404 \quad \text{kN}$

$M_\theta = -0.404 \times (2.2 + 0.6) - 0.202 \times (2.35 + 0.65) = -1.735 \quad \text{kNm}$

Dynamic forces transferred at point O are shown in Figure D 9.1-4.

Amplitudes of Vibration

For amplitude computation (see § 9.1.6). Rewriting parameters required for computation of amplitudes:

Stiffness and mass moment of inertia

$k_x = 38.32 \times 10^4; \quad k_y = 76.65 \times 10^4; \quad k_z = 38.32 \times 10^4 \quad \text{kN/m}$

$k_\theta = 3.45 \times 10^6; \quad k_\psi = 1.53 \times 10^6; \quad k_\phi = 0.62 \times 10^6 \quad \text{kNm/rad}$

$m = 151.93 \text{ t}; \quad h = \bar{y}_O = 2.8477 \text{ m}; \quad \zeta = 0.1$

$H = 4.5 \text{ m}; \quad L = 5.2 \text{ m}; \quad B = 2.2 \text{ m}$

$M_{mox} = 2049.6$ t m^2; $M_{moy} = 409.12$ t m^2; $M_{moz} = 1763.7$ t m^2

$M_{mx} = 817.5$ t m^2 $M_{my} = 409.12$ t m^2 $M_{mz} = 531.64$ t m^2

$\gamma_x = 0.399$; $\gamma_y = 1.0$; $\gamma_z = 0.30$

Limiting Frequencies:

$p_x = 50.22$ rad/s; $p_y = 71.03$ rad/s; $p_z = 50.22$ rad/s

$p_\theta = 41.02$ rad/s; $p_\psi = 61.15$ rad/s; $p_\phi = 18.75$ rad/s

Natural Frequencies:

For motion in X-Y Plane $p_1 = 17.9$ rad/s; $p_2 = 96.22$ rad/s

For motion in Y-Z Plane $p_1 = 33.6$ rad/s; $p_2 = 96.99$ rad/s

Frequency Ratios:

Limiting Frequency Ratios:

$\beta_x = (\omega/p_x) = 1.25$; $\beta_y = (\omega/p_y) = 0.88$; $\beta_z = (\omega/p_z) = 1.25$

$\beta_\theta = (\omega/p_\theta) = 1.53$; $\beta_\psi = (\omega/p_\psi) = 1.03$; $\beta_\phi = (\omega/p_\phi) = 3.35$

Natural Frequency Ratios:

For motion in X-Y Plane $\beta_1 = (\omega/p_1) = 3.52$; $\beta_2 = (\omega/p_2) = 0.65$

For motion in Y-Z Plane $\beta_1 = (\omega/p_1) = 1.87$; $\beta_2 = (\omega/p_2) = 0.65$

Case 1 Force along X in Phase

$F_x = 1.21$ kN; $M_\phi = -6.66$ kNm; $M_\psi = 0.52$ kNm

Excitation Frequency (Operating Speed of 600 rpm) = 62.83 rad/s

Natural Frequencies $p_1 = 17.9$ rad/s; $p_2 = 96.22$ rad/s

$\beta_1 = (\omega/p_1) = 3.52$; $\beta_2 = (\omega/p_2) = 0.65$

Since $\beta_1 > 1.2 \,\&\, \beta_2 < 0.8$ hence no resonance; use equation (9.1-9 & 10) for undamped amplitudes. Further, since $0.8 < \beta_\psi < 1.2$, use equation 9.1-14 for amplitude with damping.

Note: For amplitude computation, it is more convenient to consider one force at a time, evaluate amplitudes and finally obtain the resultant by taking the sum of the amplitudes.

i) $F_x = 1.21$ kN

$$\delta_x = (F_x/k_x) = \{1.21/(38.32\times10^4)\} = 3.16\times10^{-6} \text{ m}$$

$$x_o = \left[\delta_x \frac{(1-\beta_\phi^2)}{(1-\beta_1^2)(1-\beta_2^2)}\right] = \left[3.16\times10^{-6}\frac{(1-3.35^2)}{(1-3.52^2)(1-0.65^2)}\right] = 4.96\times10^{-6} \text{ m}$$

$$\phi_o = -\left[\delta_x \frac{mh}{M_{moz}} \times \frac{\beta_\phi^2}{(1-\beta_1^2)\times(1-\beta_2^2)}\right]$$

$$\phi_o = -\left[3.16\times10^{-6}\times\frac{151.93\times2.848}{1763.7}\times\frac{3.35^2}{(1-3.52^2)\times(1-0.65^2)}\right] = 1.33\times10^{-6} \text{ rad}$$

ii) $M_\phi = -6.66$ kNm

$$\delta_\phi = (M_\phi/k_\phi) = (-6.66/0.62\times10^6) = -1.07\times10^{-5} \text{ rad}$$

$$x_o = -\left[h\delta_\phi \frac{\beta_x^2}{(1-\beta_1^2)\times(1-\beta_2^2)}\right]$$

$$x_o = -\left[2.84\times(-1.07\times10^{-5})\frac{1.25^2}{(1-3.52^2)\times(1-0.65^2)}\right] = -7.32\times10^{-6} \text{ m}$$

$$\phi_o = \left[\delta_\phi \frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)}\right] = \left[(-1.07\times10^{-5})\times\frac{(1-1.25^2)}{(1-3.52^2)(1-0.65^2)}\right] = -9.31\times10^{-7} \text{ rad}$$

iii) Amplitudes for Moment $M_\psi = 0.52$ kNm

$p_\psi = 61.15\, rad/\text{sec}$; $\beta_\psi = 1.03$

$$\delta_\psi = (M_\psi/k_\psi) = (0.52/1.53\times10^6) = 3.43\times10^{-7} \text{ rad}$$

$$\psi_o = 3.43\times10^{-7}\times\frac{1}{\sqrt{\left(1-1.03^2\right)^2+\left(2\times1.03\times0.1\right)^2}} = 1.61\times10^{-6}\ \text{rad}$$

Total Amplitudes: Total amplitudes $x_o, \phi_o \,\&\, \psi_o$ @ point O

$$\sum x_o = \left(4.96\times10^{-6} - 7.32\times10^{-6}\right) = -2.37\times10^{-6}\ \text{m}$$

$$\sum \phi_o = \left(1.33\times10^{-6} - 9.31\times10^{-7}\right) = 4.0\times10^{-7}\ \text{rad}$$

$$\sum \psi_o = 1.61\times10^{-6}\ \text{rad}$$

Amplitudes @ Foundation Top

Rewriting equations 9.1-25 to 9.1-27 and substituting $\sum x_0, \sum y_0, \sum z_0, \sum \theta_0, \sum \psi_0 \,\&\, \sum \phi_0$ in place of $x_0, y_0, z_0, \theta_0, \psi_0 \,\&\, \phi_0$, we get:

$$x_{f(\max)} = \left|\left(\sum x_0 - H\sum \phi_0\right)\right| + \left|(L/2)\sum \psi_0\right|$$

$$y_{f(\max)} = \left|\sum y_0\right| + \left|(L/2)\sum \theta_0\right| + \left|(B/2)\sum \phi_0\right|$$

$$z_{f(\max)} = \left|\left(\sum z_o + H\sum \theta_o\right)\right| + \left|(B/2)\sum \psi_0\right|$$

Substituting values, we get .

$$x_{f(\max)} = \left|\left\{-2.37\times10^{-6} - 4.5\times\left(4.00\times10^{-7}\right)\right\}\right| + \left|(5.2/2)\times1.61\times10^{-6}\right| = 8.35\times10^{-6}\ \text{m}$$

$$y_{f(\max)} = 0 + 0 + \left|(2.2/2)\times\left(4.00\times10^{-7}\right)\right| = 4.4\times10^{-7}\ \text{m}$$

$$z_{f(\max)} = \left|(B/2)\sum \psi_o\right| = \left|(2.2/2)\times1.61\times10^{-6}\right| = 1.77\times10^{-6}\ \text{m}$$

Case 2 Force along X Out of Phase

$F_x = 0.4$ kN ; $M_\phi = -2.22$ kNm ; $M_\psi = 1.74$ kNm

$\omega = 62.83$ rad/s $p_1 = 17.9$ rad/s ; $p_2 = 96.22$ rad/s

$\beta_1 = (\omega/p_1) = 3.52;$ $\beta_2 = (\omega/p_2) = 0.65;$ $\beta_\phi = (\omega/p_\phi) = 3.35$

i) $F_x = 0.4 \quad \text{kN}$

Following computations on similar lines, substituting the values, we get

$$\delta_x = (F_x/k_x) = \{0.4/(38.32\times10^4)\} = 1.05\times10^{-6}\ \text{m}$$

$$x_o = \left[\delta_x \frac{(1-\beta_\phi^2)}{(1-\beta_1^2)(1-\beta_2^2)}\right] = 1.65\times10^{-5} \quad \text{m}$$

$$\phi_o = -\left[\delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{(1-\beta_1^2)(1-\beta_2^2)}\right] = 4.44\times10^{-7} \quad \text{rad}$$

ii) $M_\phi = -2.22 \quad \text{kNm}$

$$\delta_\phi = (M_\phi/k_\phi) = (-6.66/0.62\times10^6) = -3.58\times10^{-6}\ \text{rad}$$

$$x_o = -\left[h\delta_\phi \frac{\beta_x^2}{(1-\beta_1^2)(1-\beta_2^2)}\right]$$

$$= -\left[2.84\times(-3.58\times10^{-6})\frac{1.25^2}{(1-3.52^2)(1-0.65^2)}\right] = -2.44\times10^{-6} \quad \text{m}$$

$$\phi_o = \left[\delta_\phi \frac{(1-\beta_x^2)}{(1-\beta_1^2)(1-\beta_2^2)}\right] = -\left[(-3.58\times10^{-6})\times\frac{(1-1.25^2)}{(1-3.52^2)(1-0.65^2)}\right] = -3.10\times10^{-7} \quad \text{rad}$$

i) **Amplitudes for Moment** $M_\psi = 0.52 \quad \text{kNm}$

$$p_\psi = 61.15 \quad \text{rad/sec}; \quad \beta_\psi = 1.03$$

$$\delta_\psi = (M_\psi/k_\psi) = (1.74/1.53\times10^6) = 1.14\times10^{-6}\ \text{rad}$$

$$\psi_o = 1.14\times10^{-6}\times\frac{1}{\sqrt{(1-1.03^2)^2+(2\times1.03\times0.1)^2}} = 5.33\times10^{-6}\ \text{rad}$$

Total Amplitudes: Total amplitudes $x_o, \phi_o \,\&\, \psi_o$ @ point O

$$\sum x_o = (1.65\times10^{-6} - 2.44\times10^{-6}) = -7.89\times10^{-7} \quad \text{m}$$

$$\sum \phi_o = \left(4.44\times10^{-7} - 3.10\times10^{-7}\right) = 1.33\times10^{-7} \text{ rad}$$

$$\sum \psi_o = 5.33\times10^{-6} \text{ rad}$$

Amplitudes @ Foundation Top:

Substituting values, we get

$$x_{f(\max)} = 1.52\times10^{-5} \text{ m}; \quad y_{f(\max)} = 1.47\times10^{-7} \text{ m}; \quad z_{f(\max)} = 5.86\times10^{-6} \text{ m}$$

Case 3 Force along Y in Phase (Motion in Y –Z Plane)

$$F_y = 1.21 \text{ kN}; \quad M_\theta = -0.52 \text{ kNm}$$

$$\omega = 62.83 \text{ rad/s} \quad p_1 = 33.62 \text{ rad/s}; \quad p_2 = 96.99 \text{ rad/s}$$

$$\beta_1 = (\omega/p_1) = 1.87; \quad \beta_2 = (\omega/p_2) = 0.65$$

$$\beta_x = (\omega/p_x) = 1.25; \quad \beta_y = (\omega/p_y) = 0.88; \quad \beta_z = (\omega/p_z) = 1.25$$

$$\beta_\theta = (\omega/p_\theta) = 1.53; \quad \beta_\psi = (\omega/p_\psi) = 1.03; \quad \beta_\phi = (\omega/p_\phi) = 3.35$$

Since $\beta_1 > 1.2 \;\&\; \beta_2 < 0.8$ hence no resonance; use equation (9.1-11 & 12) for undamped amplitudes. Further, since $0.8 < \beta_y < 1.2$, use equation 9.1-13 for amplitude with damping.

i) $F_y = 1.21 \text{ kN}$

$$\delta_y = \left(F_y/k_y\right) = \left\{1.21/\left(7.67\times10^5\right)\right\} = 1.58\times10^{-6} \text{ m}$$

$$y_o = \delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta\right)^2}} = 1.58\times10^{-6} \frac{1}{\sqrt{\left(1-0.88^2\right)^2 + \left(2\times0.88\times0.1\right)^2}} = 5.63\times10^{-6} \text{ m}$$

ii) $M_\theta = -0.52 \text{ kNm}$ $\quad \delta_\theta = \dfrac{M_\theta}{k_\theta} = \dfrac{-0.52}{3.45\times10^6} = -1.52\times10^{-7}$

$$z_o = \left[h\delta_\theta \frac{\beta_z^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right]$$

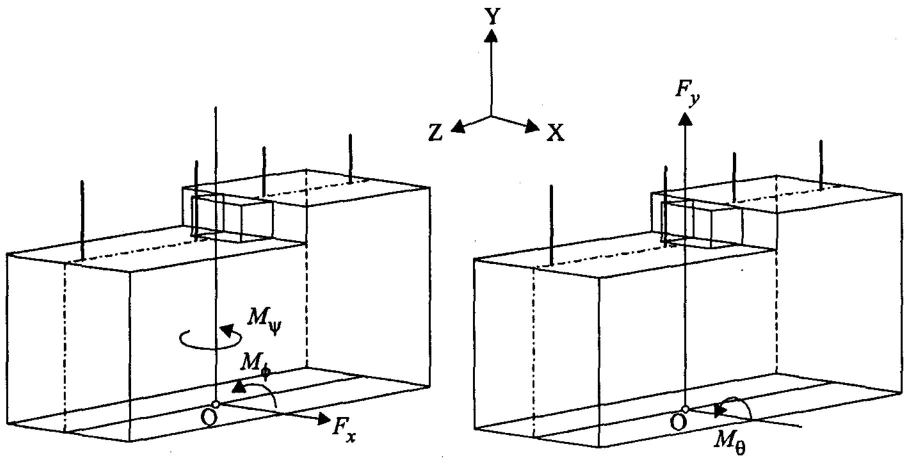
$$z_0 = \left[2.84 \times \left(-1.52 \times 10^{-7}\right) \times \frac{1.25^2}{\left(1-1.87^2\right)\left(1-0.65^2\right)}\right] = 4.67 \times 10^{-7} \text{ m}$$

$$\theta_o = \left[\delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}\right]$$

$$\theta_0 = \left[\left(-1.52 \times 10^{-7}\right) \times \frac{\left(1-1.25^2\right)}{\left(1-1.87^2\right)\left(1-0.65^2\right)}\right] = -5.94 \times 10^{-8} \text{ rad}$$

Total Amplitudes: Total amplitudes $x_o, \phi_o \,\&\, \psi_o$ @ point O

$$\sum y_o = 5.63 \times 10^{-6} \text{ m}; \quad \sum z_o = 4.67 \times 10^{-7} \text{ m}; \quad \sum \theta_o = -5.94 \times 10^{-8} \text{ rad}$$



F_x = 1.212 kN	F_x = 0.4 kN		
M_ϕ = – 6.66 kNm	M_ϕ = 2.22 kNm	F_y = 1.21 kN	F_y = 0.40 kN
M_ψ = 0.52 kNm	M_ψ = 1.74 kNm	M_θ = – 0.52 kNm	M_θ = – 1.74 kNm
In Phase	Out of Phase	In Phase	Out of Phase

Case 1 & 2 Dynamic Forces along X

Case 3 & 4 Dynamic Forces along Y

Figure D 9.1-4 Dynamic Forces and Moments Transferred @ Point O

Amplitudes @ Foundation Top: Substituting values, we get

$$y_{f(\max)} = \left|5.63\times10^{-6}\right| + \left|(5.2/2)\times\left(-5.94\times10^{-8}\right)\right| = 5.79\times10^{-6} \text{ m}$$

$$z_{f(\max)} = \left|(4.67\times10^{-7} + 4.5\times(-5.94)\times10^{-8})\right| + 0 = 2.0\times10^{-7} \text{ m}$$

Case 4 Force along Y Out of Phase (Motion in Y –Z Plane)

$$F_y = 0.40 \text{ kN}; \quad M_\theta = -1.74 \text{ kNm}$$

Following procedure same as for case 3, substituting values, we get

i) $F_y = 0.40$ kN ; $\delta_y = 5.26\times10^{-7}$ m ; $y_o = 1.88\times10^{-6}$ m

ii) $M_\theta = -1.74$ kNm ; $\delta_\theta = -5.03\times10^{-7}$

$z_0 = 1.55\times10^{-6}$ m; $\theta_0 = -1.97\times10^{-7}$ rad

Total Amplitudes: Total amplitudes $x_O, \phi_O \& \psi_O$ @ point O

$$\sum y_o = 1.88\times10^{-6} \text{ m}; \quad \sum z_o = 1.55\times10^{-6} \text{ m}; \quad \sum\theta_o = -1.97\times10^{-7} \text{ rad}$$

Amplitudes @ Foundation Top: Substituting values, we get

$$y_{f(\max)} = \left|1.88\times10^{-6}\right| + \left|(5.2/2)\times(-1.97)\times10^{-7}\right| = 2.39\times10^{-6} \text{ m}$$

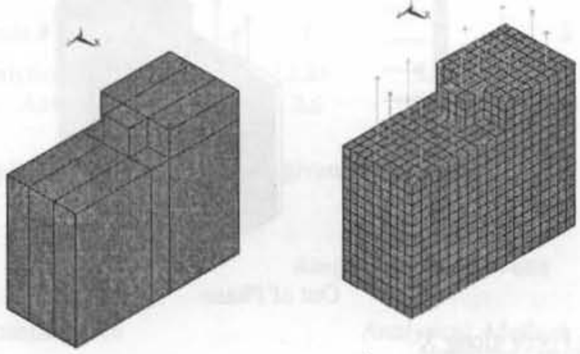
$$z_{f(\max)} = \left|(1.55\times10^{-6} + 4.5\times(-1.97)\times10^{-7})\right| = 6.6\times10^{-7} \text{ m}$$

Finite Element Analysis

This very problem is modeled and analyzed using Finite Element Method. Solid Model and FE Model are shown in Figure D 9.1-5. The results are presented herewith through Figures D 9.1-6 to D 9.1-8.

- Mode Shapes and associated frequencies are shown in Figure D 9.1-6
- Steady State Amplitudes are shown in Figure D 9.1-7
- Transient Amplitudes are shown in Figure D 9.1-8

Comparison of FE results with Analytical results



Solid Model FE Model

Figure D 9.1-5 Foundation Block – Solid Model & FE Model

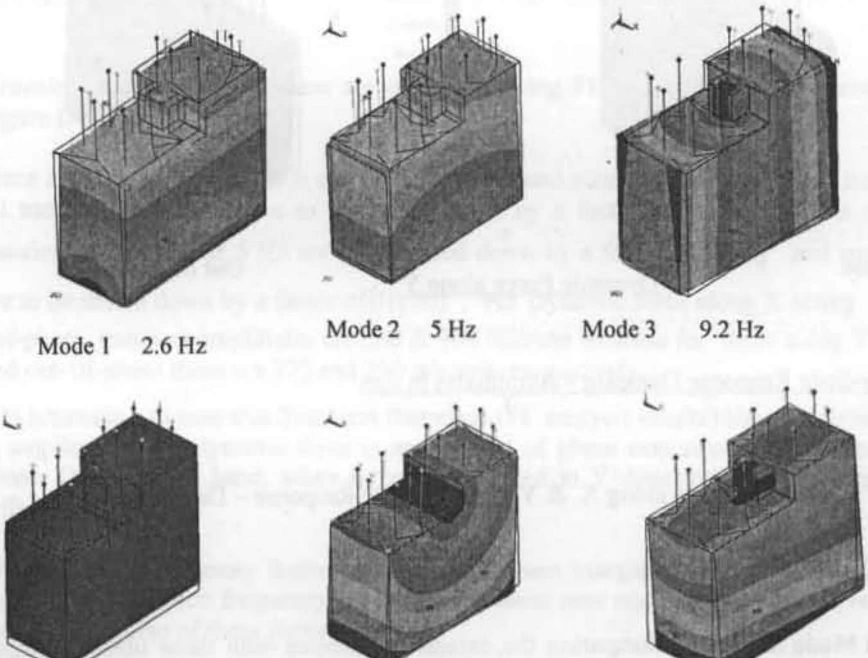
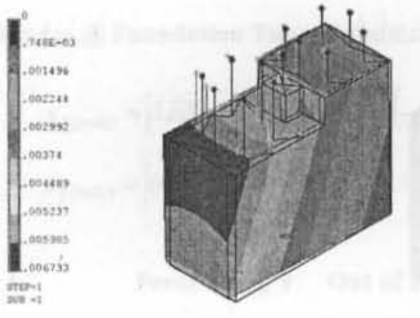
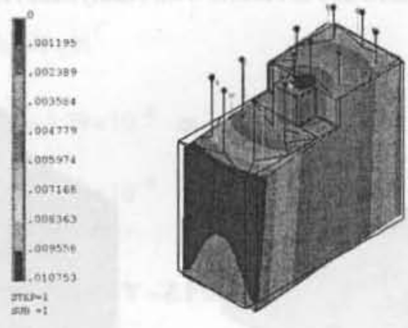


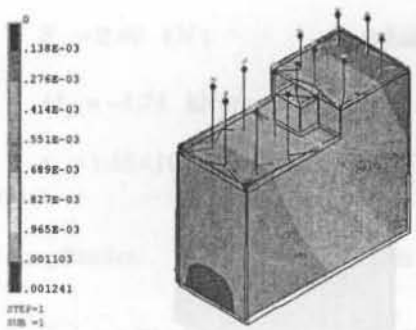
Figure D 9.1-6 Foundation Block – Mode Shapes & Frequencies

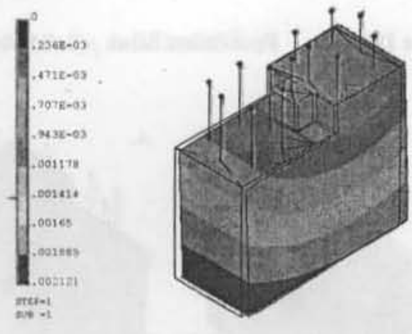




In Phase Out of Phase

Dynamic Force along X





In Phase Dynamic Force along Y Out of Phase

Steady-State Response Damping - Amplitudes in mm

Figure D 9.1-7 Dynamic Force along X **& Y** - Steady-State Response – Damping 10 %

Frequencies and Mode Shapes: Comparing the natural frequencies with those obtained using manual computational method (**see 9.1.5.1**), it is noticed that frequency evaluated by both the analysis methods show a good agreement and variation is of the order of 10 % for all the six modes of vibration. A good agreement is noticed in Frequencies and Mode Shapes by both the methods given as under:

Natural Frequencies - Hz

Mode #	1	2	3	4	5	6
Analytical Method	2.85	5.35	9.73	11.3	15.31	15.44
FE Analysis	2.6	5	9.1	10	13.8	13.8

Amplitudes of Vibration are given as under:

Amplitudes in microns

Dynamic Force	Analytical Method	FE Analysis
Along X In-phase	8.4	6.7
Along X Out of Phase	15.2	10.8
Along Y In-phase	5.8	1.3
Along Y Out of Phase	2.4	2.1

Transient amplitudes: These are computed using FE analysis only and have been shown in Figure D 9.1-8.

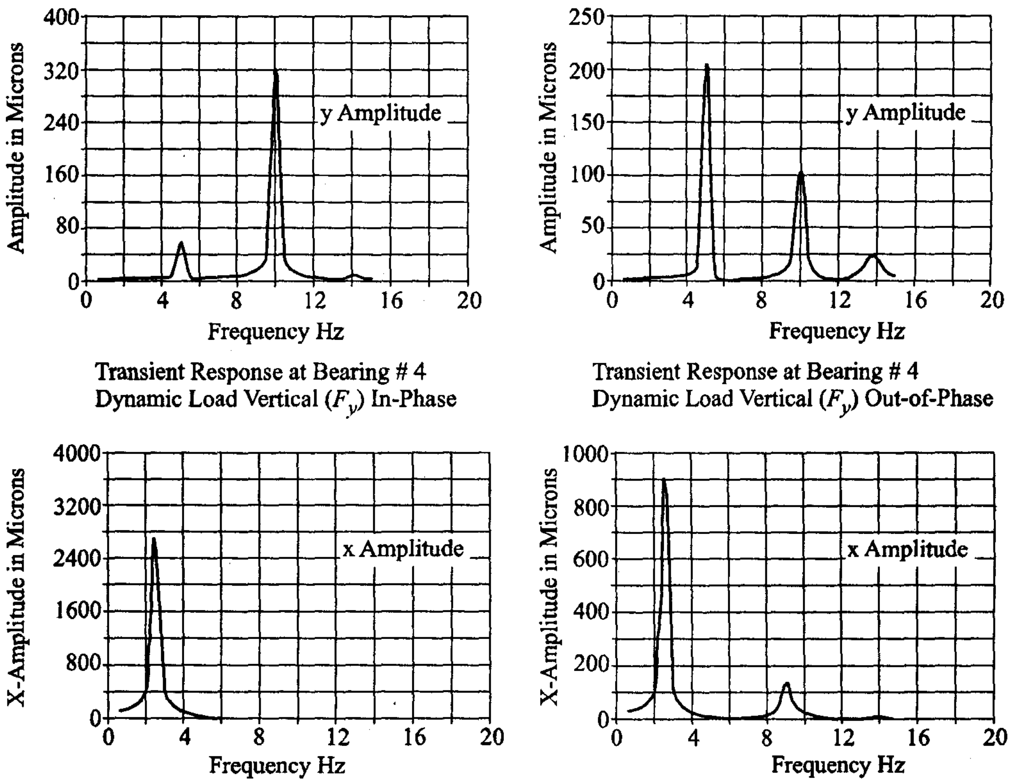
Since applied dynamic force is computed at 50 Hz and same force is applied for the sweep analysis, all transient amplitudes are to be scaled down by a factor of square of ratio of frequency i.e. transient amplitudes at 5 Hz are to be scaled down by a factor of $(5/50)^2$ and amplitude at 10 Hz are to be scaled down by a factor of $(10/50)^2$. For Dynamic force along X acting in-phase and out –of-phase, transient amplitudes are 280 & 900 microns whereas for force along Y acting in-phase and out–of-phase these are 320 and 200 microns respectively.

It is interesting to note that Transient Response (FE analysis results) shows a three fold increase in X amplitude when dynamic force is applied out of phase compared to when force applied is in phase. On the other hand, when forces are applied in Y direction, such a sharp increase is not reflected.

There are may be many factors that may influence margins between natural frequency of the system and excitation frequency. These in turn cause near resonance conditions resulting in higher amplitudes. Some of these factors are:

i) Variation in soil stiffness properties with time
ii) For electrically operated machines drawing power from the grid, variation in the grid frequency (a very common factor) results in changed excitation frequency
iii) Variation in machine parameters given at the design stage to the actual one at the time of supply

It is to be noted that both these methods suffer with inaccuracies on account of ignoring soil surrounding the foundation at all vertical interfaces. The soil effect tends to result in reduced amplitudes than those evaluated by analysis



Transient Response at Bearing # 4
Dynamic Load Vertical (F_y) In-Phase

Transient Response at Bearing # 4
Dynamic Load Vertical (F_y) Out-of-Phase

Transient Response at Bearing # 4
Dynamic Load Lateral (F_x) In-Phase

Transient Response at Bearing # 4
Dynamic Load Lateral (F_x) Out-of-Phase

Figure D 9.1-8 Transient Response at Bearing # 4

Strength Design

1. Block foundations are rigid body mass and have sufficient strength to withstand all possible force exerted by machine and as such do not need design computations for strength except those parts of the foundation which are overhang or cantilever.
2. Minimum reinforcement to be provided is 25 to 50 kg/m^3. It is recommended that bar diameter shall not be less than 12 mm and spacing shall not be more than 200 mm. For thick concrete blocks, it is desirable to provide intermediate cross reinforcement layers along the height.

3. Though not necessary, check for **Safe Bearing Pressure** and stability, due to normal as well as abnormal loading conditions is desirable.
4. Check for Strength & Embeddement of Anchor Bolts for applicable forces is a must

The foundation is designed using applicable codes of practice. Typical reinforcement arrangement for the foundation is shown in Figure D 9.1-9

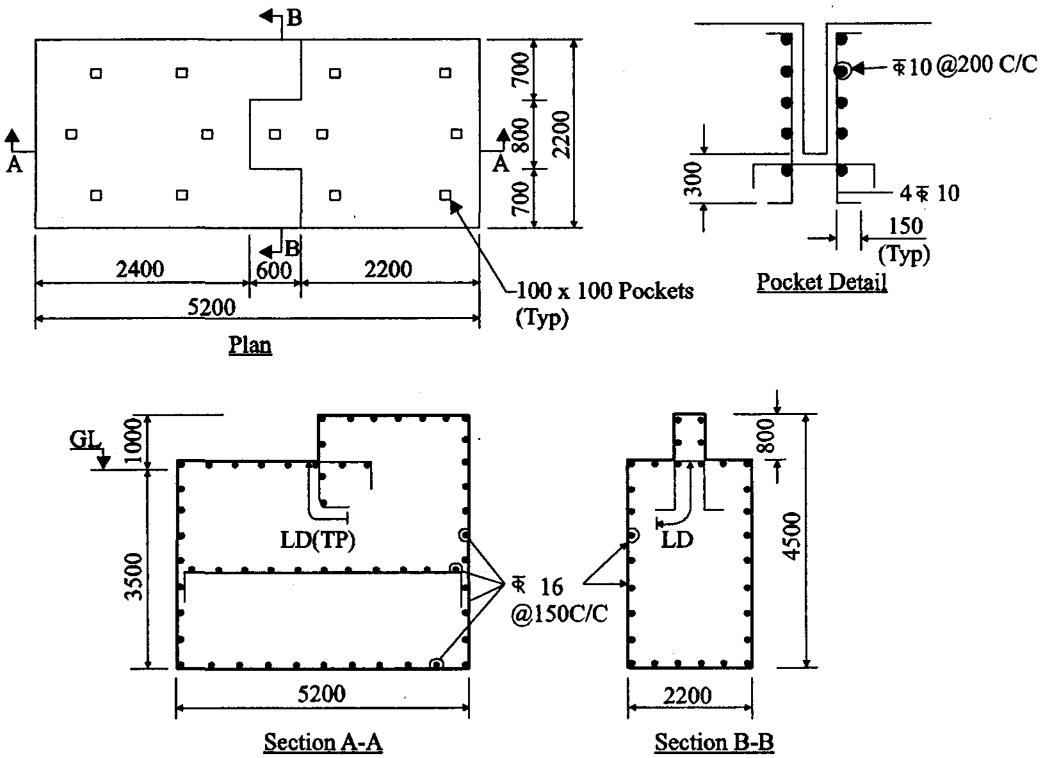


Figure D 9.1-9 Typical R/F For Block Foundation

9.2 DESIGN OF FRAME FOUNDATION

Design of Frame Foundation is relatively a complex task compared to Block foundation. There are many parameters that influence machine-foundation response. The stiffness of **Frame Structure** plays a vital role and more often than not becomes **The Governing Parameter**. Individual vibration characteristics of columns, beams, cantilever projections etc, besides being part of the system, have also been found to significantly influence the response.

Author has been associated for about three decades with Turbo-Generators and associated machinery for Thermal Power Plants, Nuclear Power Plants and Petrochemical Plants. Studies on the dynamic behaviour of Turbo Generator Foundations of various ratings have shown that there are various parameters that influence machine foundation response. Though it may not be possible to account for all of these effects in the design, it may still be desirable to take note of these and take precautionary measures, as far as possible, at the design stage itself. Some of these are listed hereunder:

i) Similar machines on the similar foundations have been observed to behave differently on **different soil.** Amplitudes of vibration on a 200 MW (3000 rpm) TG Foundation show high amplitudes on foundation built on hard rock compared to the other built on alluvial soil (see Chapter 14).

ii) Identical machines on identical foundations built on **identical soil** have also been found to exhibit different responses. Response of two identical foundations housing identical machines built side by side has been found to be different (see Chapter 14).

iii) Variation in Load (in case of Turbo-Generators), at times, has been found to influence response.

iv) Variation in grid frequency has also been found to influence response.

v) Deterioration of grout under sole plate, with time, has also been found to influence vibration response.

vi) Tightening torque of holding down bolts does influence the response.

vii) Honey-combing in the frame beams and columns.

viii) Loss of Contact underneath machine support plates supported embedded in the concrete.

ix) Resonance with elements of foundation (beam, column, cantilever projection etc has been found to influence the response significantly.

It is interesting to note that some of these parameters are machine related; some are installation related; some are construction related; some are design related and some may be attributed to combination of these. **Author strongly recommends** that the designer **must** touch upon all such issues that are possible to be included at the design stage itself.

A **Frame Foundation** typically consists of a Top Deck, a set of Frames/Columns and a Base Raft. In certain cases, a mid level platform is provided, on need basis, for supporting certain equipment. In some cases, equipment like condenser is supported over pedestals raised from the base raft and connected to machine at the top deck. Such equipment is either rigidly or flexibly mounted over the pedestals depending upon its connection to the turbine at the top deck. A typical Foundation is shown in Figure 9.2-1

The complete system is mathematically modeled and analyzed for natural frequencies and amplitudes. The extent, to which machine and foundation elements are modeled, depends upon machine and foundation characteristics.

Summary of Design Steps

1. Sizing of Foundation
2. Locating machine load points over the foundation top deck
3. Locating machine components supported at base raft, mid height deck/platform etc and their connections with machine at the top deck
4. Evaluation of Design Soil Stiffness Parameters
5. Identification/evaluation of Dynamic Forces
6. Analysis
   I. Dynamic Analysis
      a. Natural Frequencies and Mode Shapes
      b. Identification of modes likely to be in resonance with machine speeds (engine orders and harmonics)
      c. Evaluation of Dynamic Amplitudes
         - Steady State Amplitudes
         - Transient Amplitudes
   II. Strength and Stability Analysis
      a. Equivalent Static Forces (Normal Operating Conditions)
      b. Bearing Failure Loads (Abnormal conditions)
      c. Handling loads
      d. Short Circuit Loads
      e. Environmental Loads e.g. Earthquake Loads, Wind Loads etc.
      f. Thermal Loads (if any)

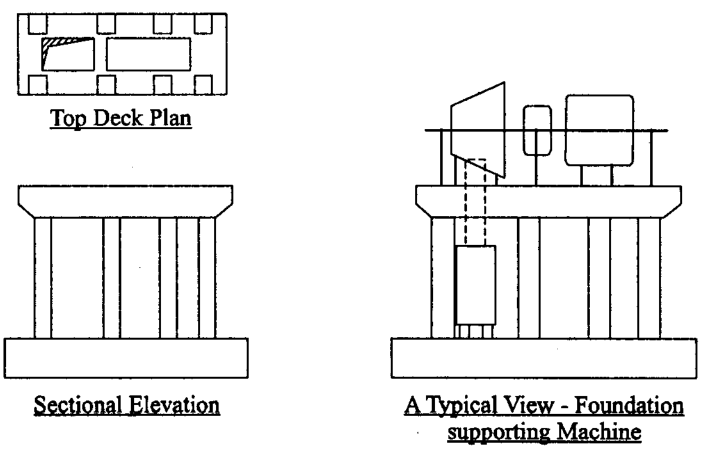


Figure 9.2-1 A Typical Frame Foundation

Required Input Data

A) Foundation Data

i) Foundation outline geometry, Levels etc
ii) Cut-outs, pockets, trenches, notches, projections etc

B) Machine Data

iii) Machine Layout
iv) Machine Load Points
v) Machine Dynamic Loads
vi) Associated excitation Frequencies
vii) Emergency Loads e.g. Short Circuit Torque, Bearing Failure Loads, Earthquake Loads, loss of blade etc.
viii) Allowable Amplitudes at Bearing Locations

C) Soil Data

ix) Site Specific Dynamic Soil Data
x) Soil type and its basic characteristic properties
xi) Bearing capacity
xii) Depth of water table
xiii) Liquefaction potential

D) Environmental Data

xiv) Site related Seismic data
xv) Wind Load Data

At this stage it is implied that Design Sub-grade Parameters, Design Machine Parameters and Design Foundation Parameters have duly been evaluated **in line with provisions given in** Chapters 5, 6, 7 and intricacies of **Modeling and Analysis,** as given in Chapter 8, have been well understood.

9.2.1 Dynamic Analysis:

From the point of view of dynamic amplitudes, following modes of vibration are of interest to designer:

i) Transverse Mode (perpendicular to rotor axis)

ii) Vertical Mode

iii) Lateral Vibrations Coupled with Torsional Vibrations

Frame foundation, being a 3-D Structural system, all these linear modes get associated with corresponding rotational modes of vibration. Whereas it is simple to evaluate all these modes with the help of computational tool, it is next to impossible to evaluate response of such a 3-D structure by manual method of analysis. Thus for manual computations, lot many assumptions and approximations are made to be able to tackle such foundations. In view of the limitations of the manual analysis procedures, the following practices are generally employed:

i) Foundation is split in to as many number of portal frames as present

ii) Transverse and Vertical vibrations are evaluated for these portal frames

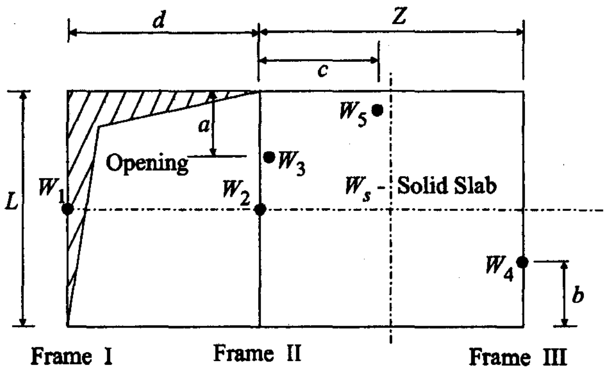
iii) Top deck being rigid, lateral vibrations coupled with torsional vibrations are evaluated using lateral stiffness properties of each portal frame

Note: Longitudinal vibration is generally not attempted using manual method of analysis

These cases are discussed one by one.

9.2.1.1 Loads on Frame Beam

From the overall machine and structural mass at the top deck and keeping in view the dynamics of the problem, the most important part is to identify mass associated with each frame for the purpose of frame analysis.



$W_1, W_2, W_3, W_4 \& W_5$ are machine load points

W_s - Weight of Solid Slab

Figure 9.2-2a Machine Loads @ Top Deck & Deck Self Weight - Typical Top Deck Plan

Consider a typical top deck plan with three frames showing distribution of machine loads on deck slab as shown in Figure 9.2-2a. For load nomenclature, refer a representative typical portal frame as shown in Figure 9.2-2b. In order to evaluate loads associated with each frame, it requires:

i) Identify machine loads at the deck and allocate the same to the nearest frame beam or longitudinal beam as the case may be using law of statics
ii) Evaluate self weight of each member at the top deck and transfer the same on to the frame beams/ longitudinal beams using law of static
iii) Evaluate self weight of each column

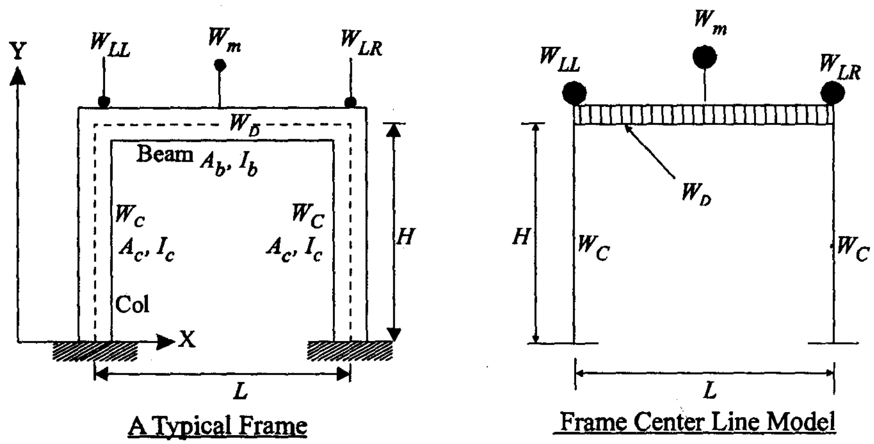


Figure 9.2-2b A representative Portal Frame showing Machine Loads &Top Deck Self Weight

Thus on each frame we get machine loads, self weight of transverse and longitudinal beams, self weight of deck slab etc. Load nomenclature associated with each frame is defined as under:

Let us denote the loads on the portal frame as under:

W_m **Total Machine Weight on Frame Beam** (9.2-1)

At times, depending on machine layout over foundation, weight of machine may be located at beam center or off-center location. It is therefore essential to compute equivalent machine weight (based on KE equivalence) at beam center

W_{mB} **Equivalent Machine Weight at Frame Beam Center**

This includes:
i) Weight of Machine directly located at frame beam center
ii) For Machine loads at off-center locations, equivalent machine weight at Frame Beam center using principles of kinetic energy equivalence as described in § 9.2.1.2 (see also Figure 9.2-3)

W_{mC} **Remaining Machine Weight** transferred to Column Top i.e. $W_{mC} = W_m - W_{mB}$

W_D **Total Distributed Weight on Frame Beam** (9.2-2)

This includes:
i) Beam self weight
ii) Concrete Pedestal weight (if any)
iii) Weight of other structural elements Transferred to frame beam through Deck slab
iv) Weight of other machine elements (if any) transferred to frame beam through deck slab

W_{LL} **Weight Transferred from Longitudinal beams @ top of Left column** (9.2-3)

This includes:
i) Weight of structural members transferred through longitudinal beams@ column top
ii) Weight of machine elements transferred through longitudinal beams @ column top

W_{LR} **Weight Transferred from Longitudinal beams @ top of Right column** (9.2-4)

This includes:
i) Weight of structural members transferred through longitudinal beams@ column top
ii) Weight of machine elements transferred through longitudinal beams @ column top

W_C **Weight of Each Column** (9.2-5)

This is to account for the cases when LHS column size is different than RHS column of the same frame.

9.2.1.2 Machine mass at off-center location

In many cases, machine mass may not be at frame beam center location. This requires equivalent generalised mass placed at frame beam center to be evaluated using principle of energy equivalence (see Chapter 2). The graph giving mass participation factor α as shown in Figure 2.1.1-12 is reproduced here for convenience in Figure 9.2-3. For machine mass m_m at beam center and another mass m_1 placed at a distance a from one end, equivalent mass at frame beam center becomes:

$$m^* = m_m + \alpha\, m_1 \qquad (9.2\text{-}6)$$

Here α is machine mass participation factor as given in Figure 9.2-3.

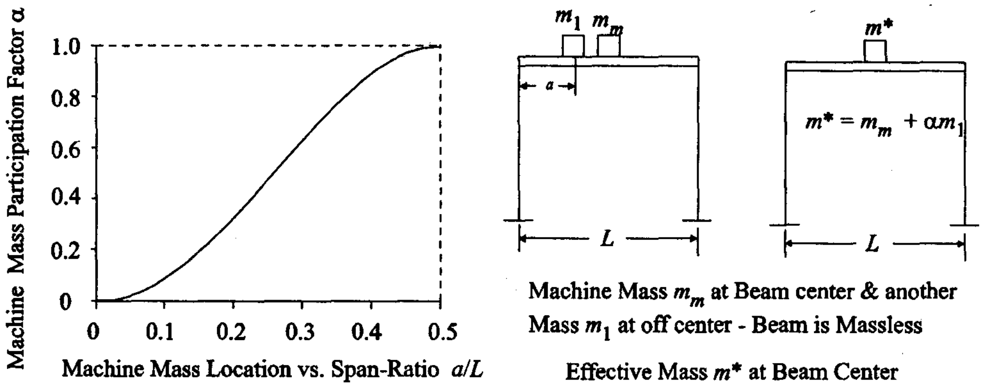
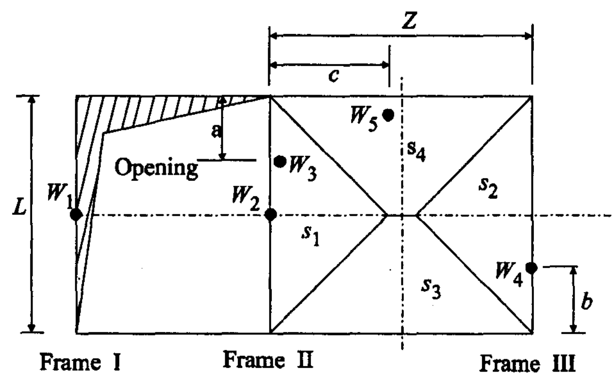


Figure 9.2-3 Mass participation Factor α - when Machine mass is at Off-centerLocation -Beam is Massless- Equivalent Mass of System

9.2.1.3 Computation of Loads on Frame Beams and Column Top

Consider a representative plan of a typical top deck (SAMPLE EXAMPLE) having three transverse frames i.e. Frame I, Frame II & frame III as shown in Figure 9.2-4



$W_1, W_2, W_3, W_4 \& W_5$ are machine load points
$s_1, s_2, s_3, \& s_4$ are weight of deck slab

Figure 9.2-4 Machine Loads @ Top Deck & Deck Self Weight

Here W_1, W_2, W_3, W_4 & W_5 represent machine loads and S_1, S_2, S_3 & S_4 represent self weight of deck slab segments. Member self weights (like beams) are also there but not indicated in the figure for clarity.

Let us consider load associated with each Frame one by one:

Frame I: This frame has opening on deck side i.e. no deck slab weight transferred to frame beam. Machine weight is located at Frame beam center. This gives:

$W_m = W_1$ (Total Machine Load on Frame Beam)
$W_{mB} = W_1$ (Machine loads at Frame Beam center)
$W_{mC} = W_m - W_{mB} = 0$ (Mchine Loads at Column Top)

Distributed Loads
$W_D =$ self weight of beam only

$W_{LL} =$ Reaction from Self weight of Longitudinal Beam on top of Left side column
$W_{LR} =$ Reaction from Self weight of Longitudinal Beam on top of Right side column

Frame II: This frame has opening on one side, deck slab on other side, machine weight W_2 located at Frame beam center and machine weight W_3 adjacent to beam located on the deck slab at a distance a from one end of the frame. This gives:

$W_m = W_2 + W_3$
$W_{mB} = W_2 + \alpha W_3$
$W_{mC} = W_m - W_{mB} = (1-\alpha)W_3$
$W_D =$ Beam self weight + Slab weight S_1

$W_{LL} =$ Reaction from LHS Longitudinal Beam
This includes its self weight, slab weight S_4 & Machine weight W_5

$W_{LR} =$ Reaction from RHS Longitudinal Beam
This includes its self weight & slab weight S_3

Frame III: This frame has deck slab on one side, machine weight W_4 located on the deck slab adjacent to beam at a distance b from one end of the frame. This gives:

$W_m = W_4$
$W_{mB} = \alpha W_4$
$W_{mC} = (1-\alpha)W_4$

W_D = Beam self weight + Slab weight S_2

W_{LL} = Reaction from LHS Longitudinal Beam

This includes its self weight, slab weight S_4 & Machine weight W_5

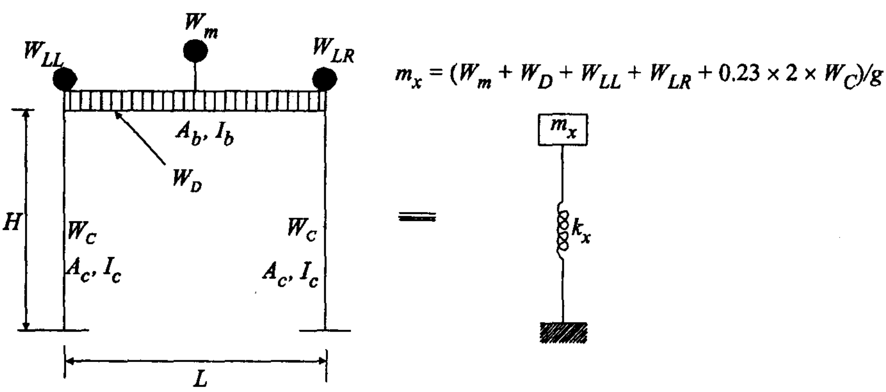
W_{LR} = Reaction from RHS Longitudinal Beam

This includes its self weight & slab weight S_3

Here α represents mass participation factor taken from Figure 9.2-3 for ratio b/L

9.2.2 Lateral Mode of Vibration along - X

Consider a Typical Frame as shown in Figure 9.2-5 showing loads as well as masses on the frame. Here W_m, W_D, W_C, W_{LL} & W_{LR} are as given by equations 9.2-1 to 9.2-5.



(i) Frame - Center Line Model (ii) Mathematical Model - Lateral Vibration

Figure 9.2-5 A Typical Frame - Mathematical Model - Lateral Vibration

Mass
$$m_x = \frac{W_m + W_D + W_{LL} + W_{LR} + 0.23 \times 2 \times W_C}{g} \tag{9.2-7}$$

Note: For mass participation factor 0.23, refer equation 2.1.1-21.

Stiffness: (see equation 2.1.1-39)

Lateral Stiffness
$$k_x = \frac{12EI_c}{H^3}\frac{1+6k}{2+3k} \tag{9.2-8}$$

Here $k = \frac{I_b/L}{I_c/H}$ represents ratio of beam to column stiffness (9.2-9)

Natural Frequency $p_x = \sqrt{\frac{k_x}{m_x}}$ (9.2-10)

Amplitude: Under influence of Dynamic Force $F_x \sin \omega t$ applied to the mass, maximum steady-state amplitude of mass m is given equation 2.2.2-5.

$$x(t) = \delta_x \frac{1}{\sqrt{\left(1-\beta_x^2\right)^2 + \left(2\beta_x \zeta_x\right)^2}}; \quad \delta_x = \frac{F_x}{k_x} \qquad (9.2\text{-}11)$$

Here δ_x represents static deflection, $\beta_x = \frac{\omega}{p_x}$ represents frequency ratio and ζ_x represents damping constant.

9.2.3 Vertical mode of Vibration along -Y

Consider a Typical Frame as shown in Figure 9.2-6 showing loads as well as masses on the frame. Here W_m, W_D, W_C, W_{LL} & W_{LR} are as given by equations 9.2-1 to 9.2-5.

For vertical motion (motion along Y), system can be represented as SDOF System or Two DOF System. Let us consider these two systems one by one.

i) Portal Frame represented as SDOF System: (Figure 9.2-6 (a))

Mass $$m_y = \frac{W_m + W_D + W_{LL} + W_{LR} + 0.33 \times 2 \times W_C}{g} \qquad (9.2\text{-}12)$$

Vertical Stiffness k_y:

Beam to Column Stiffness ratio **(see § 2.1.1.4.5)** $k = \frac{k_b}{k_c} = \frac{I_b/L}{I_c/H}$

a) Flexural Deformation of frame beam under unit load (see equation 3.1.6-4)

$$\overbrace{y_2}^{\text{Flexural}} = \frac{L^3}{96 E I_b} \times \frac{2k+1}{k+2} \qquad (9.2\text{-}13)$$

b) Shear deformation of Frame beam under unit load (see equation 7.9-2)

$$\overbrace{y_2}^{\text{Shear}} = \frac{3L}{8GA_b} \tag{9.2-14}$$

c) Vertical deformation of columns under unit load (see equation 3.1.6-5)

$$\overbrace{y_2}^{\text{Shear}} = \frac{3L}{8GA_b} \quad \overbrace{y_1}^{\text{Column}} = \frac{H}{E\left(2\times A_c\right)} = \frac{H}{2E\,A_c} \tag{9.2-15}$$

Total deformation at Frame Beam Center under Unit Load

$$y = \overbrace{y_2}^{\text{Flexural}} + \overbrace{y_2}^{\text{Shear}} + \overbrace{y_1}^{\text{Column}} = \left(\frac{L^3}{96EI_b}\times\frac{2k+1}{k+2}\right) + \frac{3L}{8GA_b} + \frac{H}{2E\,A_c} \tag{9.2-16}$$

Vertical Stiffness $\qquad k_y = \frac{1}{y}$ (9.2-17)

Natural Frequency $\qquad p_y = \sqrt{\frac{k_y}{m_y}}$ (9.2-18)

Amplitude: Under influence of Dynamic Force $F_y \sin\omega t$ applied to the mass, steady - state amplitude of mass m is given equation 2.2.2-5.

$$y(t) = \delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta_y\right)^2}} \sin(\omega t - \phi) \tag{9.2-19}$$

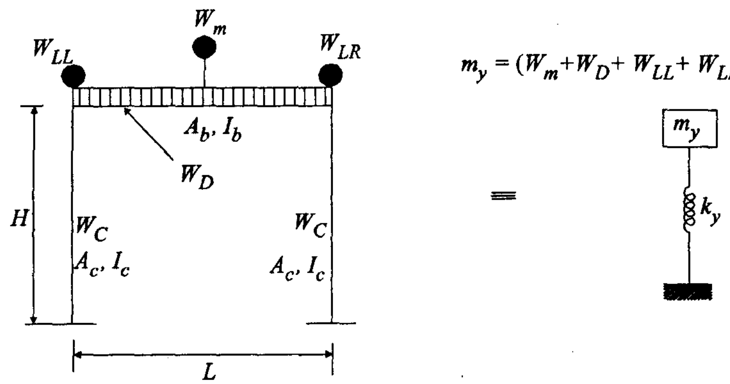
Here $\delta_y = \frac{F_y}{k_y}$ represents static deflection, $\beta_y = \frac{\omega}{p_y}$ represents frequency ratio, ζ_y represents damping constant and $\phi = \tan^{-1}\left(\frac{2\beta_y\zeta_y}{(1-\beta_y^2)}\right)$ represents the phase angle.

ii) Portal Frame represented as Two - DOF System (Figure 9.2-6 (b))

Mass m_2 $\qquad m_2 = \frac{\left(W_{mB} + 0.45\times W_D\right)}{g}$ (9.2-20)

Mass m_1 $\qquad m_1 = \frac{W_{mC} + 0.55\times W_D + W_{LL} + W_{LR} + 2\times 0.33\times W_C}{g}$ (9.2-21)

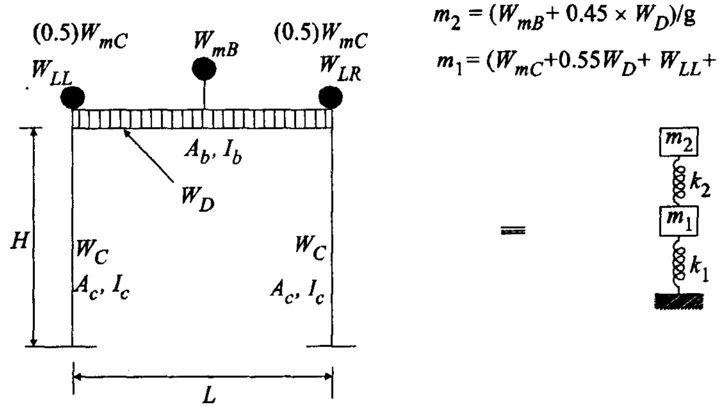
$$m_y = (W_m + W_D + W_{LL} + W_{LR} + 2 \times 0.33 W_C)/g$$



(a) Vertical Vibration - SDOF System

$$m_2 = (W_{mB} + 0.45 \times W_D)/g$$

$$m_1 = (W_{mC} + 0.55 W_D + W_{LL} + W_{LR} + 2 \times 0.33 W_C)/g$$



(b) Vertical Vibration - Two DOF System

Figure 9.2-6 A Typical Frame - Mathematical Model -Vertical Vibration

Stiffness:

Stiffness k_2

a) Flexural Deformation of frame beam under Unit Load

$$\overbrace{y_2}^{\text{Flexural}} = \frac{L^3}{96 E I_b} \times \frac{2k+1}{k+2} \qquad (9.2\text{-}22)$$

b) Shear deformation of Frame beam under Unit Load

$$\overbrace{y_2}^{\text{Shear}} = \frac{3L}{8GA_b} \qquad (9.2\text{-}23)$$

Total deformation at Frame Beam Center under Unit Load

$$y_2 = \overbrace{y_2}^{\text{Flexural}} + \overbrace{y_2}^{\text{Shear}} = \left(\frac{L^3}{96 E I_b} \times \frac{2k+1}{k+2} \right) + \frac{3L}{8 G A_b} \qquad (9.2\text{-}24)$$

$$\text{Stiffness} \qquad k_2 = \frac{1}{y_2} \qquad (9.2\text{-}25)$$

Stiffness k_1

Vertical deformation of columns under Unit Load

$$\overbrace{y_1}^{\text{Column}} = \frac{H}{E(2 \times A_c)} = \frac{H}{2 E A_c} \qquad (9.2\text{-}26)$$

$$\text{Stiffness} \qquad k_1 = \frac{1}{y_1} = \frac{2 E A_c}{H} \qquad (9.2\text{-}27)$$

Natural Frequencies:

Limiting Frequencies & Mass ratio (see equations 3.1.6-6)

$$p_{L1} = \sqrt{\frac{k_1}{m_1}};\ p_{L2} = \sqrt{\frac{k_2}{m_2}};\ \lambda = \frac{m_2}{m_1} \qquad (9.2\text{-}28)$$

Frequency equation (see equation 3.1.6-6)

$$p_{1,2}^2 = \frac{1}{2}\left\{ \left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) \mp \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)} \right\} \qquad (9.2\text{-}29)$$

Substituting for p_{L1}, p_{L2} & λ , roots of this equation give two natural frequencies p_1 & p_2.

Amplitude:

Under influence of Dynamic Force $F_y \sin \omega t$ applied to the mass m_2, maximum steady - state response of masses m_1 & m_2 are given by equations (3.2.4-7 & 8)

Maximum Response:

$$y_{1\,(\max)} = \frac{F_0}{k_1} \frac{1}{\left|\left(1-\beta_1^2\right)\right| \left|\left(1-\beta_2^2\right)\right|} \tag{9.2-30}$$

$$y_{2\,(\max)} = \frac{F_0}{k_2} \frac{\left(1+\lambda \dfrac{\beta_{L1}^2}{\beta_{L2}^2} - \beta_{L1}^2\right)}{\left|\left(1-\beta_1^2\right)\right| \left|\left(1-\beta_2^2\right)\right|} \tag{9.2-31}$$

Amplitude at Resonance: (see equation 3.2.4-9)

In case of resonance, taking advantage of the derivation done for damped SDOF system, it can be said that in case of resonance with vertical natural frequency p_1, the response to the system at resonance is obtained by replacing the term $\left|\left(1-\beta_1^2\right)\right|$ in denominator by $\sqrt{\left(1-\beta_1^2\right)^2 + \left(2\beta_1\zeta\right)^2}$.

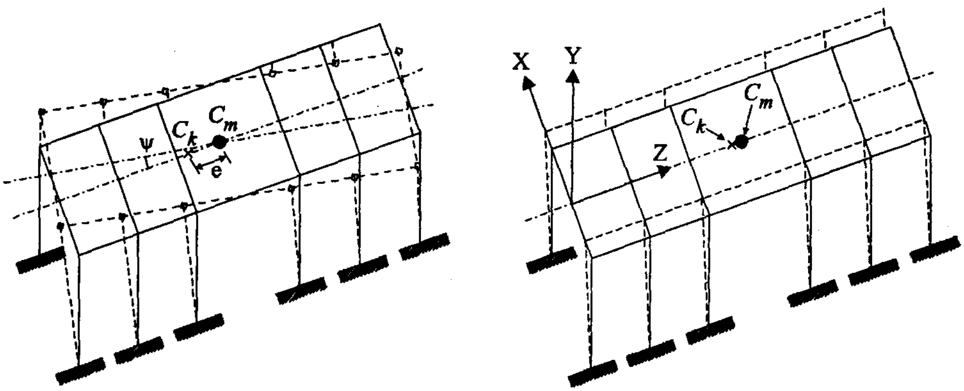
Similarly, in case of resonance with vertical natural frequency p_2, the response to the system at resonance is obtained by replacing the term $\left|\left(1-\beta_2^2\right)\right|$ in denominator by $\sqrt{\left(1-\beta_2^2\right)^2 + \left(2\beta_2\zeta\right)^2}$ in equations 9.2-30 & 31.

9.2.4 Lateral Vibrations Coupled with Torsional Vibrations

Consider a typical frame foundation as shown in Figure 9.2-7. System consists of n frames. Rotor center line is oriented towards Z axis. Figure 9.2-7 (i) shows coupling of lateral mode with torsional mode of vibration. This mode of vibration occurs due to presence of Top Deck Eccentricity i.e. eccentricity between center of mass and center of lateral stiffness of Frame foundation. In the absence of eccentricity, foundation exhibits pure translational vibration as shown in Figure 9.2-7 (ii).

Let m_{xi} & k_{xi} represent mass and stiffness associated with frame 'i' as given by equations 9.2-7 & 8 respectively. Let C_m represent center of mass and C_k represent center of stiffness. Top Deck Eccentricity e is the distance between center of mass C_m and center of stiffness C_k.

Mathematical representation of the frame foundation in Z-X plane is shown in Figure 9.2-8. Two coordinates namely translation x (along X) and rotation ψ (about Y) represent two degrees of freedom that define displaced position of the system.



(i) System with eccentricity e -Motion along X causes Torsional Vibration about Y

(ii) Motion along X - No eccentricity

Figure 9.2-7 Lateral Vibration of a typical frame foundation - Foundation with and without Top Deck Eccentricity

Let $\bar{z}_m$ & $\bar{z}_k$ denote distances of CG of overall mass m_x (point C_m) and stiffness k_x (point C_k) from Frame 1 and z_i represent distance of i^{th} frame from frame 1.

$$m_x = \sum m_{xi} \; ; \qquad \bar{z}_m = \frac{\sum m_{xi} z_i}{m_x} \tag{9.2-32}$$

$$k_x = \sum k_{xi} \; ; \qquad \bar{z}_k = \frac{\sum k_{xi} z_i}{k_x} \tag{9.2-33}$$

$$\text{Eccentricity } e \qquad e = \bar{z}_m - \bar{z}_k \tag{9.2-34}$$

It is desirable to restrict this eccentricity in Plan to be $\leq 1\%$ of corresponding top deck dimension **(See Chapter 7 - § 7.9.1).**

Equation of Motion

Let us consider **center of mass** C_m **as origin** (DOF location). Let a_i & b_i represent distance of i^{th} frame from center of mass point C_m and center of stiffness point C_k as shown in part (ii) of the figure. **Analysis for such a system vibrating in X -Y plane is given in § 3.1.5 and response is given in § 3.2.3.** Interchanging y with x and ϕ with ψ, we get solution the lateral and torsional vibration of frame foundation.

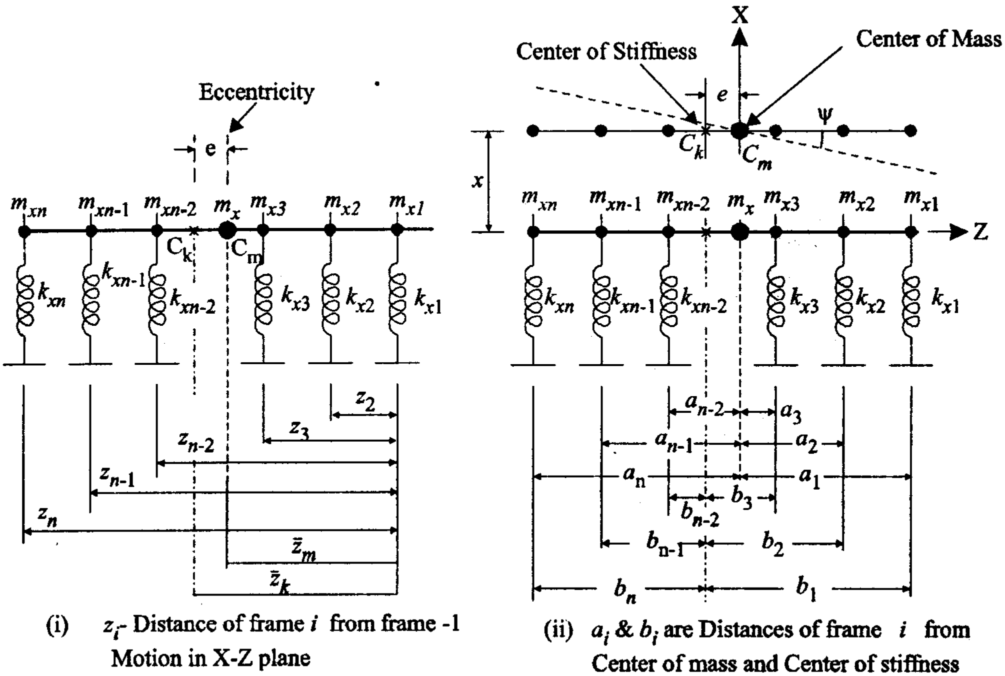


Figure 9.2-8 Mathematical Representation of A Typical Frame Foundation with n Frames - Motion in Z-X plane - Lateral vibration along X and Torsional vibration about Y

Total Mass Moment of Inertia of the system about Y @ DOF (see equation 3.1.5-6)

$$M_{my} = \sum m_{xi} a_i^2 \tag{9.2-35}$$

Total Torsional Stiffness of the system about Y @ DOF (see equation 3.1.5-7)

$$k_\psi = \sum k_{xi} b_i^2 \tag{9.2-36}$$

Equations of motion (see equations 3.1.5-10 & 11)

$$\begin{gathered} \ddot{x} + p_x^2\, x + e\, p_x^2\, \psi = 0 \\ \ddot{\psi} + p_x^2\, x \frac{e}{r^2} + p_x^2 \frac{e^2}{r^2} \psi + p_\psi^2\, \psi = 0 \end{gathered} \tag{9.2-37}$$

Terms $p_x = \sqrt{\dfrac{k_x}{m_x}}$; $p_\psi = \sqrt{\dfrac{k_\psi}{M_{my}}}$ & $r = \sqrt{\dfrac{M_{my}}{m_x}}$ represent limiting translational frequency, limiting torsional frequency and equivalent radius of gyration respectively.

It is also noted that both these equations are coupled through eccentricity term e. If eccentricity becomes zero, i.e. $e = 0$, both these equations get uncoupled and the limiting frequencies become natural frequencies.

Frequency equation (see equation 3.1.5-14)

$$p^4 - p^2\left(\alpha\, p_x^2 + p_\psi^2\right) + p_x^2 p_\psi^2 = 0$$

$$\text{Here } \alpha = \left(1 + \frac{e^2}{r^2}\right) \tag{9.2-38}$$

Roots of the equation 9.2-38 will yield two natural frequencies. We get

$$p_{1,2}^2 = \frac{1}{2}\left\{\left(\alpha\, p_x^2 + p_\psi^2\right) \pm \sqrt{\left(\alpha\, p_x^2 + p_\psi^2\right)^2 - 4 p_x^2 p_\psi^2}\right\} \tag{9.2-39}$$

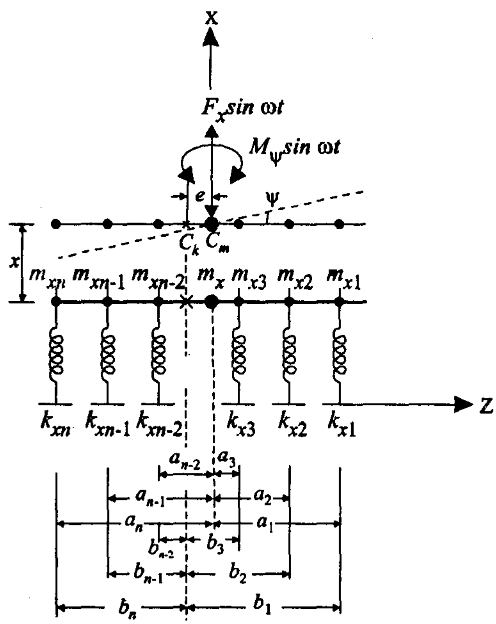


Figure 9.2-9 Frame Foundation with n Frames subjected to Dynamic Force & Moment applied at Center of Mass - Motion in Z-X Plane - Lateral vibration along X and Torsional vibration about Y

Amplitudes: Foundation in X-Z plane subjected to dynamic force $F_x \sin\omega t$ and moment $M_\psi \sin\omega t$ applied at center of mass is shown in Figure 9.2-9. Amplitudes are given by equation 3.2.3-5 & 6 by interchanging terms y by x & ϕ by ψ. We get response as:

$$x = \frac{\delta_{x(static)}\left(1 + \frac{\beta_\psi^2}{\beta_x^2}\frac{e^2}{r^2} - \beta_\psi^2\right) - \delta_{\psi(static)} \times e}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|} \tag{9.2-40}$$

$$\psi = \frac{\delta_{\psi(static)}\left(1-\beta_x^2\right) - \delta_{x(static)}\frac{\beta_\psi^2}{\beta_x^2}\frac{e}{r^2}}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|} \tag{9.2-41}$$

Here
$$\delta_x = \frac{F_x}{k_x};\ \delta_\psi = \frac{M_\psi}{k_\psi};\quad \beta_1 = \frac{\omega}{p_1}\ \&\ \beta_2 = \frac{\omega}{p_2};$$
$$\beta_x = \frac{\omega}{p_x}\ \&\ \beta_\psi = \frac{\omega}{p_\psi};\quad r^2 = \frac{M_{my}}{m_x} \tag{9.2-42}$$

When eccentricity is negligible, $(e/r)^2 \approx 0$, there is no coupling and we get limiting frequencies p_1 & p_2 same as p_x & p_ψ. With this, equations 9.2-41 & 42 become:

$$x = \frac{\delta_{x(static)}\left(1-\beta_\psi^2\right)}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|} = \frac{\delta_x}{\left|\left(1-\beta_x^2\right)\right|} = \delta_x\mu_x;\quad \mu_x = \frac{1}{\left|\left(1-\beta_x^2\right)\right|} \tag{9.2-43}$$

$$\psi = \frac{\delta_\psi\left(1-\beta_x^2\right)}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|} = \frac{\delta_\psi}{\left|\left(1-\beta_\psi^2\right)\right|} = \delta_\psi\mu_\psi;\quad \mu_\psi = \frac{1}{\left|\left(1-\beta_\psi^2\right)\right|} \tag{9.2-44}$$

It is seen that equation 9.2-43 is same as equation 9.2-11 for uncoupled lateral vibration.

Maximum Lateral amplitude due to translational motion and torsional motion shall occur at extreme end of the foundation.

Maximum amplitude $$x_{max} = x + a_n \times \psi \tag{9.2-45}$$

Here a_n is the maximum distance from center of mass to extreme end of foundation.

DESIGN EXAMPLE

D 9.2 Foundation for Turbo Generator

Design a frame foundation for a turbo generator. General arrangement and section of TG Foundation is shown in Figure D 9.2-1. Frame Plan and elevation showing center line dimensions is shown in Figure D 9.2-2. Machine Loads and unbalance forces are shown in Figure D 9.2-3. Data for machine and foundation is listed as under:

Machine Data

Machine Weight (Total including Rotor)

Turbine @ Bearing 1	400.00	kN
Turbine @ Bearing 2	360.00	kN
Generator Seating Plate location 3 -1	100.00	kN
Generator Seating Plate location 3 -2	100.00	kN
Generator Seating Plate location 4 -1	100.00	kN
Generator Seating Plate location 4 -2	100.00	kN
Total Machine weight	1160.00	kN

Weight of Rotor

Turbine Rotor weight @ Bearing 1	25.00	kN
Turbine Rotor weight @ Bearing 2	35.00	kN
Generator Rotor weight at generator Seating Plate location 3 -1	35.00	kN
Generator Rotor weight at generator Seating Plate location 3 -2	35.00	kN
Generator Rotor weight at generator Seating Plate location 4 -1	35.00	kN
Generator Rotor weight at generator Seating Plate location 4 -2	35.00	kN
Total Rotor Weight	200.00	kN

Machine Operating Speed 50.00 Hz

Unbalance Force

Along Y (Vertical)

Turbine @ Bearing 1	5.00	kN
Turbine @ Bearing 2	7.00	kN
Generator Seating Plate location 3 -1	7.50	kN
Generator Seating Plate location 3 -2	7.50	kN
Generator Seating Plate location 4 -1	7.50	kN
Generator Seating Plate location 4 -2	7.50	kN
Total Unbalance Force along Y (Vertical)	42.00	kN

Along X (Lateral)

Turbine @ Bearing 1	5.00	kN
Turbine @ Bearing 2	7.00	kN
Generator Seating Plate location 3 -1	7.50	kN
Generator Seating Plate location 3 -2	7.50	kN
Generator Seating Plate location 4 -1	7.50	kN

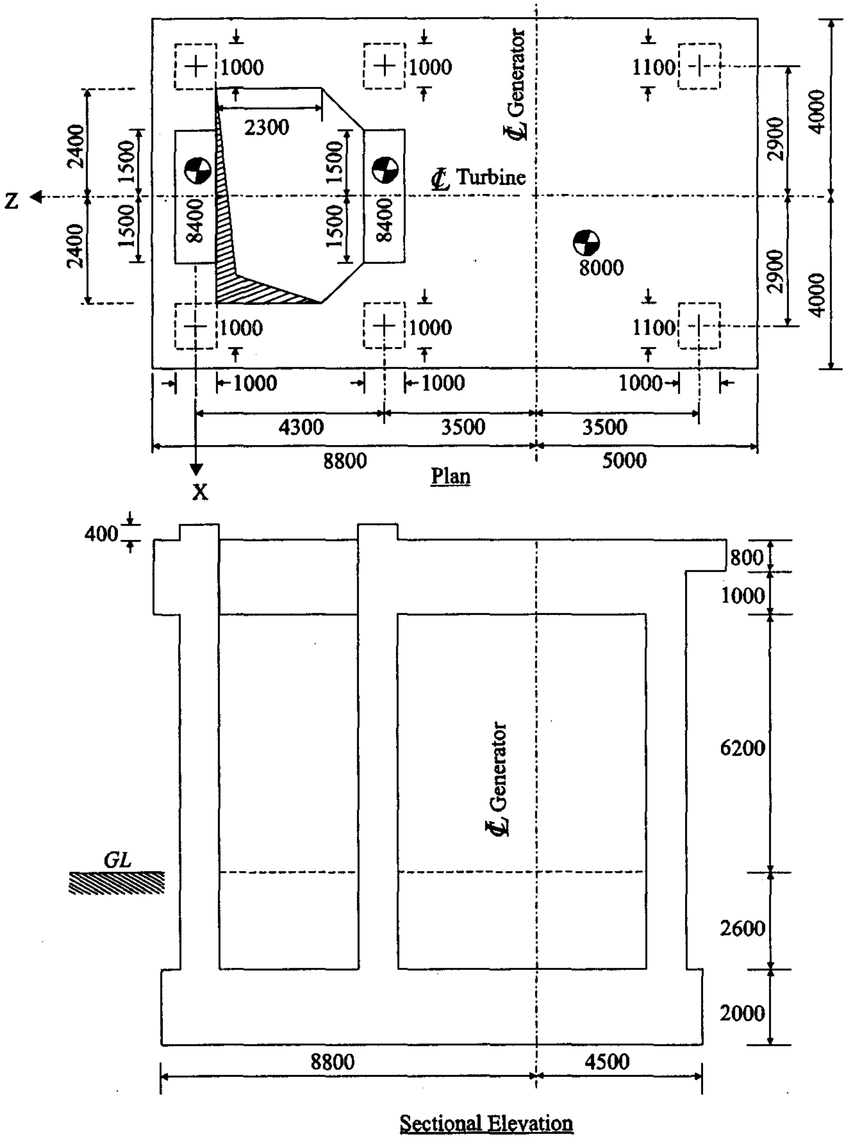


Figure D 9.2-1 General Arrangement TG Foundation

Generator Seating Plate location 4 -2	7.50	kN
Total Unbalance Force along X (Lateral)	42.00	kN

Force due to Blade Loss along X /Y

Turbine @ Bearing 1	3.00	kN
Turbine @ Bearing 2	11.00	kN
Total	14.00	kN
Short circuit Torque	2160.00	kNm
Distance between Seating Plate (along X)	2.40	m
Vertical Reaction @ seating Plate 3-1	450.00	kN
Vertical Reaction @ seating Plate 3-2	450.00	kN
Vertical Reaction @ seating Plate 4-1	450.00	kN
Vertical Reaction @ seating Plate 4-2	450.00	kN

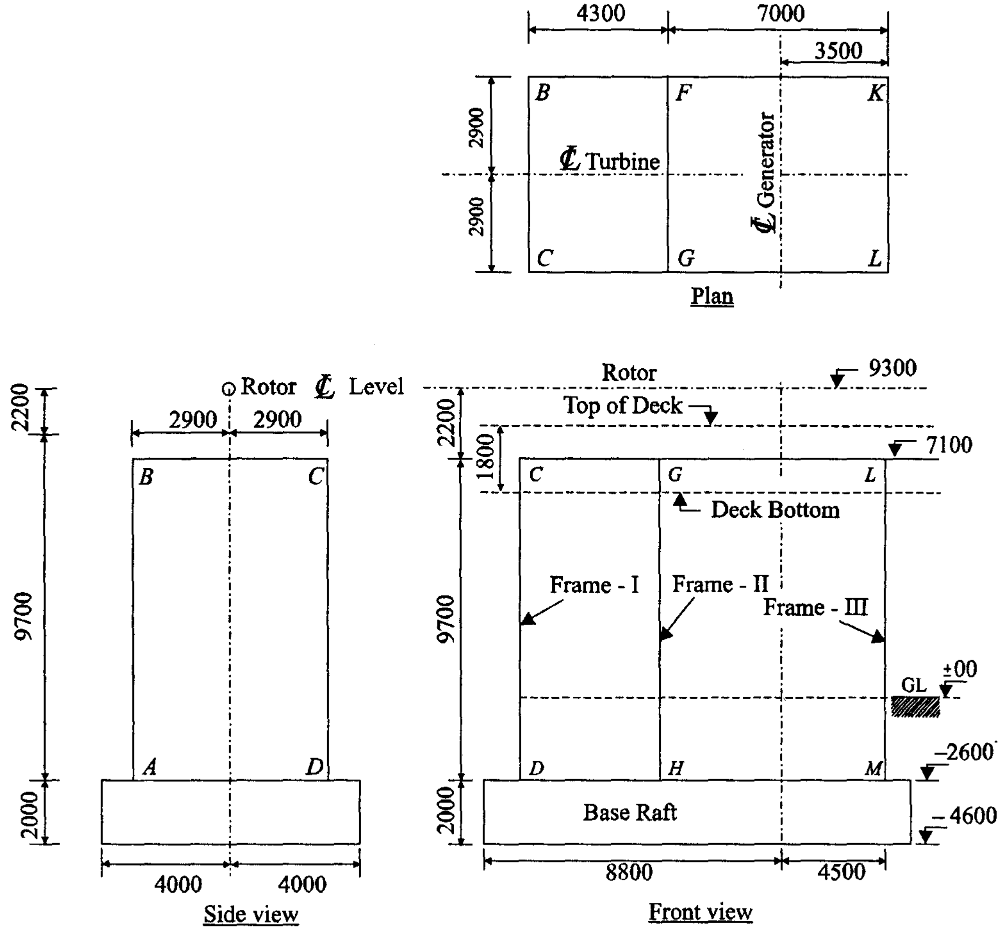
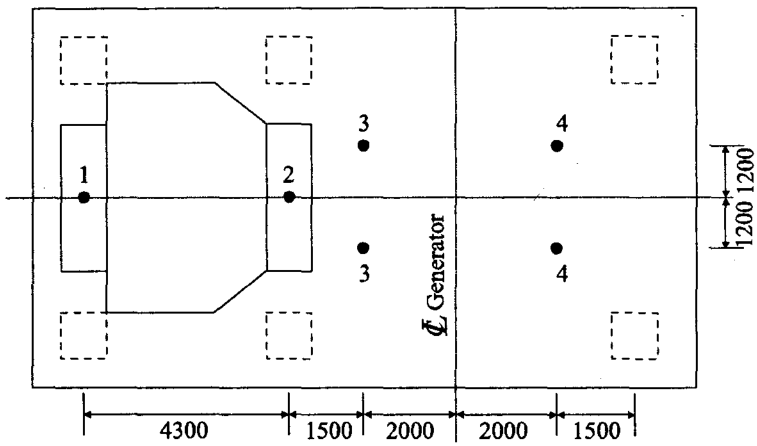


Figure D 9.2-2 Frame Plane & Elevation (center line)



Machine Load points

Load point	1 ●	2 ●	3 ●	4 ●	Total (kN)
Total M/C WT	400	360	200	200	1160 kN
Rotor WT	25	35	70	70	200 kN
Unbalance Lateral/Vertical	5	7	15	15	42 kN
Longitudinal	2	3	6	6	17 kN
Blade loss force	3	11	—	—	14 kN

Short Circuit Torque 2160 kNm

Machine Loads

Figure D 9.2-3 Machine Loads & Unbalance Forces at Top Deck

Foundation Data

Foundation material properties			
Concrete Grade		M25	
Mass density of concrete		2.50 t/m3	
Elastic Modulus E		3.00E+07 kN/m2	
Poisson's ratio		0.15 #	
Shear Modulus G		1.30E+07 kN/m2	
Top Deck	L=13.80 m	B = 8.00 m	Thickness =1.80 m
Base Raft	L =13.30 m	B = 8.00 m	Thickness = 2.00 m

Opening on Turbine Side (Trapezoidal Shape 4.8 m x 3.3 m as shown in Figure)

Frame Sizes:

	Frame 1	Frame 2	Frame 3	
Frame Beam width	1	1	1	m
Frame Beam depth	1.8	1.8	1.8	m
Frame span	9.7	9.7	9.7	m
Beam Moment of Inertia	0.49	0.49	0.49	m4
Column Moment of Inertia	0.08	0.08	0.11	m4

Soil Data

Coefficient of Uniform Compression	$C_u = 4\times10^4$ kN/m^3
Coefficient of Non-Uniform Compression	$C_\phi = 8\times10^4$ kN/m^3
Coefficient of Uniform Shear	$C_\tau = 2\times10^4$ kN/m^3
Coefficient of Non-Uniform Shear	$C_\psi = 3\times10^4$ kN/m^3

Other Loads

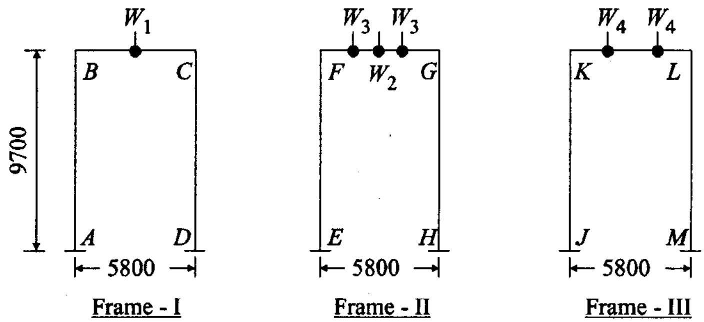
i) Earthquake loads — Equivalent seismic coefficient = 0.05 g

ii) Bearing Failure loads — 5 times rotor weight acting at bearing locations

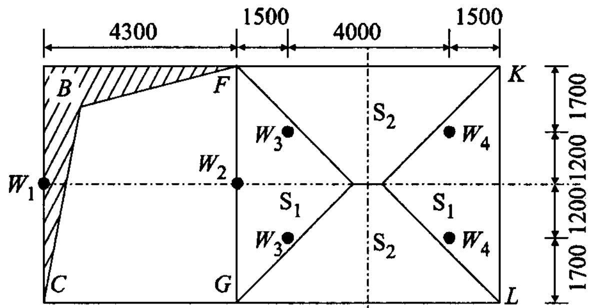
iii) Thermal Loads — Temperature differential of 25 degree C applied as a body force at the top surface of the top deck as well as inside surface of cut-out

Machine Mass on Frames: **(see Figure D 9.2-3 &4)**

Frame 1	
Mass @ frame Beam center	W1=400 kN
Total Mass on Frame 1	= 400 kN
Frame 2	
Mass @ frame Beam center	W2 =360 kN
Mass W3 @ 1.7 m from Left column	W3 = 100 kN
Mass W3 @ 1.7 m from Right column	W3 = 100 kN
Total Mass on Frame 2	= 560 kN
Frame 3	
Mass @ frame Beam center	Nil
Mass W4 @ 1.7 m from Left column	W4 = 100 kN
Mass W4 @ 1.7 m from Right column	W4 = 100 kN
Total Mass on Frame 3	= 200 kN
Total Machine Mass	1160 kN

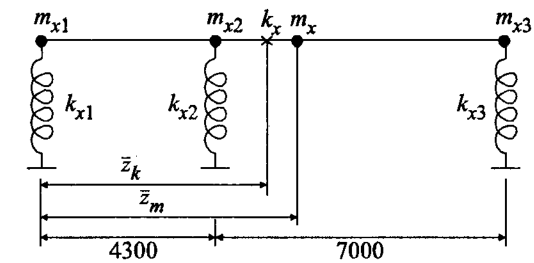


(i) Machine Loads on Frames



$S1$ & $S2$ show Partitioned Deck Slab

(ii) Machine Loads @ Top Deck



(iii) Eccentricity - Center of mass & Center of Shiftness

Figure D 9.2-4 Machine Loads and Eccentricity

Design

Sizing of Foundation

Top deck total weight (without cut-out) $1.8 \times 13.8 \times 8 \times 2.5 \times 9.81 = 4874$ kN

Weight of Opening size @ turbine side (Trapezium shape)

$$\left\{\frac{1}{2}(4.8+3)\times 1+2.3\times 4.8\right\}\times 1.8\times 2.5\times 9.81 = 660 \text{ kN}$$

Net weight of top deck $4874 - 660 = 4214$ kN

Weight ratio of top deck to machine $4214/1160 = 3.63$

Weight ratio is very high. For the present case i.e. real life Turbo- Generator Foundation, top deck thickness of 1.8 m is required by supplier. At this stage it is considered OK. This, however, needs to be checked from dynamic consideration i.e. frequency of frame beams.

Top Deck Eccentricity: Frame Lateral stiffness:

Frame 1 & Frame 2 $k = \dfrac{(I_b/L)}{(I_c/H)} = \dfrac{(0.49/5.8)}{(0.08/9.7)} = 9.75$; Frame 3 $k = \dfrac{(0.486/5.8)}{(0.111/9.7)} = 7.33$

Lateral Stiffness $k_x = \dfrac{12EI_c}{H^3}\dfrac{1+6k}{2+3k}$

Frame 1 $k_x = \dfrac{12\times 3\times 10^7\times 0.08}{9.7^3}\left(\dfrac{1+6\times 9.75}{2+3\times 9.75}\right) = 6.01\times 10^4$ kN/m

Frame 2 $k_x = \dfrac{12\times 3\times 10^7\times 0.08}{9.7^3}\left(\dfrac{1+6\times 9.75}{2+3\times 9.75}\right) = 6.01\times 10^4$ kN/m

Frame 3 $k_x = \dfrac{12\times 3\times 10^7\times 0.11}{9.7^3}\left(\dfrac{1+6\times 7.33}{2+3\times 7.33}\right) = 8.14\times 10^4$ kN/m

Total Lateral Stiffness

$$k_x = (6.01+6.01+8.14)\times 10^4 = 2.02\times 10^5 \text{ kN/m}$$

Center of Stiffness with respect frame 1

$$\bar{z}_k = (6.01\times 4.3+8.14\times(4.3+7))\times 10^4/2.02\times 10^5 = 5.83 \text{ m}$$

Masses associated with each Frame - see Figure D 9.2-4
(For Weight Nomenclature refer Figure 9.2-2b)

Machine weight on Frame Beam	W_m
Distributed loads on Frame Beam	W_D
Column weight (each)	W_C
Weight at Column Top (Left) transferred from Longitudinal Beam	W_{LL}
Weight at Column Top (Right) transferred from Longitudinal Beam	W_{LR}

Frame 1

W_m **Machine weight on Frame Beam** $W_m = 400\ \text{kN}$

W_D **Distributed Load on Frame Beam**

Self weight of Frame Beam BC $1.0 \times 1.8 \times 5.8 \times 2.5 \times 9.81 = 256\ \text{kN}$

Weight of Cantilever slab projection $0.5 \times 1.8 \times 5.8 \times 2.5 \times 9.81 = 128\ \text{kN}$

$W_D = 256 + 128 = 384\ \text{kN}$

W_{LL} **Load Transferred from Longitudinal Beam on column Top - Left**

Self Weight of Beam BF + Projection $(4 - 2.4) \times 1.8 \times 0.5 \times 4.3 \times 2.5 \times 9.81 = 151.8\ \text{kN}$

Portion of the slab at corner $(4 - 2.4) \times 1.8 \times 1.0 \times 2.5 \times 9.81 = 70.6\ \text{kN}$

$W_{LL} = 222\ \text{kN}$

W_{LR} **Load Transferred from Longitudinal Beam on column Top - Left**

Reaction from Beam BF (Self Weight of Beam BF + Projection)

$(4 - 2.4) \times 1.8 \times 0.5 \times 4.3 \times 2.5 \times 9.81 = 152\ \text{kN}$

Portion of the slab at corner $(4 - 2.4) \times 1.8 \times 1.0 \times 2.5 \times 9.81 = 70\ \text{kN}$

$W_{LR} = 152 + 70 = 222\ \text{kN}$

W_C **Self Weight of each Column** $W_C = 1.0 \times 1.0 \times 9.7 \times 2.5 \times 9.81 = 238\ \text{kN}$

Total Mass of Frame 1 $m_x = (400 + 384 + 222 + 222 + 0.23 \times 2 \times 238)/9.81 = 136\ \text{t}$

Frame 2

W_m **Machine weight on Frame Beam** $W_m = 560\ \text{kN}$

W_D **Distributed Load on Frame Beam**

Self weight of Frame Beam BC $1.0 \times 1.8 \times 5.8 \times 2.5 \times 9.81 = 256\ \text{kN}$

Weight of deck slab portion S1 $0.5 \times 4.8 \times 2.4 \times 1.8 \times 2.5 \times 9.81 = 254\ \text{kN}$

$W_D = 256 + 224 = 510\ \text{kN}$

W_{LL} **Load Transferred from Longitudinal Beam on column Top - Left**

Reaction from Beam BF & Beam FK - Self Weight of Beams + cantilever Projection

$$(4-2.4)\times 1.8\times (0.5\times (4.3+7))\times 2.5\times 9.81 = 399 \text{ kN}$$

Half of deck slab S2 $$0.5\times (0.5\times (6+6-4.8))\times 2.4\times 1.8\times 2.5\times 9.81 = 191 \text{ kN}$$

$$W_{LL} = 399+191 = 590 \text{ kN}$$

W_{LR} **Load Transferred from Longitudinal Beam on column Top - Left**

Reaction from Beam CG & Beam GL – (Self Weight of Beams + cantilever Projection)

$$(4-2.4)\times 1.8\times (0.5\times (4.3+7))\times 2.5\times 9.81 = 399 \text{ kN}$$

Half of deck slab S2 $$0.5\times (0.5\times (6+6-4.8))\times 2.4\times 1.8\times 2.5\times 9.81 = 191 \text{ kN}$$

$$W_{LR} = 399+191 = 590 \text{ kN}$$

W_C **Self Weight of each Column** $$W_C = 1.0\times 1.0\times 9.7\times 2.5\times 9.81 = 238 \text{ kN}$$

Total Mass of Frame 2 $$m_x = (560+510+590+590+0.23\times 2\times 238)/9.81 = 240 \text{ t}$$

Frame 3

W_m **Machine weight on Frame Beam** $$W_m = 200 \text{ kN}$$

W_D **Distributed Load on Frame Beam**

Self weight of Frame Beam KL $$1.0\times 1.8\times 5.8\times 2.5\times 9.81 = 256 \text{ kN}$$

Weight of deck slab portion S1 $$0.5\times 4.8\times 2.4\times 1.8\times 2.5\times 9.81 = 254 \text{ kN}$$

Weight of cantilever projection of slab $$1.0\times 1.8\times 5.8\times 2.5\times 9.81 = 256 \text{ kN}$$

$$W_D = 256+254+256 = 766 \text{ kN}$$

W_{LL} **Load Transferred from Longitudinal Beam on column Top - Left**

Reaction from Beam FK + cantilever Projection

$$(4-2.4)\times 1.8\times (0.5\times 7+1.5)\times 2.5\times 9.81 = 353 \text{ kN}$$

Half of deck slab S2 $$0.5\times (0.5\times (6+6-4.8))\times 2.4\times 1.8\times 2.5\times 9.81 = 191 \text{ kN}$$

$$W_{LL} = 353+191 = 544 \text{ kN}$$

W_{LR} **Load Transferred from Longitudinal Beam on column Top - Left**

Reaction from Beam GL + cantilever Projection

$$(4-2.4)\times 1.8\times (0.5\times 7+1.5)\times 2.5\times 9.81=353 \text{ kN}$$

Half of deck slab S2 $\quad 0.5\times (0.5\times (6+6-4.8))\times 2.4\times 1.8\times 2.5\times 9.81=191 \text{ kN}$

$$W_{LR}=353+191=544 \text{ kN}$$

W_C **Self Weight of each Column** $\quad W_C=1.1\times 1.0\times 9.7\times 2.5\times 9.81=262 \text{ kN}$

Total Mass of Frame 3 $\quad m_x=(200+766+544+544+0.23\times 2\times 262)/9.81=222 \text{ t}$

Total Mass of all the three frames $\quad m_x=\sum(136+240+222)=598 \text{ t}$

Center of Mass

CG of Masses from Frame 1 $\quad \bar{z}_m=(240\times 4.3+222\times 11.3)/597=5.93 \text{ m}$

Top Deck Eccentricity $\quad e=\bar{z}_m-\bar{z}_k=5.93-5.83=0.10 \text{ m}; \quad e=(0.10/13.8)\times 100=0.72\%$ OK

Dynamic Analysis

Lateral Vibration (along X)

Total lateral stiffness $\quad k_x=\sum(6.01+6.01+8.14)\times 10^4=2.02\times 10^5 \text{ kN/m}$

Natural Frequency $\quad p_x=\sqrt{\dfrac{2.02\times 10^5}{222}}=30.16 \text{ rad/s}$

Vertical Vibration **(Two DOF System Model)**

Frame 1

Mass

Total Machine weight on frame 1 400 kN

Machine weight at Frame Beam center 400 kN

Machine weight @ off center location Nil

Total Machine weight at Frame Beam center $W_{mB}=400 \text{ kN}$

Machine weight transferred to column top $W_{mC}=W_m-W_{mB}=0$

$$W_D=384 \text{ kN};\quad W_{LL}=222 \text{ kN};\quad W_{LR}=222 \text{ kN};\quad W_C=238 \text{ kN}$$

$$m_2 = \frac{(W_{mB} + 0.45\,W_D)}{g} = \frac{(400 + 0.45 \times 384)}{9.81} = 58.4 \quad \text{t}$$

$$m_1 = \frac{(W_{mC} + 0.55W_D + W_{LL} + W_{LR} + 0.33 \times 2 \times W_C)}{g} = \frac{(0 + 0.55 \times 384 + 222 + 222 + 0.33 \times 2 \times 238)}{9.81} = 83 \quad \text{t}$$

Stiffness

$$k_2 = \frac{1}{y_2}; \quad y_2 = \left(\frac{L^3}{96\,E\,I_b} \times \frac{2k+1}{k+2}\right) + \frac{3L}{8GA_b}$$

Deflection @ beam center under unit load

$$y_2 = \left(\frac{5.8^3}{96 \times 3 \times 10^7 \times 0.49} \times \frac{2 \times 9.75 + 1}{9.75 + 2}\right) + \frac{3 \times 5.8}{8 \times 1.3 \times 10^7 \times (1 \times 1.8)} = 3.34 \times 10^{-7} \quad \text{m}$$

$$k_2 = \frac{1}{y_2} = \frac{1}{3.34 \times 10^{-7}} = 3 \times 10^6 \quad \text{kN/m}$$

$$k_1 = \frac{(2 \times E \times A_c)}{h} = \frac{2 \times 3 \times 10^7 \times 1}{9.7} = 6.18 \times 10^6 \quad \text{kN/m}$$

Limiting Frequencies and Mass Ratio

$$p_{L2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{3 \times 10^6}{58.4}} = 226.4 \quad \text{rad/s}; \qquad p_{L1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{6.18 \times 10^6}{83}} = 273 \quad \text{rad/s}$$

$$\lambda = \frac{m_2}{m_1} = \frac{58.4}{83} = 0.7$$

Frequency Equation
$$p_{1,2}^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) \mp \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\}$$

Substituting values, we get two natural frequencies

$$p_1 = \sqrt{\frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) - \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\}} = 169.4 \ \text{rad/s}$$

$$p_2 = \sqrt{\frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) + \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\}} = 365 \ \text{rad/s}$$

Frame 2

Mass

Total Machine weight on frame 2 360+100+100 = 560 kN

Machine weight at Frame Beam center 360 kN

Machine weight @ off center location

100 kN @ 1.7 m from Left end column & 100 kN @ 1.7 m from Right end column

Mass Participation Factor for a/L = 1.7/5.8 = 0.29 (see Figure 9.2-3) $\alpha = 0.6$

Total Machine weight at Frame Beam center $W_{mB} = 360 + 120 = 480$ kN

Machine weight transferred to column top $W_{mC} = W_m - W_{mB} = 560 - 480 = 80$ kN

$$W_D = 510 \text{ kN}; \quad W_{LL} = 590 \text{ kN}; \quad W_{LR} = 590 \text{ kN}; \quad W_C = 238 \text{ kN}$$

$$m_2 = \frac{(480 + 0.45 \times 510)}{9.81} = 72.3 \text{ t}$$

$$m_1 = \frac{(80 + 0.55 \times 510 + 590 + 590 + 0.33 \times 2 \times 238)}{9.81} = 173 \text{ t}$$

Stiffness (same as for frame 1)

$k_2 = 3 \times 10^6$ kN/m; $k_1 = 6.18 \times 10^6$ kN/m

Limiting Frequencies and Mass Ratio

$$p_{L2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{3 \times 10^6}{72.3}} = 203.7 \text{ rad/s}; \qquad p_{L1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{6.18 \times 10^6}{173}} = 189.3 \text{ rad/s}$$

$$\lambda = \frac{m_2}{m_1} = \frac{72.3}{173} = 0.42$$

Substituting values in to Frequency Equation, we get two natural frequencies as

$$p_1 = 140.8 \text{ rad/s}; \quad p_2 = 273.1 \text{ rad/s}$$

Frame 3

Mass

Total Machine weight on frame 3 100+100 = 200 kN

Machine weight at Frame Beam center Nil

Machine weight @ off center location

100 kN @ 1.7 m from Left end column & 100 kN @ 1.7 m from Right end column

Mass Participation Factor for a/L = 1.7/5.8 = 0.29 (see Figure 9.2-3) $\alpha = 0.6$

Effective Mass at Beam center $0.6 \times (100 + 100) = 120$ kN

Total Machine weight at Frame Beam center $W_{mB} = 120$ kN

Machine weight transferred to column top $W_{mC} = W_m - W_{mB} = 200 - 120 = 80$ kN

$$W_D = 766 \text{ kN}; \quad W_{LL} = 544 \text{ kN}; \quad W_{LR} = 544 \text{ kN}; \quad W_C = 262 \text{ kN}$$

$$m_2 = \frac{(120 + 0.45 \times 766)}{9.81} = 47.4 \text{ t}$$

$$m_1 = \frac{(80 + 0.55 \times 766 + 544 + 544 + 0.33 \times 2 \times 262)}{9.81} = 179.6 \text{ t}$$

Stiffness

$$y_2 = \left(\frac{5.8^3}{96 \times 3 \times 10^7 \times 0.49} \times \frac{2 \times 7.33 + 1}{7.33 + 2} \right) + \frac{3 \times 5.8}{8 \times 1.3 \times 10^7 \times (1 \times 1.8)} = 3.25 \times 10^{-7} \text{ m}$$

$$k_2 = \frac{1}{y_2} = \frac{1}{3.25 \times 10^{-7}} = 3.08 \times 10^6 \text{ kN/m}$$

$$k_1 = \frac{(2 \times E \times A_c)}{h} = \frac{2 \times 3 \times 10^7 \times 1.1}{9.7} = 6.8 \times 10^6 \text{ kN/m}$$

Limiting Frequencies and Mass Ratio

$$p_{L2} = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{3.08 \times 10^6}{47.4}} = 254.8 \text{ rad/s}; \quad p_{L1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{6.8 \times 10^6}{179.6}} = 194.6 \text{ rad/s}$$

$$\lambda = \frac{m_2}{m_1} = \frac{47.4}{179.6} = 0.26$$

Substituting values in to Frequency Equation, we get two natural frequencies as

$$p_1 = 162 \text{ rad/s} \quad p_2 = 306 \text{ rad/s}$$

Coupled Lateral and Torsional Vibration

Since eccentricity is practically absent (within 1 %) these shall not be any coupling between translational and torsional mode. However, just for academic interest, computations are presented for coupled mode of vibration.

Distances of each frame from center of mass C_m :

$a_1 = 0 - 5.93 = -5.93$ m; $a_2 = 4.3 - 5.93 = -1.63$ m; $a_3 = (4.3 + 7) - 5.93 = 5.37$ m

Here a_1, a_2 & a_3 represent distance of Frame 1, 2 & 3 from Center of Mass C_m respectively (see Figure 9.2-9)

Distances of each frame from center of stiffness C_k : (refer Figure 9.2-9)

$b_1 = 0 - 5.83 = -5.83$ m; $b_2 = 4.3 - 5.83 = -1.53$ m; $b_3 = (4.3 + 7) - 5.83 = 5.47$ m

Here b_1, b_2 & b_3 represent distance of Frame 1, 2 & 3 from Center of Stiffness C_k respectively (see Figure 9.2-9)

Rewriting mass, stiffness and distances associated with each frame, we get

	Frame 1	Frame 2	Frame 3	
k_x	6.01×10^4	6.01×10^4	8.14×10^4	kN/m
m_x	136	240	222	t
a_i	-5.93	-1.63	5.37	m
b_i	-5.83	-1.53	5.47	m

$$M_{my} = \sum m_i a_i^2; \quad k_\psi = \sum k_i b_i^2; \quad k_x = \sum k_{xi}; \quad m_x = \sum m_{xi}$$

Substituting values, we get

$$M_{my} = 1.18 \times 10^4 \text{ tm}^2; \quad k_\psi = 4.62 \times 10^6 \text{ kNm/rad}; \quad k_x = 2.02 \times 10^5 \text{ kN/m}; \quad m_x = 598 \text{ t}$$

Eccentricity $e = 0.1$ m; radius of gyration $r = \sqrt{\frac{M_{my}}{m_x}} = 4.45$ m; $\alpha = 1 + (e/r)^2 = 1.0$

Limiting Frequencies

$$p_x = \sqrt{\frac{k_x}{m_x}} = \sqrt{\frac{2.02 \times 10^5}{598}} = 18.36 \text{ rad/s}; \quad p_\psi = \sqrt{\frac{k_\psi}{M_{my}}} = \sqrt{\frac{4.62 \times 10^6}{1.18 \times 10^4}} = 19.77 \text{ rad/s}$$

Frequency Equation $$p_{1,2}^2 = \frac{1}{2}\left\{\left(\alpha\, p_x^2 + p_\psi^2\right) \mp \sqrt{\left(\alpha\, p_x^2 + p_\psi^2\right)^2 - 4\, p_x^2\, p_\psi^2}\right\}$$

Substituting values, we get $p_1 = 18.33$ rad/s $p_2 = 19.77$ rad/s

It is worth noticing that natural frequencies are same as limiting frequencies because there is no eccentricity and hence no coupling of modes.

Amplitudes of Vibration

Machine operating Speed 50 Hz = 314 rad/s

A) Coupled lateral & Torsional Vibration

Since there is no coupling because eccentricity is negligible, we use equations 9.2-43 & 44.

$$x = \frac{\delta_{x(static)}\left(1-\beta_\psi^2\right)}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|} = \frac{\delta_x}{\left|\left(1-\beta_x^2\right)\right|} = \delta_x\mu_x; \quad \mu_x = \frac{1}{\left|\left(1-\beta_x^2\right)\right|} \tag{9.2-43}$$

$$\psi = \frac{\delta_\psi\left(1-\beta_x^2\right)}{\left|\left(1-\beta_1^2\right)\right|\left|\left(1-\beta_2^2\right)\right|} = \frac{\delta_\psi}{\left|\left(1-\beta_\psi^2\right)\right|} = \delta_\psi\mu_\psi; \quad \mu_\psi = \frac{1}{\left|\left(1-\beta_\psi^2\right)\right|} \tag{9.2-44}$$

i) Unbalance forces in-phase

Consider forces along (+) X

$F_x = 5+7+4\times7.5 = 42$ kN;
$M_\psi = 5\times5.93+7\times(5.93-4.3)-(4\times7.5\times(4.3+2+1.5-5.93)) = -15.04$ kNm

$k_x = 2.02\times10^5$ kN/m; $k_\psi = 4.62\times10^6$ kNm/rad;
$\delta_x = F_x/k_x = 2.08\times10^{-4}$ m; $\delta_\psi = M_\psi/k_\psi = -3.25\times10^{-6}$ m

$p_x = 18.36$ rad/s; $\beta_x = \omega/p_x = 17.1$; $\mu_x = \frac{1}{\left|\left(1-\beta_x^2\right)\right|} = 0.004$

$p_\psi = 19.77$ rad/s; $\beta_\psi = \omega/p_x = 15.88$; $\mu_\psi = \frac{1}{\left|\left(1-\beta_x^2\right)\right|} = 0.004$

Amplitude $x = \delta_x\times\mu_x = 2.08\times10^{-4}\times0.004 = 0.832\times10^{-6}$ m = 0.8 microns

Amplitude $\psi = \delta_\psi\times\mu_\psi = -3.25\times10^{-6}\times0.004 = 1.3\times10^{-8}$ rad

This torsional amplitude shall result in lateral amplitudes along X & Z.

Total Lateral amplitude along X $x = 0.8+1.3\times10^{-8}\times5.93\times10^6 = 0.88$ microns

This number is as good as zero and hence of no significance.

ii) Unbalance forces out-of-phase

$F_x = 5+7-4\times 7.5 = -18$ kN;

$M_\psi = 5\times 5.93 + 7\times(5.93-4.3) + (4\times 7.5\times(4.3+2+1.5-5.93)) = 97.16$ kNm

Moment is about 6.5 times that when forces are in phase

$\psi = 6.5\times 1.3\times 10^{-8} = 8.45\times 10^{-8}$ rad; $x = 8.45\times 10^{-8}\times 5.93\times 10^{6} = 0.5$ microns

This also is too small a value and hence of no significance.

B) Vertical Vibration along Y

Maximum Response: (see equations 9.2-30 & 31)

$$y_{1\,(\max)} = \frac{F_0}{k_1}\frac{1}{\left(1-\beta_1^2\right)\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}}; \qquad y_{2\,(\max)} = \frac{F_0}{k_2}\frac{\left(1+\lambda\frac{\beta_{L1}^2}{\beta_{L2}^2}-\beta_{L1}^2\right)}{\left(1-\beta_1^2\right)\sqrt{\left(1-\beta_2^2\right)^2+\left(2\beta_2\zeta\right)^2}}$$

Unbalance Forces on each frame:

Transferring unbalance forces from the machine to individual frames, we get;

Force on Frame 1 = 5 kN

Force on Frame 2 = 7 + 4 × 7.5 × (3.5/7) = 22 kN

Force on Frame 3 = 15 kN

Frame 1

$k_1 = 6.18\times 10^6$ kN/m; $k_2 = 3\times 10^6$ kN/m; $\lambda = 0.7$

$p_{L1} = 273$ rad/s; $p_{L2} = 226.4$ rad/s; $p_1 = 169.4$ rad/s; $p_2 = 365$ rad/s

$\omega = 314$ rad/s; $\beta_{L1} = 1.15$; $\beta_{L2} = 1.38$; $\beta_1 = 1.85$; $\beta_2 = 0.86$

Substituting values into amplitude equation, we get:

$y_1 = -1.21\times 10^{-6}$ m $= -1.21$ microns; $y_2 = -0.4\times 10^{-6}$ m $= -0.4$ microns

Total amplitude $y = \sqrt{1.21^2+0.4^2} = 1.27$ microns

Frame 2

$k_1 = 6.18\times 10^6$ kN/m; $k_2 = 3\times 10^6$ kN/m; $\lambda = 0.42$

$p_{L1} = 189.3$ rad/s; $p_{L2} = 203.7$ rad/s; $p_1 = 140.8$ rad/s; $p_2 = 273.1$ rad/s

$\omega = 314$ rad/s; $\beta_{L1} = 1.66$; $\beta_{L2} = 1.54$; $\beta_1 = 2.23$; $\beta_2 = 1.15$

Substituting values into amplitude equation, we get:

$y_1 = -5.62$ microns; $y_2 = 14.65$ microns

Total amplitude $y = \sqrt{5.62^2 + 14.65^2} = 15.7$ microns

Frame 3

$k_1 = 6.8 \times 10^6$ kN/m; $k_2 = 3.08 \times 10^6$ kN/m; $\lambda = 0.26$

$p_{L1} = 194.6$ rad/s; $p_{L2} = 254.8$ rad/s; $p_1 = 162$ rad/s; $p_2 = 306$ rad/s

$\omega = 314$ rad/s; $\beta_{L1} = 1.61$; $\beta_{L2} = 1.23$; $\beta_1 = 1.94$; $\beta_2 = 1.03$

Substituting values into amplitude equation, we get:

$y_1 = -6.93$ microns; $y_2 = 17.71$ microns

Total amplitude $y = \sqrt{6.93^2 + 17.71^2} = 19$ microns

Overall Total Vertical vibration of the top deck $= y = \sqrt{1.27^2 + 15.7^2 + 19^2} = 24.7$ microns

Strength Design

D) Other Loads

i) Earthquake loads:

Equivalent seismic coefficient = $0.05\,g$

Weight of Top deck + machine + 23% of Column weight = 5452 kN

Total seismic force (Considered along X) $= 0.05 \times 5452 = 273$ kN

ii) Bearing Failure loads

Bearing Failure loads equal to 5 times rotor weight acting at bearing locations

Total rotor weight = 200 kN

Bearing Failure Load (acting at bearing level) along X = 5 x 200 = 1000 kN

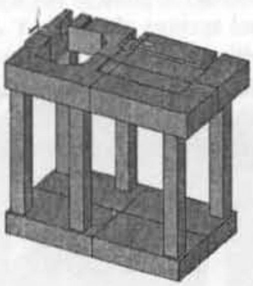
This is much higher than earthquake load, hence, governs the design. Design the foundation for this force using normal design procedures. Readers may use codes as applicable in their respective countries.

iii) Thermal Loads

Temperature differential applied as a body force at the top surface of the top deck as well as inside surface of cut-out is 25° C. Manual computations for thermal loads are quite complex hence not presented here, but these are included in Finite Element analysis and presented in the following section.

FINITE ELEMENT ANALYSIS

The TG Foundation as designed above by manual method of analysis has also been analysed using Finite Element (FE) Method. Salient results are presented here. Comparison with manual method of analysis is presented at the end of the analysis.

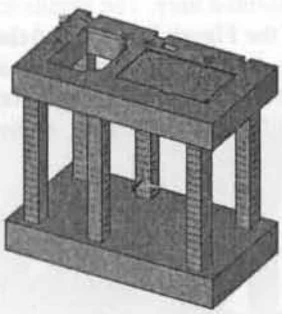


Actual TG Foundation

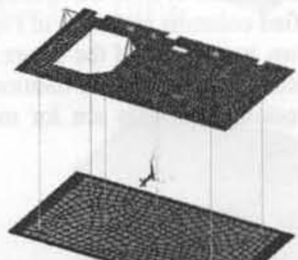
Solid Model Simplified

FE Mesh – Solid Elements

Solid Element Model



Model with element thickness ON



FE Mesh

Shell-Beam –Element Model

Figure D 9.2-7 TG Foundation Solid Model and Shell beam Model

Mathematical Model

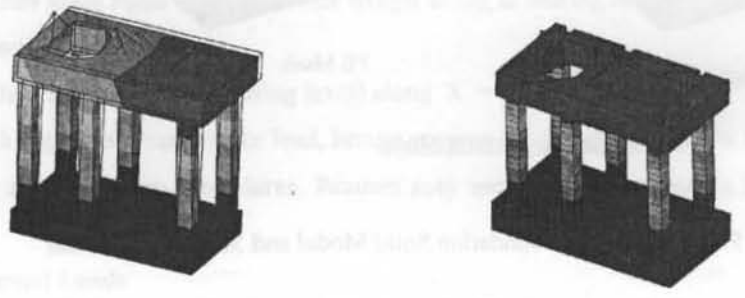
Mathematical model has been generated based on the foundation and machine data. Actual Foundation with all openings, cut-outs, recesses, notches etc as shown in Figure D 9.2-7 becomes too complex to model and analyze and moreover it is not necessary to analyze such complex model. Necessary assumptions and simplifications have been made to arrive at a model that is good

enough to represent the actual system. All major openings and depressions/recesses have been included in the mathematical model whereas all minor cutouts, notches, depressions etc have been excluded. Turbine and Generator masses are lumped at four bearing location at the top deck. Solid Model and FE Mesh of the foundation are shown in Figure D 9.2-7.

Soil is represented by six equivalent springs (3 translational and 3 rotational) applied at the CG of the base area of the base raft along respective DOF's i.e. 3 translational springs along X, Y & Z axes and 3 rotational springs about X, Y & Z axes. Since there are neither any haunches nor any depression/recess in deck slab, the system could as well be modeled using Shell and Beam elements. Just for academic interest, the system is modeled using both these element types namely i) Solid Elements & ii) Shell Beam elements. Comparison of natural frequencies and associated mode shapes, by both the models, indicates a fairly good agreement.

ANALYSIS

Initially, the foundation dimension suggested by the supplier had column sizes same for all the frames. The results of 1g-X static analysis revealed higher top deck eccentricity than permissible. To overcome the problem, the generator side columns were made stiffer by increasing their dimensions along frame. The results indicated a uniform movement when subjected to 1 'g' X load. Just for academic interest, results of both the analysis cases are presented here. The results of 1g-X analysis is with unmodified columns is shown in Figure part (a) of the Figure D 9.2-8 and that with modified column is shown in part (b) of the figure. The difference in color code at top deck as in part (a) indicates translation associated with rotation whereas uniform color as in part (b) indicates true translation. The remaining results are for model with modified columns only as used for manual method of analysis.



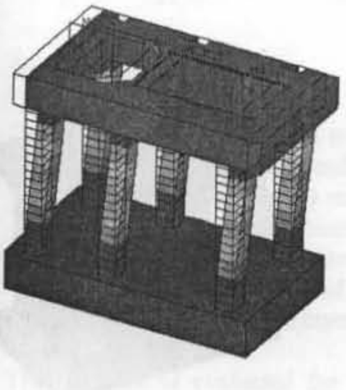
Max Displacement 38.7 mm — Columns Original

Max Displacement 28.8 mm — Columns Modified

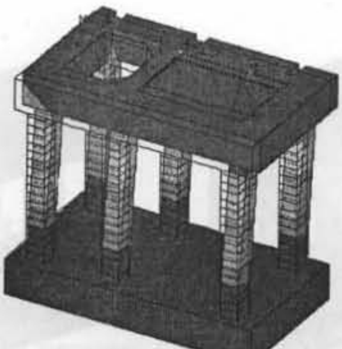
Figure D 9.2-8 Transverse Displacement – 1 'g' X Load

Free Vibration Analysis:

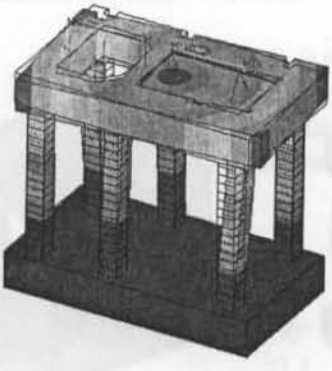
Natural Frequencies for various modes is listed in Table D 9.2-1. Fist four mode shapes are shown in Figure D 9.2-9 and some of the modes associated with column deformation mode are shown in Figure D 9.2-10. From the mode shapes it is seen that the first two modes are translational modes in Z & X direction, 3rd mode represents torsional mode of the top deck about Y and 4th mode represents vertical mode of vibration along Y.



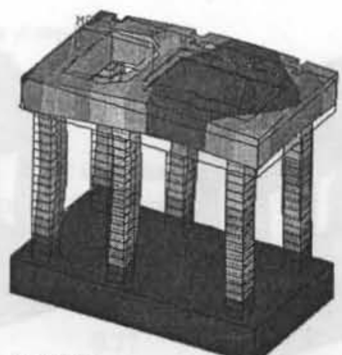
Mode 1 - 2.95 Hz



Mode 2 - 3.02 Hz



Mode 3 - 3.67 Hz



Mode 4 - 26.47 Hz

Figure D 9.2-9 First Four Modes of Vibration

It is also seen from above frequency table that modes 5 to 17 show frequencies lying in a close cluster. Study of mode shapes reveal that these modes correspond to column deformation mode along X & z directions.

From manual computation, it is seen that lateral translational frequency along X axis is 4.8 Hz as against 3.02 Hz given by FE Analysis. Comparing vertical natural frequency, it is seen that lower vertical natural frequencies, obtained by manual computation, for Frames 1, 2 & 3 are 27, 22.4 & 25.8 Hz and FE analysis gives vertical mode frequency as 26.5 Hz, which is in the same range.

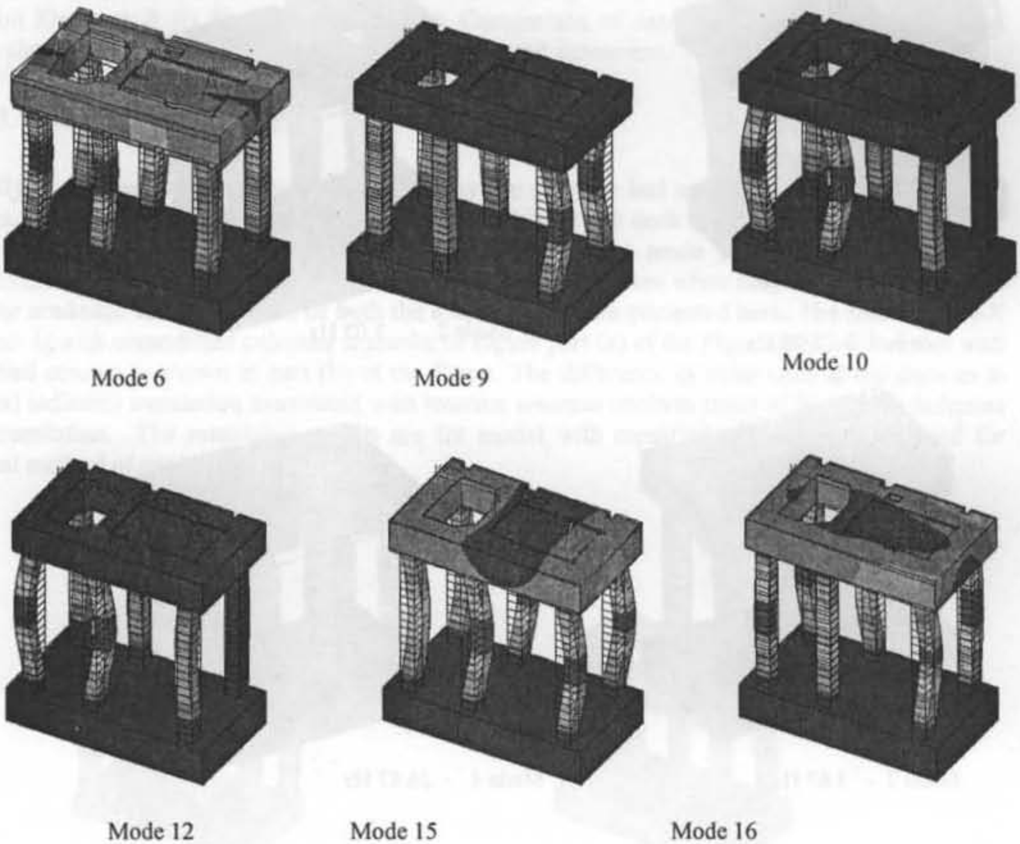


Figure D 9.2-10 Modes representing column vibration

Table D 9.2-1 Modes and Natural Frequencies

Mode #	Frequency Hz	Mode #	Frequency Hz	Mode #	Frequency Hz	Mode #	Frequency Hz
1	2.95	6	33.2	11	36.75	16	39.43
2	3.02	7	35.57	12	36.78	17	39.83
3	3.67	8	36.4	13	36.8	18	42.67
4	26.48	9	36.45	14	37:05	19	45.81
5	32.36	10	36.62	15	38.7	20	58.88

Response Analysis

Dynamic forces (see manual analysis above) are applied at the respective bearing level locations. These Forces are applied simultaneously at all the bearings but in one direction at a time. Steady State response is evaluated at salient locations, as under:

1. At all the bearing locations — 4 points
2. At all the corners of the top deck — 4 points
3. At all the mid points of columns — 6 points
4. At all the corners of the base raft — 4 points

The response is evaluated for ±5% of operating frequency i.e. from 47.5 Hz to 52.5 Hz and maximum value is reported. Damping used for response evaluation is considered as 5 % of critical. Maximum amplitudes are listed is in Table D 9.2-2.

Table D 9.2-2 Maximum Amplitudes

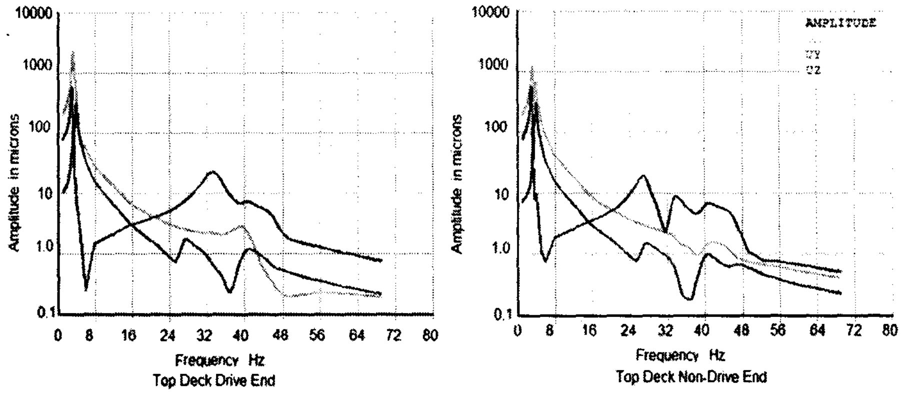
Excitation Frequency Hz	Dynamic Forces	
	In Phase	Out of Phase
	Amplitude - Microns	
47.5	3.19	4.6
50	1.75	2.3
52.5	0.65	0.77

Transient response

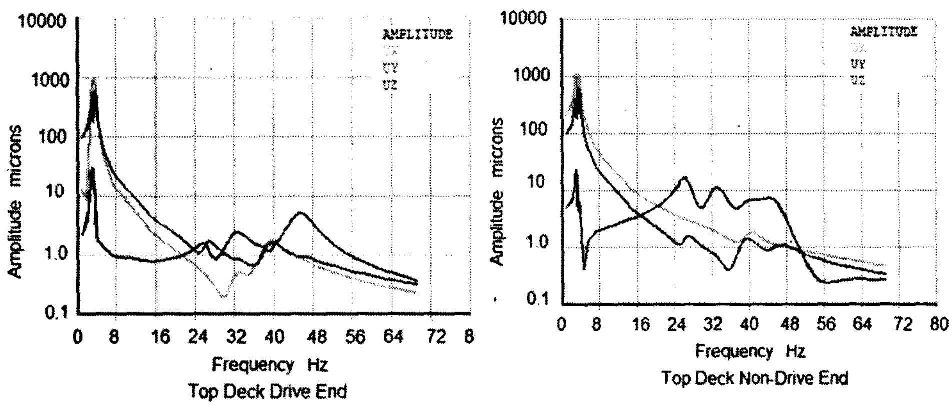
During Machine startup and coast-down conditions, all components of machine and foundation get excited at their respective natural frequencies resulting in enhanced amplitudes. For the transient response, the dynamic forces generated by the machine are applied at the respective bearing locations and a sweep run is performed for frequency 1 Hz to 52.5 Hz (in the present case sweep run is performed up to 65 Hz). Amplitudes are evaluated at desired locations of interest.

It is to be noted that the magnitude of the dynamic force is same as that computed for full operating speed. This force is however applied at frequencies from 1 to 65 Hz at an increment of about ¼ Hz.

The transient amplitudes so evaluated at transient resonant frequency (as shown in Figure D 9.2-11) are to be scaled down by square of the ratio of resonant frequency to operating frequency. In other words, amplitude say at transient resonance of 3 Hz is to be scaled down by a factor of square (3/50). The sweep response is shown in Figures D 9.2-11.



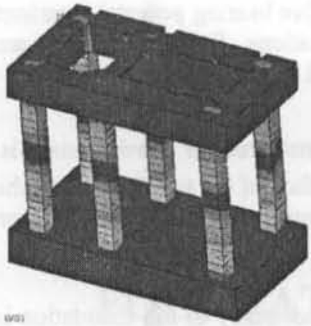
i) Dynamic Forces In-Phase



ii) Dynamic Forces Out of Phase

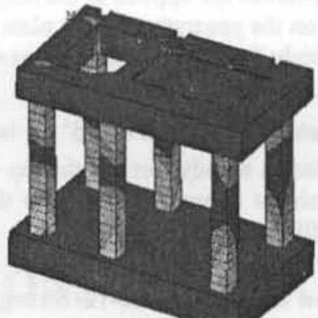
NOTE: These transient amplitudes are to be scaled down by square of the ratio of resonant frequency to operating frequency i.e. for transient resonance at 3 Hz, amplitudes to be scaled down by a factor of square (3/50).

Figure D 9.2-11 Transient Response



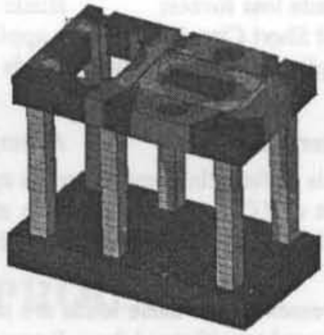
Bearing Failure Loads - X

Stress 3.3 MPa



Blade Loss - X

Stress 0.1 MPa

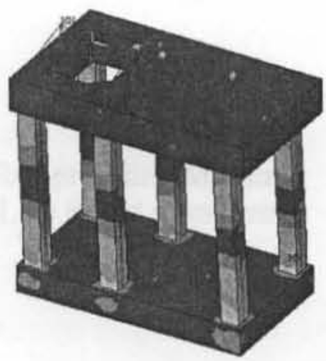


Short Circuit Loading

Stress 0.5 MPa

Thermal Loading

Stress 7 MPa



Earthquake Loading

Stress 5.2 MPa

Figure D 9.2-12 Stresses due to Bearing Failure, Blade Loss, Short circuit, Thermal Loading & Earthquake Loading

Strength Analysis

Foundation is analysed for equivalent static forces besides normal machine loads and self-weight of foundation.

Bearing Failure Loads: Bearing Failure Loads, equal to 5 times the rotor weight, are applied at the respective bearing locations along transverse and longitudinal directions, one at a time and stresses are computed in the foundation.

Blade loss forces: Blade loss forces are applied at the respective bearing pedestal locations and Short Circuit Forces are applied on the generator seating plate locations. Earthquake Loads are applied as equivalent static loads as body force on the machine as well as foundation.

Thermal Loads: A temperature differential of $25^\circ C$ is considered for thermal analysis. This differential temperature is applied as a body force at the top surface of the top deck while the rest of TG is considered to be at ambient temperature. Further the surface inside the opening on turbine side is also subjected to this differential temperature.

Stresses due to these loads are shown in Figure D 9.2-12. Strength adequacy of the foundation is ensured to withstand these forces.

In addition to the above stresses are also computed due to operating dynamic loads. Since the stresses on account of these dynamic loads are much smaller than bearing failure loads, these no longer remain governing loads.

10

FOUNDATIONS FOR RECIPROCATING MACHINES

- **Design Examples**

  - Block Foundation for a Typical Reciprocating Machine
  - Frame Foundation for a Typical Low Speed Compressor

For better clarity, all Figures related to FE analysis, including animations of frequencies and mode shapes, in color, are given in the CD attached at the end of the handbook

FOUNDATIONS FOR RECIPROCATING MACHINES

Different types of reciprocating machines that come under this category these have been adequately addressed in Chapter 6. Normally, both Block and Frame foundations are used to support these machines and these have been addressed in Chapter 7. Modeling aspects have been covered in Chapter 8.

Dynamic Forces developed by **Reciprocating Machines** are much higher compared to those generated by Rotary Machines. These dynamic forces are predominant along piston axis and dynamic forces are generated at operating speed as well as its 1^{st} Harmonic i.e. twice the operating speed. Allowable limits for amplitudes are higher for reciprocating machines compared to those for rotary machines.

The system vibrates in all six DOFs and thus requires computation of frequencies and amplitudes corresponding to all six DOF's. **Procedures** for design of foundations for machines supported on a) **Block Foundation** and b) **Frame Foundations** are given hereunder. The application of these design methodic for evaluation of natural frequencies and amplitudes are common for all types of machines irrespective of their speed.

10.1 DESIGN OF BLOCK FOUNDATION

Machine is considered supported by a block foundation resting directly over soil. The complete system is mathematically modeled and analyzed for natural frequencies and amplitudes. Mathematical treatment and Design steps are same as those for Rotary Machines given in Chapter 9. Representation of a typical foundation is shown in Figure 9.1-1 and necessary formulae required for computation of natural frequencies and response are given by equations 9.1-1 to 9.1-27.

Significant steps are reproduced for convenience.

Summary of Design Steps

1. Sizing of Foundation
2. Equivalent Soil Stiffness
3. Dynamic Forces
4. Analysis
   I. Dynamic Analysis
      i. Natural Frequencies
      ii. Dynamic Amplitudes
   II. Strength and Stability Analysis

Required Input Data

a) Machine Data
   1. Machine Layout
   2. Machine Load Distribution at Load Points
   3. Dynamic Loads
      a. Magnitude of Dynamic Loads
      b. Point of application and associated excitation Frequencies
   4. Allowable Amplitudes
b) Foundation Data
   1. Foundation outline geometry, Levels etc
   2. Cut-outs, pockets, trenches, notches, projections etc
c) Soil Data
   1. Site Specific Dynamic Soil Data
   2. Bearing capacity

At this stage it is implied that a) Site Soil data b) Machine data & c) Foundation data are converted to respective Design Parameters in line with provisions given in Chapter 5, 6 & 7. It is also anticipated that intricacies of Modeling and Analysis, as given in Chapter 8, have been well understood.

Design Data: The design data at this stage is summarized as under:

CG of Base area of Foundation, marked O represents DOF Location and is considered as Origin analysis and design.

Mass & Mass Moment of Inertia

Total Mass of Machine and Foundation	m
Height of Overall Centroid C from O	h

Mass Moment of Inertia (Machine+ Foundation) @ Overall Centroid C

Mass Moment of Inertia about X axis	M_{mx}

Mass Moment of Inertia about Y axis	M_{my}
Mass Moment of Inertia about Z axis	M_{mz}

Mass Moment of Inertia (Machine+ Foundation) @ DOF Location O

Mass Moment of Inertia about X axis	M_{mox}
Mass Moment of Inertia about Y axis	M_{moy}
Mass Moment of Inertia about Z axis	M_{moz}

Area and Moment of Inertia of Foundation Base in contact with soil

Area of Foundation	A
Moment of Inertia about X	I_{xx}
Moment of Inertia about Y	I_{yy}
Moment of Inertia about Z	I_{zz}

Equivalent Soil Stiffness at the foundation base level (at DOF location point O**) duly corrected for a) area effect and b) overburden pressure effect**

Translational Soil Stiffness along X	k_x
Translational Soil Stiffness along Y	k_y
Translational Soil Stiffness along Z	k_z
Rotational Soil Stiffness about X	k_θ
Rotational Soil Stiffness about Y	k_ψ
Rotational Soil Stiffness about Z	k_ϕ

Dynamic Loads:

- For FE Analysis, Dynamic Forces need to be specified only at respective bearing locations.
- For manual method of computation, Dynamic Forces acting at bearing locations are transferred at DOF Location point O in terms of Forces and Moments.

- One can have as many sets of forces and moments as number of excitation frequencies

Here we describe forces and moments @ DOF location **point O for manual method of computation.**

Forces @ DOF location **point** O along X, Y & Z direction F_x , F_y & F_z

Moments about X, Y & Z @ DOF location **point** O M_x , M_y & M_z

10.1.1 Dynamic Analysis

The dynamic analysis of a machine foundation system involves computation of natural frequencies and amplitudes of vibration.

From this stage onwards, one can choose either Finite Element Method of Analysis (Chapter 8) or Manual Method of Analysis (Chapters 2 &3).

Natural Frequencies: The machine foundation system undergoes Six Modes of Vibration i.e. three Translational Modes and three Rotational Modes (see chapter 3). Natural frequencies corresponding to these six modes of vibration are reproduced as under:

1. Motion along Y (Vertical direction): This mode is uncoupled (see equation 9.1-1)

Vertical Natural frequency $$p_y = \sqrt{\frac{k_y}{m}} \tag{10.1-1}$$

2. Rotation about Y (Torsional): This vibration mode is also uncoupled (see equation 9.1-2).

Torsional Natural frequency $$p_\psi = \sqrt{\frac{k_\psi}{M_{moy}}} \tag{10.1-2}$$

3. Motion in X-Y Plane - (Translation along X and Rocking about Z – i.e. $x \,\&\, \phi$ modes) - These modes are always coupled (see equation 9.1-3).

$$p_1^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) - \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2} \tag{10.1-3}$$

$$p_2^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) + \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2} \tag{10.1-4}$$

Here $\gamma_z = \dfrac{M_{mz}}{M_{moz}}$; $p_x^2 = \dfrac{k_x}{m}$; $p_\phi^2 = \dfrac{k_\phi - mgh}{M_{moz}}$ and

Here

p_1 & p_2 represent lower & higher natural frequencies

p_1 & p_2 represent lower & higher natural frequencies

4. Motion in Y-Z Plane – (Translation along Z and Rocking about X – i.e. $z\,\&\,\theta$ modes) – These modes are always coupled.

$$p_1^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) - \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x\, p_z^2 p_\theta^2} \qquad (10.1\text{-}5)$$

$$p_2^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) + \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x\, p_z^2 p_\theta^2} \qquad (10.1\text{-}6)$$

Here $\gamma_x = \dfrac{M_{mx}}{M_{mox}}$; $p_z^2 = \dfrac{k_z}{m}$; $p_\theta^2 = \dfrac{k_\theta - mgh}{M_{mox}}$

As far as possible, effort is made to ensure that these frequencies are not in direct resonance with operating speed/speeds of the machine. In fact these frequencies should preferably be away by a margin of $\pm 20\%$ from operating speed/speeds. In case resonance is noticed, it may be desirable to suitably alter the foundation dimensions and repeat the computations till the natural frequencies are found to be away from operating speed/speeds of the machine.

10.1.2 Amplitudes of Vibration

Vibration Amplitude is the response of the Machine Foundation System subjected to unbalance force acting on the machine. When the natural frequencies are in resonance with excitation frequency, damping plays a significant role and amplitudes need to be computed considering system with damping.

Response Computation using FE Analysis: For response computation, these unbalance forces are applied directly at the bearing level locations. Amplitudes at desired locations viz. Foundation top or bearing levels are obtained directly.

Response Computation using Manual Methods of Analysis: While evaluating response using manual method of analysis, these unbalance forces are transferred at the DOF location (CG of base area of foundation in contact with the soil i.e. **point** O). Thus we get three force components $F_x, F_y\,\&\,F_z$ and three moment components $M_\theta, M_\psi\,\&\,M_\phi$ @ **point** O. Undamped response is evaluated using equations 9.1-7 to 9.1-12 whereas equations 9.1-13 to 9.1-18 are used for evaluating damped response.

Amplitudes are evaluated at DOF location **point** O. Amplitudes at any other location viz. at foundation top or at bearing locations are computed using geometrical relationships given by equations 9.1-19 to 9.1-27.

DESIGN EXAMPLES

Design Examples are those which are encountered in real life practice. Comparison with **Finite Element** Analysis (**FEA**) is also given for specific cases to build up the confidence level. Effort is made to highlight the influence of certain slips commonly committed while computing response of the foundation.

EXAMPLE D 10.1: Foundation for a Reciprocating Engine

Design a foundation for a Single Cylinder Horizontal Reciprocating engine coupled with motor through gear box. Foundation outline showing machine-loading diagram, sectional elevation showing machine cg line, rotor-center line and bearing locations, is given in Figure D 10.1-1. Machine, foundation and soil parameters are as under:

A. Machine Data

Machine Weight

Compressor	220 kN
Motor (excluding Rotor)	100 kN
Motor Rotor	14 kN
Weight of Motor Bearing Pedestal (2kN each)	4 kN
Weight of Operating Gear	8 kN

Machine Speed

Operating Speed of engine	360 rpm
Operating Speed of Motor	720 rpm
Height of Rotor Centerline above Ground level	2000 mm
Height of Machine Centroid below rotor centerline	100 mm

Unbalance Forces Generated by Reciprocating Machine (engine)

Reciprocating engine is mounted over a base Frame. Unbalance Forces generated by engine are given at point Q (Point Q represents CG of Base Frame in contact with the foundation as shown in Figure 10.1-1)

Dynamic forces at point Q

Force along Z @ engine frequency F_{1z} @ $\cos\omega t = 10$ kN

Wm 1 = 25 kN Wm 2 = 25 kN Wm 3 = 25 kN Wm 4 = 25 kN
Wm 5 = 9 kN Wm 6 = 9 kN Wm 7 = 8 kN Wm 8 = 55 kN
Wm 9 = 55 kN Wm10 = 55 kN Wm11 = 55 kN

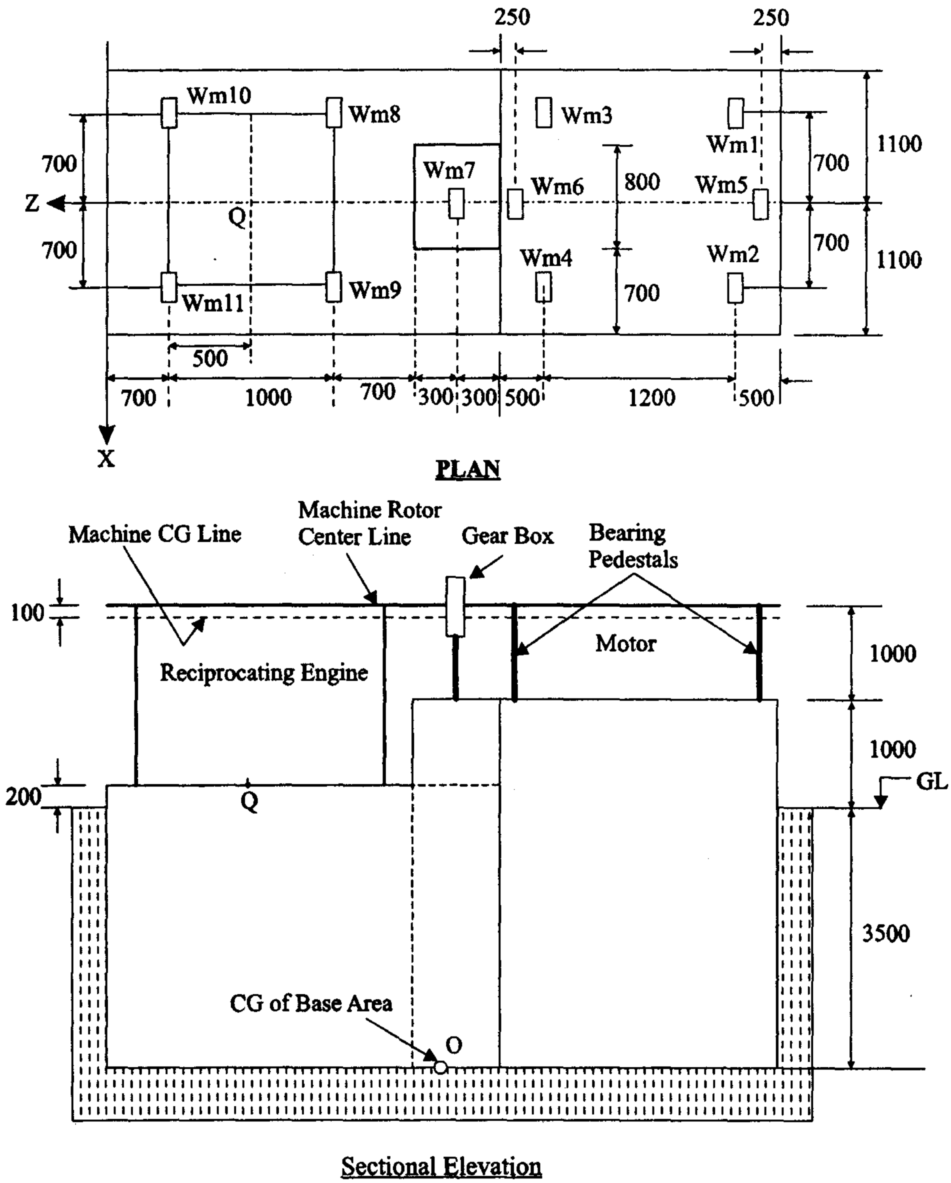


Figure D 10.1-1 Reciprocating Machine Coupled with Motor Through Gear Box supported on Block Foundation

Force along Z @ twice the engine frequency	F_{2z} @$\cos 2\omega t = 2.6$ kN
Moment about X at engine frequency	M_{1x} @$\cos \omega t = 13.5$ kNm
Moment about X @ twice the engine frequency	M_{2x} @$\cos 2\omega t = 6.7$ kNm

Unbalance Forces generated by Motor

Dynamic forces considered along Y (Vertical) @ center of rotor

Motor Excitation Frequency (speed of motor)	75.40 rad/s
Balance Grade for motor rotor	G 16 i.e. $e\omega = 16$ mm/s

Unbalance force @ center of rotor considered along Y

$$F_y = me\omega^2 = \frac{14}{9.81} \times 16 \times 10^{-3} \times \frac{720}{60} \times 2 \times \pi = 1.72 \text{ kN}$$

B. Foundation Data

Mass Density of Concrete	2.5 t /m^3
Foundation Length	5.2 m
Foundation width	2.2 m
Foundation depth below Ground level	3.5 m
Foundation Part above Ground level	
Supporting Drive machine	1.0 m
Supporting Compressor	0.2 m

Gear Box Pedestal Along length = 0.60 m; Along Width = 0.8 m; Height = 0.8 m

Foundation Plan and Section is shown in Figure D 10.1-2

C. Soil Data

Basic Soil Data	
Mass density of soil	2.0 t /m^3
Poisson's Ratio	0.25
Soil Damping Constant	0.1
Foundation depth for bearing capacity evaluation	3.5 m
Bearing capacity	250 kN /m^2
Coefficient of Uniform compression normalized for Area	10 m^2
Site Coefficient of Uniform Compression	$C_u = 5 \times 10^4$ kN/m^3
Corresponding Static Stress	100 kN/m^2

DESIGN

Foundation Sizing

Consider Foundation as shown in Figure D 10.1 & D 10.2

Consider Foundation to Machine Weight Ratio 3

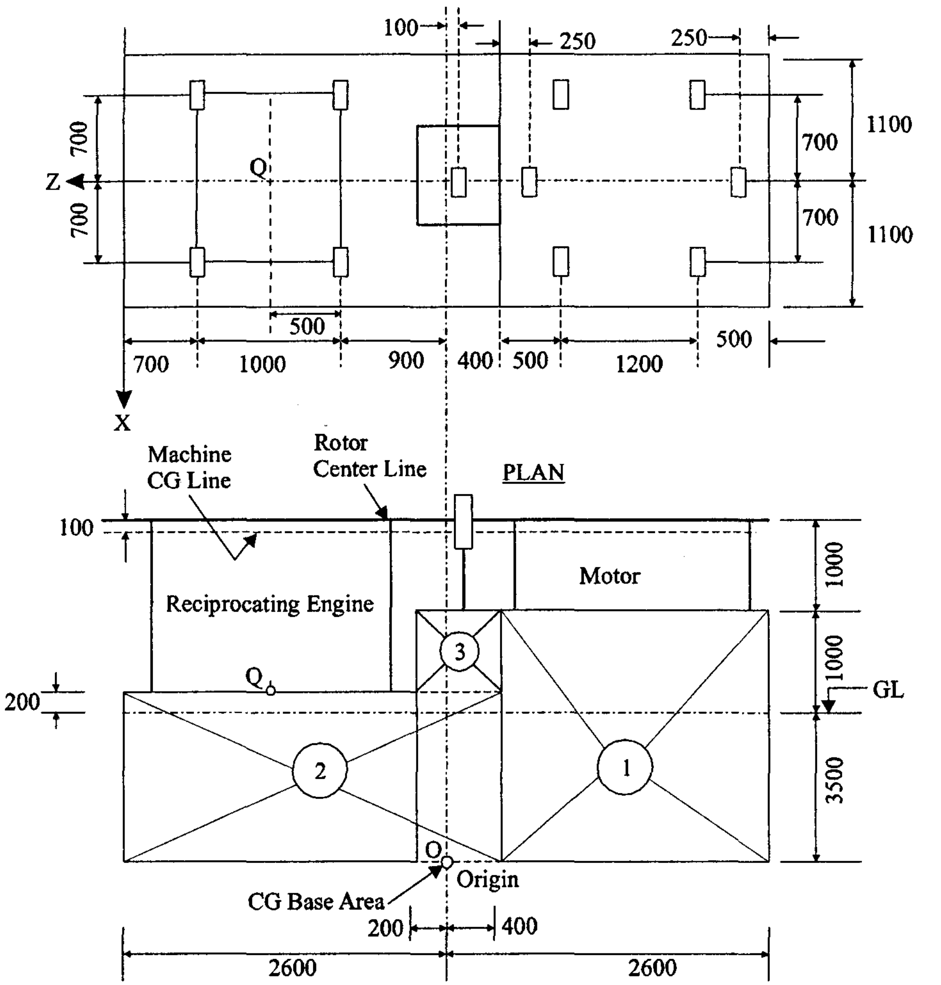


Figure D 10.1-2 Machine Layout with Respect to CG of Base Area O (Origin)

Weight of Machine	346 kN
Desired weight of Foundation	1038 kN

Overall Centroid

Overall Centroid with respect to CG of Base area: Consider CG of Base area point O (also termed as DOF location) as shown in Figure D 10.1-2.

a) Machine

	Drive M/c				Bearings + Pedestals		Coupling	Non-Drive M/c				
W_i	25.0	25.0	25.0	25.0	9.0	9.0	8.0	55.0	55.0	55.0	55.0	kN
x_i	0.7	-0.7	0.7	-0.7	0.0	0.0	0.0	0.7	-0.7	0.7	-0.7	m
y_i	5.4	5.4	5.4	5.4	5.5	5.5	5.5	5.4	5.4	5.4	5.4	m
z_i	-2.1	-2.1	-0.9	-0.9	-2.35	-0.65	-0.1	0.9	0.9	1.9	1.9	m

Let $\bar{x}_{mo}, \bar{y}_{mo}, \bar{z}_{mo}$ represent Machine Centroid with respect to CG of Base Area point O. We get

$$\sum W_i = 100+18+8+220 = 346\ kN; \quad \sum W_i x_i = 0.0; \quad \sum W_i y_i = 1871; \quad \sum W_i z_i = 130.2$$

$$\bar{x}_{mo} = \frac{\sum W_i x_i}{\sum W_i} = 0; \quad \bar{y}_{mo} = \frac{\sum W_i y_i}{\sum W_i} = 5.41; \quad \bar{z}_{mo} = \frac{\sum W_i z_i}{\sum W_i} = 0.38$$

b) Foundation

Block	Dimension x	y	z	Distance of CG from Point O xi	yi	zi
1	2.2	4.5	2.2	0.0	2.25	-1.5
2	2.2	3.7	3.0	0.0	1.85	1.1
3	0.8	0.8	0.6	0.0	4.10	-0.1

Weight of Foundation

$W_{f1} = 2.2 \times 2.2 \times 4.5 \times 2.5 \times 9.81 = 534$ kN

$W_{f2} = 3.0 \times 2.2 \times 3.7 \times 2.5 \times 9.81 = 599$ kN

$W_{f3} = 0.6 \times 0.8 \times 0.8 \times 2.5 \times 9.81 = 9.5$ kN

Total weight $= 1142$ kN

Let $\bar{x}_{fo}, \bar{y}_{fo}, \bar{z}_{fo}$ represent Foundation Centroid with respect to CG of Base Area point O. We get

$$\sum Wf_i = 534+599+9.5 = 1142.5\text{ kN}; \quad \sum W_{fi} x_i = 0.0; \quad \sum W_{fi} y_i = 2360.4; \quad \sum W_{fi} z_i = -143$$

$$\bar{x}_{fo} = \frac{\sum W_i x_i}{\sum W_i} = 0; \quad \bar{y}_{fo} = \frac{\sum W_i y_i}{\sum W_i} = 2.066; \quad \bar{z}_{fo} = \frac{\sum W_i z_i}{\sum W_i} = -0.125$$

Let $\bar{x}_o, \bar{y}_o$ & $\bar{z}_o$ represent overall centroid of Machine + Foundation system with respect to CG of Base Area point O. Substituting values we get:

$$\bar{x}_o = \frac{\sum\left(W_m \bar{x}_{mo} + W_f \bar{x}_{fo}\right)}{\sum\left(W_m + W_f\right)} = \frac{346 \times 0 + 1142 \times 0}{346 + 1142} = 0$$

$$\bar{y}_o = \frac{\sum\left(W_m \bar{y}_{mo} + W_f \bar{y}_{fo}\right)}{\sum\left(W_m + W_f\right)} = \frac{346 \times 5.41 + 1142 \times 2.066}{1488} = 2.84$$

$$\bar{z}_o = \frac{\sum\left(W_m \bar{z}_{mo} + W_f \bar{z}_{fo}\right)}{\sum\left(W_m + W_f\right)} = \frac{346 \times 0.38 + 1142 \times (-0.125)}{1488} = -0.008$$

Eccentricity

Eccentricity in X-Z plane:

Eccentricity along X-direction $e_x = \left(\bar{x}_o / B\right) \times 100 = \frac{0.0}{2.2} \times 100 = 0.0\% \quad < 5\% \quad$ OK

Eccentricity along Z-direction $e_z = \left(\bar{z}_o / L\right) \times 100 = \frac{0.008}{5.2} \times 100 = 0.15\% \quad < 5\% \quad$ OK

Both the values of eccentricity are less than 5 %, hence OK

Bearing Pressure

Foundation Base Area $5.2 \times 2.2 = 11.44 \text{ m}^2$

Soil bearing pressure $(1142 + 346)/11.44 = 130 \text{ kN/m}^2$

Margin for other loads $100 \times (1 - (130/250)) = 48\%$

Greater than 30 % Hence OK

Dynamic Analysis

Site Soil Parameters

Site Coefficient of Uniform Compression (as given) $C_{u01} = 5 \times 10^4 \text{ kN/m}^3$

Corresponding base area (given) $A_{01} = 10\ \text{m}^2$

Site Static Stress @ **3.5 m depth (as given)** $\bar{\sigma}_{01} = 100\ \text{kN/m}^2$

Design Soil Parameters

Width of Foundation $B = 2.2\ \text{m}$

Foundation depth Below GL $D = 3.5\ \text{m}$

Effective depth **(See $5.4)** $d_{02} = 0.5 \times 2.2 + 3.5 = 4.6\ \text{m}$

Overburden pressure due to soil at depth d_{02} $\sigma_1 = 2 \times 4.6 \times 9.81 = 90.25\ \text{kN/m}^2$

Area of Foundation $A_{02} = 11.44\ \text{m}^2$

Total weight of Machine + Foundation 1490 kN

Overburden pressure due to foundation + machine $\sigma_2 = \dfrac{1488}{11.44} = 130.07\ \text{kN/m}^2$

Design Static Stress $\bar{\sigma}_{02} = \sigma_1 + \sigma_2 = (90.25 + 130.07) = 220.32\ \text{kN/m}^2$

Design Coefficient of Uniform Compression $C_{u02} = C_{u01} \times \sqrt{\left(\dfrac{\bar{\sigma}_{02}}{\bar{\sigma}_{01}}\right)} \times \sqrt{\left(\dfrac{A_{01}}{A_{02}}\right)}$

Since $A_{02} = 11.44\ m^2 > 10\ \text{m}^2$; effective $A_{02} = 10\ \text{m}^2$

$$C_{u02} = 5 \times 10^4 \times \sqrt{\left(\frac{220.32}{100}\right)} \times \sqrt{\left(\frac{10}{10}\right)} = 7.42 \times 10^4\ \text{kN/m}^3$$

$$C_u = C_{u02} = 7.42 \times 10^4\ \text{kN/m}^3$$

Other design coefficients:

Coefficient of Uniform Shear $C_\tau = 0.5 \times C_u = 0.5 \times 7.42 \times 10^4 = 3.71 \times 10^4\ \text{kN/m}^3$

Coefficient of Non-Uniform Compression

$$C_\theta = C_\phi = 2 \times C_u = 2 \times 7.42 \times 10^4 = 14.84 \times 10^4\ \text{kN/m}^3$$

Non-Uniform Shear $C_\psi = 0.75 \times C_u = 0.75 \times 7.42 \times 10^4 = 5.565 \times 10^4\ \text{kN/m}^3$

Soil Stiffness (Equivalent Springs):

Moment of Inertia of Base Area:

About X-axis $I_{xx} = \frac{1}{12} \times 2.2 \times 5.2^3 = 25.78 \quad m^4$

About Z-axis $I_{zz} = \frac{1}{12} \times 2.2^3 \times 5.2 = 4.614 \quad m^4$

About Y-axis $I_{yy} = I_{xx} + I_{zz} = 25.78 + 4.614 = 30.4 \quad m^4$

Substituting values, we get:

$$k_x = C_\tau \times A = 3.71 \times 10^4 \times 11.44 = 4.25 \times 10^5 \quad kN/m$$

$$k_y = C_u \times A = 7.42 \times 10^4 \times 11.44 = 8.49 \times 10^5 \quad kN/m$$

$$k_z = k_x = C_\tau \times A = 4.25 \times 10^5 \quad kN/m$$

$$k_\theta = C_\theta \times I_{xx} = 14.84 \times 10^4 \times 25.78 = 3.83 \times 10^6 \quad kNm/rad$$

$$k_\psi = C_\psi \times I_{yy} = 5.565 \times 10^4 \times 30.4 = 1.69 \times 10^6 \quad kNm/rad$$

$$k_\phi = C_\phi \times I_{zz} = 14.84 \times 10^4 \times 4.614 = 6.85 \times 10^5 \quad kNm/rad$$

Mass and Mass Moment of Inertia

a) Mass Moment of Inertia about CG of Base Point O

Machine load distribution and locations with respect to point O (see Figure 10.1-2)

i) Machine

	Drive M/c				Bearings		Coupling	Non-Drive M/c				
W_i	25.0	25.0	25.0	25.0	9.0	9.0	8.0	55.0	55.0	55.0	55.0	kN
x_i	0.7	-0.7	0.7	-0.7	0.0	0.0	0.0	0.7	-0.7	0.7	-0.7	m
y_i	5.4	5.4	5.4	5.4	2.7	2.7	2.7	5.4	5.4	5.4	5.4	m
z_i	-2.1	-2.1	-0.9	-0.9	-2.35	-0.65	-0.1	0.9	0.9	1.9	1.9	m

Total machine Mass $= 346/9.81 = 35.27$ t

Mass Moment of Inertia of Machine

$$M_{\text{mox\_machine}} = \sum \{(W_i / g) \times (y_i^2 + z_i^2)\} = 1113 \quad \text{t m}^2$$

$$M_{\text{moy\_machine}} = \sum \{(W_i / g) \times (x_i^2 + z_i^2)\} = 97.6 \quad \text{t m}^2$$

$$M_{\text{moz\_machine}} = \sum \{(W_i / g) \times (x_i^2 + y_i^2)\} = 1047.4 \quad \text{t m}^2$$

ii) Foundation

	Dimension			Distance of CG from Point O				
Block	x	y	z	xi	yi	zi	Density	Mass
1	2.2	4.5	2.2	0.0	2.25	-1.5	2.5	$2.2 \times 4.5 \times 2.2 \times 2.5 = 54.45$
2	2.2	3.7	3.0	0.0	1.85	1.1	2.5	$2.2 \times 3.7 \times 3.0 \times 2.5 = 61.05$
3	0.8	0.8	0.6	0.0	4.10	-0.1	2.5	$0.8 \times 0.8 \times 0.6 \times 2.5 = 0.96$
								Total Mass = 116.46 t

Mass Moment of Inertia of Foundation

$$M_{\text{mox - foundation}} = \sum \{(m_i / 12)(y^2 + z^2) + m_i (y_i^2 + z_i^2)\} = 926.49 \text{ t m}^2$$

$$M_{\text{moy - foundation}} = \sum \{(m_i / 12)(x^2 + z^2) + m_i (x_i^2 + z_i^2)\} = 310.81 \quad \text{t m}^2$$

$$M_{\text{moz - foundation}} = \sum \{(m_i / 12)(y^2 + x^2) + m_i (y_i^2 + x_i^2)\} = 708.95 \quad \text{t m}^2$$

Total Mass and Mass Moment of Inertia about CG of Base Point O

$$m = 35.27 + 116.46 = 151.73 \quad \text{t}$$

$$M_{mox} = 1113 + 926.49 = 2039.5 \text{ t m}^2$$

$$M_{moy} = 97.6 + 310.81 \quad = 408.42 \text{ t m}^2$$

$$M_{moz} = 1047.4 + 708.95 = 1756.35 \text{ t m}^2$$

b) Mass Moment of Inertia about Overall Centroid

Coordinates of Overall Centroid with respect to CG of Base Area point O

$\bar{x}_o = 0; \quad \bar{y}_o = 2.8; \quad \bar{z}_o = -0.008$

$M_{mx} = M_{mox} - m\left(\bar{y}_o^{\,2} + \bar{z}_o^{\,2}\right) = \left\{2039.5 - 151.73 \times \left(2.8^2 + (-0.008)^2\right)\right\} = 820.2 \text{ t m}^2$

$M_{my} = M_{moy} - m\left(\bar{x}_o^{\,2} + \bar{z}_o^{\,2}\right) = \left\{408.42 - 151.93 \times \left(0 + (-0.008)^2\right)\right\} = 408.4 \text{ t m}^2$

$M_{mz} = M_{moz} - m\left(\bar{y}_o^{\,2} + \bar{x}_o^{\,2}\right) = \left\{1756.35 - 151.93 \times \left(2.8^2 + 0\right)\right\} = 537.04 \text{ t m}^2$

Ratio of Mass Moment of Inertia at overall centroid to Mass Moment of Inertia at CG of base area point O

$$\gamma_x = \frac{M_{mx}}{M_{mox}} = \frac{820.2}{2039.5} = 0.4; \quad \gamma_y = \frac{M_{my}}{M_{moy}} = \frac{408.4}{408.4} = 1.0; \quad \gamma_z = \frac{M_{mz}}{M_{moz}} = \frac{537.04}{1756.35} = 0.31$$

Natural Frequencies

Limiting Frequencies:

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{4.25 \times 10^5}{151.73}} = 52.89 \text{ rad/s}$$

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{8.49 \times 10^5}{151.73}} = 74.8 \text{ rad/s}$$

$$p_z = \sqrt{\frac{k_z}{m}} = \sqrt{\frac{4.25 \times 10^5}{151.73}} = 52.89 \text{ rad/s}$$

$$p_\theta = \sqrt{\frac{k_\theta}{M_{mox}}} = \sqrt{\frac{3.83 \times 10^6}{2039.5}} = 43.31 \text{ rad/s}$$

$$p_\psi = \sqrt{\frac{k_\psi}{M_{moy}}} = \sqrt{\frac{1.69 \times 10^6}{408.42}} = 64.36 \text{ rad/s}$$

$$p_\phi = \sqrt{\frac{k_\phi}{M_{moz}}} = \sqrt{\frac{6.85 \times 10^5}{1756.35}} = 19.75 \text{ rad/s}$$

Uncoupled Modes: Since vertical and torsional modes (corresponding to $y \,\&\, \psi$ deformation) are uncoupled modes, $p_y \,\&\, p_\psi$ also represent the natural frequencies in respective modes.

$p_y = 74.8$ rad/sec ; $f_y = 11.90$ Hz

$p_\psi = 64.36$ rad/sec ; $f_\psi = 10.24$ Hz

Coupled Modes are:

Modes corresponding to $x \& \phi$ deformation (X-Y Plane)

Modes corresponding to $z \& \theta$ deformation (Y-Z Plane)

Natural Frequencies corresponding to $x \& \phi$ deformation

Frequency Equation $$p_{1,2}^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) \mp \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$

Substituting Values, we get

$p_1 = 18.82$ rad/s ; $f_1 = 3.0$ Hz
$p_2 = 100.35$ rad/s ; $f_2 = 15.97$ Hz

Natural Frequencies corresponding to $z \& \theta$ deformation (Y-Z Plane)

Frequency Equation $$p_{1,2}^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) \mp \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x\, p_z^2 p_\theta^2}$$

Substituting values, we get

$p_1 = 35.49$ rad/s ; $f_1 = 5.65$ Hz
$p_2 = 101.8$ rad/s ; $f_2 = 16.2$ Hz

Unbalance Forces

Unbalance forces generated by machine areas given above (see machine data).

Amplitude computations are done in two stages:

a) Dynamic forces acting at frequency of 6 Hz correspond to load case 1

b) Dynamic forces acting at frequency of 6 Hz correspond to load case 2

Load Case 1 **Dynamic forces @ 6 Hz (37.7 rad/s)**

Point of Application - Point Q (see Figure D 10.1-1&2)

Force along Z $F_{1z} = 10$ kN

Moment about X $M_{1x} = 13.5$ kNm

Load Case 2 Dynamic forces @ 12 Hz (75.4 rad/s)

Point of Application - Point Q

Force along Z $F_{2z} = 2.6$ kN

Moment about X $M_{2x} = 6.7$ kNm

Point of Application – Motor Rotor Center

Force along Y $F_y = 1.72$ kN

Transferring these forces at CG of Base area point O , we get

Load Case 1 $\omega_1 = 37.7$ rad/s; $F_z = 10$ kN; $M_x = 13.5 + 10 \times 3.7 = 50.5$ kNm

Load Case 2 $\omega_2 = 75.4$ rad/s; $F_y = 1.72$ kN; $F_z = 2.6$ kN

$M_x = 6.7 + 2.6 \times 3.7 + 1.72 \times 1.5 = 18.9$ kNm

Amplitudes of Vibration

Rewriting parameters required for computation of amplitudes:

Stiffness

$k_x = 4.25 \times 10^5$; $k_y = 8.49 \times 10^5$; $k_z = 4.25 \times 10^5$ kN/m

$k_\theta = 3.83 \times 10^6$; $k_\psi = 1.69 \times 10^6$; $k_\phi = 6.85 \times 10^5$ kNm/rad

Limiting Frequencies

$p_x = 52.9$ rad/s; $p_y = 74.8$ rad/s; $p_z = 52.9$ rad/s

$p_\theta = 43.3$ rad/s; $p_\psi = 64.3$ rad/s; $p_\phi = 19.75$ rad/s

Natural Frequencies

In X - Y Plane $p_1 = 18.82$ rad/s & $p_2 = 100.35$ rad/s

In Y - Z Plane $p_1 = 35.49$ rad/s & $p_2 = 101.80$ rad/s

Mass and mass moment of inertia

$M_{mox} = 2039.5$ t m^2; $M_{moy} = 408.42$ t m^2; $M_{moz} = 1756.35$ t m^2

$\gamma_x = 0.4$; $\gamma_y = 1.0$; $\gamma_z = 0.31$; $m = 151.73$ t

Foundation size, height of centroid and damping constant

$H = 4.5$ m; $L = 5.2$ m; $B = 2.2$ m; $h = \bar{y}_o = 2.84$ m; $\zeta = 0.1$

Machine operating speeds **6 Hz & 12 Hz**

Load Case 1 Forces and Moments @ 6 Hz

$\omega_1 = 37.7$ rad/s; $F_z = 10$ kN; $M_x = 13.5 + 10 \times 3.7 = 50.5$ kNm

These dynamic forces i.e. F_z & M_x correspond to motion in Y-Z Plane.

Frequency ratios corresponding to Y-Z plane are:

$$\beta_z = (\omega_1 / p_z) = (37.7/52.9) = 0.71; \quad \beta_\theta = (\omega_1 / p_\theta) = (37.7/43.3) = 0.87$$
$$\beta_1 = (\omega_1 / p_1) = (37.7/35.49) = 1.06; \quad \beta_2 = (\omega_1 / p_2) = (37.7/101.8) = 0.37$$

It is seen that only β_1 lies in $\pm 20\,\%$ range. Thus amplitudes corresponding to β_1 shall be computed considering damping and for other, undamped amplitude shall be good enough.

Note: For amplitude computation, it is more convenient to consider one force at a time, evaluate amplitudes and finally obtain the resultant by taking the sum of the amplitudes.

i) **Force** $F_z = 10$ kN (For amplitude, see equation 9.1-17a)

Displacement z_o along Z and Rotation θ_o about X @ O

$$z_o = \delta_z \frac{\left(1-\beta_\theta^2\right)}{\left((-)\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right) \times \left(1-\beta_2^2\right)} \quad \& \quad \theta_o = \delta_z \frac{mh}{M_{mox}} \frac{\beta_\theta^2}{\left((-)\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right) \times \left(1-\beta_2^2\right)}$$

It may be noted that since $\left(1-\beta_1^2\right)$ is negative, sign of the term $\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}$ shall also be negative (see Note 2 § 9.1.2.2). Accordingly (-) sign is applied to the radical in denominator.

$$\delta_z = (F_z / k_z) = 10/4.25 \times 10^5 = 2.35 \times 10^{-5} \text{ m}$$

$$z_o = 2.35 \times 10^{-5} \frac{\left(1-0.87^2\right)}{\left((-)\sqrt{\left(1-1.06^2\right)^2 + (2 \times 1.06 \times 0.1)^2}\right) \times \left(1-0.37^2\right)} = -2.69 \times 10^{-5} \text{ m}$$

$$\theta_o = 2.35\times10^{-5}\,\frac{151.73\times2.8}{2039.5}\,\frac{0.87^2}{\left((-)\sqrt{\left(1-1.06^2\right)^2+(2\times1.06\times0.1)^2}\right)\times\left|\left(1-0.37^2\right)\right|} = -1.75\times10^{-5}\ \text{rad}$$

ii) **Moment** $M_\theta = M_x = 50.5$ kNm (For amplitude, see equation 9.1-17b)

Displacement z_0 along Z and Rotation θ_0 about X @ O

$$z_o = h\delta_\theta\,\frac{\beta_z^2}{\left(1-\beta_2^2\right)\times\left((-)\sqrt{\left(1-\beta_1^2\right)^2+(2\beta_1\zeta)^2}\right)};\quad \theta_o = \delta_\theta\,\frac{\left(1-\beta_z^2\right)}{\left(1-\beta_2^2\right)\times\left((-)\sqrt{\left(1-\beta_1^2\right)^2+(2\beta_1\zeta)^2}\right)}$$

$$\delta_\theta = M_\theta/k_\theta = 50.5/\left(3.83\times10^6\right) = 1.32\times10^{-5}\ \text{rad}$$

$$z_o = 2.84\times1.32\times10^{-5}\,\frac{0.71^2}{\left(1-0.37^2\right)\times\left((-)\sqrt{\left(1-1.06^2\right)^2+(2\times1.06\times0.1)^2}\right)} = -8.86\times10^{-5}\ \text{m}$$

$$\theta_o = 1.32\times10^{-5}\,\frac{\left(1-0.71^2\right)}{\left(1-0.37^2\right)\times\left((-)\sqrt{\left(1-1.06^2\right)^2+(2\times1.06\times0.1)^2}\right)} = -3.09\times10^{-5}\ \text{rad}$$

Total amplitudes @ O

$$z_0 = (-2.69-8.86)\times10^{-5} = -1.16\times10^{-4}\ \text{m}$$
$$\theta_0 = (-1.75-3.09)\times10^{-5} = -4.84\times10^{-5}\ \text{rad}$$

Amplitudes @ Foundation Top:

Amplitude $\bar{z}_f$ due to $z_0\ \&\ \theta_0$ (see equations 9.1-21)

$$\bar{z}_{f(\max)} = \left|(z_o + H\,\theta_o)\right| = \left|-1.16\times10^{-4} + 4.5\times\left(-4.85\times10^{-5}\right)\right|$$
$$= 3.4\times10^{-4}\ \text{m} = 340\ \text{microns}$$

Amplitude y_{fc} due to θ_0 (see equations 9.1-23)

$$y_{fc} = |(L/2)\theta_o| = |(5.2/2)\times(-4.84\times10^{-5})|$$
$$= 1.25\times10^{-4} \text{ m} = 125 \text{ microns}$$

Total Amplitude = $\sqrt{(340)^2 + (125)^2} = 362$ microns

Load Case 2 $\omega_2 = 75.4$ rad/s; $F_y = 1.72$ kN; $F_z = 2.6$ kN; $M_x = 18.9$ kNm

Dynamic forces F_z & M_x relate to coupled motion in Y-Z Plane whereas Dynamic force F_y relates to uncoupled motion along Y. Corresponding frequency ratios are:

$$\beta_z = (\omega_2/p_z) = (75.4/52.9) = 1.42; \ \beta_\theta = (\omega_2/p_\theta) = (75.4/43.3) = 1.74$$
$$\beta_y = (\omega_2/p_y) = (75.4/74.8) = 1.008$$
$$\beta_1 = (\omega_2/p_1) = (75.4/35.49) = 2.12; \ \beta_2 = (\omega_2/p_2) = (75.4/101.8) = 0.74$$

Since only β_y lies in $\pm 20\%$ range, amplitudes corresponding to β_y shall be computed considering damping and for others, undamped amplitude is good enough.

i) **Force** $F_z = 2.6$ kN (For amplitude, see equation 9.1-11)

Amplitude $$z_o = \delta_z \frac{(1-\beta_\theta^2)}{(1-\beta_1^2)(1-\beta_2^2)}; \quad \theta_o = \delta_z \frac{mh}{M_{mox}} \frac{\beta_\theta^2}{(1-\beta_1^2)(1-\beta_2^2)}$$

$$\delta_z = (F_z/k_z) = 2.6/4.25\times10^5 = 6.1\times10^{-6} \text{ m}$$

$$z_o = 6.1\times10^{-6}\times\frac{(1-1.74^2)}{(1-2.12^2)(1-0.74^2)} = 7.8\times10^{-6} \text{ m}$$

$$\theta_o = 6.1\times10^{-6}\times\frac{151.73\times2.84}{2039.5}\frac{1.74^2}{(1-2.12^2)(1-0.74^2)} = -2.43\times10^{-6} \text{ rad}$$

ii) **Moment** $M_\theta = M_x = 18.9$ kNm (For amplitude, see equation 9.1-12)

Amplitudes @ O - z_0 along Z and θ_0 about X

$$z_o = h\delta_\theta \frac{\beta_z^2}{(1-\beta_1^2)(1-\beta_2^2)}; \quad \theta_o = \delta_\theta \frac{(1-\beta_z^2)}{(1-\beta_1^2)(1-\beta_2^2)}$$

$$\delta_\theta = M_\theta / k_\theta = 18.9/\left(3.83\times10^{6}\right) = 4.93\times10^{-6} \quad \text{rad}$$

$$z_o = 2.84\times4.93\times10^{-6}\frac{1.42^2}{\left(1-2.12^2\right)\left(1-0.74^2\right)} = -1.78\times10^{-5} \quad \text{m}$$

$$\theta_o = 4.93\times10^{-6}\frac{\left(1-1.42^2\right)}{\left(1-2.12^2\right)\left(1-0.74^2\right)} = 3.17\times10^{-6}$$

Total amplitudes @ O

$$z_0 = \left(7.8\times10^{-6} - 1.78\times10^{-5}\right) = -1.0\times10^{-5} \quad \text{m}$$
$$\theta_0 = \left(-2.43\times10^{-6} + 3.17\times10^{-6}\right) = 7.4\times10^{-7} \quad \text{rad}$$

iii) Force F_y $F_y = 1.72$ kN (For amplitude, see equation 9.1-13)

Resonance in vertical mode of vibration

$$y_o = \delta_y \frac{1}{\sqrt{\left(1-\beta_y^2\right)^2 + \left(2\beta_y\zeta\right)^2}} = \frac{1.72}{8.49\times10^{5}}\frac{1}{\sqrt{\left(1-1.008^2\right)^2 + \left(2\times1.008\times0.1\right)^2}} = 1\times10^{-5} \quad \text{m}$$

Amplitude @ O 1×10^{-5} m

Amplitudes @ Foundation Top:

Amplitude $\bar{z}_f$ due to $z_0 \,\&\, \theta_0$ (see equation 9.1-21)

$$\bar{z}_{f(\max)} = \left|(z_o + H\,\theta_o)\right| = \left|(-1.0\times10^{-5} + 4.5\times7.4\times10^{-7})\right|$$
$$= 6.67\times10^{-6} \quad \text{m} = 6.7 \text{ microns}$$

Amplitude y_{fc} due to θ_0 (see equation 9.1-23)

$$y_{fc} = \left|(L/2)\theta_o\right| = (5.2/2)\times7.4\times10^{-7} = 1.9\times10^{-6}$$

Maximum amplitude along Y (see equation 9.1-26)

$$y_{f(\max)} = \bar{y}_{f(\max)} + \bar{y}_{fc(\max)} = \left|y_o\right| + \left\{\left|(L/2)\theta_o\right|\right\} = \left|1\times10^{-5}\right| + \left\{\left|(5.2/2)\times7.4\times10^{-7}\right|\right\}$$
$$= 1.19\times10^{-5} \quad \text{m} = 12 \text{ microns}$$

Total amplitude at foundation top $\sqrt{(6.7)^2 + (12)^2} = 13.74$ microns

Finite Element Analysis

This very problem is modeled and analyzed using Finite Element Method. Frequencies and mode shapes are shown in figure D 10.1-3 and steady state response is shown in Figure D 10.1-4.

From the mode shapes it is noticed that 3[rd] mode and 4[th] mode represent pure torsional and vertical vibration whereas other modes are coupled modes. 1[st] and 5th modes are coupled modes and represent rocking about Z and translation along X axis respectively. 2[nd] and 6[th] modes are coupled and represent rocking about X and translation along Z axis respectively.

Amplitudes: Dynamic forces are applied at machine locations at center line of machine axis. Amplitudes obtained are shown in Figure D 10.1-4.

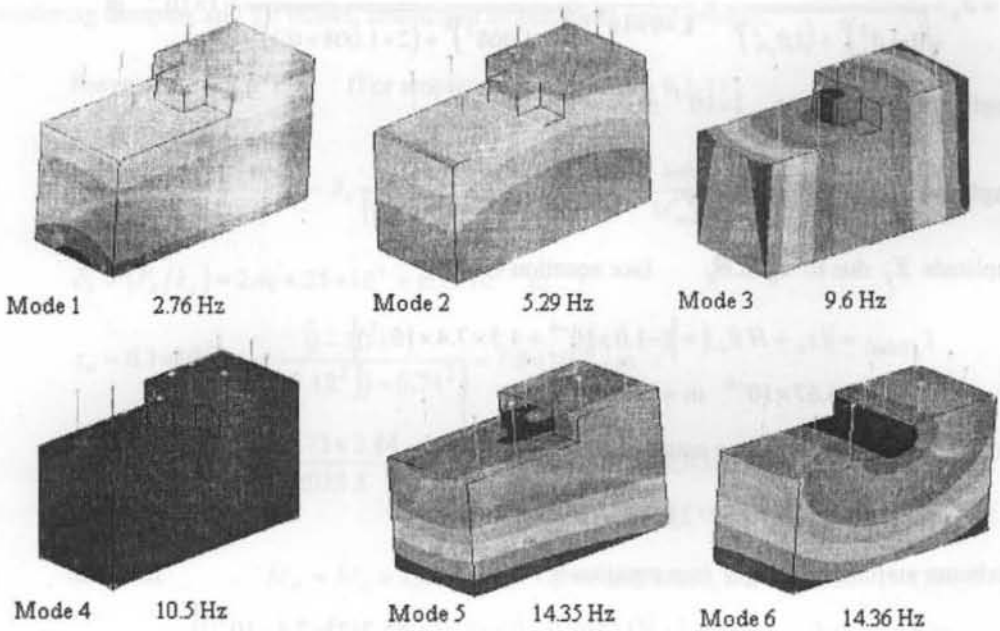


Figure D 10.1-3 Frequencies and Mode Shapes

Comparison of FE Method of Analysis with Manual method of Analysis

Table 10.1-1 Natural Frequencies (Hz)

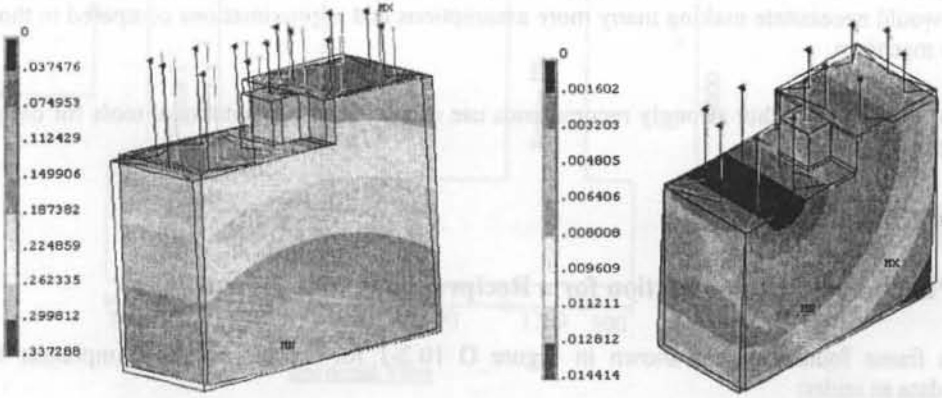
Mode #	1	2	3	4	5	6
Manual Method of Analysis	3.00	5.65	10.24	11.90	15.97	18.82
FE Analysis	2.76	5.29	9.60	10.50	14.35	14.36

Amplitudes of Vibration are given as under:

Table 10.1-2 Amplitudes in microns

Amplitude in Microns	@ 6 Hz	@ 12 Hz
Manual Method of Analysis	362	13.8
FE Analysis	337	14.4

Both the results show a good degree of agreement.



Amplitudes in mm @ frequency 6 Hz | Amplitudes in mm @ Frequency 12 Hz

Figure D 10.1-4 Vibration Amplitudes @ 6 Hz and @ 12 Hz

Transient amplitudes: Since system frequencies are low, transient amplitude need not be evaluated

Strength Design

1. Block foundations are rigid body mass and have sufficient strength to withstand all possible force exerted by machine and as such do not need design computations for strength except those parts of the foundation which are overhang or cantilever.

2. Minimum reinforcement to be provided is 25 to 50 kg/m^3 subject to condition that bar diameter shall not be less than 12 mm and spacing shall not be more than 200 mm. For thick concrete blocks, it is desirable to provide intermediate cross reinforcement layers along the height.
3. Though not necessary, check for **Safe Bearing Pressure** and stability, due to normal as well as abnormal loading conditions is desirable.
4. Check for Strength & Embeddement of Anchor Bolts for applicable forces is a must

10.2 DESIGN OF FRAME FOUNDATION

Design of Frame Foundation for reciprocating machines is relatively more complex compared to those for rotary machines. These machines are generally low frequency machines and unbalance force developed is very large. The member sizes i.e. sizes of beams and columns are relatively heavier compared to similar foundations for rotary machines. Analysis procedure, using manual method of analysis, is same as that for rotary machine with modifications as necessitated by the problem. In certain cases, columns of the same frame may have different sizes. This makes computation little more complex. Contribution of soil to the response of the foundation is significant for such foundations. For horizontal reciprocating machines, dynamic forces are along piston axis which invariably is the longitudinal direction of the frame foundation and response evaluation of frame foundations with 3 or more sets of transverse frames, using manual method of analysis, would necessitate making many more assumptions and approximations compared to those for rotary machines.

In view of the above, author strongly recommends use of advanced computational tools for design of such foundations.

DESIGN EXAMPLE

EXAMPLE D 10.2: Foundation for a Reciprocating Compressor

Design a frame foundation, as shown in Figure D 10.2-1 for a reciprocating compressor for machine data as under:

Machine Data

Weight of Motor @ Point A, B, P, Q			440.00 kN
Weight of Compressor @ point R			1200.00 kN
Speed of Motor Compressor			200.00 rpm

Dynamic Force @ point R

i)	At Engine Frequency	F_x (Lateral along X)	130.00 kN
		F_y (Vertical along Y)	40.00 kN

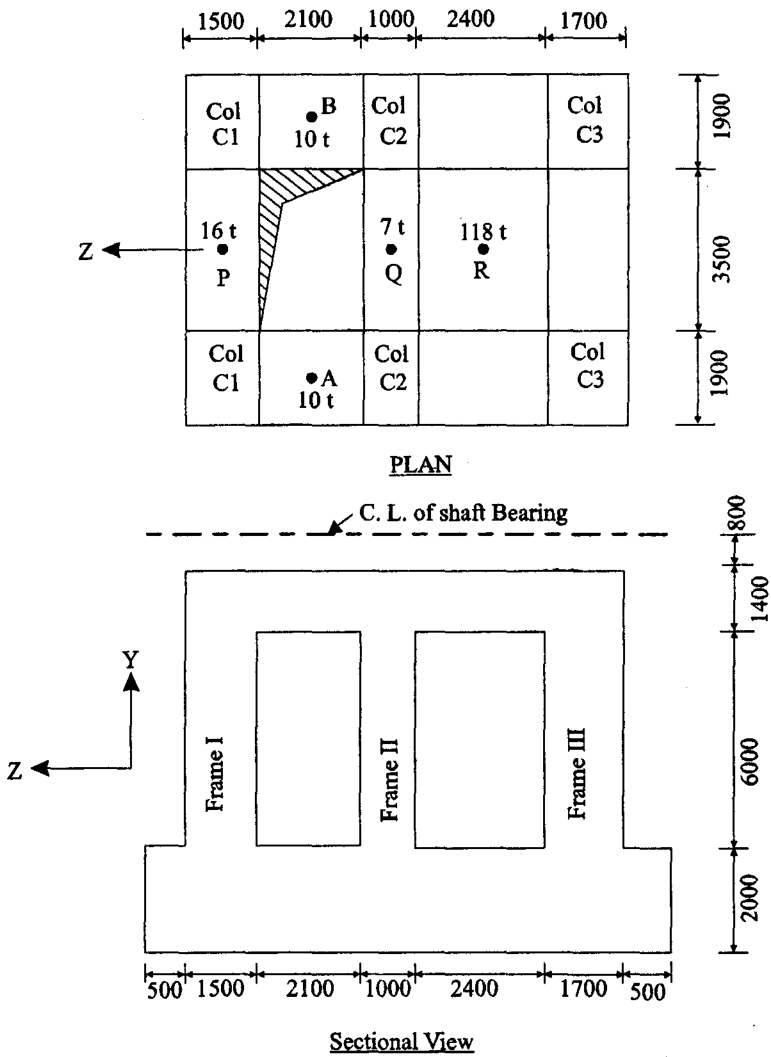


Figure D 10.2-1 Reciprocating Compressor on Frame Foundation

M_x (About X)	240.00	kNm
M_y (About Y)	800.00	kNm

ii) At 1st harmonic @ 400 rpm

F_x	40.00	kN
M_y (About Y)	130.00	kNm

Short Circuit Force Motor @ point A & B	240.00	kN
Maximum permissible half amplitude	0.10	mm

Foundation:

Concrete Grade M20

Top Deck: Length = 8.7 m Width = 7.3 m Thickness = 1.4 m

Columns (B x D) Frame 1 1.5×1.9 m; Frame 2 1.0×1.9 m; Frame 3 1.7×1.9 m

Column Height (From top of raft up to beam bottom) 6.0 m

Raft: Length = 9.7 m Width = 8.3 m Thickness = 2.0 m

Material properties: $E=3\times10^7$ kN/m^2; $\nu=0.15$; $G=1.3\times10^7$ kN/m^2; $\rho=2.5$ t/m^3

Depth of Foundation below GL 2.5 m

Soil:

Dynamic Soil Parameters

$$C_u=4\times10^4 \text{ kN/m}^3;\quad C_\tau=2\times10^4 \text{ kN/m}^3$$
$$C_\theta=C_\phi=8\times10^4 \text{ kN/m}^3;\quad C_\psi=8\times10^4 \text{ kN/m}^3$$

Design:

Sizing of Foundation: For Layout see Figure D10.2-1

Overall Eccentricity

Eccentricity between Center of Mass and CG of Base area of Foundation

Top deck Mass (without opening) $=8.7\times7.3\times1.4\times2.5=222$ t

Mass of opening $=2.1\times3.5\times1.4\times(-2.5)=-26$ t

Mass of Base Raft $=9.7\times8.3\times2.0\times2.5=403$ t

Column Mass (both column inclusive)

Frame I $2\times(1.5\times1.9\times6\times2.5)=86$ t

Frame II $2\times(1.0\times1.9\times6\times2.5)=57$ t

Frame III $\quad 2\times(1.7\times1.9\times6\times2.5)=97\ \text{t}$

Machine Mass

Motor $\quad 440/9.81=45\ \text{t}$; Compressor $\quad 1200/9.81=122\ \text{t}$

Let us denote CG of Base area of Raft as **point O**

Overall Centroid

CG of machine and foundation with respect to point O

		Mass (t)	$\bar{x}_i$	$\bar{z}_i$	$\bar{y}_i$
Motor		45	0.0	-1.80	10.2
Compressor		122	0.0	1.45	10.2
Top Deck (Without cut-out)		222	0.0	0.0	8.7
Cut-out		-26	0.0	-1.80	8.7
Raft		403	0.0	0.0	1.0
Columns	Frame 1	86	0.0	-3.6	5.0
	Frame 2	57	0.0	-0.25	5.0
	Frame 3	97	0.0	3.5	5.0

Here $\bar{x}_i, \bar{y}_i$ & $\bar{z}_i$ represent CG of mass element m_i from point O

Centroid of Mass from Point O

Total Mass $m=\sum m_i = 222-26+86+57+97+403+45+122=1006\ \text{t}$

$$\bar{x}_m = \frac{\sum m_i x_i}{\sum m_i} = 0.0\ \text{m}$$

$$\bar{z}_m = \frac{\{45\times(-1.8)+122\times1.45-26\times(-1.8)+86\times(-3.6)+57\times(-0.25)+97\times3.5\}}{1006} = \frac{158.35}{1006} = 0.16\ \text{m}$$

$$\bar{y}_m = \frac{\{45\times10.2+122\times10.2+222\times8.7-26\times8.7+403\times1+(86+57+97)\times5\}}{1006} = \frac{5012}{1006} = 4.98\ \text{m}$$

Eccentricity $\quad e_z = (0.16/9.7)\times100 = 1.65\% < 5\%$ OK

DYNAMIC ANALYSIS

Vibrations on account of i) Structural Effect and ii) Soil Effect influence response of machine foundation system. For manual computation, it may turn out to be too complex to combine both these effects and therefore these are evaluated independently. Overall response is the summation of the two individual responses.

STRUCTURAL VIBRATIONS

Top Deck Eccentricity

Eccentricity between Center of Mass (Machine weight +top deck weight + 23 % of column weight) & Center of lateral stiffness of Frames:

Center of Mass (Machine weight +top deck weight + 23 % of column weight)

$$\bar{z}_m = \frac{\{45\times(-1.8)+122\times1.45-26\times(-1.8)+0.23\times(86\times(-3.6)+57\times(-0.25)+97\times3.5)\}}{45+122+222-26+0.23\times(86+57+97)}$$

$$=\frac{146.3}{418.2}=0.35 \text{ m}$$

CG of Lateral Stiffness of Frames with respect to point O:

	Frame 1	Frame 2	Frame 3
Frame span (m) (center line)	5.4	5.4	5.4
Frame Height (m) (Up to beam center line)	6+ (1.4/2) = 6.7	6.7	6.7
Beam Moment of Inertia I_b	0.34	0.23	0.39
Column Moment of Inertia I_c	0.86	0.57	0.97
$k=\frac{(I_b/L)}{(I_c/H)}$	0.50	0.50	0.50
$k_x=\frac{12EI_c}{H^3}\frac{1+6k}{2+3k}$ (kN / m)	1.17×10^6	7.78×10^5	1.33×10^6
CG with respect to Point O	-1.8	-0.25	+3.5

$$\bar{z}_k=\frac{\sum k_{xi}z_i}{\sum k_{xi}}=\frac{1.17\times10^6\times(-3.6)+7.78\times10^5\times(-0.25)+1.33\times10^6\times3.5}{(1.17+.778+1.33)\times10^6}=0.08 \text{ m}$$

Eccentricity $e_z = \dfrac{0.35 - 0.08}{8.7} \times 100 = 3.1\% \quad > 1\%$ Hence Not OK

This calls for changes in foundation size. There are many ways of implementing this change. This is best done by the close interaction with machine group. In the present case it is achieved by increasing column depth of Frame 3 from 1900 mm to 2100 mm keeping overall foundation size same.

Foundation Plan with modified column is shown in Figure D 10.2-2.

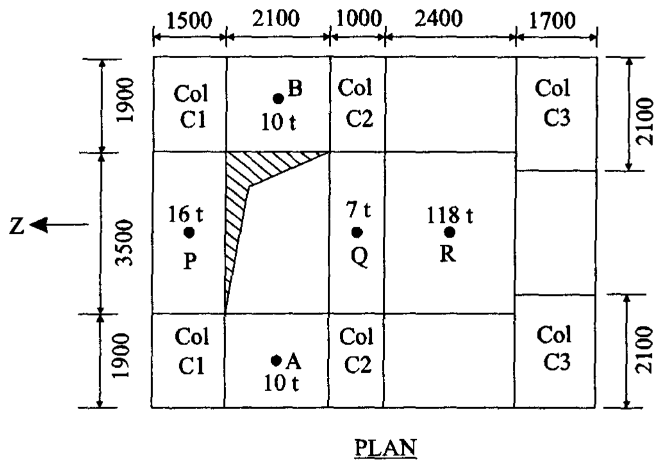


Figure D 10.2-2 Foundation Plan - Frame III Columns Modified

Design with revised sizes: Re-doing the computations with the revised column sizes of frame III, we get the changed parameters as:

Frame III

Span $L = 5.2\,\text{m}$; $I_c = 1.31\ \text{m}^4$; $k = 0.38$; $k_x = 1.64 \times 10^6$ kN/m

Mass of frame III columns (both columns)

$= 1.7 \times 2.1 \times 6 \times 2.5 \times 2 = 107$ t i.e. increase of 10 t

Check for Overall Eccentricity

Total mass (M/c + Foundation + Columns + Raft) = 1006+10=1016 t

Centre of Mass $\bar{z}_m = \dfrac{\sum m_i z_i}{\sum m_i} = \dfrac{158.35 + 10 \times 3.5}{1006 + 10} = \dfrac{193.35}{1016} = 0.19$ m

$$e_z = \left|\frac{0.19}{9.7}\right| \times 100 = 1.96\% \quad < 5\% \quad \text{Hence OK}$$

Check for Top Deck Eccentricity:

$$\bar{z}_m = \frac{\{45\times(-1.8)+122\times1.45-26\times(-1.8)+0.23\times(86\times(-3.6)+57\times(-0.25)+107\times3.5)\}}{45+122+222-26+0.23\times(86+57+107)}$$

$$= \frac{154.35}{420.5} = 0.37 \text{ m}$$

Lateral Stiffness: (with modified frame III columns)

Frame 1 1.17×10^6 kN/m; Frame 2 7.78×10^5 kN/m; Frame 3 1.64×10^6 kN/m
$\sum k_{xi} = 3.59\times10^6$ kN/m;

Total Lateral stiffness

$$\bar{z}_k = \frac{1.17\times10^6\times(-3.6)+7.78\times10^5\times(-0.25)+1.64\times10^6\times3.5}{3.59\times10^6} = \frac{1.33\times10^6}{3.59\times10^6} = 0.37 \text{ m}$$

$$e_z = \frac{(\bar{z}_k - \bar{z}_m)}{8.7}\times100 = \frac{(0.37-0.37)}{8.7}\times100 = 0.0\% \quad <1\% \quad \text{Hence OK}$$

Natural Frequencies

a) Lateral & Torsional Vibration

Since top deck eccentricity is less than 1 %, there shall not be any coupling between translational and Torsional Mode of Vibration.

i) Lateral Vibration

Total Lateral stiffness	3.59×10^6 kN/m
Total Mass (M/c + Top deck + 23 % column)	420.5 t
Natural Frequency	$p_x = \sqrt{\frac{3.59\times10^6}{420.5}} = 92.34$ rad/s

ii) Torsional Vibration

CG of machine and foundation with respect to center of top deck mass

		Mass (t)	$\bar{z}_i$
Motor		45	-1.8-0.37=-2.17
Compressor		122	1.45-0.37=1.08
Top Deck (Without cut-out)		222	0.0-0.37=-0.37
Cut-out		-26	-1.80-0.37=-2.17
23 % of Columns	Frame 1	19.8	-3.6-0.37=-3.97
	Frame 2	13.1	-0.25-0.37=-0.62
	Frame 3	24.6	3.5-0.37=3.13

Total Mass 420.5 t

Mass Moment of Inertia $M_{my(\text{Deck})}$ about Y axis passing through top deck center of mass

$$M_{my(\text{Deck})} = 45 \times 2.17^2 + 122 \times 1.08^2 + \left\{(222/12) \times \left(8.7^2 + 7.3^2\right) + 222 \times 0.37^2\right\}$$
$$+ \left\{(-26/12) \times \left(2.1^2 + 3.5^2\right) + (-26) \times 2.17^2\right\} = 2612 \text{ tm}^2$$

Torsional Stiffness

$$k_{\psi(\text{Deck})} = 1.17 \times 10^6 \times (-3.6 - 0.37)^2 + 7.78 \times 10^5 \times (-0.25 - 0.37)^2$$
$$+ 1.64 \times 10^6 \times (3.5 - 0.37)^2 = 3.48 \times 10^7 \text{ kNm/rad}$$

Center of Stiffness is 3.95 m from Frame I center line. Let b_i denote distance of each frame from center of stiffness. Substituting values, we get:

Natural Frequency $$p_\psi = \sqrt{\frac{3.48 \times 10^7}{2612}} = 115.4 \text{ rad/s}$$

b) Vertical Vibration

Mass associated with each frame (M/c + top deck + 33 % columns)

	Frame I	Frame II	Frame III	Total
Motor	22.5	22.5	0	45 t
Compressor	0	61	61	122 t
Deck (without opening)	$222\times\frac{2.55}{8.7}=65$	$222\times\frac{3.25}{8.7}=83$	$222\times\frac{2.9}{8.7}=74$	222 t
Opening	-13	-13	0	-26 t
33 % of Columns	28.7	19	35.7	83.4 t
Mass on each frame	103.2	172.5	170.7	446.4 t

Vertical Stiffness associated with each frame (equation 9.2-17)

Deflection of portal frame under unit (see equation 9.2-16)

$$y = \underbrace{y_2}_{\text{Flexural}} + \underbrace{y_2}_{\text{Shear}} + \underbrace{y_1}_{\text{Column}} = \left(\frac{L^3}{96EI_b}\times\frac{2k+1}{k+2}\right)+\frac{3L}{8GA_b}+\frac{H}{2E\,A_c}$$

	Frame 1	Frame 2	Frame 3
Frame span (m)	5.4	5.4	5.2
Frame Height (m) (Up to beam center line)	6.7	6.7	6.7
Beam area m^2	2.1	1.4	2.38
Beam Moment of Inertia I_b	0.34	0.23	0.39
Area of column m^2	2.85	1.9	3.57
$k=\frac{(I_b/L)}{(I_c/H)}$	0.50	0.50	0.38

Frame I Deflection under unit load

$$y=\left(\frac{5.4^3}{96\times3\times10^7\times0.34}\times\frac{2\times0.5+1}{0.5+2}\right)+\frac{3\times5.4}{8\times1.3\times10^7\times(1.5\times1.4)}+\frac{6.7}{2\times3\times10^7\times(1.5\times1.9)}$$

$$=6.4\times10^{-8}+7.42\times10^{-8}+3.92\times10^{-8}=1.77\times10^{-7}\ \text{m}$$

Vertical Stiffness Frame I $k_{1y}=\frac{1}{y}=\frac{1}{1.77\times10^{-7}}=5.65\times10^6$ kN/m

Similarly for frame II & III, we get

Frame II $y=2.66\times10^{-7}$ m; $k_y=3.76\times10^6$ kN/m

Frame III $y=1.47\times10^{-7}$ m; $k_y=6.8\times10^6$ kN/m

Total Vertical Stiffness $\sum k_y = 1.62 \times 10^7$ kN/m

Vertical Natural Frequency $p_y = \sqrt{\dfrac{1.62 \times 10^7}{446.4}} = 190.4$ rad/s

Amplitudes

Dynamic Force @ point R

	F_x kN	F_y kN	M_x kNm	M_y kNm
@200 rpm	130	40	240	800
@400 rpm	40			130

Center of mass at top deck is at 0.35 m along Z direction from point O.

Transferring forces @ center of mass location, we get:

Dynamic Force @ center of mass

	F_x kN	F_y kN	M_x kNm	M_y kNm
@200 rpm	130	40	240	$800 + 130 \times (1.45 - 0.35) = 943$
@400 rpm	40			$130 + 40 \times (1.45 - 0.35) = 174$

i) Lateral Vibration

a) Amplitude @ engine frequency of 200 rpm i.e. 20 94 rad/s

$F_x = 130$ kN; $k_x = 3.59 \times 10^6$ kN/m; $p_x = 92.34$ rad/s; $\omega = 20.94$ rad/s; $\beta_x = 0.23$

Amplitude $x = \dfrac{130}{3.59 \times 10^6} \times \dfrac{1}{\left|\left(1 - 0.23^2\right)\right|} = 3.83 \times 10^{-5}$ m = 38 microns

b) Amplitude @ 1st harmonic (400 rpm) i.e. 41.84 rad/s

$F_x = 40$ kN; $k_x = 3.59 \times 10^6$ kN/m; $p_x = 92.34$ rad/s; $\omega = 41.84$ rad/s; $\beta_x = 0.45$

Amplitude $x = \dfrac{40}{3.59 \times 10^6} \times \dfrac{1}{\left|\left(1 - 0.45^2\right)\right|} = 1.4 \times 10^{-5}$ m = 14 microns

ii) Torsional Vibration

a) Amplitude @ engine frequency of 200 rpm i.e. 20 94 rad/s

$M_\psi = 943$ kNm; $k_\psi = 3.48 \times 10^7$ kNm/rad; $p_\psi = 115.4$ rad/s; $\omega = 20.94$ rad/s; $\beta_\psi = 0.18$

Amplitude $\psi = \dfrac{943}{3.48 \times 10^7} \times \dfrac{1}{\left|\left(1 - 0.18^2\right)\right|} = 2.8 \times 10^{-5}$ rad

b) Amplitude @ 1st harmonic (400 rpm) i.e. 41.84 rad/s

$M_\psi = 174$ kNm; $k_\psi = 3.48 \times 10^7$ kNm/rad; $p_\psi = 115.4$ rad/s; $\omega = 41.84$ rad/s; $\beta_\psi = 0.36$

Amplitude $\psi = \dfrac{174}{3.48 \times 10^7} \times \dfrac{1}{\left|\left(1 - 0.36^2\right)\right|} = 5.8 \times 10^{-6}$ rad

iii) Vertical Vibration

a) Amplitude @ engine frequency of 200 rpm i.e. 20 94 rad/s

$F_y = 40$ kN; $k_y = 1.62 \times 10^7$ kN/m; $p_y = 190.4$ rad/s; $\omega = 20.94$ rad/s; $\beta_x = 0.1$

Amplitude $y = \dfrac{40}{1.62 \times 10^7} \times \dfrac{1}{\left|\left(1 - 0.1^2\right)\right|} = 2.5 \times 10^{-6}$ m = 2.5 microns

iv) Rotational Vibration about X

Computation of amplitudes due to rocking moment about X axis is a complex task from the point of view of manual computations hence not attempted here.

Total Amplitudes

Total amplitudes are as under:

Amplitudes @ engine order frequency

Along X $3.83 \times 10^{-5} + 2.8 \times 10^{-5} \times (8.7/2) = 1.6 \times 10^{-4}$ m = 160 microns

Along Y 2.5 microns

Along Z $2.8 \times 10^{-5} \times (7.3/2) = 1.02 \times 10^{-4}$ m = 102 microns

Amplitudes @ 1st harmonic

Along X $1.4\times10^{-5}+5.8\times10^{-6}\times(8.7/2)=3.9\times10^{-5}$ m = 39 microns

Along Y 0.0 microns

Along Z $5.8\times10^{-6}\times(7.3/2)=2.1\times10^{-5}$ m = 21 microns

Total Amplitudes (SRSS) microns

Direction	engine order	1st harmonic	Total
Along X	160	39	165
Along Y	2.5	0	2.5
Along Z	102	21	104

SOIL EFFECT

Computation of Frequencies and Amplitudes considering overall system along with soil is a complex and involved task using manual method of analysis. We need to make some approximations for its simplification.

a) **Consider foundation as rigid:** Let us consider the foundation including base raft as rigid body resting over soil and let us evaluate parameters required for frequency estimation.

Base Raft Size 9.7 x 8.3 m

$$\text{Area } 9.7\times8.3=80.5 \text{ m}^2;\ I_{xx}=\frac{80.5\times9.7^2}{12}=631.2 \text{ m}^4;\ I_{zz}=\frac{80.5\times8.3^2}{12}=462.2 \text{ m}^4$$

Soil Stiffness

$$k_x=k_z=C_\tau\times A;\quad k_y=C_u\times A;\quad k_\theta=C_\theta\times I_{xx};\quad k_\phi=C_\phi\times I_{zz};\quad k_\psi=C_\psi\times I_{xx}$$

Substituting values we get

$$k_x=k_z=80.5\times2\times10^4=1.61\times10^6 \text{ kN/m};\ k_y=80.5\times4\times10^4=3.22\times10^6 \text{ kN/m}$$

$$k_\theta=631.2\times8\times10^4=5.05\times10^7 \text{ kN/m};\ k_\phi=462.2\times8\times10^4=3.7\times10^7 \text{ kN/m}$$

$$k_\psi=(631.2+462.2)\times3\times10^4=3.28\times10^7 \text{ kN/m}$$

Mass & Mass Moment of Inertia @ CG of Base Area about X, Y & Z axes

Total Mass $m = 1016 \text{ t}$

Height of overall centroid from point O

$$\bar{y}_m = \frac{\{45\times 10.2 + 122\times 10.2 + 222\times 8.7 - 26\times 8.7 + 403\times 1 + (86+57+107)\times 5\}}{1016} = \frac{5062}{1016} = 4.98 \text{ m}$$

$$h = \bar{y}_m = 4.98 \text{ m}$$

Mass moment of Inertia M_{mox}, M_{moy} & M_{moz} about X, Y & Z axes respectively

$$\begin{aligned} M_{mox} &= (222/12)\times(8.7^2+1.4^2) + 222\times(8.7^2) + (-26/12)\times(2.1^2+1.4^2) + (-26)\times(8.7^2+1.8^2) \\ &+ (403/12)\times(9.7^2+2^2) + 403\times(1^2) + 86\times(3.6^2+5^2) + 57\times(0.25^2+5^2) + 107\times(3.5^2+5^2) \\ &+ 45\times(1.8^2+10.2^2) + 122\times(1.45^2+10.2^2) \\ &= 46326 \text{ tm}^2 \end{aligned}$$

$$\begin{aligned} M_{moz} &= (222/12)\times(7.3^2+1.4^2) + 222\times(8.7^2) + (-26/12)\times(3.5^2+1.4^2) + (-26)\times(8.7^2) \\ &+ (403/12)\times(8.3^2+2^2) + 403\times(1^2) + 86\times(5^2) + 57\times(5^2) + 107\times(5^2) \\ &+ 45\times(10.2^2) + 122\times(10.2^2) \\ &= 42302 \text{ tm}^2 \end{aligned}$$

$$\begin{aligned} M_{moy} &= (222/12)\times(7.3^2+8.7^2) + (-26/12)\times(2.1^2+3.5^2) + (-26)\times(3.6^2) + (403/12)\times(8.3^2+9.7^2) \\ &+ 86\times(3.6^2) + 57\times(0.25^2) + 107\times(3.5^2) + 45\times(1.8^2) + 122\times(1.45^2) \\ &= 10317 \text{ tm}^2 \end{aligned}$$

Note: In the present case Mass Moment of Inertia of columns about their own centroid has been ignored just for the purpose of simplification.

Motion in Y- Z Plane (Only due to soil influence):

Translational motion along Z and Rotational motion about X axis passing through point O

$m = 1016 \text{ t}$; $M_{mox} = 46326 \text{ tm}^2$; $k_z = 1.61\times 10^6$ kN/m; $k_\theta = 5.05\times 10^7$ kNm/rad

Limiting Frequencies:

Translational Frequency along Z $p_z = \sqrt{\dfrac{1.61\times10^6}{1016}} = 39.8$ rad/s

Rotational Frequency about X $p_\theta = \sqrt{\dfrac{5.05\times10^7}{46326}} = 33$ rad/s

Coupled Frequencies:

Frequency equation is (refer equations 9.1-5 & 6):

$$p_{1,2}^2 = \frac{1}{2\gamma_x}\left(p_z^2 + p_\theta^2\right) \mp \frac{1}{2\gamma_x}\sqrt{\left(p_z^2 + p_\theta^2\right)^2 - 4\gamma_x\, p_z^2 p_\theta^2}$$

Here $\gamma_x = \dfrac{M_{mx}}{M_{mox}}$; $p_z^2 = \dfrac{k_z}{m}$; $p_\theta^2 = \dfrac{k_\theta}{M_{mox}}$

$M_{mx} = 46326 - 1016\times4.98^2 = 21129$; $\gamma_x = \dfrac{M_{mx}}{M_{mox}} = \dfrac{21129}{46326} = 0.46$

Substituting in frequency equation, we get

$p_1 = 27$ rad/s; $p_2 = 71$ rad/s

Motion in X- Y Plane (Only due to soil influence):

Translational motion along X and Rotational motion about Z axis passing through point O

$m = 1016$ t; $M_{moz} = 42302$ tm^2; $k_x = 1.61\times10^6$ kN/m; $k_\phi = 3.7\times10^7$ kNm/rad

Limiting Frequencies:

Translational Frequency along Z $p_x = \sqrt{\dfrac{1.61\times10^6}{1016}} = 39.8$ rad/s

Rotational Frequency about X $p_\phi = \sqrt{\dfrac{3.7\times10^7}{42302}} = 29.6$ rad/s

Coupled Frequencies:

Frequency equation is (refer equations 9.1-3 & 4):

$$p_{1,2}^2 = \frac{1}{2\gamma_z}\left(p_x^2 + p_\phi^2\right) \mp \frac{1}{2\gamma_z}\sqrt{\left(p_x^2 + p_\phi^2\right)^2 - 4\gamma_z\, p_x^2 p_\phi^2}$$

Here $\gamma_z = \frac{M_{mz}}{M_{moz}}$; $p_x^2 = \frac{k_x}{m}$; $p_\phi^2 = \frac{k_\phi}{M_{moz}}$

$M_{mz} = 42302 - 1016 \times 4.98^2 = 17105$; $\gamma_z = \frac{M_{mz}}{M_{moz}} = \frac{17105}{42302} = 0.4$

Substituting in frequency equation, we get

$p_1 = 25$ rad/s; $p_2 = 74$ rad/s

Motion in X-Z Plane

Torsional vibration about Y axis passing through point O

$k_\psi = 3.28 \times 10^7$ kNm/rad; $M_{moy} = 10317$ tm^2

Natural Frequency $p_\psi = \sqrt{\frac{3.28 \times 10^7}{10317}} = 56.4$ rad/s

Vertical Vibration along Y

$m = 1016$ t; $k_y = 3.22 \times 10^6$ kN/m; $p_y = \sqrt{\frac{3.22 \times 10^6}{1016}} = 56.3$ rad/s

Amplitude:

Dynamic Force @ point R

	F_x	F_y	M_x	M_y
	kN	kN	kNm	kNm
@200 rpm	130	40	240	800
@400 rpm	40			130

Transferring forces and moments @ point O, we get dynamic forces and moments @ point O as:

i) Dynamic loads @ engine frequency @ 20.94 rad/s (200 rpm)

$$F_x = 130\ kN;\quad Fy = 40\ kN;\quad Fz = 0$$
$$M_\theta = 240 + 40\times 1.45 = 298\ \text{kNm}$$
$$M_\psi = 800 + 130\times 1.45 = 989\ \text{kNm}$$
$$\text{M}_\phi = 130\times 10.2 = 1326\ \text{kNm}$$

ii) Dynamic loads @ 1st harmonic i.e. twice the engine frequency @ 41.88 rad/s

$$F_x = 40\ kN;\quad M_\psi = 130 + 40\times 1.45 = 188\ \text{kNm};\quad \text{M}_\phi = 40\times 10.2 = 408\ \text{kNm}$$

i) Amplitudes @ Engine Frequency

a) Dynamic Force F_y

$$F_y = 40\ \text{kN};\quad k_y = 3.22\times 10^6\ \text{kN/m}$$
$$\omega = 20.94\ \text{rad/s};\ p_y = 56.3\ \text{rad/s (away from resonance range)}$$
$$\beta_y = (20.94/56.3) = 0.37$$

$$\text{Amplitude}\quad y = \frac{40}{3.22\times 10^6}\times\frac{1}{(1-0.37^2)} = 1.44\times 10^{-7}\ \text{m}$$

b) Dynamic Moment M_ψ

$$M_\psi = 989\ \text{kNm};\qquad k_\psi = 3.28\times 10^7\ \text{kNm/rad}$$
$$\omega = 20.94\ \text{rad/s};\ p_\psi = 56.4\ \text{rad/s (away from resonance)}$$
$$\beta_\psi = (20.94/56.4) = 0.37$$

$$\text{Amplitude}\quad \psi = \frac{989}{3.28\times 10^7}\times\frac{1}{(1-0.37^2)} = 3.5\times 10^{-5}\ \text{rad}$$

c) Dynamic Force F_x and Moment M_ϕ (motion in X-Y plane)

$$F_x = 130\ \text{kN};\ M_\phi = 1326\ \text{kNm};\ k_x = 1.61\times 10^6\ \text{kN/m};\quad k_\phi = 3.7\times 10^7\ \text{kNm/rad}$$
$$\text{h} = 4.98;\quad m = 1016\ \text{t};\ M_{moz} = 42302\ \text{tm}^2;\ \omega = 20.94\ \text{rad/s}$$
$$p_x = 39.8\ \text{rad/s};\quad p_\phi = 29.6\ \text{rad/s}\quad p_1 = 25\ \text{rad/s};\quad p_2 = 74\ \text{rad/s}$$
$$\beta_x = 0.53;\quad \beta_\phi = 0.70;\quad \beta_1 = 0.84;\quad \beta_2 = 0.28;\quad \zeta = 0.1$$

Frequency ratio $\beta_1 = 0.84$ lies in resonance range.

Amplitude @ O due to **Force** $F_x = 130$ kN (see equation 9.1-15a)

$$x_o = \delta_x \frac{\left(1-\beta_\phi^2\right)}{\left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right)\times\left(1-\beta_2^2\right)}; \quad \phi_o = -\delta_x \frac{mh}{M_{moz}} \frac{\beta_\phi^2}{\left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right)\times\left(1-\beta_2^2\right)}$$

Substituting values, we get $x_o = 1.32\times10^{-4}$ m; $\phi_o = -1.51\times10^{-5}$ rad

Amplitude @ O due to **Moment** $M_\phi = 1326$ kNm (see equation 9.1-16a)

$$x_o = -h\delta_\phi \frac{\beta_x^2}{\left(1-\beta_2^2\right)\times\left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right)}; \quad \phi_o = \delta_\phi \frac{\left(1-\beta_x^2\right)}{\left(1-\beta_2^2\right)\times\left(\sqrt{\left(1-\beta_1^2\right)^2 + (2\beta_1\zeta)^2}\right)}$$

Substituting values, we get $x_o = -1.60\times10^{-4}$ m; $\phi_o = 8.25\times10^{-5}$ rad

d) Dynamic Moment M_θ (motion in Y-Z plane)

$M_\theta = 298$ kNm; $k_\theta = 5.05\times10^7$ kNm/rad; h = 4.98; $\omega = 20.94$ rad/s
$p_z = 39.8$ rad/s; $p_\theta = 33$ rad/s $p_1 = 27$ rad/s; $p_2 = 71$ rad/s
$\beta_z = 0.53$; $\beta_\theta = 0.63$; $\beta_1 = 0.77$; $\beta_2 = 0.3$; $\zeta = 0.1$

Amplitude @ O due to M_θ (see equation 9.1-12)

$$z_o = h\delta_\theta \frac{\beta_z^2}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}; \quad \theta_o = \delta_\theta \frac{\left(1-\beta_z^2\right)}{\left(1-\beta_1^2\right)\left(1-\beta_2^2\right)}$$

Substituting values, we get $z_o = -2.08\times10^{-5}$ m; $\theta_o = 1.07\times10^{-5}$ rad

Total amplitude @ O

$x_o = 1.32\times10^{-4} + \text{m}$; $y_o = 1.44\times10^{-7}$; $z_o = -2.08\times10^{-5}$ m
$\theta_o = 1.07\times10^{-5}$ rad; $\psi_o = 3.5\times10^{-5}$ rad; $\phi_o = -1.51\times10^{-5}$ rad

Amplitudes at foundation top (equations 9.1-25 to 27)

Let x_{fc}, y_{fc} & z_{fc} represent amplitudes at corner of Top of Foundation (Figure 9.1-2). Let L, B & H represent length, width and height of the foundation along Z, X and Y axes respectively.

$$x_{f(\max)} = |(x_o - H\phi_o)| + |(L/2)\psi_o|$$
$$= |(1.32\times10^{-4} - 9.4\times(-1.51\times10^{-5}))| + |(8.7/2)\times3.5\times10^{-5}|$$
$$= 4.24\times10^{-4} \quad \text{m} = 424 \text{ microns}$$
$$y_{f(\max)} = |y_o| + \{|(L/2)\theta_o| + |(B/2)\phi_o|\}$$
$$= |1.44\times10^{-7}| + \{|(8.7/2)\times1.07\times10^{-5}| + |(7.3/2)\times(-1.51\times10^{-5})|\}$$
$$= 1.017\times10^{-4} \quad \text{m} = 102 \text{ microns}$$
$$z_{f(\max)} = |(z_o + H\,\theta_o)| + |(B/2)\psi_o|$$
$$= |(-2.08\times10^{-5} + 9.7\times1.07\times10^{-5})| + |(7.3/2)\times3.5\times10^{-5}|$$
$$= 2.11\times10^{-4} \quad \text{m} = 211 \text{ microns}$$

ii) Amplitudes @ 1st harmonic (@ Twice Engine Frequency)

On the similar lines, as above, amplitudes can be computed for dynamic loads at 1st harmonics. Computations are not shown here and readers may attempt on their own.

Overall amplitudes

Overall amplitudes are obtained by combining both the effects i.e. structural effect as well as soil effect. Combination is preferably obtained using Square Root of Sum of Squares (SRSS) method.

From the results as above, following observations are made:

a) Amplitudes are marginally higher than permissible (100 microns) even for structural vibration alone. This calls for structural stiffening of columns along X
b) Amplitudes are much beyond permissible limit due to soil effect alone. It implies that even if structural stiffening of columns is done, amplitudes on account of soil effect shall still be much higher. The only answer to such problem is either to go in for
   i. Soil strengthening / stabilization
   ii. Vibration Isolation
   iii. Use of high stiffness material under base raft of Foundation

11

FOUNDATIONS FOR IMPACT & IMPULSIVE LOAD MACHINES

- **Design Examples**

    - Foundation for a Forge Hammer
    - Foundations For Machines Producing Impulsive Loads

FOUNDATIONS FOR IMPACT & IMPULSIVE LOAD MACHINES

Machines producing repeated impacts are i) Forge Hammers & ii) Drop Hammers whereas Machines Producing Impulse/Pulse Loading are i) Forging and Stamping Press, ii) Drop Weight Crushers, iii) Crushing, Rolling and Grinding Mills etc. **Design Procedures** given hereunder are for:

a) **For Impact Loading Machines - Hammer Foundations**
b) **For Impulse Loading Machines – Coverage is restricted only to Typical Foundations falling under this category**

11.1 HAMMER FOUNDATIONS

Machines producing repeated impacts are Forge Hammers & Drop Hammers. The Tup falls from a height and strikes the material, to be forged, placed on the Anvil. The Anvil is invariably placed over an elastic pad and the pad rests on the Foundation Block supported over soil. The elastic pad helps in preventing the bouncing of the Anvil over the Foundation. The force produced during the strike is termed as the Impact Force. The energy imparted by the impact force results in motion of the Anvil. The energy from the Anvil is then transmitted to the soil through the foundation.

Summary of Design Steps

1. Sizing of Foundation
2. Equivalent Soil Stiffness
3. Dynamic Forces
4. Analysis

I. Dynamic Analysis
   a. Natural Frequencies
   b. Dynamic Amplitudes

II. Strength and Stability Analysis

Required Input Data

Typical parameters required are as under.

a) Foundation Data

i) Foundation outline geometry
ii) Details of cut-out for anvil
iii) Support locations of frame on the foundation, if applicable

Any other parameter that is specific to machine may also be needed.

Foundation data viz. Foundation Type, Size, Mass, Stiffness, Material Properties is translated in to **Design Foundation Parameters** in line with the procedures given in **Chapters 7**. Necessary provisions must be made for Isolation of foundation from adjoining Structures. Wherever Isolation Pads are used below the Anvil &/or below the foundation, **Stress-Strain properties of these Isolation Pads** should also be listed.

b) Machine data

i) Total Mass of the Hammer i.e. Mass of the Tup, Anvil, Die & Frame
ii) Mass of falling part i.e. Tup (also Mass of Upper Die in case of Drop Hammers)
iii) Height of fall for the Tup/ Energy of Impact
iv) Area of the Piston
v) Pressure in the Cylinder
vi) Frequency of Impact
vii) Mass of Anvil (also Mass of Guide Frame if attached to Anvil)
viii) Frequency i.e. Number of Blows/min
ix) Base area of the Anvil
x) Details of Anchor Bolts connecting frame base to the foundation
xi) Thickness of Elastic Pad placed below Anvil and its Elastic Modulus
xii) Coefficient of Restitution/Impact
xiii) Allowable Amplitudes at Anvil and Foundation

Machine data, as above, is translated in to **Design Machine Parameters** in line with the procedures given in **Chapters 6**.

c) Soil Data

i) Site Specific **Dynamic Soil Data**
ii) Soil type and its basic characteristic properties
iii) Bearing capacity
iv) Depth of water table
v) Liquefaction potential

Site Soil Parameters as above are converted to Design Soil Parameters in line with provisions of **Chapter 5.**

In addition, environmental data like **Site related Seismic data,** presence of any industry/ housing clusters sensitive to ill effects of vibration may also be needed.

11.1.1 Foundation Sizing

Design of Hammer Foundation is as good an **Art** as **Science**. The decision regarding mass ratio (Foundation/Anvil) is very subjective. It depends upon various factors e.g. mass of Tup, its initial velocity, desired foundation amplitude, depth of foundation, Coefficient of Uniform Compression etc. For **initial sizing**, procedure recommended is as under:

1. Based on **Tup** mass, **Anvil** mass, **Anvil** size, and **Soil** parameters, select suitable depth of foundation D and approximate base area of the foundation
2. **Based on soil data,** evaluate **Design Coefficient of Uniform Compression of soil** C_u for this depth and foundation base area
3. Assign allowable amplitude value y_f for the foundation as given in this book or as per applicable codes of practice
4. Corresponding to the **Design** C_u **value**, compute factor λ (Factor for computing Mass Ratio) using the graph shown in Figure 11.1-1
5. Compute **Foundation Mass** (first approximate size) using the relation ship as given below

$$\frac{m_f + m_a}{m_a} = \lambda \times \left\{ \frac{v_0' \times \sqrt{D}}{y_f} \right\} \tag{11.1-1}$$

Here, m_f is mass of foundation, m_a is mass of anvil, D is foundation depth in m, y_f is desired foundation amplitude in mm and v_0' is tup initial velocity in m/sec (equations 11.1-2, 3 & 4)

6. Proceed with the detailed design process

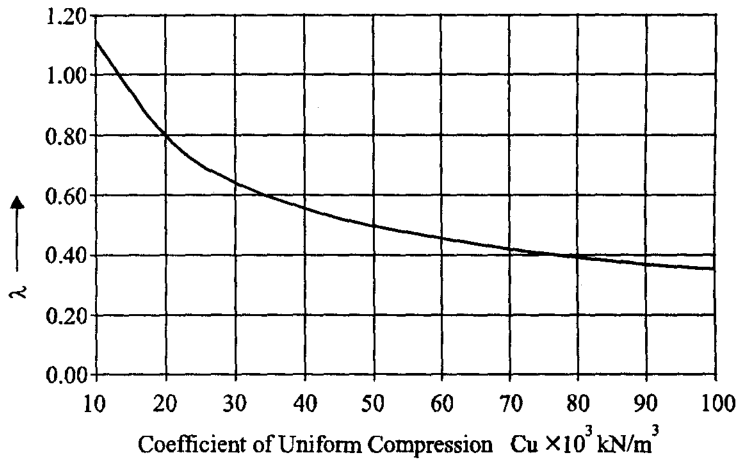
Consider, for example a hammer foundation with anvil mass of 40 t and initial tup velocity of 4 m/sec, desired foundation amplitude is 1.1 mm and Design Coefficient of Uniform Compression is $C_u = 20 \times 10^4$ kN/m^3. Mass of the foundation works out as under:

Using the graph for $C_u = 20000$ kN/m^3,	$\lambda = 0.8$
Tup initial velocity	4.0 m/sec
Foundation Depth	D = 3.5 m

Desired foundation amplitude	1.1 mm
Mass of Anvil	40 t

We get $$\frac{m_f + m_a}{m_a} = 0.8 \times \left\{ \frac{4 \times \sqrt{3.5}}{1.1} \right\} = 5.44$$

This gives $m_f = 40 \times (5.44 - 1) = 178$ t, which is about 4.5 time's mass of anvil.



$$\frac{m_f + m_a}{m_a} = \lambda \times \left\{ \frac{v'_0 + \sqrt{D}}{y_f} \right\}$$

m_f is foundation mass & m_a anvil mall;
v'_0 is initial velocity of Tup before impact in *m*/sec;
D is depth of foundation in m; y_f is foundation amplitude in mm

Figure 11.1-1 Factor for Foundation to Anvil Mass ratio for different Cu Values

11.1.2 Dynamic Analysis

Vibration of the foundation subjected to impact by the hammer is basically an **Initial Velocity Problem**. We can represent the complete **Hammer-Foundation System** in two parts:

i. A falling Mass m_0 from a height h producing Impact
ii. Remaining System that withstands this impact

Summary of Formulae:

Initial Velocity of the Falling Mass

Initial Velocity of the **freely falling mass** m_0 from a height h

$$v_0' = \sqrt{2\,g\,h} \tag{11.1-2}$$

(a) Single Acting Drop Hammers

For a single acting Drop Hammer, initial velocity of the falling mass **(mass of Tup and mass of Top Die)** from a height h is written as:

$$v_0' = \eta\sqrt{2\,g\,h} \tag{11.1-3}$$

- h Represents **total height of fall** of the falling mass
- g Represents **acceleration due to gravity**
- η Represents factor for **Efficiency of Drop**

Factor η depends upon energy lost in overcoming the friction to the Tup's movement and the resistance of the steam/air counter pressure. From practical considerations, the recommended value of **Efficiency of Drop** η is 0.65.

(b) Double Acting Hammers

In this case hammer is operated by pneumatic/steam pressure and initial velocity is given as:

$$v_0' = \eta\sqrt{2\,g\,h \times \left(\frac{m_0 \times g + p_s \times A_p}{m_0 \times g}\right)} \tag{11.1-4}$$

The quantity in the bracket represents influence of force on the piston to the initial velocity.

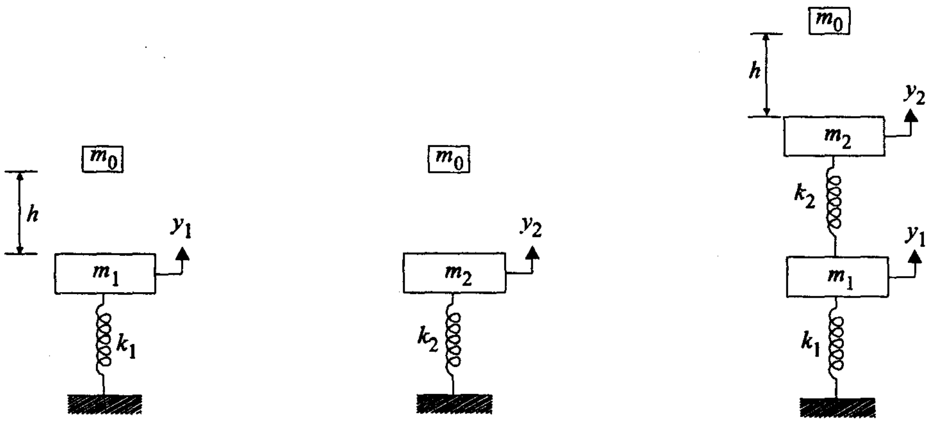
Here:

- A_p Represents area of the piston
- p_s Represents pressure (steam/air) acting on the piston
- m_0 Represents total mass of the falling parts
- g Represents acceleration due to gravity

Analysis:

Analysis is done using manual method of computation as given in Chapters 2 & 3. Dynamic response is evaluated at anvil and at foundation. Recommended steps for analysis are as under:

1. Consider anvil and foundation as one mass supported on soil i.e. no elastic pad between anvil and foundation. The system is represented as SDOF system as shown in Figure 11.1-2a
2. Consider elastic pad between anvil and foundation. Assume foundation as rigid. The system is represented as SDOF system as shown in Figure 11.1-2b
3. Consider total system as Two DOF System as shown in Figure 11.1-2c



(a) Single DOF System

m_0 = Mass of Tup + Upper Die
m_1 = Mass of Anvil +Foundation
k_1= Soil Stiffness

(b) Single DOF System

m_0 = Mass of Tup + Upper Die
m_2 = Mass of Anvil
k_2= Stiffness of Elastic Pad

(c) Two DOF System

m_0 = Mass of Tup + Upper Die
m_2 = Mass of Anvil
k_2= Stiffness of Elastic Pad
m_1 = Mass of Foundation
k_1= Soil Stiffness

Figure 11.1-2 Typical Hammer Foundation Represented as Single DOF System and as Two DOF System

1. Anvil and Foundation as One Mass supported over Soil – System represented as Single DOF System: (see Figure 11.1-2a).

Velocity of foundation just after impact is given as

$$v_1 = v_0' \times \frac{(1+e)}{(1+\lambda_1)} \tag{11.1-5}$$

Here $\lambda_1 = \dfrac{m_1}{m_0}$ represents ratio of mass m_1 to mass m_0, m_1 represents total mass of foundation & anvil, m_0 represents mass of falling part, e represents coefficient of restitution and v_0' is velocity of Tup just before impact as given by equations 11.1-3 & 4.

Coefficient of restitution (for real bodies in practice)	$e = 0.6$	(11.1-6)
Desired Foundation amplitude	y_f mm; $y_1 = y_f \times 10^{-3}$ m	
Required soil stiffness	$k_1 = m_1 \times \left(\dfrac{v_1}{y_1}\right)^2$	
Required Base area of foundation	$A_f = \dfrac{k_1}{C_u}$	(11.1-7)

Here A_f represents Base Area of Foundation.

Note: Ensure that base area provided is equal to or higher than this value. If area provided is less than this value, increase the base dimensions suitably to match this value.

2. Elastic pad between anvil and foundation. Assume foundation as rigid – System represented as Single DOF System: (see Figure 11.1-2b).

In this segment, we evaluate required **Stiffness of Elastic Pad**.

Velocity of Anvil just after impact is given as

$$v_2 = v_0' \times \frac{(1+e)}{(1+\lambda_2)} \tag{11.1-8}$$

Here $\lambda_2 = \dfrac{m_2}{m_0}$ represents ratio of mass m_2 to mass m_0, m_2 represents mass of anvil and m_0, v_0' & e are as defined above.

Note: If frame is attached to anvil, it's mass to be added to anvil mass

Desired anvil amplitude	y_a mm; $y_2 = y_a \times 10^{-3}$ m	
Required stiffness of elastic pad	$k_2 = m_2 \times \left(\dfrac{v_2}{y_2}\right)^2$	(11.1-9)

Required thickness of pad $$t = \frac{E_p \times A_a}{k_2} \times 10^3 \text{ mm} \qquad (11.1\text{-}10)$$

Here E_p represents Elastic modulus of pad and A_a represents Area of Anvil Base

3. Elastic pad between anvil and foundation and foundation supported over soil – System represented as Two DOF System: (see Figure 11.1-2c).

Velocity of Anvil just after impact (as given by equation 11.1-8) v_2

Mass of Anvil m_2

Stiffness of Elastic pad below Anvil (as given by equation 11.1-9) k_2

Mass of Foundation (alone) m_1

Note: If frame is attached to foundation, it's mass to be added to foundation mass

Foundation Base area (As provided) A_f

Stiffness of soil (at foundation base) $k_1 = C_u \times A_f$ (11.1-11)

Frequency Equation:

$$p_{1,2}^2 = \frac{1}{2}\left\{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right) \mp \sqrt{\left(p_{L2}^2(1+\lambda) + p_{L1}^2\right)^2 - 4\left(p_{L1}^2 p_{L2}^2\right)}\right\} \qquad (11.1\text{-}12)$$

$$\text{Here,} \quad p_{L1} = \sqrt{\frac{k_1}{m_1}} \;;\; p_{L2} = \sqrt{\frac{k_2}{m_2}} \quad \& \quad \lambda = \frac{m_2}{m_1} \qquad (11.1\text{-}13)$$

Amplitude:

Foundation Amplitude

$$y_1 = \frac{v_2}{p_1}\frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)}\sin p_1 t - \frac{v_2}{p_2}\frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)}\sin p_2 t \qquad (11.1\text{-}14)$$

Anvil Amplitude

$$y_2 = \frac{v_2}{p_1} \times \frac{\left(p_{L2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)}\sin p_1 t - \frac{v_2}{p_2} \times \frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)}\sin p_2 t \qquad (11.1\text{-}15)$$

Here coefficient of term $\sin p_1 t$ represents **First Mode Response** and coefficient of term $\sin p_2 t$ represents **Second Mode Response**.

Denoting $y_1' = \frac{v_2}{p_1}\frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)}$ & $y_1'' = -\frac{v_2}{p_2}\frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)}$, we represent foundation amplitude as:

$$y_1 = y_1' \sin p_1 t - y_1'' \sin p_2 t \quad (11.1\text{-}16)$$

Denoting $y_2' = \frac{v_2}{p_1}\times\frac{\left(p_{L2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)}$ & $y_2'' = -\frac{v_2}{p_2}\times\frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)}$, we represent anvil amplitude as:

$$y_2 = y_2' \sin p_1 t - y_2'' \sin p_2 t \quad (11.1\text{-}17)$$

DESIGN EXAMPLE

Design Examples are those encountered in real life practice. Comparison with Finite Element Analysis (FEA) is also given for specific cases to build up the confidence level. Effort is made to highlight the influence of certain slips commonly committed while computing response of the foundation.

Example D 11.1: Foundation for Drop Hammer

Design the foundation for 1.1 t Drop Hammer, as shown in Figure D 11.1-1 having the following details:

Machine Data:

Mass of Tup + Upper Die	1.38 t
Mass of Anvil	34.00 t
Mass of Anvil with Frame	34.00 t
Anvil Base dimension	
Length = 1.90 m; Width = 1.45 m;	Anvil Height = 1.30 m
Anvil Height above FFF	0.66 m
Factor for Impact Efficiency	0.65 #
Coefficient of Restitution	0.60

Operating Conditions

Strike rate of hammer	
Case 1 Blows per minute	36.00 #
Stroke length (Ht. fall of Tup)	1.70 m

Case 2 Blows per minute	75.00 #
Stroke length (Ht. fall of Tup)	1.00 m

Max Permitted Amplitudes

Anvil Amplitude	2.00 mm
Foundation Amplitude	1.50 mm

Soil Data

Dynamic Soil Properties	
Design Coefficient of Uniform Compression	4×10^4 kN/m^3
Poisson's ratio	0.30 #
Soil Density (Mass Density)	2.00 t/m^3
Damping Constant	0.10
Safe Gross Bearing Capacity @ 3.5 m depth	230.00 kN/m^2
Allowable Load Intensity at 3.5 m depth	184.00 kN/m^2

Foundation Data

Concrete Grade	M 25
Concrete Density	2.50 t/m^3
Depth of foundation (proposed)	3.50 m

Design of Foundation

Design Machine Parameters

Mass of Tup	$m_0 = 1.38$ t
Mass of Anvil with frame	34.00 t
Stroke (Height of fall of Tup)	1.70 m

Velocity of Tup before Impact

$$v_0' = \eta\sqrt{2gl} = 0.65 \times \sqrt{2 \times 9.81 \times 1.7} = 3.75 \text{ m/s}$$

Velocity of Anvil after Impact

$$v_2 = \frac{(1+e)}{(1+m_a/m_0)} v_0' = \frac{(1+0.6)}{(1+34/1.38)} \times 3.75 = 0.23 \text{ m/s}$$

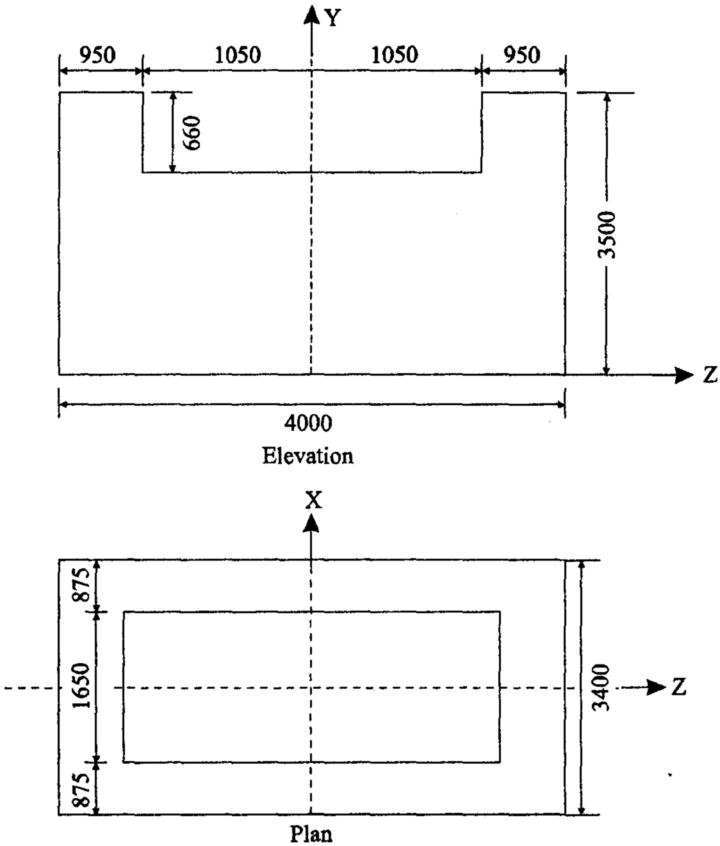


Figure D 11.1-1 Foundation Plan & Elevation for 1.1 t Drop Hammer

From Figure 11.1-1, for $C_u = 4\times10^4$ kN/m^3, we get factor $\lambda = 0.58$

$$\frac{v_0' \times \sqrt{D}}{y_f} = \frac{3.75\times\sqrt{3.5}}{1.1} = 7.02$$

$$\frac{m_f + m_a}{m_a} = 7.02\times0.58 = 4.07$$

Mass of anvil + foundation = $34\times4.07 = 138$ t

Mass of Foundation $m_f = 138 - 34 = 104$ t

This gives Foundation to Anvil Mass ratio as $\frac{104}{34} = 3.05$

Foundation sizing

Depth of Foundation	3.50 m
Desired Foundation Base Area	11.89 m
Provide Foundation Length	4.00 m
Provide Foundation Width	3.40 m
Area of Foundation as provided	13.60 m^2

Design Soil parameters

$C_u = 4 \times 10^4$ kN/m^3

Allowable Load Intensity at 3.5 m depth	184.00 kN/m^2
Soil Mass density	2.00 t/m3

Design of Foundation

a) Foundation Response to Impact Loading:

Let us consider that there is no pad below anvil i.e. anvil and foundation behave like a single unit. Foundation rests on the soil and the entire impact is borne by combined anvil-foundation system (see Figure 11.1-2a).

Representing Anvil + Foundation as SDOF System, we get

Total Mass Foundation + Anvil $m_1 = 114 + 34 = 148$ t

Mass Ratio $\lambda_1 = \frac{m_1}{m_0} = \frac{148}{1.38} = 1.07$

Velocity of foundation after impact

$$v_2 = \frac{(1+e)}{(1+\lambda_1)} v_0' = \frac{(1+0.6)}{(1+107.25)} \times 3.75 = 0.055 \text{ m/s}$$

Desired Foundation Amplitude (consider 80 % of allowable) $= 0.8 \times 1.5 = 1.2$ mm

Required soil stiffness $k_1 = 148 \times \left(\frac{0.055}{1.2 \times 10^{-3}} \right)^2 = 3.16 \times 10^5$ kN/m

Required Base Area of Foundation $A_f = \frac{3.16 \times 10^5}{4 \times 10^4} = 7.91$ m^2

Provided Base Area of Foundation 13.60 m^2, which is more than required hence OK

b) Anvil Response due to Impact Loading:

Let us consider an Elastic pad placed below the anvil and the foundation is fixed i.e. no deformation of foundation or soil. Anvil is considered as a SDOF System (see Figure 11.1-2b)

Mass of Anvil $m_2 = 34$ t

Mass ratio $\lambda_2 = \frac{34}{1.38} = 24.64$

Velocity of Anvil just after impact $v_2 = \frac{(1+e)}{(1+\lambda_2)} v_0' = \frac{(1+0.6)}{(1+24.64)} \times 3.75 = 0.23$ m/s

Desired Anvil Amplitude (80 % of allowable) 0.8 x 2 = 1.60 mm

Required stiffness of pad below anvil $k_2 = 34 \times \left(\frac{0.23}{1.6 \times 10^{-3}} \right)^2 = 7.3 \times 10^{-5}$ kN/m

Stress developed in pad $\sigma = \frac{7.29 \times 10^5 \times 1.6 \times 10^{-6}}{1.9 \times 1.45} = 423.35$ kN/m^2 = 0.423 MPa

Elastic Modulus at this stress is obtained from stress strain plot for pad material as supplied by manufacturer / supplier (see Figure D 11.1-2).

Stress =0.42 MPa; Strain = 0.0150; Elastic Modulus (stress/strain) = 28.22 MPa

Multiplying Factor for Dynamic Modulus (as recommended by pad supplier) = 1.1

Dynamic Elastic Modulus of pad $E_{pad} = 31.05 \text{ MPa} = 3.1 \times 10^4$ kN/m^2

Required Pad Thickness $t = \frac{E_{pad} \times A_{anvil}}{k_2} = \frac{3.1 \times 10^4 \times 1.9 \times 1.45}{7.29 \times 10^5} \times 1000 = 117.33$ mm

Provide Pad Thickness 120 mm

Modified Stiffness with this pad thickness

$$k_2 = \frac{E_{pad} \times A_a}{t} = \frac{3.1 \times 10^4 \times (1.9 \times 1.45)}{120 \times 10^{-3}} = 7.12 \times 10^5 \text{ kN/m}$$

Damping ratio of Pad material (data from supplier) = 0.25

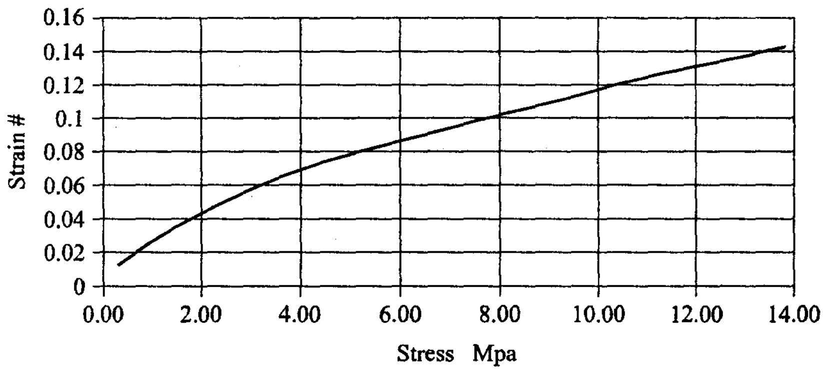


Figure D 11.1-2 Stress Strain Plot of Elastic Pad Material

c) Anvil + Foundation response to impact loading:

Consider an elastic pad between anvil and foundation and foundation supported over soil – System is represented as Two DOF System (see Figure 11.1-2c):

From cases a) & b), for Two DOF system, we get:

$$m_2 = 34 \text{ t}; \quad k_2 = 7.13E+05 \text{ kN/m}; \quad m_1 = 114 \text{ t}; \quad k_1 = 5.44E+05 \text{ kN/m}$$

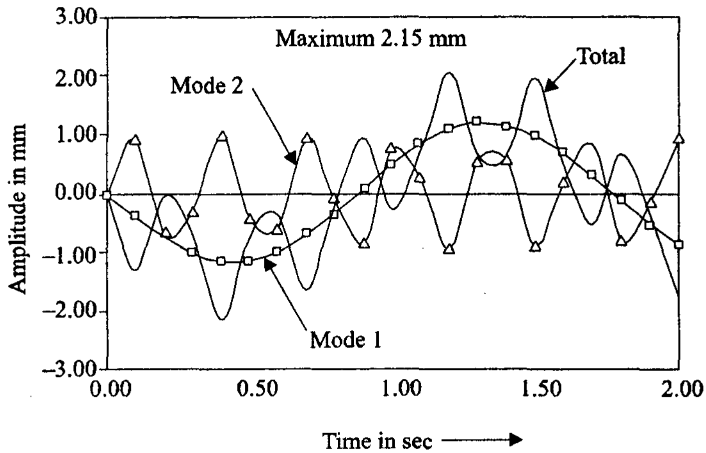
Limiting Frequencies and mass ratio

$$p_{l2}^2 = k_2/m_2 = 20963; \quad p_{l1}^2 = k_1/m_1 = 4772;$$
$$p_{l1} = 69.08 \text{ rad/s}; \quad p_{l2} = 144.79 \text{ rad/s}; \quad \lambda = m_2/m_1 = 34/114 = 0.3$$

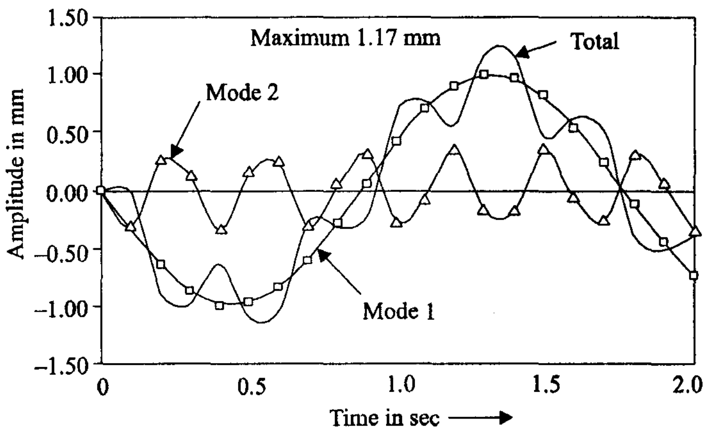
Frequency Equation (see equation 11.1-12)

$$p_{1,2}^2 = \frac{1}{2}\left\{\left(p_{l2}^2(1+\lambda) + p_{l1}^2\right) \mp \sqrt{\left(p_{l2}^2(1+\lambda)+p_{l1}^2\right)^2 - 4\left(p_{l1}^2 p_{l2}^2\right)}\right\}$$

Substituting Values, we get $p_1 = 59.27$ rad/s; $p_2 = 168.74$ rad/s



Anvil Response



Foundation Response

Figure D 11.1-3 Amplitude Response of Anvil and Foundation

Response of Anvil and Foundation – (Modal Response)

Maximum Anvil Response in first mode (see equation 11.1-15)

$$y_2' = \frac{v_2}{p_2}\frac{p_2}{p1} \times \frac{\left(p_{l,2}^2 - p_2^2\right)}{\left(p_1^2 - p_2^2\right)} = 1.19 \times 10^{-3} \text{ m}$$

Anvil Response in second mode $y_2'' = \frac{v_2}{p_2} \times \frac{\left(p_{L2}^2 - p_1^2\right)}{\left(p_1^2 - p_2^2\right)} = -9.71 \times 10^{-4}$ m

Foundation response in first mode $y_1' = \frac{v_2}{p_2} \frac{p_2}{p_1} \frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)} = 9.9 \times 10^{-4}$ m

Foundation response in second mode $y_1'' = \frac{v_2}{p_2} \frac{\left(p_{L2}^2 - p_1^2\right)\left(p_{L2}^2 - p_2^2\right)}{p_{L2}^2\left(p_1^2 - p_2^2\right)} = 3.48 \times 10^{-4}$ m

Total response for anvil as well as for Foundation is plotted and is as shown in Figure D 11.1-3

Total Anvil Response:	2.15 mm
Total Foundation Response:	1.17 mm

Isolation Efficiency

Impact Energy from Tup transferred to Anvil	9.72 kNm
Energy Transmitted from Anvil to Foundation	1.64 kNm
Isolation Efficiency at Anvil Base level	83.12 %
Energy Transmitted to Soil thro Foundation	0.37 kNm
Isolation Efficiency at Foundation Base level	96.19 %

Check for Bearing Pressure underneath foundation

Dynamic Load on the foundation $F_1 = k_1 \times y_1 = 5.44 \times 10^5 \times 1.17 \times 10^{-3} = 636$ kN

Soil pressure due to dynamic Load $= 636/13.6 = 46.7$ kN/m^2

Soil Pressure due to Static Load $= 148 \times 9.81/13.6 = 106.75$ kN/m^2

Total Bearing Pressure =46.7 + 106.7 = 153.4 kN/m2

Allowable bearing capacity = 184.00kN/m^2

Ratio of Bearing Pressure to Bearing Capacity 153/184 = 0.83 OK

Dynamic response under Impact Frequency

Usually Dynamic Analysis is not required for number of blows less than 150 per minute because there is hardly any dynamic interaction dynamic loading and the system experiencing dynamic loads. However just for academic interest, response is computed for such a loading in order to demonstrate steps involved to do similar analysis for higher number of blows per minute.

Case 1 Blows per minute 36

Stroke length (Ht. fall of Tup)	1.70 m
Damping Constant of soil	$\zeta = 0.1$
Mass of Tup	$m_0 = 1.38$ t
Factor for Impact Efficiency	0.65
Velocity of Tup before Impact	$v_0' = \eta\sqrt{2gl} = 0.65 \times \sqrt{2 \times 9.81 \times 1.7} = 3.75$ m/s
Frequency of Impact (36 Blows/min)	$\omega = 3.77$ rad/s
Acceleration	$\alpha = v_0' \times \omega = 3.75 \times 3.77 = 14.15$ m/s^2
Force of impact	$m_0 \times \alpha = 1.38 \times 14.15 = 19.53$ kN
Stiffness (Lower of k_1 or k_2)	5.44×10^5 kN/m
Natural freq (lower of two values for Two DOF System)	59.27 rad/sec
Frequency ratio	$\beta = \omega/p = 3.77/59.27 = 0.06$
Dynamic magnification factor	$\dfrac{1}{\left\|\left(1-(0.06)^2\right)\right\|} = 1.0$

Amplitude of foundation = $\dfrac{19.53}{5.44 \times 10^5} \times 1.0 = 3.6 \times 10^{-5}$ m $= 0.036$ mm

Case 2 Blows per minuet 75.00 #

Stroke length (Ht. fall of Tup) 1.00 m

Following steps as above, we get amplitude of foundation as 0.06 mm.

11.2 FOUNDATIONS FOR MACHINES PRODUCING IMPULSIVE LOADS

Machines, Producing Impulse/Pulse Loading, cover Forging and Stamping Press, Drop Weight Crushers, Crushing, Rolling and Grinding Mills etc.

In the absence of any real life machine data available with author, coverage is oriented towards broad spectrum of machines falling under this head.

Summary of Design Steps

1. Sizing of Foundation
2. Equivalent Soil Stiffness
3. Dynamic Forces
4. Analysis

II. Dynamic Analysis
 a. Natural Frequencies
 b. Dynamic Amplitudes

II. Strength and Stability Analysis

Required Input Data

Typical parameters required are as under.

a) Foundation Data

i) Foundation outline geometry, Machine Support locations on the foundation
ii) Any other parameter that is specific to machine may also be needed

b) Machine data

In view of high level of automation, it is difficult to generalize specific machine data for this class of machines. A host of machine data is listed. User may select the applicable data for the machine to be designed.

i) Mass Parameters

a. Total Mass of machine
b. Mass of Anvil
c. Mass of cross head
d. Mass of material to be forged
e. Mass of Ram

ii) Dynamic Force Parameters

f. Stroke of the press/ height of fall of ram
g. Pressure exerted by the press
h. Load time history of the pulse
i. Frequency of Impact i.e. Number of Blows/min
j. Unbalance force (in case of mills)
k. Dynamic force and moments in case of eccentric presses
l. Height of steel columns (in case of Press)
m. Cross section area of steel columns (in case of Press)

n. Details of Anchor Bolts and other embedded parts
o. Properties of isolation pads placed below the machine if any
p. Coefficient of Restitution/Impact
q. Allowable Amplitudes of vibration

c) Soil Data

i) Soil type and its basic characteristic properties
ii) Site Specific **Dynamic Soil Data**
iii) Bearing capacity
iv) Depth of water table
v) Liquefaction potential

Site Soil data, Machine data and Foundation data are converted to respective Design Parameters in line with provisions of **Chapter 5, 6 & 7**.

Necessary provisions must be made for Isolation of foundation from adjoining Structures. Wherever Isolation Pads are used, **Stress-Strain properties of these Isolation Pads** should also be listed.

In addition, environmental data like **Site related Seismic data,** presence of any industry/ housing clusters sensitive to ill effects of vibration may also be needed.

11.2.1 Foundation Sizing

Such machines are likely to cause overstressing. Hence, soil bearing pressure under normal loads should be kept at 50 % of bearing capacity keeping 100 % margin for overstressing. There should be no eccentricity between center of gravity of mass and center of stiffness i.e. combined CG of mass of machine and foundation system should coincide with the CG of Base area of the foundation.

11.2.2 Dynamic Analysis

Dynamic Forces

Impulsive loading is a special class of dynamic loading and generally consists of a single impulse of short duration. The force produced during operation is termed as the Impulsive Force. Two types of pulse loading are considered:

i) Short duration Impulse Loading
ii) Long duration Pulse Loading

Short Duration Impulse Loading: Dynamic magnification depends upon ratio of Frequency of Impact to natural frequency.

Long Duration Pulse Loading: Dynamic magnification factor depends upon ratio of pulse duration to natural time period of foundation.

Analysis: Analysis is done using manual method of computation as given in **Chapter 2**. Dynamic response is evaluated at the foundation.

Summary of Formulae:

For a SDOF System (Single spring mass system):

Stiffness of the system (Equivalent soil springs) $k = C_u \times A_f$ (11.2-1)

Here A_f is area of foundation base in contact with soil

Mass of the system $m = m_1 + m_2$ (11.2-2)

Here m_1 represents mass of foundation and m_2 represents mass of machine supported over foundation

Natural frequency $p = \sqrt{\frac{k}{m}}$ rad/s (11.2-3)

1. **Machines producing Short duration Impulses applied as repeated blows with frequency ω**

Load Impulse I

Number of strike N rpm

Frequency of Repeated blows $\omega = \frac{2\pi}{60} N$ rad/s

Frequency ratio $\beta = \frac{\omega}{p}$ (11.2-4)

Dynamic Magnification $\mu_y = \frac{1}{\left|(1 - \beta^2)\right|}$ (11.2-5)

Static Deflection $\delta_{st} = \frac{I \times p}{m \times p^2} = \frac{I \times p}{k}$ (11.2-6)

Amplitude $y = \delta_{st} \times \mu_y$ (11.2-7)

For certain machines having falling mass m_0 and height of fall being h , Impulse is given as:

$$I = m_0 \times v_0 \qquad (11.2\text{-}8)$$

Here $v_0 = \sqrt{2gh}$ is the terminal velocity

Substituting in equation 11.2-6, we get dynamic amplitudes as given by equation 11.2-7.

2. Machines producing Long Duration Pulse:

Response Magnification due to Pulse loading of a SDOF System, as given in Chapter 2, is reproduced here for convenience and is shown in Figure 11.2-1

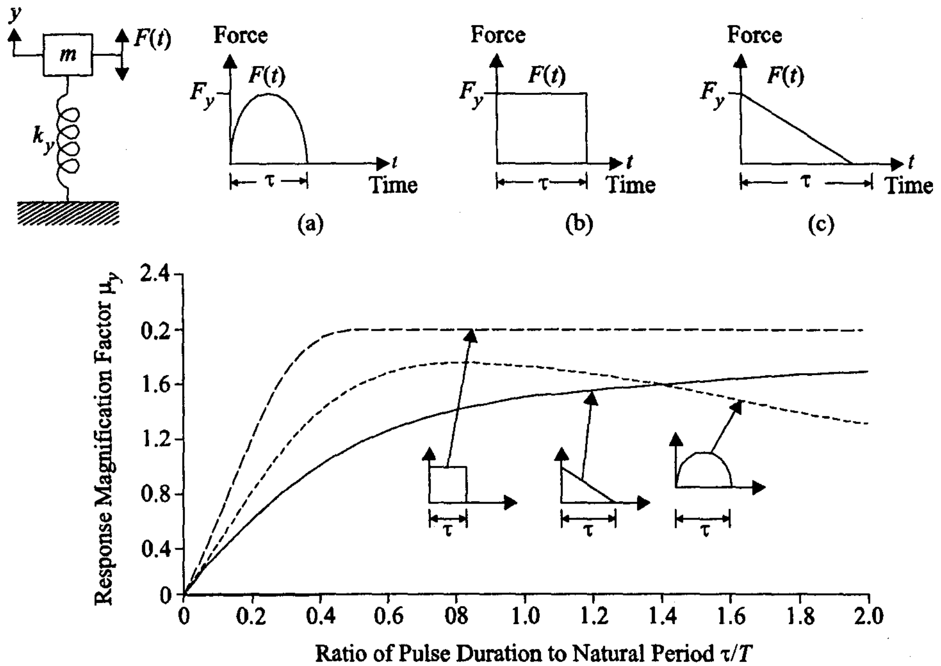


Figure 11.2-1 SDOF System subjected to Pulse Loading -Magnification Factor vs. ratio of pulse duration to Time period of the system

Dynamic Force induced by machine as (Peak of the pulse loading) F_y

Static deflection $\delta_{st} = \dfrac{F_y}{k}$ (11.2-9)

Magnification Factor μ_y vs. τ / T (from Figure 11.2-1) (11.2-10)

Amplitude thus becomes $y = \delta_{st} \times \xi \times \mu_y$ (11.2-11)

Here ξ is the Fatigue Factor. In case fatigue factor is not defined, it may be taken equal to 2.

DESIGN EXAMPLE

Example D 11.2: Machine Producing Impulsive Loads Applied at Repeated Interval

System Data

Machine Weight	1500 kN
Impulse produced by machine	$I = 5000$ Ns
Foundation weight	2500 kN
Foundation Base Area	35 m^2
Depth of Foundation	3.5 m
Gross Bearing Capacity @ 3.5 m depth	225 kN/m^2
Coefficient of uniform compression of soil	4×10^4 kN/m^3
Frequency of repeated blows	90 rpm

Design

Total weight of Machine + foundation	2500+1500 = 4000 kN
Soil Bearing pressure	4000/35 = 114.29kN/m^2
Margin for overload	49.21% Hence OK
Excitation frequency (90 rpm)	9.42 rad/s
Mass (Machine + foundation)	$m = (4000/9.81) = 407.75$ t
Soil Stiffness	$k_y = C_u \times A = 4 \times 10^4 \times 35 = 1.4 \times 10^6$ kN/m

Natural frequency of SDOF system

$$p_y = \sqrt{k_y/m} = \sqrt{1.4 \times 10^6/407.75} = 58.6 \text{ rad/s}$$

Frequency ratio $\beta_y = 9.42/58.6 = 0.16$

Force due to Impulse $F_y = I \times p_y = 5000 \times 58.6 = 293 \text{ kN}$

Static deflection $\delta_{st} = \dfrac{F_y}{k_y} = \dfrac{293}{1.4 \times 10^6} = 2.09 \times 10^{-4} \text{ m}$

Magnification $\dfrac{1}{\left|\left(1 - 0.16^2\right)\right|} = 1.026$

Amplitude of foundation $y = \delta_{st} \times \mu = 2.09 \times 10^{-4} \times 1.026 = 2.15 \times 10^{-4} \text{ m} = 0.215 \text{ mm}$

Example D 11.3: Machine producing long duration Impulsive Loads

System Data:

Machine weight 6000 kN

Pulse Load Time History: Half Sine Pulse having pulse duration of 0.08 s and peak pulse force of 5000 N

Base Area of foundation 84 m^2

Foundation Weight 8000 kN

Depth of Foundation 4.0 m

Gross Bearing Capacity @ 4 m depth 300 kN/m2

Design Coefficient of Uniform Compression $C_u = 6 \times 10^4 \text{ kN/m}^3$

Design

Total weight of machine + foundation 6000 + 8000 = 14000 kN

Soil Bearing pressure 14000 / 84 = 166.67 kN/m^2

Margin for overload $(1 - 166.67/300) \times 100 = 44.44$ %; hence OK

Mass (Machine + foundation) 14000/ 9.81 = 1427.12 t

Soil Stiffness $k_y = 6 \times 10^4 \times 84 = 5.04 \times 10^6 \text{ kN/m}$

Natural frequency of SDOF system $p_y = \sqrt{\dfrac{5.04 \times 10^6}{(14000/9.81)}} = 59.4 \text{ rad/s} = 9.46 \text{ Hz}$

Natural Time period	$T = (1/9.46) = 0.10$ s
Pulse duration	$\tau = 0.8\,s$
Ratio of Pulse Duration to natural time period	$\tau/T = (0.08/0.1) = 0.8$
Peak pulse force	$F_y = 5000$ N
Magnification from Figure 11.2-1	$\mu = 1.78$
Static deflection	$\delta_{st} = (5.0/(5.04\times10^{6})) = 1\times10^{-6}$ m
Amplitude of foundation	$y = 1.0\times10^{-6}\times1.78 = 1.78\times10^{-6} = 2$ microns

PART - IV

VIBRATION ISOLATION DESIGN FOR REAL LIFE MACHINES

12

VIBRATION ISOLATION SYSTEM

- **Design Examples**

- Vibration Isolation Design of a Fan Foundation
- Vibration Isolation Design of a Crusher Foundation

For better clarity, all Figures related to FE analysis, including animations of frequencies and mode shapes, in color, are given in the CD attached at the end of the handbook

VIBRATION ISOLATION SYSTEM

In industrial environment, need for Vibration Isolation arises on one count or the other. Some of the obvious reasons are listed as under:

1. To control excessive vibration levels to the machine itself
2. To control propagation of vibration to adjoining machines/ systems
3. To house a new (higher rating) machine on an existing foundation
4. To control vibrations on account of locating a machine at intermediate structural floors
5. To overcome uncertainty of dynamic soil parameters, and so on .....

All machines do not need inertia block for providing vibration isolation. In certain cases of stand-alone machines like utility DG sets, Utility compressors etc, Vibration Isolators have become a part and parcel of the machine system itself and are supplied by the manufacturer/ supplier bundled with the machine. On the other hand, in industrial environment, majority of machines, that need Vibration Isolation, are supported over inertia block which in turn rests on Vibration Isolators.

Basic theory of Vibration Isolation is described in Chapter 4. Related aspects like Principle of Isolation, Transmissibility Ratio, Isolation Efficiency, Isolation Requirements and Selection of Isolators are adequately covered in Chapter 4.

Sizing of Inertia block, selection of right type of isolators for a given application and identification of placement locations of the isolators beneath the inertia block are key parameters for efficient **VIS** (Vibration Isolation System) design.

12.1 VIBRATION ISOLATION DESIGN

Design of Vibration Isolation System for a Machine or a Machine Foundation System comprises of the followings:

- Sizing of Inertia Block
- Selection of Isolators - stiffness & damping properties of isolators
- Location of Isolators

- Dynamic analysis –Evaluation of Amplitudes & Reaction forces on support structure
- Strength analysis of Inertia Block
- Evaluation of Stiffness & damping properties of sub-structure if any

12.1.1 Sizing of Inertia Block

While sizing inertia block, due consideration should be given to the following:

i) Inertia block – Plan Dimensions are sized such that it is able to house complete machine including all its components with necessary cutouts, openings & pockets required by machine for its operation
ii) It has sufficient room underneath for placement of isolators
iii) The thickness of inertia block should be such that its flexural natural frequencies (with machine mass) are sufficiently away from
 a) The operating speed of machine and its harmonics
 b) Critical speed of machine &
 c) Frequency of isolation system
iv) It should provide stability to overall centroid (machine + inertia block)
v) Net mass of inertia block should preferably be 2 to 3 times that of the machine. This is only recommendatory and not mandatory. A higher ratio, if needed, may also be provided.

12.1.2 Selection of Isolators

Criteria for selection of Isolators are as under:

i) It should be capable of supporting loads from machine and inertia block
ii) Its stiffness parameters are selected so as to achieve desired frequency ratio required for effective isolation
iii) It should have a good damping constant required to control transient vibration amplitudes

12.1.3 Location of Isolators

Isolators are located such that their **center of stiffness** (vertical stiffness along Y-direction) matches with overall **center of mass** (machine + inertia block) i.e. overall centroid location of machine + inertia block (in X-Z Plane). Needless to mention that presence of eccentricity would result in reduced isolation effectiveness.

12.1.4 Dynamic Analysis

The stiffness and damping properties of finally selected isolators to be considered while analyzing the system. More often than not, it may be possible to represent the system as SDOF system or two DOF system. In such a case, modeling and analysis is carried out in line with the procedure given in Chapter 2 and Chapter 3 respectively. For complex systems, it may be advisable to resort to computer-based analysis using commercially available packages.

Transient Resonance: For effective isolation, frequency of isolation system must always be lower than the operating speed of the machine by an order of 2 to 4. Thus the system will always undergo **Transient Resonance** with every start and shutdown operation. It is therefore desirable that **amplitudes** as well as **reaction forces,** even during transient resonance, are kept within control. Though damping is undesirable from the point of view of isolation efficiency (see Chapter 4), it very much helps in bringing down amplitudes during transient resonance. In other words damping associated with the Isolator becomes instrumental in bringing down the amplitudes and thereby reaction forces during transient resonance.

Flexural Frequencies of Inertia Block: Invariably, flexural frequencies of inertia block are sufficiently away from isolator frequencies. A frequency check, however, is recommended for the Inertia Block to ensure that none of its frequencies is in resonance with isolator frequency or machine operating speeds.

Reaction Forces: Reaction forces are transmitted from the inertia block to the support block/support structure (sub-structure) through the isolators. It is desirable to check the isolator capacity against maximum reaction force. Similar check is required for maximum deflection the isolator would experience under maximum reaction force. The maximum deflection should never exceed the permissible value of the isolator. At times, either of these checks may suggest change in isolator specifications and the complete process to be repeated.

Strength Analysis of Inertia Block: Inertia block should be strong enough to withstand the forces generated by the machine during normal operating condition as well as transient resonance condition. Inertia block is. Structural analysis of inertia block, modeled as supported on isolators, with all cutouts and openings is carried out to ensure its strength adequacy. Necessary reinforcement, required as per design codes, is provided.

Stiffness of Support Structure/sub-structure: From lay out constraints, at times, inertia block is supported on a support structure/ sub-structure. Such support structure should be strong enough to withstand the reaction forces transmitted through isolators. Further it must be sufficiently rigid in both vertical and lateral direction and not interfere with system frequencies desired for effective isolation.

The support structure, if not sufficiently rigid, tends to reduce the effective stiffness of isolator. Let us understand this with the help of following example:

Consider two springs having stiffness k_1 & k_2 attached in series subjected to axial force F.

The deflection of spring k_1 is $$\delta_1 = \frac{F}{k_1}$$

The deflection of spring k_2 is $\delta_2 = \dfrac{F}{k_2}$

Total deflection $\delta = \delta_1 + \delta_2$ $\delta = \dfrac{F}{k_1} + \dfrac{F}{k_2} = F\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right)$

Equivalent stiffness of combined springs in series = $k_{eq} = \dfrac{F}{\delta} = \dfrac{k_1 k_2}{k_1 + k_2}$

Equivalent stiffness is plotted with respect to stiffness ratio k_2/k_1 .

It is seen that for $k_2/k_1 > 20$, k_{eq} becomes more or less constant. Thus it is clear that stiffness of support structure at the isolator support location should at least be 20 times that of isolator stiffness. If the support structure stiffness is less than 20 times the isolator stiffness, the computed frequency of the supported machine foundation system would be higher than actual. Thus one may not achieve the desired isolation efficiency.

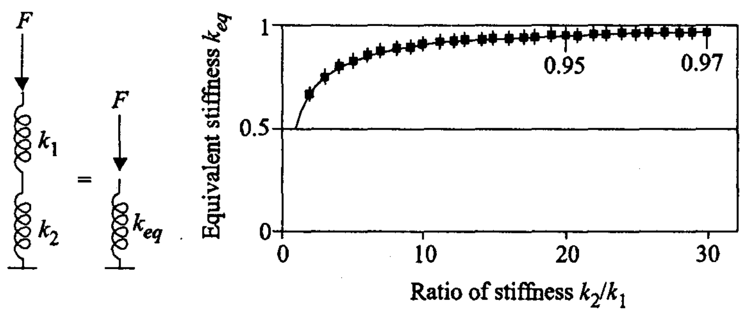


Figure 12.1-1 Equivalent Stiffness for springs in series

DESIGN EXAMPLES

Design Examples are selected so as to cover up most of the conditions encountered in real life practice. Comparison with Finite Element Analysis (**FEA**) is also given for specific cases to build up the confidence level. Effort is made to highlight the influence of certain slips commonly committed while computing response of the foundation.

Example D 12.1 Vibration Isolation for Fan Foundation

Design Vibration Isolation System for FD Fan Foundation as shown in Figure D 12.1-1. Machine and Foundation data is given as under:

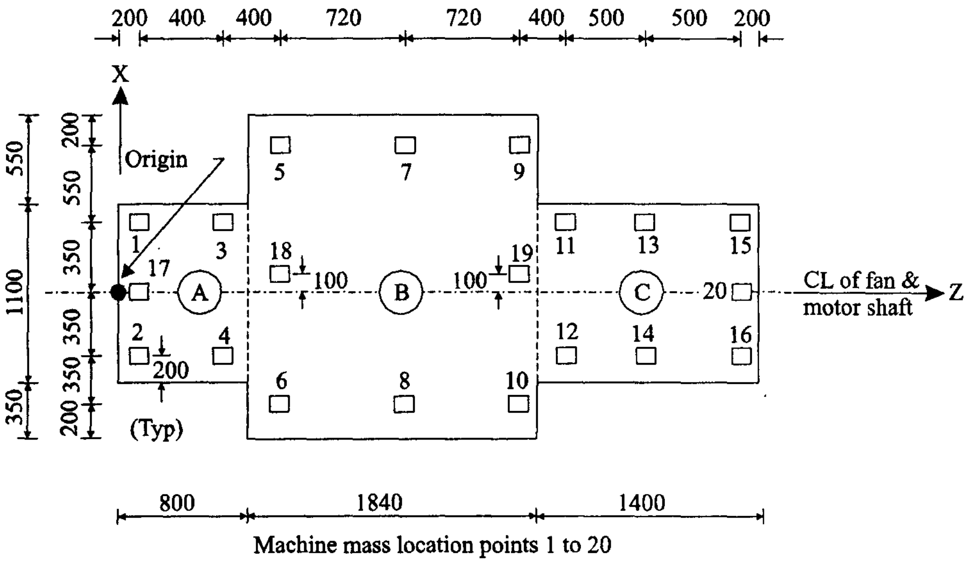


Figure D 12.1-1 Vibration Isolation for FD Fan Foundation out line and Machine Mass location

Machine Data

Machine weight	18.20 kN
Number of support points	20
Height of center of shaft above foundation	1230 mm
Speed	960 rpm

Dynamic Force

Rotor mass	0.19 t
Rotor eccentricity	200 microns
Dynamic force @ center line of shaft level	0.38 kN

Layout of machine mass point over foundation is shown in Figure 12.1-1. For computation, consider Origin on left side top of inertia block as shown in the Figure.

Machine Mass Points and coordinates with respect to Origin

Mass Point	Machine Weight	Coordinates with respect to Origin		
#	kN	mm	mm	mm
	W_{mi}	x_{mi}	z_{mi}	y_{mi}
1	0.80	350.00	200.00	1230.00
2	0.80	-350.00	200.00	1230.00
3	0.80	350.00	600.00	1230.00
4	0.80	-350.00	600.00	1230.00
5	1.50	900.00	1000.00	1230.00
6	1.50	-700.00	1000.00	1230.00
7	1.50	900.00	1720.00	1230.00
8	1.50	-700.00	1720.00	1230.00
9	1.50	900.00	2440.00	1230.00
10	1.50	-700.00	2440.00	1230.00
11	0.70	350.00	2840.00	1230.00
12	0.70	-350.00	2840.00	1230.00
13	0.70	350.00	3340.00	1230.00
14	0.70	-350.00	3340.00	1230.00
15	0.70	350.00	3840.00	1230.00
16	0.70	-350.00	3840.00	1230.00
17	0.70	0.00	200.00	1230.00
18	1.10	100.00	1000.00	1230.00
19	1.10	100.00	2440.00	1230.00
20	0.70	0.00	3840.00	1230.00

Here x_{mi}, y_{mi} & z_{mi} represent coordinates of CG of machine load points with respect to Origin.

Total Machine Weight 20.00 kN

CG of Machine weight with respect to Origin

$$\bar{x}_m = \frac{\sum(W_{mi}x_{mi})}{\sum W_{mi}} = \frac{1120}{20} = 56 \text{ mm}; \quad \bar{z}_m = \frac{\sum(W_{mi}z_{mi})}{\sum W_{mi}} = \frac{37400}{20} = 1870 \text{ mm} \ \& \ \bar{y}_m = 1230 \text{ mm}$$

Foundation Data

Foundation Geometry is as shown in Figure 12.1-1. Inertia block is divided in to 3 sub blocks marked 'A', 'B' & 'C' as shown in the Figure. Thickness of Inertia block (as considered for initial sizing) is 300 mm.

Foundation Self Weight and its CG with respect to Origin

Block #	Dimensions			Weight	CG Coordinates		
	mm	mm	mm	kN			
	X	Z	Y	W_{fi}	x_{fi}	z_{fi}	y_{fi}
A	1100	800	300	6.47	0	400	-150
B	2000	1840	300	27.08	100	1720	-150
C	1100	1400	300	11.33	0	3340	-150

Here x_{fi}, y_{fi} & z_{fi} represent coordinates of CG of each foundation block element.

Total Foundation Weight $W_f = 44.8 \approx 45$ kN

$$\bar{x}_f = \frac{\sum(W_{fi}x_{fi})}{\sum W_{fi}} = \frac{2707.56}{44.88} = 60.33 \text{ mm}; \quad \bar{z}_f = \frac{\sum W_{fi}z_{fi}}{\sum W_{fi}} = \frac{87003.91}{44.88} = 1938.56 \text{ mm}$$

$$\bar{y}_f = -150 \text{ mm}$$

Total weight of Machine + Foundation $W = 20 + 44.8 \approx 65$ kN

Overall Centroid

Let us represent Overall centroid by **point C**

Center of Mass of Machine-Foundation System in X-Z Plane

$$\bar{x} = \frac{W_m\bar{x}_m + W_f\bar{x}_f}{W_m + W_f} = \frac{20\times 56 + 45\times 60.33}{65} = 59 \text{ mm}$$

$$\bar{y} = \frac{W_m\bar{y}_m + W_f\bar{y}_f}{W_m + W_f} = \frac{20\times 1230 + 45\times(-150)}{65} = 275 \text{ mm}$$

$$\bar{z} = \frac{W_m\bar{z}_m + W_f\bar{z}_f}{W_m + W_f} = \frac{20\times 1870 + 45\times 1938.56}{65} = 1917 \text{ mm}$$

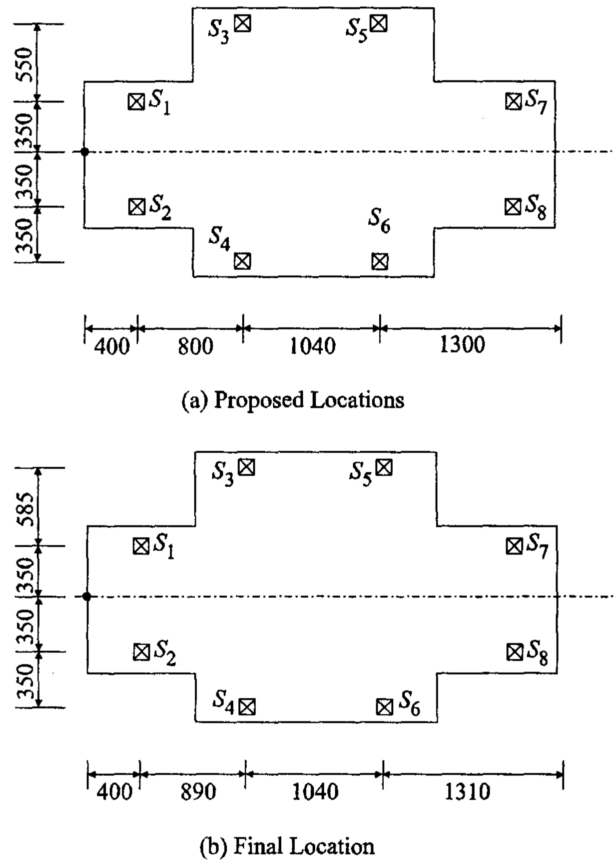


Figure D 12.1-2 Isolator location points (S1-S2) underneath the foundation Block

Selection of Isolators

Let us consider Target Isolation Efficiency $\eta = 90\,\%$. Consider isolator damping $\zeta = 0$.

For $\eta = 90\,\%$ & $\zeta = 0$ frequency ratio (Table 4.1.1) = 3.2

Operating Speed (960 rpm) $\omega = 100.5$ rad/s

Required Isolator Frequency $100.5/3.2 = 31.4$ rad/s

Static Deflection of Isolator $\sqrt{\frac{g}{\delta}} = p_y = 31.4 \text{ rad/s}$ $\delta = \frac{g}{(31.4)^2} = \frac{9810}{986.3} = 9.95 \text{ mm}$

Let us select $\delta = 10 \text{ mm}$

Total weight of machine foundation system $W = 65 \text{ kN}$

Consider number of isolators $= 8$

Load per Isolator $= \frac{65}{8} \approx 8 \text{ kN}$

Stiffness of Isolator (vertical) $k_y = \frac{8 \times 1000}{10} = 800 \text{ N/mm}$

Isolator stiffness (Lateral) $k_x = k_z = 0.6 \times 800 = 480 \text{ N/mm}$

Isolator Placement Locations Proposed Initial Location Shown In Figure D 12.1-2a

Center of Stiffness of Isolators (Vertical Stiffness k_y)

Isolator #	Stiffness N/mm k_{yi}	Coordinates with respect to origin mm x_{ki}	mm z_{ki}	mm y_{ki}
S1	800	350	400	-300
S2	800	-350	400	-300
S3	800	900	1200	-300
S4	800	-700	1200	-300
S5	800	900	2240	-300
S6	800	-700	2240	-300
S7	800	350	3540	-300
S8	800	-350	3540	-300

Total Stiffness 6400 N

Center of Isolator Stiffness

$$\bar{x}_k = \frac{\sum (k_{yi} \times x_{ki})}{\sum k_{yi}} = \frac{3.2 \times 10^5}{800 \times 8} = 50 \text{ mm}; \quad \bar{z}_k = \frac{\sum (k_{yi} \times z_{ki})}{\sum k_{yi}} = \frac{1.18 \times 10^7}{800 \times 8} = 1845 \text{ mm}$$

Eccentricity with Center of Mass $e_x = \frac{59 - 50}{2000} \times 100 = 0.45\,\%$; $e_z = \frac{1917 - 1845}{4040} \times 100 = 1.8\,\%$

For better performance, it is desirable to keep the eccentricity less than 0.5 %.

Modified Locations of Isolators

Isolator #	Stiffness N/mm	Coordinates with respect to origin mm	mm	mm
	k_{yi}	x_{ki}	z_{ki}	y_{ki}
S1	800	350	400	-300
S2	800	-350	400	-300
S3	800	935	1290	-300
S4	800	-700	1290	-300
S5	800	935	2330	-300
S6	800	-700	2330	-300
S7	800	350	3640	-300
S8	800	-350	3640	-300

With these modified locations, we get $\bar{x}_k = 58.8$ mm; $\bar{z}_k = 1915$ mm; $\bar{y}_k = -300$ mm

$$e_x = \frac{59 - 58.8}{2000} \times 100 = 0.01\,\%; \quad e_z = \frac{1917 - 1915}{4040} \times 100 = 0.05\,\%$$

Dynamic Analysis

Mass and Mass moment of Inertia

For Vibration Isolation Design, DOF location remains as center of mass only.

Mass and Mass moment of Inertia at Centroid (Center of Mass)

i) Machine

Point	Mass	Coordinates wrt Origin			Centroid coordinates			Mass Moment of Inertia		
#	t	mm	mm	mm	mm	mm	mm	tm^2	tm^2	tm^2
	m_{mi}	x_{mi}	z_{mi}	y_{mi}	$\bar{x}$	$\bar{z}$	$\bar{y}$	M_{mix}	M_{miz}	M_{miy}
1	0.08	350	200	1230	59	1917	275	0.31	0.08	0.25
2	0.08	-350	200	1230	59	1917	275	0.31	0.09	0.25
3	0.08	350	600	1230	59	1917	275	0.22	0.08	0.15
4	0.08	-350	600	1230	59	1917	275	0.22	0.09	0.16
5	0.15	900	1000	1230	59	1917	275	0.27	0.25	0.24
6	0.15	-700	1000	1230	59	1917	275	0.27	0.23	0.22
7	0.15	900	1720	1230	59	1917	275	0.15	0.25	0.11
8	0.15	-700	1720	1230	59	1917	275	0.15	0.23	0.09
9	0.15	900	2440	1230	59	1917	275	0.18	0.25	0.15
10	0.15	-700	2440	1230	59	1917	275	0.18	0.23	0.13
11	0.07	350	2840	1230	59	1917	275	0.13	0.07	0.07
12	0.07	-350	2840	1230	59	1917	275	0.13	0.08	0.07
13	0.07	350	3340	1230	59	1917	275	0.21	0.07	0.15
14	0.07	-350	3340	1230	59	1917	275	0.21	0.08	0.16
15	0.07	350	3840	1230	59	1917	275	0.33	0.07	0.27
16	0.07	-350	3840	1230	59	1917	275	0.33	0.08	0.28
17	0.07	0	200	1230	59	1917	275	0.28	0.07	0.21
18	0.11	100	1000	1230	59	1917	275	0.20	0.10	0.09
19	0.11	100	2440	1230	59	1917	275	0.13	0.10	0.03
20	0.07	0	3840	1230	59	1917	275	0.33	0.07	0.26

$m_m = \sum m_{mi} = 2.04$ t; $M_{mx(\text{machine})} = \sum M_{mix} = 4.51$ tm^2

$M_{mz(\text{machine})} = \sum M_{miz} = 2.54$ tm^2; $M_{my(\text{machine})} = \sum M_{miy} = 3.34$ tm^2

ii) Foundation

		Foundation Blocks		
Dimensions		A	B	C
X	mm	1100	2000	1100
Z	mm	800	1840	1400
Y	mm	300	300	300
Mass	t	0.66	2.76	1.16

CG with respect to Origin

x_{fi}	mm	0	100	0
z_{fi}	mm	400	1720	3340
y_{fi}	mm	-150	-150	-150

Overall Centroid coordinates with respect to Origin

$\bar{x}$	mm	59	59	59
$\bar{z}$	mm	1917	1917	1917
$\bar{y}$	mm	275	275	275

Mass Moment of Inertia @ centroid

$M_{mix(\text{Foundation})}$	tm2	1.64	0.61	2.55
$M_{miz(\text{Foundation})}$	tm2	0.12	0.50	0.21
$M_{miy(\text{Foundation})}$	tm2	1.52	0.11	2.34

$m_f = 4.58$ t; $M_{mx(\text{Foundation})} = \sum M_{mix} = 5.83 \text{ tm}^2$

$M_{mz(\text{Foundation})} = \sum M_{miz} = 1.97 \text{ tm}^2$; $M_{my(\text{Foundation})} = \sum M_{miy} = 6.08 \text{ tm}^2$

Total for Machine and Foundation (Mass and Mass Moment of Inertia)

$m = 2.04 + 4.58 = 6.62$ t; $\quad M_{mx} = 4.51 + 5.83 = 10.34$ tm^2

$M_{mz} = 2.54 + 1.97 = 4.51$ tm^2; $\quad M_{my} = 3.34 + 6.08 = 9.42$ tm^2

Stiffness

Translational and Rotational stiffness at **Center of Stiffness**

$$k_x = \sum k_{xi}; \quad k_y = \sum k_{yi}; \quad k_z = \sum k_{zi}$$

$$k_{\theta i} = k_{yi} \times (\bar{z}_k - z_i)^2; \quad k_{\phi i} = k_{yi} \times (\bar{x}_k - x_i)^2; \quad k_{\psi i} = k_{xi} \times (\bar{z}_k - z_i)^2 + k_{zi} \times (\bar{x}_k - x_i)^2$$

$$k_\theta = \sum k_{\theta i}; \; k_\psi = \sum k_{\psi i}; \; k_\phi = \sum k_{\phi i}$$

Stiffness computations are tabulated as under:

Isolator	Stiffness	Coordinates wrt Origin			Stiffness Centroid			Stiffness (kNm /rad)		
#	kN /m	mm	mm	mm	mm	mm	mm			
	k_{yi}	x_{ki}	z_{ki}	y_{ki}	$\bar{x}_k$	$\bar{z}_k$	$\bar{y}_k$	$k_{\theta i}$	$k_{\psi i}$	$k_{\phi i}$
S1	800	350	400	-300	58.8	1915	-300	1841	1145	68
S2	800	-350	400	-300	58.8	1915	-300	1841	1185	134
S3	800	935	1290	-300	58.8	1915	-300	315	557	614
S4	800	-700	1290	-300	58.8	1915	-300	315	465	461
S5	800	935	2330	-300	58.8	1915	-300	136	450	614
S6	800	-700	2330	-300	58.8	1915	-300	136	358	461
S7	800	350	3640	-300	58.8	1915	-300	2375	1466	68
S8	800	-350	3640	-300	58.8	1915	-300	2375	1505	134

We get overall stiffness as:

$k_x = 3.84 \times 10^3$ kN/m; $\quad k_y = 6.4 \times 10^3$ kN/m; $\quad k_z = 3.84 \times 10^3$ kN/m

$k_\theta = 9.334 \times 10^3$ kNm/rad; $\quad k_\psi = 7.132 \times 10^3$ kNm/rad; $\quad k_\phi = 2.553 \times 10^3$ kNm/rad

Natural Frequency

$$p_x = \sqrt{\frac{k_x}{m}}; \quad p_y = \sqrt{\frac{k_y}{m}}; \quad p_z = \sqrt{\frac{k_z}{m}}$$

$$p_\theta = \sqrt{\frac{k_\theta}{M_{mx}}}; \quad p_\phi = \sqrt{\frac{k_\phi}{M_{mz}}}; \quad p_\psi = \sqrt{\frac{k_\psi}{M_{my}}}$$

Rewriting Mass and Mass moment of Inertia

$$m = 6.62; \quad M_{mx} = 10.34\ \text{tm}^2; \quad M_{mz} = 4.51\ \text{tm}^2; \quad M_{my} = 9.42\ \text{tm}^2$$

Substituting values, we get

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{3.84 \times 10^3}{6.62}} = 24 \text{ rad/s}$$

$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{6.4 \times 10^3}{6.62}} = 31.1 \text{ rad/s}$$

$$p_z = p_x = 24 \text{ rad/s}$$

$$p_\theta = \sqrt{\frac{k_\theta}{M_{mx}}} = \sqrt{\frac{9.334 \times 10^3}{10.34}} = 30 \text{ rad/s}$$

$$p_\psi = \sqrt{\frac{k_\psi}{M_{moy}}} = \sqrt{\frac{7.132 \times 10^3}{9.42}} = 27.5 \text{ rad/s}$$

$$p_\phi = \sqrt{\frac{k_\phi}{M_{moz}}} = \sqrt{\frac{2.553 \times 10^3}{4.51}} = 23.8 \text{ rad/s}$$

Amplitudes

Dynamic force @ center line of shaft level $F_x = 0.38$ kN; $F_y = 0.38$ kN

Excitation Frequency (960 rpm) $\omega = 100.5$ rad/s

Dynamic force transferred @ DOF location point O

Height of rotor center line from Overall Centroid $s = 1.23 - 0.275 = 0.955$ m

Moment about Z axis $M_\phi = F_x \times s = 0.38 \times 0.955 = 0.36 \text{ kNm}$

Net forces acting at Centroid $F_x = 0.38 \text{ kN}; \quad F_y = 0.38 \text{ kN}; \quad M_\phi = 0.36 \text{ kNm}$

Amplitude @ Centroid

i) Force $F_x = 0.38 \text{ kN}$

Frequency Ratio $\beta_x = \dfrac{\omega}{p_x} = \dfrac{100.5}{24} = 4.18$

Amplitude $$x_c = \frac{F_x}{k_x} \times \frac{1}{\left|\left(1-\beta_x^2\right)\right|} = \frac{0.38}{3.84\times10^3} \times \frac{1}{\left|\left(1-4.18^2\right)\right|} = 6\times10^{-6} \text{ m}$$

ii) Force $F_y = 0.38 \text{ kN}$

Frequency Ratio $\beta_y = \dfrac{\omega}{p_y} = \dfrac{100.5}{31.1} = 3.23$

Amplitude $$y_c = \frac{F_y}{k_y} \times \frac{1}{\left|\left(1-\beta_y^2\right)\right|} = \frac{0.38}{6.4\times10^3} \times \frac{1}{\left|\left(1-3.23^2\right)\right|} = 6.3\times10^{-6} \text{ m}$$

iii) Moment $M_\phi = 0.36 \text{ kNm}$

Frequency Ratio $\beta_\phi = \dfrac{\omega}{p_\phi} = \dfrac{100.5}{23.8} = 4.22$

Amplitude $$\phi_c = \frac{M_\phi}{k_\phi} \times \frac{1}{\left|\left(1-\beta_\phi^2\right)\right|} = \frac{0.36}{2.553\times10^3} \times \frac{1}{\left|\left(1-4.22^2\right)\right|} = 8.4\times10^{-6} \text{ rad}$$

Amplitude @ Foundation top

Maximum amplitude along X

Height of foundation top from Point C H = -0.275 m

$$x_{f(\max)} = x_c + H\phi_c = 6\times10^{-6} + \left|8.4\times10^{-6}\times(-0.275)\right| = 8.31\times10^{-6} \text{ m} = 8.3 \text{ microns}$$

Maximum amplitude along Y

Maximum foundation width along X B = 2.0 m

$$y_{f(\max)} = y_c + (B/2)\phi_c = 6.3\times10^{-6} + 8.4\times10^{-6}\times(2.0/2) = 14.7\times10^{-6}\text{ m} = 15\text{ microns}$$

Maximum Vertical Dynamic Force Transmitted by Isolator

Maximum Isolator spacing along X (see Figure 12.1-2) = 350+350+350+585 = 1635 mm

Maximum Vertical Dynamic force experienced by one isolator (placed along Z)

$$\frac{F_y}{8} + \frac{1}{3}\frac{M_\phi}{(1.635/2)} = \frac{0.38}{8} + \frac{1}{3}\times\frac{0.58}{(1.635/2)} = 0.0475 + 0.237 = 0.285\text{ kN} = 285\text{ N}$$

Maximum Vertical Dynamic force transmitted by single isolator

Stiffness of single isolator = 0.8×10^3 kN/m

$$F_{Ty} = k_{yi}\times(y_c + \phi_c\times x_s)$$
$$= 0.8\times10^3\times(6.3\times10^{-6} + 8.4\times10^{-6}\times(1.635/2))$$
$$= 10.5\times10^{-3}\text{ kN} = 10.5\text{ N}$$

Transmissibility Ratio $\text{TR} = \dfrac{10.5}{285} = 0.037$

Isolation Efficiency (Equation 4.1-8)

Isolation Efficiency of Individual Isolator $\eta = (1-0.037)\times100 = 96\ \%$

Example D 12.2 Vibration Isolation of a Crusher Foundation

Design Vibration Isolation System for Crusher Foundation as shown in Figure D 12.2-1. Machine and Foundation data is given as under:

Machine Data

Machine mass		
Mass of the Crusher	18.00	t
Mass of the Material in the crusher	3.65	t
Mass of the Coupling	5.20	t
Mass of the Motor	6.30	t
Total Machine Mass m_m	33.15	t

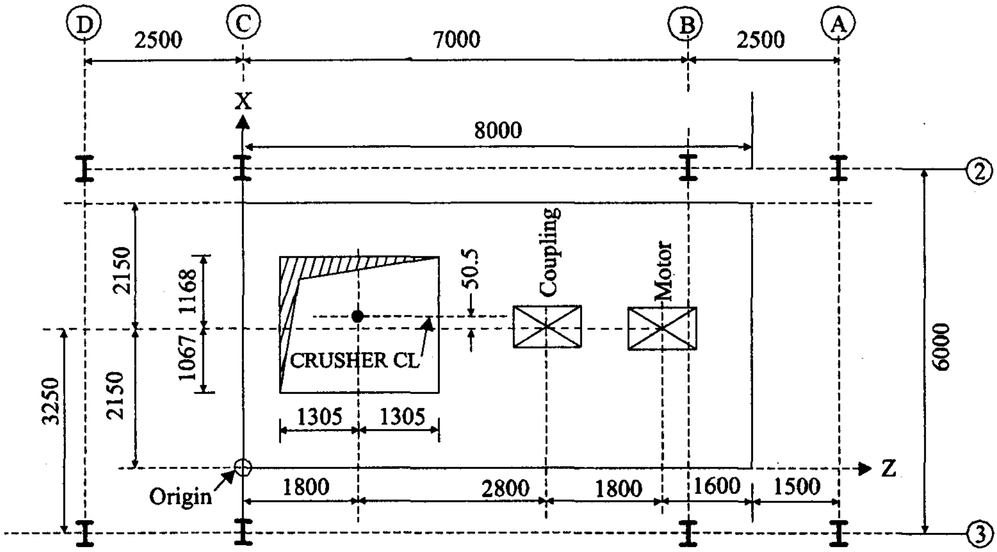


Figure D 12.2-1 Crusher Foundation Supported over Steel Structure @ Elevation of 12.75 m

Dynamic Force

Mass of Crusher Rotor	8.40	t
Mass of Motor Rotor	3.20	t
Motor rotor speed	720	rpm
Crusher speed	720	rpm
Crusher rotor unbalance grade	G16	
Motor rotor unbalance grade	G16	
Hammer tip diameter (in position)	1.220	m
Mass of hammer	41.0	kg
Max. Unbalance - one hammer missing	100	kN
Max. Unbalance - two hammer missing	200	kN
Max. Unbalance - three hammer missing	300	kN

CG of mass location (Above top of RCC Block)

Motor	0.80	m
Crusher	1.00	m
Material	1.00	m
Coupling	1.00	m

Centre to Centre distances

Between Crusher and coupling	2.8	m
Between Motor and coupling	1.8	m
Eccentricity between crusher rotor and motor rotor	50.5	mm

DESIGN:

Machine Data (Mass)

Crusher = 18 t; Material = 3.65 t; Coupling = 5.2 t; Motor = 6.3 t

Total Machine Mass 33.15 t

Sizing of Foundation

Foundation Data: Inertia block Plan, Section and Elevation are shown in Figure 12.2-1

Inertia Block size

Length = 8.00 m; Width = 4.30 m;

Opening = 2.61m (along length) x 2.235 m (along width)

Thickness of the inertia block is restricted to 1.00 m at Isolator Support Location and thickness in the middle portion can be higher. As initial proposal, thickness considered is shown in Figure 12.1-2 and is given as under:

Sides supporting mounts (as shown in Figure) 0.8 m wide 1.00 m thick

Middle portion (all along length) 1.20 m thick

Foundation Material properties

Concrete Grade M30

Elastic Modulus = 30,000 MPa; Mass Density = 2.5 t/m^3

Foundation mass

$$m_f = (\{(4.3 - 2\times0.8)\times8\times1.2\} + \{2\times0.8\times8\times1.0\} - \{2.61\times2.235\times1.2\})\times2.5 = 79.3 \text{ t}$$

Ratio of Foundation mass to Machine mass 79.3/33.15 = 2.4

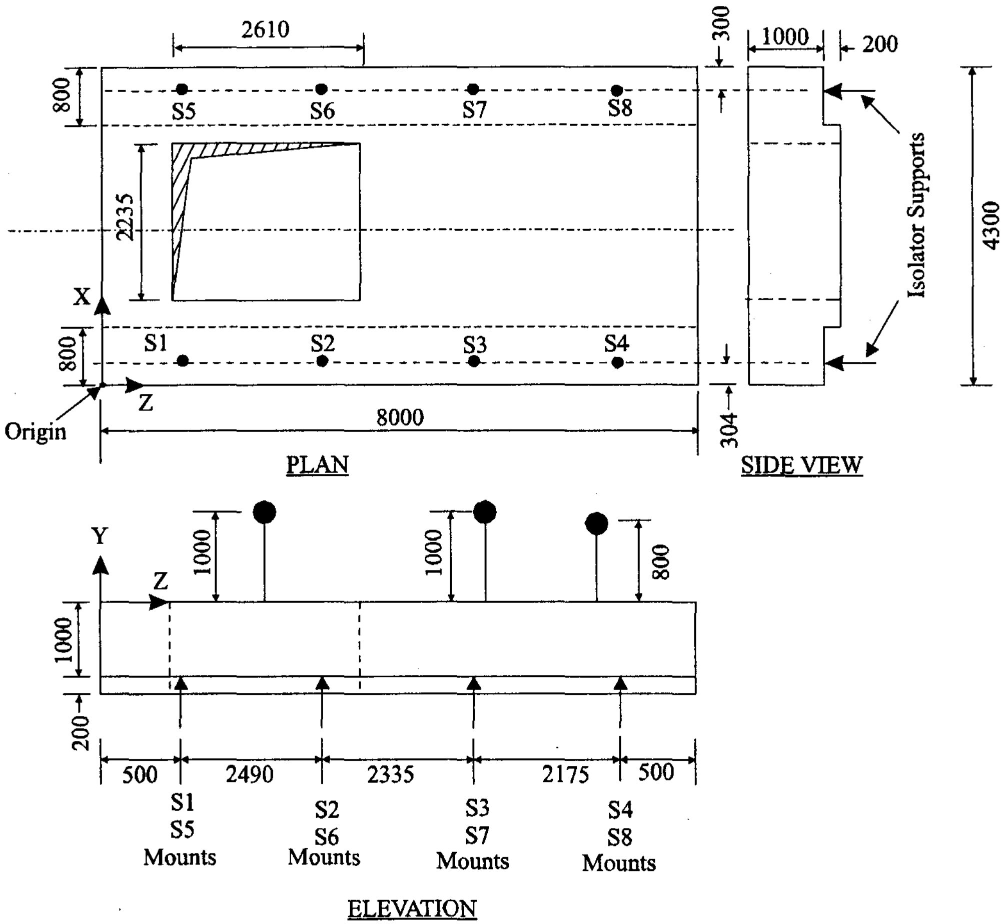


Figure D 12.2-2 Crusher Foundation - Slab Thickness and Isolation Mount Support Locations

Overall Centroid (Centre of Mass with respect to Origin)

Let us denote Overall centroid as point C. Consider Origin @ Left hand corner of the top deck as shown in Figure 12.2-1. Details of calculation steps are not given. Following the procedures similar to Example D12.1, we get coordinates of centroid with respect to origin as:

Foundation Centroid $m_f = 79.3$ t; $\bar{x}_f = 2.14$ m; $\bar{y}_f = -0.56$ m; $\bar{z}_f = 4.49$ m

Machine Centroid $m_m = 33.15$ t; $\bar{x}_m = 2.18$ m; $\bar{y}_m = 0.96$ m; $\bar{z}_m = 3.10$ m

Overall centroid $m = 112.45$ t; $\bar{x} = 2.15$ m; $\bar{y} = -0.11$ m; $\bar{z} = 4.08$ m

Here $m_f, m_m \& m$ represent Mass of foundation, machine and total mass respectively and $\bar{x}_f$, $\bar{y}_f, \bar{z}_f$ & $\bar{x}_m$, $\bar{y}_m, \bar{z}_m$ represent CG coordinates of foundation and machine elements with respect to Origin in X, Y & Z direction respectively. Terms $\bar{x}$, $\bar{y} \& \bar{z}$ represent overall centroid with respect to Origin along X, Y & Z directions respectively.

Mass Moment of Inertia about overall centroid (Center of Mass) (tm^2)

Following procedure similar to that for Example D 12.1, we get

$m = 112.45$ t; $M_{mx} = 624.47$ tm^2; $M_{my} = 693.52$ tm^2; $M_{mz} = 194.55$ tm^2

Here $M_{mx}, M_{my} \& M_{mz}$ represent Mass Moment of Inertia at centroid about X, Y & Z axes respectively

Isolator Data

Total weight of Machine + Foundation $112.45 \times 9.81 = 1103$ kN

No. of mounts 8

Load per mount $1103/8 = 138$ kN

Machine speed 720 rpm = 75.4 rad/s

Target Isolation Efficiency $\eta = 90$ to 93%

Frequency ratio for $\eta = 90$ to 93% (Table 4.1-1) 3.2 to 4

Let us select frequency ratio = 4

Isolation Frequency 75.4/ 4 = 19 rad/s

Corresponding deflection $\sqrt{\frac{g}{\delta_y}} = 19$; $\delta_y = \frac{9810}{(19)^2} = 27.17$ mm

Let us select deflection as 30 mm

Required vertical stiffness $138/30 = 4.6$ kN/mm

At this point one has to look for manufacturer's catalogue for supply of isolators.

Specs for nearest available isolator (From the catalogue) are:

Rated Load 159 kN

Effective Deflection 30 mm

Vertical Stiffness	5.3 kN/mm
Horizontal Stiffness	4.24 kN/mm
Damping Vertical	24.0 kN/m/s
Damping Horizontal	10.7 kN/m/s

Location of Spring Isolators

Spring Isolators are so located such that **Centre of Stiffness** (springs) matches closely with **Centre of Mass** (machine + foundation). Final locations of Isolators S1 to S8, given as under, are shown in Figure 12.2-2.

Along Z direction:

Isolator #	S1& S5	S2 & S6	S3 & S7	S4 & S8
Distance from origin (m)	0.50	2.99	5.325	7.50

Along X direction

Isolator #	S1, S2, S3, S4	S5, S6, S7, S8
Distance along X from origin (m)	0.304	4.00

Center of Stiffness from origin:

Computing on the same lines as for D 12.1, we get

$$k = \sum k_{yi} = 5.3 \times 8 = 42.4 \text{ kN/mm}; \; \bar{x}_k = \frac{\sum (k_{yi} x_{ki})}{\sum k_{yi}} = 2.15 \text{ m}; \; \bar{z}_k = \frac{\sum (k_{yi} z_{ki})}{\sum k_{yi}} = 4.078 \text{ m}$$

Here k_{yi} represent vertical stiffness of isolator and x_{ki} & z_{ki} represent its distance from origin. Terms $\bar{x}_k$ & $\bar{z}_k$ represent centroid of isolators with respect to origin.

Eccentricity between Center of Mass (Centroid) and Center of Stiffness

Comparing with Coordinates of Overall centroid with respect to origin $\bar{x} = 2.15$ m; $\bar{z} = 4.08$ m, we get:

$$e_x = 0; \quad e_z = \frac{4.08 - 4.078}{8} \times 100 = 0.025 \ \%$$

Dynamic Analysis

Overall Translational and Rotational Stiffness at point of Center of Stiffness

$$k_x = \sum k_{xi} = 8\times 4.24 = 33.92 \text{ kN/mm } = 3.392\times 10^4 \text{ kN/m}$$
$$k_y = \sum k_{yi} = 8\times 5.3 = 42.4 \text{ kN/m } = 4.24\times 10^{34} \text{ kN/m}$$
$$k_z = \sum k_{zi} = 8\times 4.24 = 33.92 \text{ kN/mm } = 3.392\times 10^4 \text{ kN/m}$$

$$\begin{aligned} k_\theta &= \sum k_{yi}\left(\bar{z}_k - z_{ki}\right)^2 \\ &= 2\times 5.3\times 10^3 \times \left\{(4.078-0.5)^2 + (4.078-2.99)^2 + (4.078-5.325)^2 + (4.078-7.5)^2\right\} \\ &= 2.89\times 10^5 \text{ kNm/rad} \end{aligned}$$

$$\begin{aligned} k_\phi &= \sum k_{yi}\left(\bar{x}_k - x_{ki}\right)^2 = 4\times 5.3\times 10^3 \times \left\{(2.15-0.304)^2 + (2.15-4)^2\right\} \\ &= 1.45\times 10^5 \text{ kNm/rad} \end{aligned}$$

$$\begin{aligned} k_\psi &= \sum k_{xi}\left(\bar{z}_k - z_{ki}\right)^2 + \sum k_{zi}\left(\bar{x}_k - x_{ki}\right)^2 \\ &= 2\times 4.24\times 10^3 \times \left\{(4.078-0.5)^2 + (4.078-2.99)^2 + (4.078-5.325)^2 + (4.078-7.5)^2\right\} \\ &\quad + 4\times 4.24\times 10^3 \times \left\{(2.15-0.304)^2 + (2.15-4)^2\right\} \\ &= 3.47\times 10^5 \text{ kNm/rad} \end{aligned}$$

Natural Frequency

$$p_x = \sqrt{\frac{k_x}{m}} = \sqrt{\frac{3.392\times 10^4}{112.45}} = 17.36 \text{ rad/s}$$
$$p_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{4.24\times 10^4}{112.45}} = 19.42 \text{ rad/s}$$
$$p_z = p_x = 17.36 \text{ rad/s}$$
$$p_\theta = \sqrt{\frac{k_\theta}{M_{mx}}} = \sqrt{\frac{2.89\times 10^5}{624.47}} = 21.5 \text{ rad/s}$$
$$p_\psi = \sqrt{\frac{k_\psi}{M_{my}}} = \sqrt{\frac{3.47\times 10^5}{693.52}} = 22.37 \text{ rad/s}$$
$$p_\phi = \sqrt{\frac{k_\phi}{M_{mz}}} = \sqrt{\frac{1.45\times 10^5}{194.55}} = 27.3 \text{ rad/s}$$

Amplitudes

Dynamic Force

Crusher Unbalance Force

Mass of Crusher Rotor 8.40 t

Excitation frequency (720 rpm) 75.40 rad /s

Balance Grade G16

Dynamic Force $F = 8.4 \times (16 \times 10^{-3}) \times 75.4 = 10.17\ \text{kN}$

Point of application Crusher CG $x_i = 2.2\ \text{m}; y_i = 1.0\ \text{m}; z_i = 1.8\ \text{m}$

Dynamic Force acts in X & Y direction (one at a time) $F_x = 10.17\ \text{kN};\ F_y = 10.17\ \text{kN}$

Coordinates of Overall centroid $\bar{x} = 2.15\ \text{m};\ \bar{y} = -0.11\ \text{m};\ \bar{z} = 4.08\ \text{m}$

Transferring Force at overall centroid, we get

$F_y = 10.17\ \text{kN};\ M_\phi = 10.17 \times (2.15 - 2.2) = -0.51\ \text{kNm};\ M_\theta = 10.17 \times (4.08 - 1.8) = 23.18\ \text{kNm}$

OR

$F_x = 10.17\ \text{kN};\ M_\phi = 10.17 \times (-.11 - 1) = -11.3\ \text{kNm};\ M_\psi = 10.17 \times (4.08 - 1.8) = 23.18\ \text{kNm}$

Motor Unbalance Force

Rotor Mass 3.2 t

Excitation frequency (720 rpm) 75.4 rad /s

Dynamic Force $F = 3.2 \times (16 \times 10^{-3}) \times 75.4 = 3.86\ \text{kN}$

Point of application at motor CG $x_i = 2.15\ \text{m}; y_i = 0.8\ \text{m}; z_i = 6.33\ \text{m}$

Dynamic Force acts in X & Y direction (one at a time) $F_x = 3.86\ \text{kN};\ F_y = 3.86\ \text{kN}$

Transferring Forces at overall centroid, we get

$F_y = 3.86\ \text{kN};\ M_\phi = 3.86 \times (2.15 - 2.15) = 0.0\ \text{kNm};\ M_\theta = 3.86 \times (4.08 - 6.33) = -8.7\ \text{kNm}$

OR

$F_x = 3.86\ \text{kN};\ M_\phi = 3.86 \times (-.11 - 0.8) = -3.5\ \text{kNm};\ M_\psi = 3.86 \times (4.08 - 6.33) = -8.7\ \text{kNm}$

Total Force and Moment @ Centroid due to Motor and Compressor

$F_y = 14$ kN; $M_\phi = -0.5$ kNm; $M_\theta = 14.5$ kNm

OR

$F_x = 14$ kN; $M_\phi = -14.8$ kNm; $M_\psi = 14.5$ kNm

Steady State Amplitudes

Steady State Amplitudes are obtained using procedure as that for Example D12.1.

Force F_y

$$F_y = 14 \text{ kN}; \ k_y = 4.24\times10^4 \text{ kN/m}; \ \delta_y = \left(14/4.24\times10^4\right) = 3.3\times10^{-4} \text{ m}$$

$$p_y = 19.42 \text{ rad/s}; \ \omega = 75.4 \text{ rad/s}; \ \beta_y = 3.88; \ \frac{1}{\left|1-\beta_y^2\right|} = 0.071 \text{ m}$$

Amplitude $\mathrm{y_c} = 3.3\times10^{-4}\times0.071 = 2.35\times10^{-5}$ m

Moment M_ϕ

$$M_\phi = -0.49 \text{ kNm}; \ k_\phi = 1.45\times10^5 \text{ kNm/rad}; \ \delta_\phi = \left(-0.49/1.45\times10^5\right) = -3.37\times10^{-6} \text{ rad}$$

$$p_\phi = 27.27 \text{ rad/s}; \ \omega = 75.4 \text{ rad/s}; \ \beta_\phi = 2.76; \ \frac{1}{\left|1-\beta_\phi^2\right|} = 0.15$$

Amplitude $\phi_\mathrm{c} = -3.37\times10^{-6}\times0.15 = -5.07\times10^{-7}$ rad

Similarly for $M_\theta = 14.5$ kNm, we get amplitude as $\theta_\mathrm{c} = 4.43\times10^{-6}$ rad

Combining the amplitudes due to F_y, M_ϕ & M_θ, we get

$$y_{\max} = \left\{\left|2.35\right|\times10^{-5} + \left|(-5.07\times10^{-7})\times2.15\right| + \left|(4.43\times10^{-6})\times4.08\right|\right\}\times10^6 = 42.65 \text{ microns}$$

Similarly for another force set of F_x, M_ϕ & M_ψ, we get

$$x_{\max} = 39.56 \text{ microns}; \ y_{\max} = 33.15 \text{ microns}; \ z_{\max} = 8.65 \text{ microns}$$

Support reactions are obtained using procedure as that for Example D12.1.

Force due to 3 hammer loss

This is a faulted condition. The amplitudes thus obtained should be used only to check operational safety only.

Dynamic Force of 300 kN acts in X & Y direction (one at a time) at Excitation frequency 720 rpm

Crusher CG Coordinates $x_i = 2.2$ m; $y_i = 1.0$ m; $z_i = 1.8$ m

Transferring Forces at overall centroid (on the same lines as above), we get

$F_y = 300$ kN; $M_\theta = 683$ kNm; $M_\phi = -14.6$ kNm
$F_x = 300$ kN; $M_\psi = 683$ kNm; $M_\phi = -333$ kNm

Computing on the similar lines, we get amplitudes as

Due to Force along X $x_{max} = 1.27$ mm; $y_{max} = 0.7$ mm; $z_{max} = 0.4$ mm
Due to Force along Y $y_{max} = 1.4$ mm

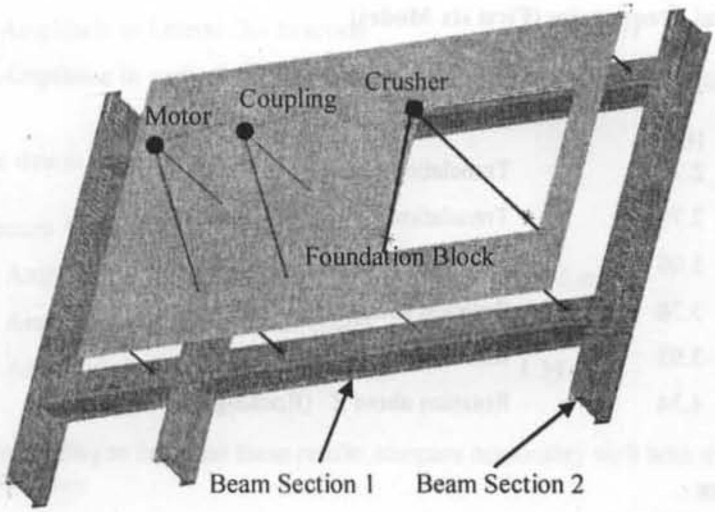


Figure D 12.2-3 Model of Crusher Foundation with support structure

FE Analysis

The foundation has been analyzed using standard FE Package. The foundation is modeled using Shell elements. Rigid links have been used to locate machine mass over the RCC block at specified height. Machine mass is lumped at its CG location. Isolation mount at each support location is represented by three spring elements with appropriate stiffness properties in X, Y & Z directions.

Free vibration analysis yields natural frequencies and mode shapes. Steady state response has been computed for unbalance forces due to crusher and motor. Amplitudes are also computed for forces due to faulted condition i.e. 3 hammer loss condition. Transient Response is obtained to simulate start-up and shut-down conditions.

FE Model of the Foundation and Frequency Plots

Crusher Foundation with steel support structure is shown in Figure D 12.2-3. FE model of the foundation supported on Isolators considering support structure as rigid is shown in Figure D 12.2-4. Frequencies and mode shapes (first six modes) are as shown in Figure D 12.2-5. Higher mode frequencies are much beyond operational range and are not of much interest from isolation point of view.

Results of FE Analysis

Table D 12.2-1 Natural Frequencies (First six Modes)

Mode #	Frequency		Mode
	rad/s	Hz	
1	17.52	2.79	Translation along X (Lateral)
2	17.52	2.79	Translation along Z (Longitudinal)
3	19.40	3.09	Translation along Y (Vertical)
4	23.78	3.78	Rotation about X (Rocking)
5	24.69	3.93	Rotation about Y (Torsion)
6	27.28	4.34	Rotation about Z (Rocking)

Crusher Unbalance Force

Maximum Amplitudes

Max Amplitude in Lateral (Z) direction	32 microns
Max Amplitude in Lateral (X) direction	34 microns
Max Amplitude in Vertical (Y) direction	54 microns

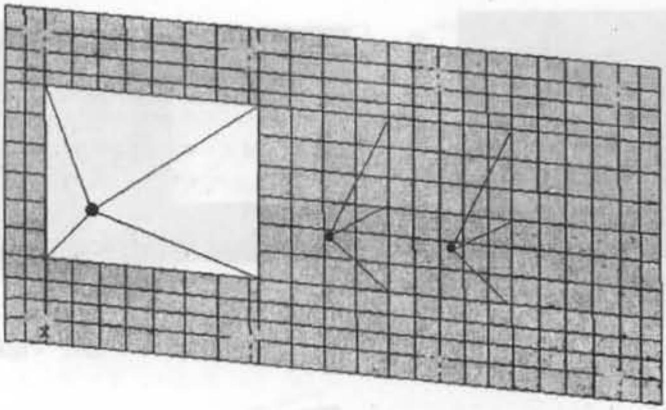


Figure D 12.2-4 Crusher Foundation on Isolators –Support Structure Rigid

Motor Unbalance Force

Maximum Amplitudes

Max Amplitude in Lateral (Z) direction	12 microns
Max Amplitude in Lateral (X) direction	13 microns
Max Amplitude in vertical (Y) direction	20 microns

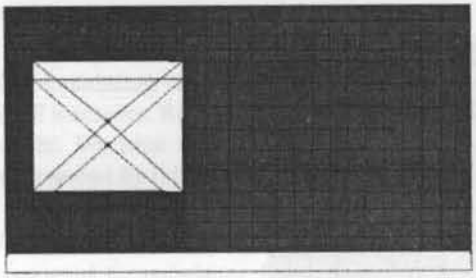
Force due to 3 hammer loss

Maximum Amplitudes

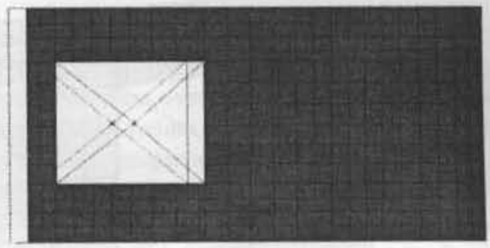
Max. Amplitude in Lateral (Z) direction	0.95 mm
Max. Amplitude in Lateral (X) direction	1.01 mm
Max. Amplitude in Vertical (Y) direction	1.34 mm

It is interesting to note that these results compare reasonably well with those obtained by manual computations.

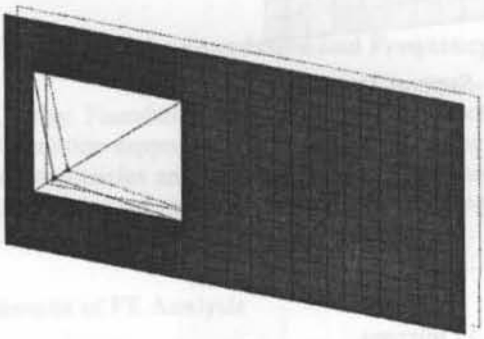
Stress and deflection due to 3 hammer loss force is shown in Figure D 12.2-6.



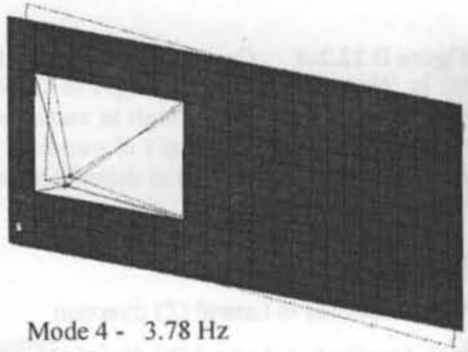
Mode 1 - 2.68 Hz



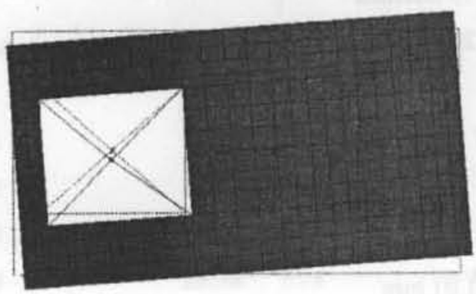
Mode 2 - 2.78 Hz



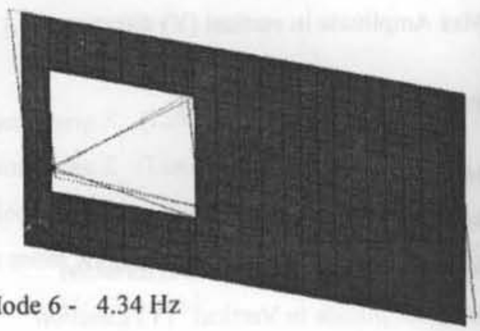
Mode 3 - 3.08 Hz



Mode 4 - 3.78 Hz



Mode 5 - 3.92 Hz



Mode 6 - 4.34 Hz

Figure D 12.2 5 Frequencies and Mode Shapes

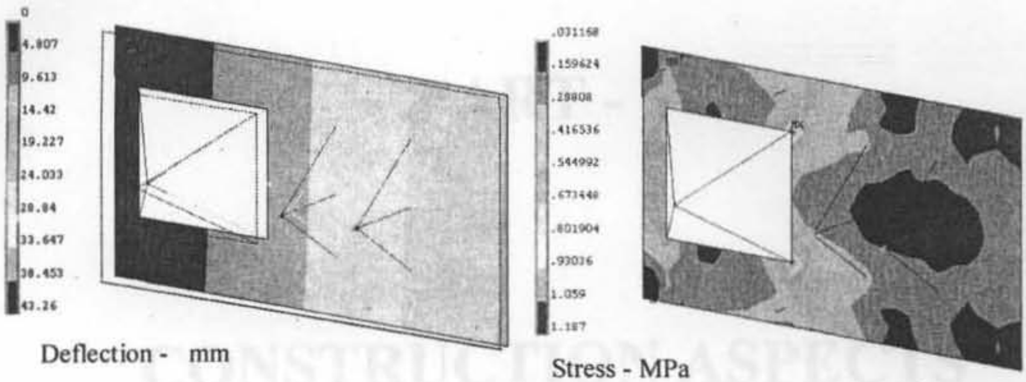


Figure D 12.2 6 3 Hammer Loss Force - 300 kN along Y

Transient Vibrations

During Start up and Shut down the machine passes through natural frequencies of the foundation. At each foundation frequency, the machine foundation system remains in transient resonance till the frequency crossover. The damping of the isolator plays a very significant role in controlling build up of high amplitudes.

The results of FE analysis show that maximum stress developed in the foundation due to three hammer loss force is of the order of 1.2 MPa. This is much below the allowable compressive stress of 8.5 MPa for un-reinforced concrete.

Strength Requirement of Support Structure

Foundation rests over isolators that in turn are supported by structural support system. Support structure stiffness should be at least 20 times that of the isolator for isolation system to be as effective as designed.

Frequency and mode shapes of Crusher Foundation with steel support structure are shown in Figure D 12.2-7. It is seen from the figure that first mode frequency is about 12 Hz which is same as operating speed. The member sizes of Support structure therefore need to be modified.

PART - V

CONSTRUCTION ASPECTS

&

CASE STUDIES

13

CONSTRUCTION ASPECTS

- Construction Joints
- Embedded Parts
- Placing / Laying of Concrete
- Grouting

CONSTRUCTION ASPECTS

Construction of a machine foundation is as important an aspect as design. Needless to mention that a machine foundation built using good construction practices becomes an asset for the industry and on the other hand, foundation built with improper construction practices becomes a liability for entire life of machine. This fact is known to one and all associated with the industry but not enough care is being exercised cap the practices responsible for shortcomings.

Honeycombing, porosity, out of plumb columns, improper bonding of embedded parts, opening up of construction joints, etc are some of the common items encountered during construction of a machine foundation. Patchwork repair of concrete beams and columns with sole objective of hiding faults and shortcomings is a common sight practically at every industrial set up. Whereas the executing agency walks away after carrying out the necessary repairs, it is the machine which has to live with the associated problems for its entire life. It then becomes a starting point of debate between customer /owner and manufacturer for all associated problems related to machine performance.

All these issues are a clear & direct pointer to;

- Lack of right infrastructure with the executing agency
- Inadequate supervision during construction
- Lack of clarity in communicating intricacies associated with each shortcomings
- Laxity in acceptance norms

It has been taken for granted that all possible **Engineering Details** required for casting a machine foundation deemed to have been transformed in to corresponding engineering drawings. All specifications regarding i) material i.e. concrete, reinforcing steel, structural steel (if any), ii) embedded parts, its material and fabrication, iii) cover to main reinforcement, etc are presumed to have been included in a ten line note placed in the body of the drawing. It is implied and taken for granted that the listed details are good enough to produce a flawless foundation. More often than not, this objective is not met and the end product i.e. foundation sufferers with one or the other form of shortcomings. It is therefore desirable that every drawing must address all the necessary issues explicitly and in detail. These are in addition to what is normally provided on each drawing. Some of the aspects that need attention are discussed hereunder.

The subject matter, in principle, is very broad and needs deliberation by experts of concrete technology and construction practices. An effort has been made to address the machine foundation related construction aspects in as explicit manner as possible. For more details readers may look for expert advice.

13.1 CONSTRUCTION JOINTS

Every construction joint must be a designed construction joint. Its specific location, details of shear dowels etc must be clearly marked on the drawing. The procedure of joining old concrete with fresh concrete covering all possible aspects like cleaning surface of old concrete, use of bonding material if any, etc must be clearly indicated in the drawing. An improper construction joint leads to change in natural frequencies of the foundation and that in turn reflects on the performance of machine. It is the designer who must analyse the implications of misbehaviour of construction joint and take due care while designing these construction joints.

13.2 EMBEDDED PARTS

Every machine foundation, especially frame foundation, has many embedded parts that are used to support auxiliary components, instrumentation, piping etc. Invariably it is noticed that a structural angle with lugs provided at suitable interval / spacing is placed at each corner of practically every column. More often than not, these lugs are welded to the reinforcing steel to hold these in position. Such a practice reflects non-engineered approach and in author's opinion must be discouraged. It indirectly speaks of no planning with regard to what to support and where to support. More often than not these angles lose contact with the main concrete and become a source of transmission of vibration from columns to supported piece of equipment.

Welding of lugs of embedded parts to reinforcing steel of beams, columns, deck it is highly undesirable. Due to shrinkage associated with concrete this process produces results in loss of contact between embedded parts and concrete and becomes a source of vibration especially when these embedded parts are machine seating plates. A note to this effect must clearly be included in the drawing.

Proper contact of seating plate with concrete bears more importance for turbo Generator sets. During Start-up and cooling cycle, turbine casing undergoes differential thermal expansion and contraction. Casing slides over seating plates and frictional force is transferred to the foundation through these seating plates. This basic process demands proper contact and bond between seating plates and concrete. Quality assurance plans must be drawn to highlight these issues

Wherever an embedded plate is just abutting the sides of the beam, more often than not, it ends up in loss of full contact with the concrete. This is perhaps due to inaccessibility to vibrator. Author has experienced such shortcomings at more than one sight. In such a case it is recommended to extend the shuttering in a tapered manner so as to get proper ligament along the edge for vibrator and remove the excessive material after opening up of the side shuttering while concrete is still green. A typical arrangement is shown in Figure 13.1-1. Author has used this arrangement and found it to be useful. The details shown are only recommendatory and not mandatory.

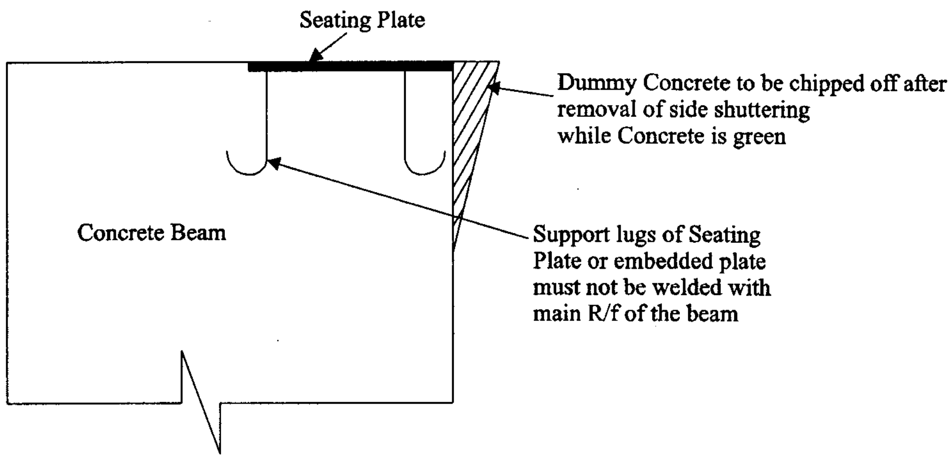


Figure 13.1-1 Sketch showing dummy concrete for proper embeddement of Seating Plate located abutting to beam edge

13.3 PLACING / LAYING OF CONCRETE

Quality plan: Quality plan of concrete production, laying of concrete, curing of concrete, field quality checks like slump, workability, cube strength etc must be drawn by the industry. Its reference must be made available to consultants for inclusion in all concerned drawings. The said document must address all the issues listed here under:

Cold Joints: All efforts must be made to avoid cold joints. While laying concrete in thick blocks like base raft / top deck of a frame foundation, large size block foundations etc, limitation on thickness layer, time lapse between laying two successive layers of concrete, must be specified. In absence of any other recommendation by the designer, the concrete layer height should be limited to 400 mm and time gap between laying two layers of concrete should be restricted to 25 – 30 minutes.

Segregation: Construction process for machine foundation, especially frame foundations, is a bit complex compared to normal building construction. Heights of columns, which are to be concreted in a single pour, are invariably larger than routine building construction jobs. Height of drop of concrete must explicitly be specified so as to avoid any segregation of concrete. This must also be specifically and clearly indicated in the design drawing.

Mass Concreting: Ambient temperature for laying concrete must be specified. In case concrete is laid in hot climate, concept of employing chillers must also be specified.

13.4 GROUTING

Various types of grouts are currently available to cover up shortcomings of concreting process. These are employed for filling honeycombs, cracks etc. These are also used for leveling under seating plates, filling up bolt pockets etc. Each grout compound has associated limitations that restrict its usage for all kind of environments. Specific Epoxy grouts show a significant variation of elastic modulus with temperature. Such grouts are not recommended for use in high temperature zones like area near HP turbine. Some grouts may show reaction with oil & chemical environments. These are unsuitable for grouting seating plates in zones where oils and chemicals are active.

14

CASE STUDIES

- Motor Compressor
- Turbo-Generator
- Reciprocating Compressor
- FD Fan

CASE STUDIES

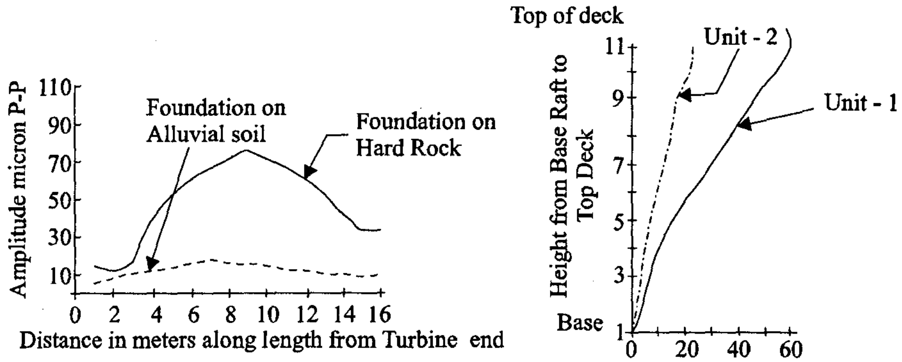
14.1 INTRODUCTION

There is enough to learn from the field performance of machine foundation systems. Feed back from failure analysis, on one hand, serves as a guiding tool to update design philosophy and on the other, builds confidence level in one's own design.

General understanding that similar machines on similar foundations and on similar soil would show similar performance behaviour no longer holds good. There are many variables linked to machine that may result in changed behaviour of one machine to another. Similarly identical foundations for two identical machines may also turn out to exhibit different behaviour because there are another set of variable parameters that may make one foundation behave differently than another. Figure 14.1-1 shows a set of vibration recordings for identical foundations for identical machines. Vibration measurements are shown on the foundation so as to include both machine and foundation aspects.

A uniform reduction of vibration amplitudes from top to bottom of bearing housing exhibits a healthy trend. In one of the case the trend was observed to be totally opposite i.e. vibration levels were found to have increasing trend from top to bottom of bearing housing. Records are shown in Figure 14.1-2.

During routine overhaul, bearing housing was removed from the supporting pedestal. The grout underneath the bearing seating plate was totally carbonized. It behaved like charcoal powdery cake having no strength and it was fully soaked in oil. It was perhaps due to chemical reaction of the grout with spilled oil. After re-grouting the pedestal the problem disappeared.



(a) Vibration Along length of Top Deck

Identical Machines on Identical Foundations
and on different soil conditions

(b) Vibration along Column Height

Identical Machines on Identical Foundations
and on same soil

Figure 14.1-1 Identical Machines having Identical Foundations

It is a common norm that vibration measurements are taken only on the machine components and inference about the health of the machine is derived from these records. It is author's considered opinion that foundation and its elements must also be included for such vibration measurements. Foundation ultimately is a part of the total system that is responsible for healthy performance of machine.

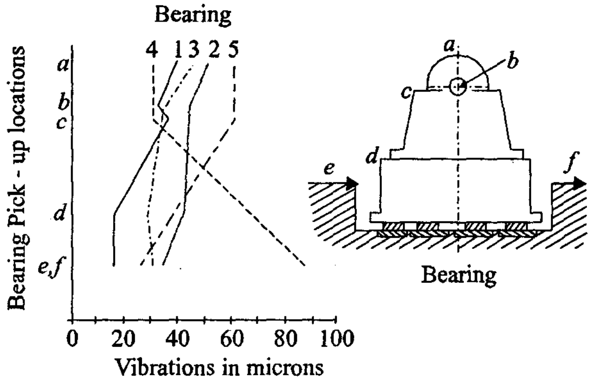


Figure 14.1 -2 Amplitude along Bearing Height 200 MW TG Foundation

Figure 14.1-3 shows records of vibration taken at the top deck of a 200 MW TG foundation deck. Only coast-up records are shown here. Figure shows high deck vibrations to the tune of 110 microns at about 40 Hz. This indirectly hints that there exist structural frequencies of the deck below operating speed. If these frequencies happen to coincide with rotor critical speeds or with sub-harmonics of engine frequency, it may result in increased vibration levels where ever these become governing parameters.

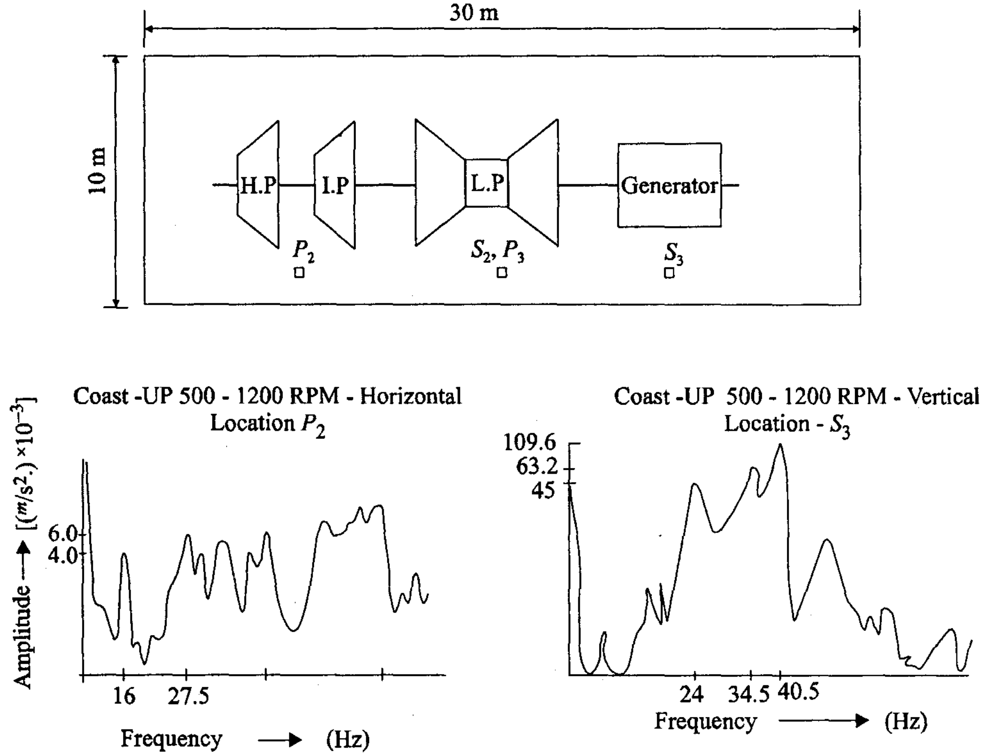


Figure 14.1-3 200 MW Turbo-Generator Unit - Coast Up Measurements at Top Deck

It is also true that in that era when these foundations were designed, computational tools were not available to evaluate foundation response to such level of specifications. But such tools are now easily available. It is strongly recommended that current design philosophy and practices must be in line with the available technology. One should not stick to age old methods of designing machine foundations

Author has spent nearly 3 decades in designing of foundations and handling critical failure analysis cases starting from designing, field measurements, fault diagnosis and remedial measures. A few case studies are presented here. Each case study is a typical in itself.

Study is presented here only to demonstrate that in 9 out of 10 cases, the blame always goes to machine manufacturer. In 50 % of the cases, the source of the problem may not be machine alone and solution may lie somewhere else. It requires **Right Attitude** to tackle all failure problems with of course **Right Team of Experts** armed with **Right Instrumentation**.

14.2 CASE STUDIES

Example 14.2 High Vibrations of a Motor Compressor Unit

Preamble: High vibrations were reported from a motor compressor unit of a plant. Schematic representation of the motor compressor unit is shown in Figure 14.2-1. Vibrations higher than permissible were reported on the foundation as well as on the motor and compressor units. As usual, manufacturers for the motor and compressor units were different agencies. Though all possible corrective steps were taken by both the manufacturers but the problem was not resolved. Each manufacturer was blaming the other for high vibrations.

Action Plan: Vibration measurements were recorded at various pick-up locations, marked on the Figure 14.2-1, both on the foundation as well as on the machine including connected piping. Recorded vibration levels (Max value in microns) on foundation, base frame, equipment and connected piping are as under:

	X	Y	Z
	Longitudinal	Transverse	Vertical
Foundation			40
Base frame Channel	40	20	25
Base Plate			60 / 420
Motor		150	125
Tank	600	150	90
Compressor	500	300	500
NRV	1500	1000	100
Piping		2600	800

Step 1: High base plate vibration indicated loosening of foundation bolts. Close examination revealed broken welding of support lug to base frame. Repair of support lug and tightening foundation bolts brought down vibration level from 420 microns to 130 microns. This also brought down compressor vibration levels to 250 /300 /250 microns in X /Y /Z direction.

Step 2: High piping vibration levels suggested need to isolate NRV. Isolation of NRV resulted in drastic reduction of vibration all through. NRV levels were still around 200 microns.

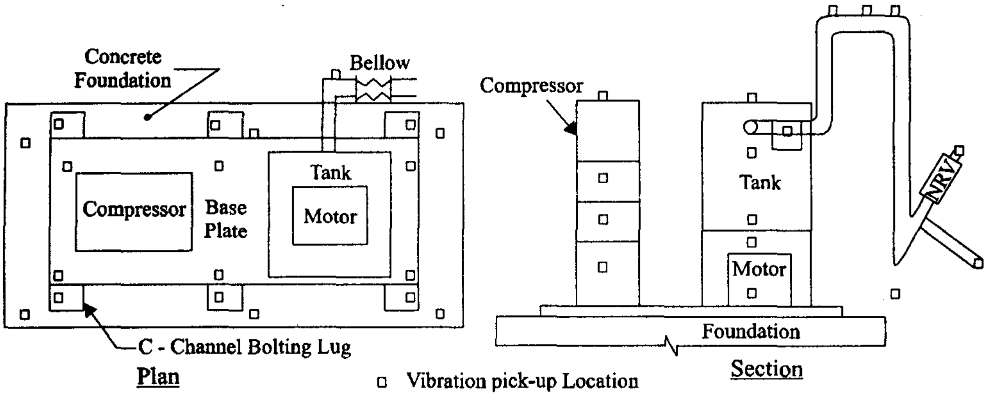


Figure 14.2-1 Motor Compressor Unit on a Block Foundation

Example 14.3 210 MW Turbo-Generator unit –High Vibration Problems

Dissimilar vibration behaviour of two Identical Units on Identical Foundations placed next to each other

Preamble: Machine is running with high vibrations since a decade and a half. High column vibrations as well as high deck vibrations are reported in one unit whereas another identical unit just adjacent to it is reported running satisfactorily. Cracks at the column top below deck have also been observed. These are perhaps locations of construction joint of column with top deck. Top deck Plan is shown in Figure 14.3-1.

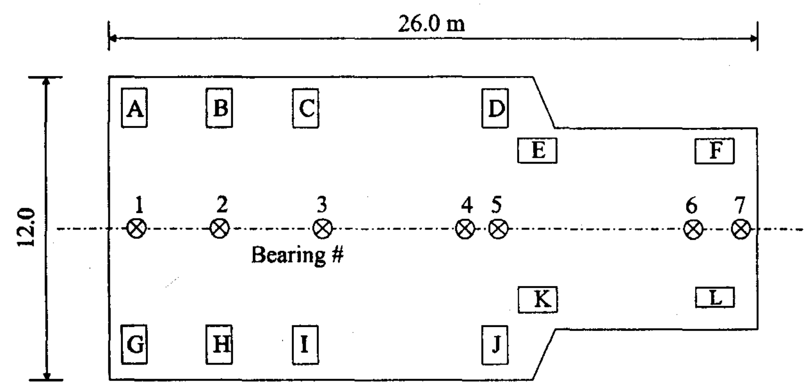


Figure 14.3-1 A 210 MW T G Foundation (Typical) - Top Deck Plan Showing Column and Bearing Locations

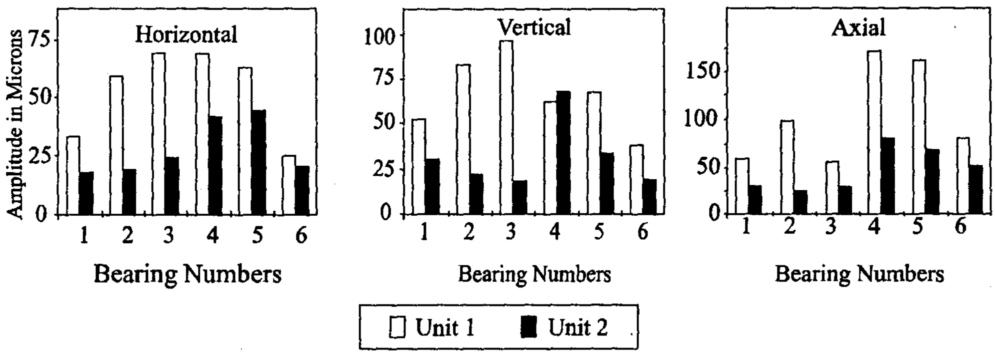


Figure 14.3-2 Bearing Level Amplitudes Turbo Generator Units 1 & 2

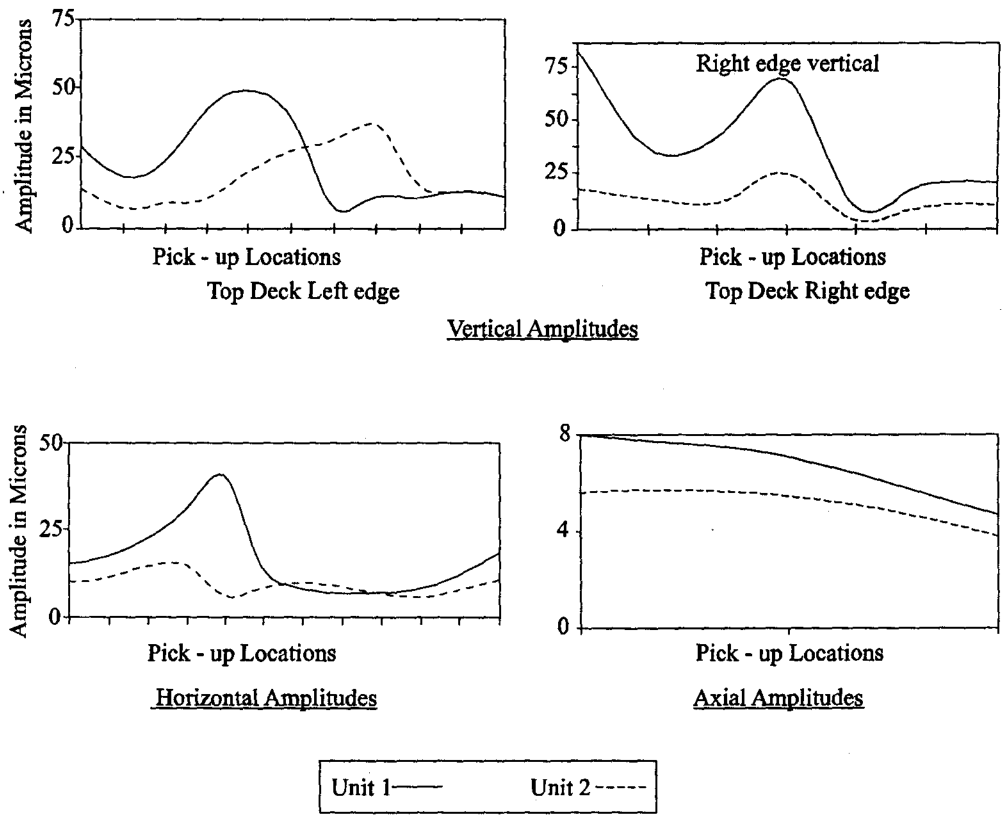


Figure 14.3-3 Amplitude at Top Deck Along Length from HP Side to Exciter Side

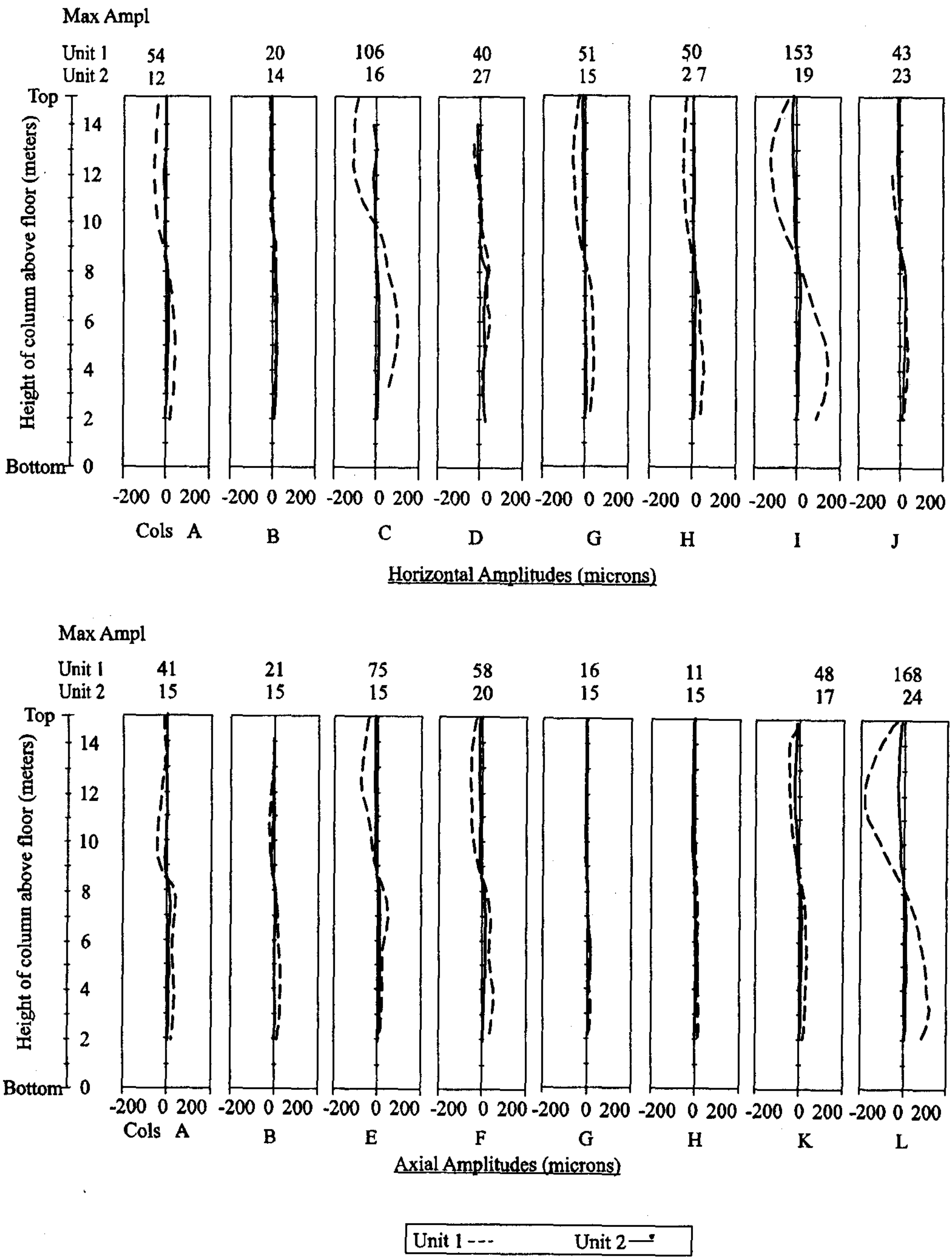


Figure 14.3-4 Amplitude Variation in Columns Along Height

Action Plan: In order to fix magnetic probes on the foundation, metal washers were fixed at 1/4th, 1/2 & 3/4th height of all the columns as well as along top deck (both on left and right side of the machine). In addition, metal washers were also fixed at specific locations of interest at beam bottom. Vibration measurements were taken at all these points in addition to bearings.

Bearing Vibrations are shown in Figure 14.3-2 and top deck vibrations are shown in Figure 14.3-3. Column vibration measurements along height of column are shown in Figure 14.3-4.

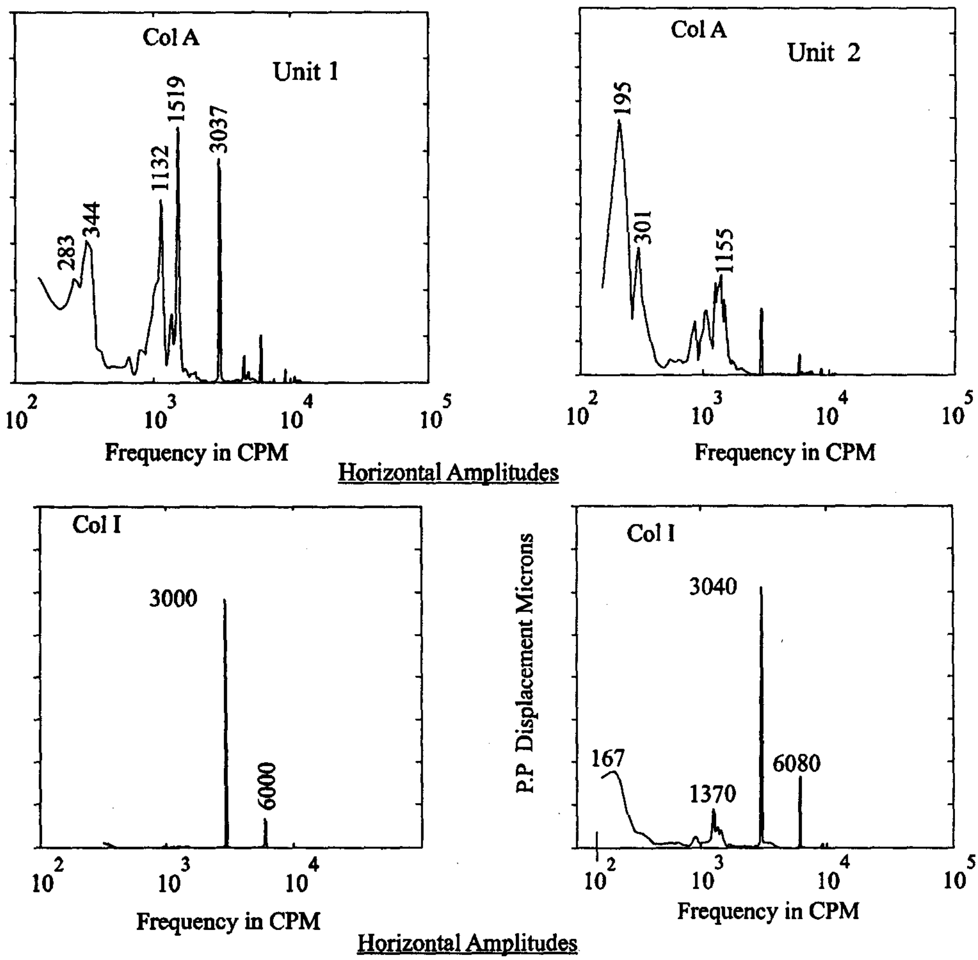


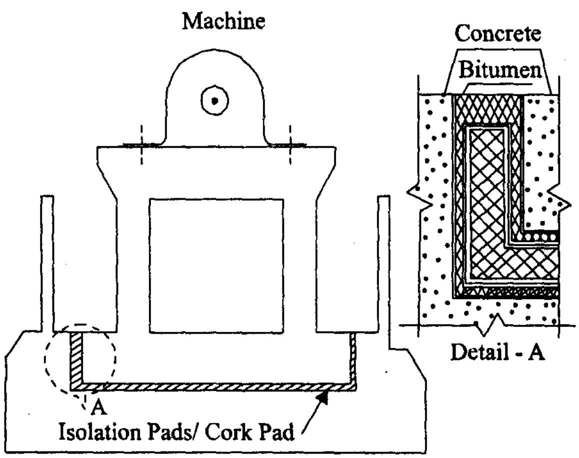
Figure 14.3-5 Column Amplitudes - FFT Record Unit 1 & 2

Observations: Ratio of bearing amplitudes of two units is of the order of 3 whereas ratio of top deck amplitudes is of the order of 2. It is a wild guess and a question mark whether cracking of the columns at top (close to deck bottom (soffit of beam) is responsible for such a behaviour or such high vibrations have resulted in cracking of the column? Visual examination of the column vibration and plot of amplitudes indicate that its 2^{nd} mode frequency of some of the individual columns in transverse direction has tendency to be in resonance with operating speed. Similar behaviour is also noticed for some columns in axial direction.

Figure 14.3-5 shows FFT analysis of column vibration records. FFT analysis of records confirm to the above observation. It is seen that the resonant frequency of some of the columns lie close to 50 Hz which is the machine running speed. Though the amplitude levels are low, the trend is not healthy. It is primarily because the analytical tools available about two decades back were not adequate to carry out such a detailed analysis and moreover the need was neither emphasized by the owner / customer nor by machine manufacturer.

Example 14.4 Reciprocating Compressor on Isolation pads

Preamble: A reciprocating compressor on a frame foundation is to be located inside a plant building. The forces developed by compressor are extremely high and supporting the foundation over the soil results in excessive amplitudes of vibration. The size of the base raft also can not be increased because of restrictions imposed by other structural foundation. The only options were i) either to strengthen the soil by whatsoever possible means, ii) resort to pile supported foundations or iii) use strong stiffness material underneath the base of the foundation so as to limit the amplitudes within permissible levels. This is with reference to the design problem discussed in Chapter 10. The decision by the company was to resort to isolation technique and design the foundation.



Action Plan: A common raft was provided spanning across width of the building. Compressor foundation was placed over the raft with Cork as Isolation device so as to minimize transmission of forces from machine to the common foundation. This system was designed by the author in 1974. Schematic arrangement is shown in Figure 14.4 -1. With this arrangement Machine was installed and has run satisfactorily keeping the amplitudes within permissible limits.

Cork thickness of 75 mm below the frame foundation was found to be adequate. Cork properties were tested at one of the national laboratory. Recommended values for computation are as under:

- Compressive strength — 500 kN/m^2
- Elastic Modulus (Static) — 10 MPa
- Elastic Modulus (Dynamic) — 15 MPa
- Coefficient of Uniform Compression — $C_u = 20 \times 10^4$ kN/m^3

The concept was used once again by the author, in 1979, for design of a Frame Foundation for a Gas Turbine. The need arose because of slip during planning. While making the layout, not adequate space was left to accommodate the GT. Unlike previous case, the machine is a high rpm machine and it was rather easy to get the required frequency ratio for achieving desired isolation.

The success has given improved confidence level for such designs.

Example 14.5-1 Vibration Isolation of FD Fan Foundation

Preamble: Vibration Isolation design of FD fan is discussed in Chapter 12. After machine installation, high vibrations were reported from the site.

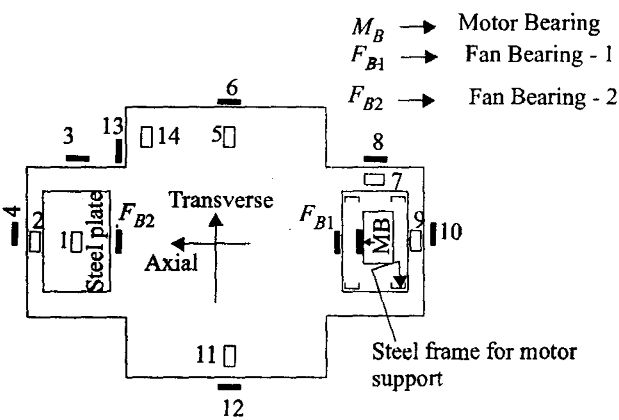


Figure 14.8-1 Pick up Location at Foundation Top and Sides

Action Plan: Vibrations were recorded at all pick up location points on the top as well as on sides of inertia block (Foundation). Pick up locations are as shown in Figure 14.5 -1. Here MB refers to motor bearing, FB1 refers to fan bearing 1 at drive end and FB2 refers to fan bearing 2 at non-drive end.

Recorded amplitudes are:

Vertical amplitude @ pick up # 1, 2, 7 & 8	about 600 microns
Axial amplitudes @ pick up # 4 & 10	50 microns
Transverse amplitudes @ pick up # 5 & 8	100 microns
Vertical amplitudes @ Steel frame for motor, fan etc	600 microns

High vertical vibration led to the conclusion that desired frequency ratio is not being achieved as designed. After a thorough examination, it emerged that the connection from fan to air outlet duct is rigid. It restrains motion of the fan and thereby motion of inertia block. The system is not able to achieve desired frequency ratio for isolation to be effective. After bellow was introduced between fan and the duct, vibration levels reduced drastically.

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FOUNDATIONS FOR INDUSTRIAL MACHINES

Handbook for Practising Engineers

The author has been engaged in design, testing and review of machine foundations for various industrial projects viz. Petrochemicals, Refineries, Power plants etc. for over three decades. The author has been associated with **Failure Analysis Studies** on various types of machines for over two decades and has conducted extensive tests on machine foundation models as well as on prototypes. This handbook shares author's long experience on the subject and focuses on the improvements needed in the design process with the sole objective of making practising engineers to have better perspective of the dynamics of machine foundation system.

The handbook covers basic fundamentals necessary for understanding and evaluating dynamic response of machine foundation system. It is anticipated that this handbook shall serve as a Reference Book for several industry segments like power, petrochemical, refineries, sugar, steel, cement, textile, fertilizer, etc. The author is confident that it shall bridge the knowledge gap and shall be beneficial to practising engineers, students, academicians/researchers as well to the industry in general.

The text is divided in to five parts spread over fourteen chapters. Part I takes care of Theoretical Aspects; Part II caters to Design Parameters of Sub-grade, Machine & Foundation; Part III deals with design of Foundations for Real Life Machines; Part IV caters to Design of Foundations with Vibration Isolation System & Part V caters to Construction Aspects and Case Studies related to machine foundation system.

Dr. K. G. Bhatia, after obtaining Bachelors Degree in Civil Engineering from BHU, (presently Institute of Technology, BHU, India), completed Masters in Structural Dynamics with specialization in Earthquake Engineering from University of Roorkee (presently Indian Institute of Technology, Roorkee) and thereafter joined International Institute of Seismology & Earthquake Engineering (IISEE), Tokyo, Japan under UNESCO for advanced research. The research work was further extended for award of PhD from Indian Institute of Technology, Delhi. He switched over to industry in 1971 and after serving Engineers India Ltd, a pioneering Consultancy Organization, he moved to power giant BHEL where he was Head of Structural Dynamics for many years. He has about three decades of rich experience in design, testing and failure analysis of machine foundations, seismic qualification of power plant equipment as well as stress & vibration analysis of structures and equipment.

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